Quantum Control Systems

From the von Neumann Architecture to Quantum Computing

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Abstract. In this paper we consider the (state) reachability and controllability problems of special two-level quantum systems, the so-called quantum bits via externally applied electro-magnetic field. The system is described by special a bi-linear right-invariant model whose state varies on the Lie group of 2×2 special unitary matrices. We show that if two or more independent controls are used, then every state can be achieved in arbitrary time using finite energy. The mathematical construction is motivated by the demand of manipulating (or logically operating on) the state of quantum bits, and the results provide some insight into the feasibility of realizing given operations in quantum computers.

1 Introduction

John von Neumann (born December 28, 1903 in Budapest, Austria-Hungary; died February 8, 1957 in Washington D.C., United States) was a Hungarian-born mathematician and polymath who made contributions to many mathematics-related fields as one of history's outstanding mathematicians. Most notably, von Neumann was a pioneer of the application of operator theory to quantum mechanics, and the commonly known von Neumann architecture is the de facto standard of nowadays computers.

In the paper we first outline von Neumann's contributions to both fields, then we show that the ever-increasing need for computer speed inevitably leads to the formulation of a new scientific and technological field, the so-called quantum information technology. However, operating on information physically associated to quantum mechanical phenomena is fundamentally different to "classical" computer technology, and several theoretical and pragmatical questions have to be answered. Out of them we aim at the (state) reachability and controllability problems of

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quantum bits using the ideas of another famous Hungarian Rudolph E. Kalman. We will show that if two or more independent controls (externally applied electro-magnetic fields) are used, then every state of the quantum bit can be achieved in arbitrary time using finite energy.

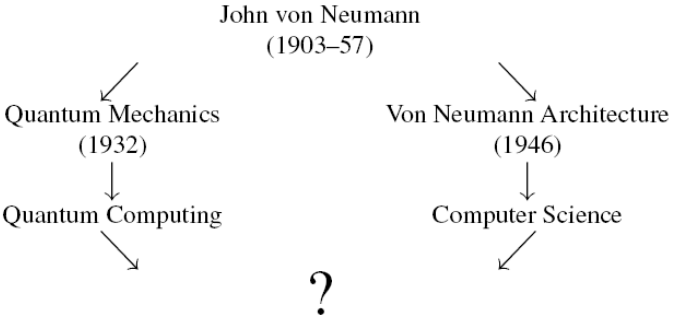

Fig. 1. The "heritage" of John von Neumann – quantum information technology

2 Quantum Mechanics

At the *International Congress of Mathematicians of 1900*, David Hilbert presented his famous list of twenty-three problems considered central for the development of the mathematics of the new century. The sixth of these was the axiomatization of physical theories. Among the new physical theories of the century the only one which had yet to receive such a treatment by the end of the 1930's was quantum mechanics. QM found itself in a condition of foundational crisis similar to that of set theory at the beginning of the century, facing problems of both philosophical and technical natures. On the one hand, its apparent non-determinism had not been reduced to an explanation of a deterministic form. On the other, there still existed two independent but equivalent heuristic formulations, the so-called *matrix mechanical formulation* due to Werner Heisenberg and the *wave mechanical formulation* due to Erwin Schrödinger, but there was not yet a single, unified satisfactory theoretical formulation.

After having completed the axiomatization of set theory, von Neumann began to confront the axiomatization of QM. He immediately realized, in 1926, that a quantum system could be considered as a point in a so-called *Hilbert-space*, analogous to the $6N$ dimension (N is the number of particles, 3 general coordinate and 3 canonical momentum for each) phase space of classical mechanics but with infinitely many dimensions (corresponding to the infinitely many possible states of the system) instead: the traditional physical quantities (e.g. position and momentum) could therefore be represented as particular linear operators operating in these spaces. The physics of quantum mechanics was thereby reduced to the

mathematics of the linear Hermitian operators on Hilbert spaces. For example, the famous *indeterminacy principle of Heisenberg*, according to which the determination of the position of a particle prevents the determination of its momentum and vice versa, is translated into the non-commutativity of the two corresponding operators.

This new mathematical formulation included as special cases the formulations of both Heisenberg and Schrdinger, and culminated in the 1932 classic *The Mathematical Foundations of Quantum Mechanics* [Grundlagen]. However, physicists generally ended up preferring another approach to that of von Neumann (which was considered elegant and satisfactory by mathematicians). This approach was formulated in 1930 by Paul Dirac and was based upon a strange type of function (the so-called *Dirac delta function*) which was harshly criticized by von Neumann.

In any case, von Neumann's abstract treatment permitted him also to confront the foundational issue of determinism vs. non-determinism and in the book he demonstrated a theorem according to which quantum mechanics could not possibly be derived by statistical approximation from a deterministic theory of the type used in classical mechanics. This demonstration contained a conceptual error, but it helped to inaugurate a line of research which, through the work of John Stuart Bell in 1964 on *Bell's Theorem* [3] and the experiments of Alain Aspect in 1982 [2], demonstrated that quantum physics requires a notion of reality substantially different from that of classical physics.

3 Computer Science

The earliest computing machines had fixed programs. Some very simple computers still use this design, either for simplicity or training purposes. To change the program of such a machine, one have to re-wire, re-structure, or even re-design the machine. Indeed, the earliest computers were not so much “programmed” as they were “designed”. “Reprogramming”, when it was possible at all, was a very manual process, starting with flow charts and paper notes, followed by detailed engineering designs, and then the often-arduous process of implementing the physical changes.

The idea of the stored-program computer changed all that. In 1945 John von Neumann published a now-famous paper, the *First Draft of a Report on the EDVAC* [4], describing a computer architecture in which data and program memory are mapped into the same address space. The *von Neumann architecture* became the de facto standard and can be contrasted with the so-called Harvard architecture, which has separate program and data memories on a separate bus. By creating an instruction set architecture and detailing the computation as a series of instructions (the program), the machine becomes much more flexible. By treating those instructions in the same way as data, a stored-program machine can easily change the program, and can do so under program control. *The majority of home computers, microcomputers, minicomputers and mainframe computers use the single-memory (a.k.a. Von Neumann) computer architecture.*

However, the ever-increasing need for computer speed drives the design of integrated circuits into the direction of using smaller and smaller units of physical state-space to represent units of information. This technological process in the near future will lead to reach atomic (or sub-atomic) dimensions, therefore there is a need for a new paradigm in computer design: the so-called *quantum information technology*.

Von Neumann worked together with other immigrant Hungarian scientists, who also participated in the development of computer science.

- János Kemény (1926-92) as the Rector of Dartmouth Collage obligated students of arts and laws to use personal computers, and for helping them formulated the *BASIC* language. Kemény is also known for the time-shared computer network that was honored by the first Robinson Prize of IBM.
- Von Neumann was also collaborating with Leó Szilárd who introduced the term *bit* as the elementary unit of information (yes/no).
- This list would not be complete without Andy Grove (Gróf András), who was nominated as "The Man of the Year" in 1997 by the journal *Time*. He was the CEO of INTEL, and almost yearly doubled the speed of its microprocessors.

The ever-increasing need for computer speed implies using *smaller and smaller units of physical state-space* to represent units of information. This technological process in the near future will lead to reach *atomic (or sub-atomic) dimensions*. There is a need for a new paradigm: *quantum information technology*.

4 Quantum Information Technology

A *quantum bit*, or *qubit* (sometimes qbit) is a unit of quantum information. That information is described by a state vector in a two-level quantum mechanical system which is formally equivalent to a two-dimensional vector space over the complex numbers.

Benjamin Schumacher discovered a way of interpreting quantum states as information. He came up with a way of compressing the information in a state, and storing the information on a smaller number of states. This is now known as *Schumacher compression*. In the acknowledgments of his paper [Schumacher], Schumacher states that the term qubit was invented in jest, during his conversations with Bill Wootters.

A bit is the base of computer information. Regardless of its physical representation, it is always read as either a ‘0’ or a ‘1’. An analogy to this is a light switch – the down position can represent ‘0’ (normally equated to *off*) and the up position can represent ‘1’ (normally equated to *on*).

A qubit has some similarities to a classical bit, but is overall very different. Like a bit, a qubit can have only two possible values – normally a ‘0’ or a ‘1’. The difference is that whereas a bit must be either ‘0’ *or* ‘1’, a qubit can be ‘0’, ‘1’, or a *superposition* of both.

4.1 Physical Representation

Any two-level system can be used as a qubit. Multilevel systems can be used as well, if they possess two states can be effectively decoupled from the rest (e.g., ground state and first excited state of a nonlinear oscillator). There are various proposals. Several physical implementations which approximate two-level systems to various degrees were successfully realized. Similarly to a classical bit where the state of a transistor in a processor, the magnetization of a surface in a hard disk and the presence of current in a cable can all be used to represent bits in the same computer, an eventual *quantum computer* is likely to use various combinations of qubits in its design. *Table tab:qubit* contains an incomplete list of possible physical implementation of qubits.

Table 1. Possible physical implementation of qubits

Physical support	Name	Information support	'0'	'1'
Single photon (Fock states)	Polarization encoding	Polarization of light	Horizontal	Vertical
	Photon number	Photon number	Vacuum	Single photon state
	Time-bin encoding	Time of arrival	Early	Late
Coherent state of light	Squeezed light	Quadrature	Amplitude-squeezed state	Phase-squeezed state
Electrons	Electronic spin	Spin	Up	Down
	Electron number	Charge	No electron	One electron
Nucleus	Nuclear spin addr. through NMR	Spin	Up	Down
Optical lattices	Atomic spin	Spin	Up	Down
Josephson junction	Superconducting charge qubit	Charge	Uncharged superconducting island (Q=0)	Charged superconducting island (Q=2e)
	Superconducting flux qubit	Current	Clockwise current	Counterclockwise current
Singly-charged quantum dot pair	Electron localization	Charge	Electron on left dot	Electron on right dot

4.2 Mathematical Representation

Each physical system is associated with a (topologically) *separable complex Hilbert-space* H with inner product $\langle \psi \mid \phi \rangle$. Physical observables are represented by densely-defined *self-adjoint operators* on H. The expected value (in the sense of probability theory) of the observable A for the system in state represented by the unit vector $|\psi\rangle \in H$ is $\langle \psi \mid A \mid \psi \rangle$.

The states a qubit may be measured in are known as basis states (or vectors). As is the tradition with any sort of quantum states, *Dirac* (or bra-ket) *notation* is used to

represent them. This means that the two computational basis states are conventionally written as $|0\rangle$ and $|1\rangle$.

The state at any time t is given by:

$$\psi(t) = X(t)\psi(0), \tag{1}$$

where X is the so-called *evolution operator* (matrix), solution of the Schrdinger-equation

$$i\hbar\dot{X} = HX. \tag{2}$$

with initial condition equal to the identity operator.

A pure qubit state is a linear superposition of those two states. This means that the qubit can be represented as a *linear combination* of $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are *probability amplitudes* and can in general both be complex numbers.

When we measure this qubit in the standard basis, the probability of outcome $|0\rangle$ is $|\alpha|^2$ and the probability that the outcome is $|1\rangle$ is $|\beta|^2$. Because the absolute squares of the amplitudes equate to probabilities, it follows that α and β must be constrained by the equation

$$|\alpha|^2 + |\beta|^2 = 1,$$

simply because this ensures you must measure *either* one state *or* the other.

In implementations of quantum computers, the operation given by the evolutionary operator X represents a (logic) operation to be performed on a quantum bit, i.e. the *reachability* question can be seen as the feasibility of logic operations on a quantum bit. In practical terms: can all operations be achieved on a quantum bit by opportunely shaping an input electro/magnetic field? (Using finite energy.)

5 Reachability of Quantum States

Consider the Schrdinger equation for the evolution operator (2) in the common situation where the Hamiltonian operator H can be written as $H = H_0 + \sum_{i=1}^{m} H_i u_i$,

$$i\hbar\dot{X} = \left(H_0 + \sum_{i=1}^{m} H_i u_i(t) \right) X. \tag{3}$$

The operators H_i ($i = 0,\ldots,m$) are Hermitian operators on a finite dimensional vector space and the overall phase of the solution of (3) does not have physical meaning. Considering the above, Eq. (3) can be transformed into a differential system [6] of the form

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} B_i X(t) u_i(t), \tag{4}$$

where A, B_i are elements of the Lie algebra of 2×2 skew-Hermitian matrices with zero trace, which is denoted by su(2).

Definition 1. Lie group is a group G that is also a differentiable manifold such that for any $a,b \in G$ the multiplication $(a,b) \mapsto ab$ and the inverse $a \mapsto a^{'}$ are smooth maps.

Proposition 1. *All compact finite dimensional Lie groups can be represented as matrix Lie groups.*

The solution of (4) with initial condition equal to identity varies in the Lie group associated to su(2), namely in the Lie group of 2×2 unitary matrices with determinant 1. This group is called the group of *special unitary matrices* and is denoted by SU(2).

Definition 2. The set of reachable states $R(T)$ consists of all the possible values for $X(T)$ (solution of (4) at time T with initial condition equal to identity) obtained varying the controls $u_1,\ldots,u_m$ in the set of all the piecewise continuous functions defined in $[0,T]$.

Theorem 1 [7]. *Consider system (4) with* $3 \geq m \geq 2$ *and assume that* $B_1,\ldots,B_m$ *are linearly independent. Then, for any time* $T > 0$ *and for any desired final state* X_f *there exist a set of piecewise continuous control functions* $u_1,\ldots,u_m$ *driving the state of the system* X *to* $X(T) = X_f$ *at time* T*. This means that in this case* $R(T) = SU(2)$ *for every* $T > 0$.

We present a general approach to derive reachability/controllability results like was given in Theorem 1.

6 Open-Loop Unconstrained Controllability

First we cite the Kalman controllability results for LTI systems. The fundamental matrix for zero initial time is

$$\Phi(t) = e^{At} = \sum_{i=1}^{n} \psi_i(t) A^{i-1},$$

and the reachability subspace is

$$\mathcal{R} = \sum_{k=0}^{n-1} \mathrm{Im} A^k B.$$

Proposition 2. *It is possible to generate linearly independent functions* $\psi_i, i = 1, \ldots, n$ *if the Kalman-rank condition*

$$rank\left[B, AB, ..., A^{n-1}B \right] = n$$

is satisfied.

We show a general method for systems over Lie groups using Lie algebraic approach. Write the fundamental matrix (locally) as exponential function of the "coordinates of second kind" associated with the equation

$$\dot{x} = \sum_{i=0}^{N} \rho_i(t) A_i x.$$

Using the Wei--Norman equation:

$$\dot{g}(t) = \left(\sum_{i=1}^{K} e^{\Gamma_1 g_1} \cdots e^{\Gamma_{i-1} g_{i-1}} E_{ii} \right)^{-1} \rho(t), \quad g(0) = 0,$$

where $\{\hat{A}_1, \ldots, \hat{A}_K\}$ is a basis of the Lie-algebra $\mathcal{L}(A_1, \ldots, A_N)$,

$$[\hat{A}_i, \hat{A}_j] = \sum_{l=1}^{K} \Gamma_{i,j}^{l} \hat{A}_l, \quad \Gamma_i = [\Gamma_{i,j}^{l}]_{j,l=1}^{K}.$$

Proposition 3 (Generalized Kalman-rank condition). *For systems* $A(\rho), B(\rho)$ *the points attainable from the origin are those from the subspace spanned by the vectors*

$$\mathscr{R}_{(\mathscr{A},\mathscr{B})} := \mathrm{span} \left\{ \prod_{j=1}^{K} A_{l_j}^{i_j} B_k \right\}$$

where $K \geq 0$, $l_j, k \in \{0, \cdots, N\}$, $i_j \in \{0, \cdots, n-1\}$, i.e., $\mathscr{R} \subset \mathscr{R}_{(\mathscr{A},\mathscr{B})}$.

Denote by $\mathcal{L}(A_0, \ldots, A_N)$ the finitely generated Lie-algebra containing the matrices $A_0, \ldots, A_N$, and let $\hat{A}_1, \ldots, \hat{A}_K$ be a basis of this algebra, then the points attainable from the origin are in the subspace

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \sum_{l=0}^{N} \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} \mathrm{Im} \left(\hat{A}_1^{n_1} \cdots \hat{A}_K^{n_K} B_l \right).$$

The question is that under what condition is $\mathcal{R} = \mathcal{R}_{\mathcal{A},\mathcal{B}}$?

The fundamental matrix can be written in exponential form:

$$\Phi(t) = \sum_{n_1=0}^{n-1} \dots \sum_{n_K=0}^{n-1} \hat{A}_1^{n_1} \dots \hat{A}_K^{n_K} \psi_{n_1,\dots,n_K}(t).$$

$$\Phi(t) = \sum_{\mathbf{j}\in\mathbf{J}} \hat{A}_{\mathbf{j}} \varphi_{\mathbf{j}}(t), \quad \hat{A}_{\mathbf{j}} := \hat{A}_1^{j_1} \dots \hat{A}_K^{j_K}.$$

The subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is the image space of the matrix $R_{\mathcal{A},\mathcal{B}} := [\hat{A}_{\mathbf{j}} B]_{\mathbf{j}\in\mathbf{J}}$.
The controllability Grammian is given as

$$W(\sigma,\tau) = R_{\mathcal{A},\mathcal{B}} \left(\int_\sigma^\tau [\varphi_{\mathbf{j}}(s)]_{\mathbf{j}\in\mathbf{J}} [\varphi_{\mathbf{j}}(s)]^*_{\mathbf{j}\in\mathbf{J}} ds \right) R^*_{\mathcal{A},\mathcal{B}}.$$

Theorem 2 [8]. *The quantum system is controllable, iff*
(i) The generalized Kalman-rank condition is satisfied:

$$\operatorname{rank} R_{\mathcal{A},\mathcal{B}} = \operatorname{rank} [\hat{A}_{\mathbf{j}} B]_{\mathbf{j}\in\mathbf{J}}$$

(ii) The set of functions $\{\varphi_j(\sigma) | \mathbf{j} \in \mathbf{J}\}$ *contains* n *linearly independent functions.*

7 Switching System's Controllability

Hybrid models characterize systems governed by continuous differential and difference equations and discrete variables. Such systems are described by several operating regimes (modes) and the transition from one mode to another is governed by the evolution of internal or external variables or events.

Depending on the nature of the events there are two big classes of hybrid systems that are considered in the control literature: *switching systems* and *impulsive systems*.

A switching system is composed of a family of different (smooth) *dynamic modes* such that the switching pattern gives continuous, piecewise smooth trajectories. Moreover, it is assumed that one and only one mode is active at each time instant.

In a broader sense every time-varying system with measurable variations in time can be cast as a switching system, therefore it is usually assumed that the number of switching modes is finite and for practical reasons the possible switching functions (sequences) are restricted to be piecewise constant, i.e. only a finite number of transition is allowed on a finite interval. Moreover, sometimes the frequency of the transitions is also bounded – dwell time.

Formally, these systems can be described as:

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)),$$

$$y(t) = h_{\sigma(t)}(x(t), u(t)), \quad x(\tau^+) = \iota(x(\tau^-), u(\tau), \tau),$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \Omega \subset \mathbb{R}^m$ is the input variable and $y \in \mathbb{R}^p$ is the output variable.

The $\sigma : \mathbb{R}^+ \to S$ is a measurable switching function mapping the positive real line into $S = \{1, \cdots, s\}$. The impulsive effect can be described by the relation $(\tau, x(\tau^-)) \in \mathcal{I} \times \mathcal{A}$ with $\mathcal{I}$ a set of time instances and $\mathcal{A} \in \mathbb{R}^n$ a certain region of the state space.

Consider a bimodal system

$$\dot{X} = A_{\sigma(t)} X, \quad X(0) = I, \quad \sigma(t) : \mathbb{R}^+ \mapsto \{1, 2\}$$

and $A_1, A_2 \in \mathcal{U}(n)$ that is SU(n).

A set of gates is called *universal* if – by switching $\{A_1, A_2\}$ – it is possible to generate all (special) unitary evolutions.

Proposition 4 [8]. *Since A_1, A_2 generate the whole Lie-algebra $u(n)$ or $su(n)$, therefore almost every couple of skew-Hermitian matrices generate $u(n)$, i.e. almost every quantum gate is universal.*

8 Outlook

We certainly know that the above results are only the first steps into the direction of physically realizing quantum computers. There are several theoretical problems to be solved, among them we outline state-estimation, state-observation, and *Kalman filtering*, that are all necessary to relax the inconsistency between the fundamental indeterministic (or statistical) character of quantum mechanics, and the natural need against computers to execute exact calculations.

Besides the above, there are practical considerations that should be taken into account in order to build functionally adequate quantum computers: the control of multi-level quantum systems, minimal-time control, minimal-energy control.

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