On Generation of Digital Fuzzy Parametric Conjunctions

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Abstract. A new method of generation of digital fuzzy parametric conjunctions by means of basic t-norms is proposed. Fuzzy conjunctions also referred to as conjunctors and semicopulas. Digital fuzzy conjunctions are defined on the set of integer membership values $L=\{0,1,2,\ldots,2^m-1\}$, where m is a number of bits used in presentation of membership values and I = 2^m-1 denotes a maximal membership value corresponding to 1 in traditional set of true values L= [0,1]. The proposed method referred to as the monotone sum of fuzzy conjunctions generalizes the method of ordinal sum of t-norms and gives possibility to construct a wide class of digital fuzzy parametric conjunctions that have effective digital hardware implementation. The classes of simplest commutative digital fuzzy parametric conjunctions obtained by this method are described. These classes of operations can constitute a part of a library of basic blocks for generation of digital fuzzy systems.

Keywords: Digital fuzzy conjunction, conjunctor, semicopula, finite scale, t-norm, commutativity.

1 Introduction

In construction of digital fuzzy systems it is convenient instead of traditional set on membership values L=[0,1] to use the set of true values L= {0,1,2,…,I}, where I= 2^m-1 and m is a number of bits used in presentation of membership values. Such replacement of the set of true values on the one hand simplifies digital hardware implementation of fuzzy logic operations. On the other hand digital representation of true values preserves most of traditional definitions and properties of fuzzy concepts.

For hardware implementation of digital systems it is important to have the basic blocks that can be used as “bricks” in construction of digital system [1]. As such blocks in construction of digital fuzzy conjunctions in [2] they are used basic t-norms and t-conorms together with the set of simple generators. These basic functions have simple and effective digital hardware implementation because they

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use only simple operations such as comparison, minimum, maximum, bounded sum and bounded difference.

In this paper we introduce a new method of generation of digital fuzzy parametric conjunctions referred to as a monotone sum of fuzzy conjunctions that generates fuzzy conjunctions by means of only basic t-norms (drastic, Lukasiewicz and minimum t-norms) and one parameter p that can take values in the set of true values L. This method generalizes the method of generation of t-norms by means of ordinal sum of t-norms [3-6]. We describe all classes of commutative digital fuzzy parametric conjunctions that can be obtained as monotone sum of basic t-norms with one parameter. These classes of operations can constitute a part of a library of simple basic blocks for generation of digital fuzzy systems.

The paper has the following structure. Section 2 contains basic definitions of digital fuzzy conjunctions. Section 3 proposes a new method of generation of digital fuzzy parametric conjunctions referred to as the monotone sum of basic t-norms. Section 4 describes classes of commutative digital fuzzy parametric conjunctions that can be obtained as monotone sum of basic t-norms with parameter p. Section 5 describes the classes of commutative digital fuzzy parametric conjunctions obtained by monotone sum of basic t-norms when the value of parameter p is used together with the value I - p in definition of partition of domain $L\ \ L$ on sections. In Conclusion we discuss obtained results and future directions of research.

2 Basic Definitions

Suppose m bits are used in digital representation of membership values L= {0,1,2,…, 2^m-1} with maximal value I = 2^m-1. This value will represent the full membership corresponding to the value 1 in traditional set of membership values [0,1]. For example, for m = 4 we have I = 15. Most of definitions of fuzzy operations have straightforward extension on digital case by replacing the set of membership values [0,1] by L={0,1,2,…, 2^m-1} and maximal membership value 1 by I.

Fuzzy conjunction operation is a function $T{:}L\ \ L \to L$ satisfying on L conditions:

$$T(x,I) = x, \qquad T(I,y) = y, \qquad \text{(boundary conditions)}$$

$$T(x,y) \le T(u,v), \qquad if\ \ x \le u,\ y \le v. \qquad \text{(monotonicity)}$$

Fuzzy conjunctions have been studied in [7, 8]. Such functions are also referred to as conjunctors [6] or semicopulas [9]. Note that from the properties of fuzzy conjunctions it follows:

$$T(x,0) = 0,\ \ T(0,y) = 0.$$

T-norms are fuzzy conjunctions satisfying on L commutativity and associativity conditions [3]:

$$T(x,y)=T(y,x), \qquad \text{(commutativity)}$$

$$T(x,T(y,z)) = T(T(x,y),z). \qquad \text{(associativity)}$$

Associativity property of fuzzy conjunctions usually does not used in applied fuzzy systems. For this reason we do not require associativity from digital fuzzy conjunctions. Note also that most of known associative parametric t-norms are sufficiently complicated for digital hardware implementation and we consider here the methods of generation of simple digital fuzzy parametric conjunctions. Commutativity of fuzzy conjunctions can be useful in some applications of fuzzy systems so we will describe further the classes of such operations obtained by proposed methods.

Below are basic *t*-norms [3] that will be used further in generation of digital fuzzy parametric conjunctions:

$$T_M(x,y) = min\{x,y\}, \qquad \text{(minimum)}$$

$$T_L(x,y)= max\{x+y-I, 0\}, \qquad \text{(Lukasiewicz t-norm)}$$

$$T_D(x,y) = \begin{cases} x, & if\ y = I \\ y, & if\ x = I \\ 0, & if\ x,y < I \end{cases}. \qquad \text{(drastic product)}$$

Lukasiewicz *t*-norm is also known as a bounded product [10]. The definition of these t-norms uses only simple operations like comparison, minimum, maximum, bounded sum and bounded difference, for this reason these t-norms have efficient digital hardware implementation [11].

It can be shown that any fuzzy conjunction T satisfies the following inequalities:

$$T_D(x,y) \le T(x,y) \le T_M(x,y). \qquad (1)$$

We will say that $T_1 \le T_2$ if $T_1(x,y) \le T_2(x,y)$ for all x,y from L. For example, we have:

$$T_D \le T_L \le T_M. \qquad (2)$$

An example of parametric t-norm having efficient digital hardware implementation is the Mayor-Torrens t-norm [3] depending on parameter $p \in L$:

$$T(x,y) = \begin{cases} \max(x+y-p,0), & if\ p>0, \quad x \le p, y \le p \\ \min(x,y), & otherwise \end{cases}. \qquad (3)$$

Another example of parametric fuzzy conjunction having efficient hardware implementation gives the following fuzzy conjunction introduced in [8] and depending on two parameters $p,q \in L$:

$$T(x,y) = \begin{cases} \min(x,y), & if \quad p \le x \ or \ q \le y \\ 0, & otherwise \end{cases}. \tag{4}$$

This conjunction will be a t-norm when $p = q$. Both parametric t-norm and fuzzy conjunction considered above are obtained by using suitable generator function in some way. The methods of generation of digital fuzzy parametric conjunctions by means of basic t-norms, t-conorms and simple generators are considered in [2]. In the following section we propose new method of generation of digital fuzzy parametric conjunctions without generator functions by means of basic t-norms and one parameter p.

3 Monotone Sum of Digital Fuzzy Conjunctions

A set X containing one number or a sequence of consecutive numbers from L is called an interval in L. Let $J = \{1,\dots,n\}$, $1 < n \le I+1$, be a set of indexes and $(X_j)_{j\in J}$, is a partition of L on pairwise disjoint intervals such that from $i < j$ it follows $x < y$ for all $x \in X_i$ and $y \in X_j$. Denote $D_{ij}=X_i \ \ X_j$. Suppose Q is some set of indexes and $(T_q, \le)_{q\in Q}$ is a partially ordered set of fuzzy conjunctions. Assign to each D_{ij} some $T_{ij} = T_q$ from this set such that

$$T_{ij}(x,y) \le T_{st}(u,v) \quad \text{if} \quad i \le s,\ j \le t, \text{ and } x \le u,\ y \le v, \quad x,y \in D_{ij},\ u,v \in D_{st}. \tag{5}$$

Define a function T on $L \ \ L$ by $T(x,y) = T_{ij}(x,y)$ if $(x,y) \in D_{ij}$, $i,j \in J$. Then T is called the monotone sum of $(D_{ij},T_{ij})_{i,j\in J}$ or monotone sum of fuzzy conjunctions T_{ij}, $i,j \in J$.

Theorem 1. A monotone sum of fuzzy conjunctions is a fuzzy conjunction.

Proof. Since all fuzzy conjunctions used in construction of T satisfy boundary conditions $T(x,1)=x$, $T(1,y)=1$ then a resulting function T also will satisfy these conditions. Monotonicity of T follows from definition (5).

Note that the monotone sum of fuzzy conjunctions can be considered as a generalization of the ordinal sum of t-norms [3-6]. The ordinal sum can be obtained from monotone sum by suitable selection of diagonal sections $(D_{ii},T_{ii})_{i\in J}$ when T_M is used in other sections.

Two conjunctions T_{ij} and T_{st} satisfying condition (5) in the definition of monotone sum may be not related to each other by ordering relation $\le$ on all domain $L \ \ L$. If in the definition above we will replace condition (5) by the more simple condition:

$$T_{ij} \le T_{st} \quad \text{if} \quad i \le s \quad \text{and} \quad j \le t \tag{6}$$

then due to monotonicity of fuzzy conjunctions from (6) it will follow (5). The function T defined by (6) will be referred to as simple monotone sum of $(D_{ij},T_{ij})_{i,j\in J}$.

Corollary 2. A simple monotone sum of fuzzy conjunctions is a fuzzy conjunction.

Based on Theorem 1 we will consider two methods of construction of monotone sum of fuzzy conjunctions using one parameter $p\in L$. Fig. 1 illustrates the methods of partition of L on intervals X_j and partition of L L on corresponding sections $D_{ij}=X_i$ X_j. In the first method we define the partition of L as follows (see Fig. 1, on the left): X_1 = [0,p], X_2= [p+1,I], $p\in\{0,1,\ldots, 2^{m}-2\}$. In the second method partitions are defined as follows (see Fig. 1, on the right): X_1 = [0,p], X_2 = [p+1,I-p], X_3 = [I-p+1,I], $p\in\{0, 1 ,\ldots, 2^{m-1}-1\}$. In the second method we have $p <$ I-p and for m= 4, I= 15, p can take values 0,1,…,7. The first method will be referred to as (p)-monotone sum and the second method as (p,I-p)-monotone sum.

4 (p)-Monotone Sum of Digital Fuzzy Conjunctions

(p)-monotone sum of fuzzy conjunctions can be defined as follows. Select a set of fuzzy conjunctions $\{T_{11}, T_{21}, T_{12}, T_{22}\}$ ordered as follows: $T_{11} \le T_{12} \le T_{22}$, $T_{11} \le T_{21} \le T_{22}$. Define fuzzy conjunction T as follows (see also Fig. 1, on the left):

$$T(x,y)=\begin{cases} T_{11}(x,y), & if \quad x\le p, \quad y\le p \\ T_{21}(x,y), & if \quad x> p, \quad y\le p \\ T_{12}(x,y), & if \quad x\le p, \quad y> p \\ T_{22}(x,y), & if \quad x> p, \quad y> p \end{cases}. \tag{7}$$

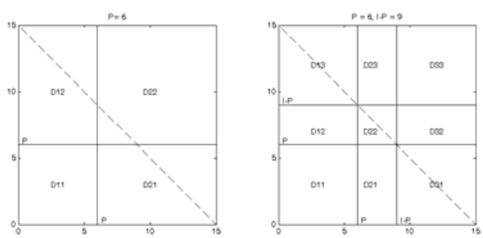

Fig. 1. Partition of L L on sections $D_{ij}=X_i$ X_j defined by parameter values: 1) p (on the left); 2) p and **I**-p (on the right)

Consider all nontrivial fuzzy commutative conjunctions that can be obtained by this method by means of basic t-norms T_D, T_L, T_M. Remember that these t-norms are ordered as follows: $T_D \le T_L \le T_M$. To obtain commutative conjunction it should be $T_{21} = T_{12}$. Fig. 2 depicts all commutative digital fuzzy parametric conjunctions obtained by considered method. On the left there are shown assignments of basic t-norms to sections D_{ij}. On the right there are shown the shapes of obtained digital fuzzy parametric conjunctions for parameter value p= 9. Here we use four bits for digital representation of conjunctions: m= 4, and I= 15. Each of obtained fuzzy conjunctions coded by the name depending on the list of t-norms used in its definition: $(T_{11}, T_{21}, T_{12}, T_{22})$. For example, conjunction T_{DLLM} is defined by the sequence of t-norms (T_D, T_L, T_L, T_M). It can be seen that some of obtained fuzzy conjunctions will be t-norms, for example conjunctions T_{DDDM} and T_{DMMM} are t-norms (see [3] and (4) above). An example of non-commutative digital fuzzy conjunction is depicted in Fig. 3.

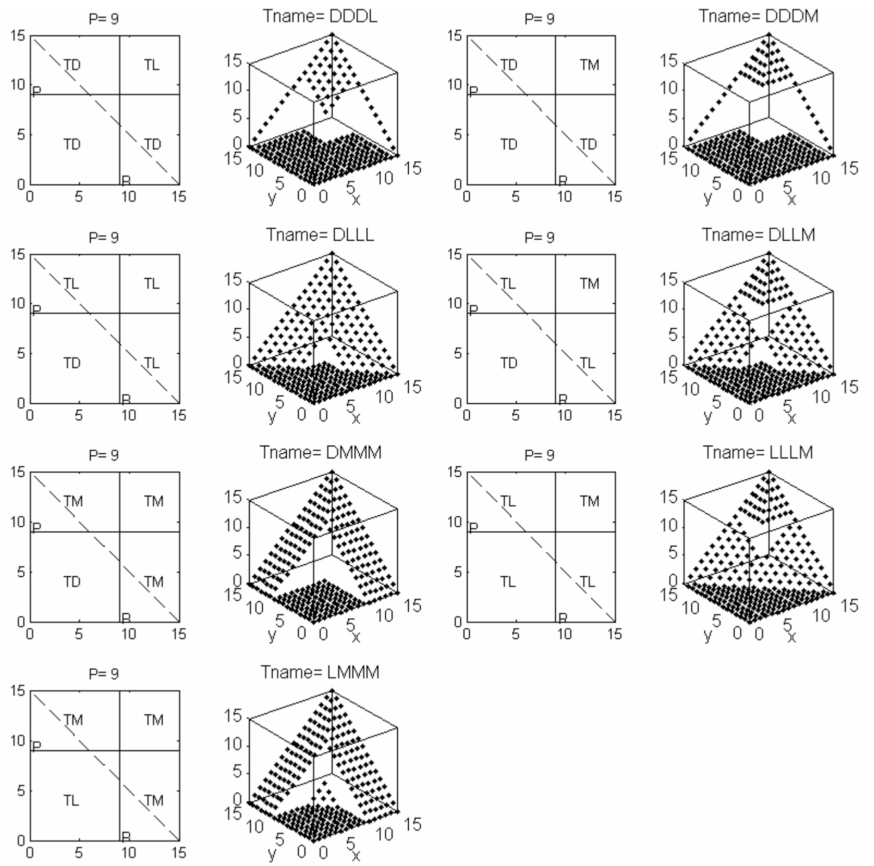

Fig. 2. Simplest commutative digital fuzzy parametric conjunctions obtained by (p)-monotone sum of basic t-norms: TD - drastic, TL - Lukasiewicz and TM – minimum t-norms

Note that both commutative and non-commutative digital fuzzy parametric conjunctions obtained from basic t-norms by means of (7) have very simple digital hardware implementation. We need to have blocks implementing basic t-norms and we need to have selector that will compare values of x and y with the value of parameter p and select corresponding t-norm given in the sequence $(T_{11}, T_{21}, T_{12}, T_{22})$.

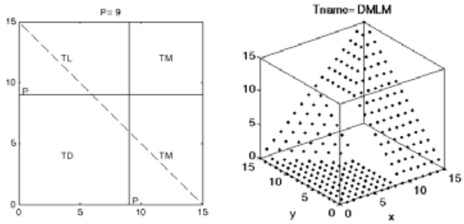

Fig. 3. Example of non-commutative digital fuzzy parametric conjunction

5 (p,I-p)-Monotone Sum of Digital Fuzzy Conjunctions

*(p,*I-*p)*-monotone sum of fuzzy conjunctions can be defined as follows (see Fig. 1 on the right). Select a set of fuzzy conjunctions {T_{11}, T_{21}, T_{31}, T_{12} T_{22}, T_{32}, T_{13}, T_{23}

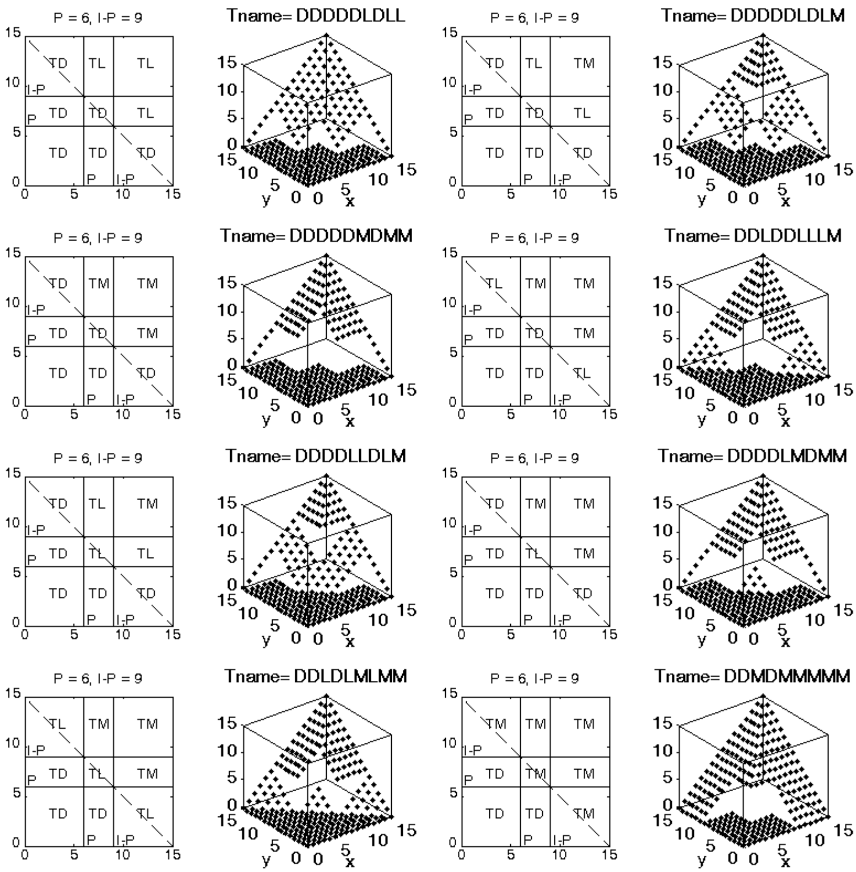

Fig. 4. Non-trivial commutative digital fuzzy parametric conjunctions obtained by (*p*,I-*p*)-monotone sum

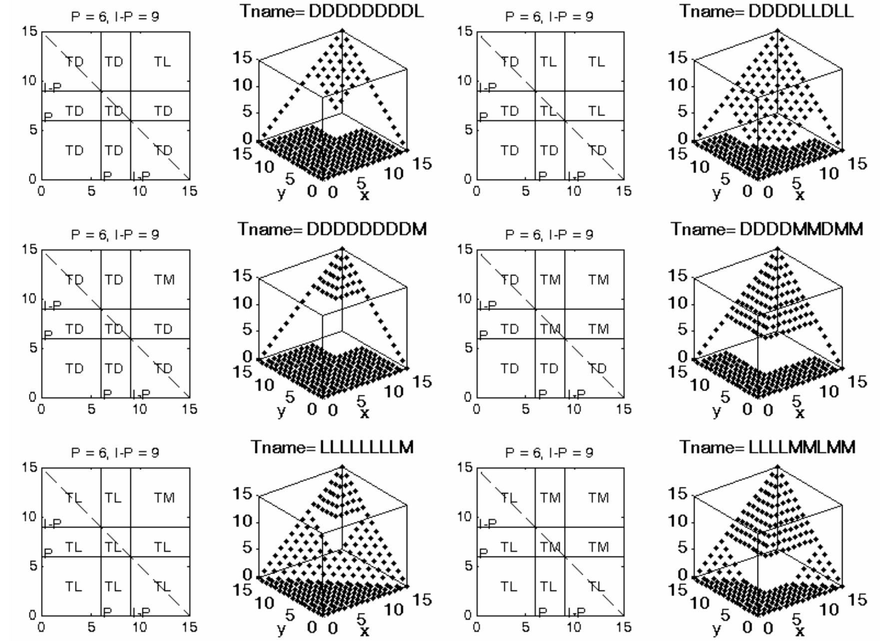

Fig. 5. Commutative digital fuzzy parametric conjunctions obtained by (p,I-p)-monotone sum that can be reduced to conjunctions obtained by (p)-monotone sum

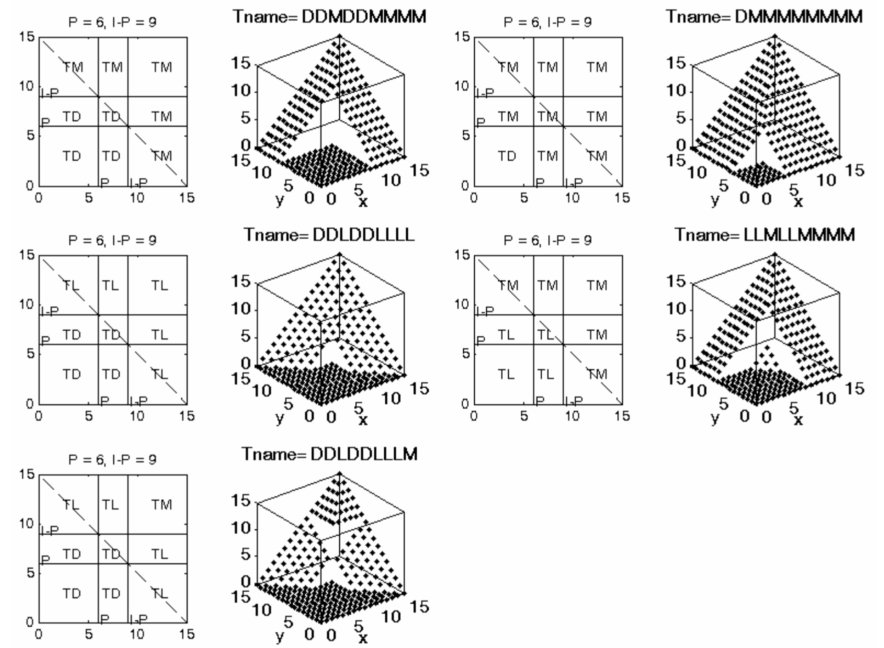

Fig. 6. Commutative digital fuzzy parametric conjunctions obtained by (p,I-p)-monotone sum that can be reduced to conjunctions obtained by (p)-monotone sum

$T_{33}\}$ ordered as follows: $T_{ij} \leq T_{st}$ if $i \leq s$ or $j \leq t$. Define fuzzy conjunction T as follows:

$$T(x,y)=\begin{cases} T_{11}(x,y), & if \quad x \leq p, \quad y \leq p \\ T_{21}(x,y), & if \quad p < x \leq I-p, \quad y \leq p \\ T_{31}(x,y), & if \quad x > I-p, \quad y \leq p \\ T_{12}(x,y), & if \quad x \leq p, \quad p < y \leq I-p \\ T_{22}(x,y), & if \quad p < x \leq I-p, \quad p < y \leq I-p \\ T_{32}(x,y), & if \quad x > I-p, \quad p < y \leq I-p \\ T_{13}(x,y), & if \quad x \leq p, \quad y > p \\ T_{23}(x,y), & if \quad p < x \leq I-p, \quad y > p \\ T_{33}(x,y), & if \quad x > p, \quad y > p \end{cases}$$

Consider all nontrivial fuzzy commutative conjunctions that can be obtained by this method by means of basic t-norms T_D, T_L, T_M. To obtain commutative conjunction it should be $T_{21} = T_{12}$, $T_{31} = T_{13}$ and $T_{32}= T_{23}$. Fig. 4 depicts all non-trivial commutative digital fuzzy parametric conjunctions obtained by this method. Note that because the maximal value of p in this method is less than the value of I-p, the values of both T_D and T_L equal 0 in sections D_{11}, D_{21} and D_{12}. For this reason any two conjunctions coinciding in all other sections and containing in sections D_{11}, D_{21} and D_{12} only t-norms T_D and T_L will be equal.

Other commutative conjunctions that can be obtained by *(p,*I-*p)*-monotone sum of basic t-norms can be reduced to some of conjunctions obtained by *(p)*-monotone sum of basic t-norms. These conjunctions are presented in Figs. 5, 6. For example, conjunctions *DDDDDDDDL* and *DDDDLLDLL* obtained by *(p,*I-*p)*-monotone sum and shown on the top of Fig. 5 are partial cases of conjunction *DDDL* obtained by *(p)*-monotone sum and shown on the top of Fig. 2. Remember that in the first two conjunctions parameter *p* is changing only till the mean of the scale *L* but in the last case this parameter can take any value from *L* less than I.

Conclusions

Digital fuzzy conjunctions considered in this work are related with conjunctors [6] and semicopulas [9] defined on the set *L*= [0,1], with t-norms defined on finite ordinal scales [12,13] and with *T*-seminorms on partially ordered sets [14]. It gives possibility to extend many results obtained for these operations on digital fuzzy conjunctions and vice versa. For example, the method of generation of fuzzy conjunctions by monotone sums introduced in this work can be directly applied to conjunctors and semicopulas. As it was mentioned above the method of generation of fuzzy conjunctions by monotone sum of fuzzy conjunctions generalizes the method of ordinal sum used for generation of t-norms [3, 5]. It would be interesting to study more carefully relationships between the method of monotone sum used here for generation of fuzzy conjunctions and existing methods of generation of t-norms, e.g. ordinal sums of t-norms.

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