Pseudo-analysis in Engineering Decision Making

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Abstract. There is presented probabilistic von Neumann-Morgenstern type approach to engineering design. Further generalization of the utility theory, using pseudo-analysis, first based on possibility theory, and second, as a common generalization through hybridization of the both preceding approaches are given. In modeling uncertainty in engineering design it is very useful the fuzzy system approach, which involves further real operations as aggregation functions.

1 Introduction

Design is a process where the human intellect with creativity produce useful artifacts, and which involves pure and applied sciences, but also behavioral and social sciences. Engineering design is recognized as a decision-making process at the core [23, 24, 35, 49, 56]. Engineering design conducted with incomplete and imperfect information, yet most traditional design approaches treat the design problem are deterministic. The proposed research is to develop tools for decision making under risk and uncertainty and apply the tools to engineering design.

The approach with probability has three main elements [24]: identification of the options, determination of expectations on each option and the expression of values. The main decision rule is: the preferred decision is the option whose expectation has the highest value. Classical decision theory [32] separates expectations and values - a common mistake is to make them equal. The decision making involve options, expectations and values.

The advantage of the pseudo-analysis [38, 40], as a generalization of the classical real analysis, based on a semiring structure (see [18, 31]) on a real interval $[a,b] \subset [-\infty,\infty]$, is the fact that it coveres as one theory and so with unified methods equations (usually nonlinear), and models with uncertainty (not only with probability) from many different fields (system theory, optimization, control theory, differential equations, difference equations, decision making, etc.).

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An engineering decision cannot be done in the absence of human values, whereas problems in the science are solved in the absence of options and values. The purpose of values in decision making is to rank order alternatives. This ranking is managed by a preference relation, which is connected by the usual real order relation through the utility function.

In Section 2 we present probabilistic von Neumann-Morgenstern type approach to engineering design. Section 3 contains further generalization of the utility theory, first based on possibility theory, and second, as a common generalization through hybridization of the both preceding approaches. In modeling uncertainty in engineering design is very useful the fuzzy system approach, presented in Section 4, and which involves further real operations as aggregation functions, presented in Section 5.

2 Probabilistic Approach Based on von Neumann–Morgenstern Theory

We start with an axiomatic approach to engineering design which guarantees a rational treatment of all information that the designer uses for the design and enables a rational decision making [23, 24, 57]. We shall use the symbol $\succ$ for the relation "is preferred to", the symbol $\sim$ for "is indifferent to" and the symbol $\succeq$ for "is preferred or indifferent to".

Axiom 1: *The axiom of deterministic making.* Given a defined set of options from which to choose, each with a known and deterministic outcome, the decision maker's preferred choice is that option whose outcome is most desired.

Axiom 2: *Ordering of alternatives.* Preference and indifference orderings hold between any two outcomes, and they are transitive.

Axiom 3: *Reduction of compound lotteries.* Any compound lottery is indifferent to a simple lottery with the same outcomes and associated probabilities.

Axiom 4: *Continuity.* Given outcomes of a lottery ordered by preference from A_1 through A_r, there exists a number u_i such that each outcome A_i is indifferent to a lottery containing only A_1 and A_r.

With mathematical symbols

$$A_i \sim [u_i A_1, (1-u_i)A_r] = \hat{A}_i,$$

where $\hat{A}_i$ is the lottery.

Axiom 5: *Substitutability.* In any lottery L, $\hat{A}_i$ is substitutable for A_i.

Axiom 6: *Transitivity.* Preference and indifference among lotteries are transitive relations.

Axiom 7: *Monotonicity.* A lottery $[pA_1,(1-p)A_r]$ is preferred or indifferent to a lottery $[p'A_1,(1-p')A_r]$ if and only if $p \geq p'$.

Axiom 8: *Reality of engineering design.* All engineering designs are selected from among the set of potential designs that are explicitly considered.

Let us mathematically formalize the preceding axioms. Let p be a simple probability measure on $X=\{x_1,...,x_n\}$, thus $p=(p(x_1),p(x_2)...,p(x_n))$, where $p(x_i)$ are probabilities of outcome $x_i \in X$ occurring, i.e., $p(x_i)\geq 0$ for all $i=1,2...,n$, and $\sum_{i=1}^{n} p(x_i)=1$. Define $\mathbf{P}(X)$ as the set of simple probability measures on X. A particular lottery p is a point in $\mathbf{P}(X)$. A compound lottery (mixture) is an operation defined on $\mathbf{P}(X)$ which combines two probability distributions p and p' into a new one, denoted $V(p,p';\alpha,\beta)$, with $\alpha,\beta\in[0,1]$ and $\alpha+\beta=1$, and it is defined by

$$V(p,p';\alpha,\beta) = \alpha p + \beta p'.$$

Note that $V(p,p';\alpha,\beta)\in \mathbf{P}(X)$. Let $\preceq f$ be a binary relation over $\mathbf{P}(X)$, where $p \preceq fq$ means that lottery q is "preferred to or equivalent to" lottery p.

The preceding system of axioms corresponds to the following utility axioms:

NM1 $\mathbf{P}(X)$ is equipped with a complete preordering structure $\preceq f$.

NM2 (Continuity): For $p \prec q \prec r$ there exists α such that

$$q \sim V(p,r;\alpha,1-\alpha).$$

NM3 (Independence): $p \sim q$ implies

$$V(p,r;\alpha,1-\alpha)\sim V(q,r;\alpha,1-\alpha) \ \ (r\in \mathbf{P}(X), \alpha\in[0,1]).$$

NM4 (Convexity): For $p \prec q$ we have

$$p \prec V(p,q;\alpha,1-\alpha) \prec q \ \ (\alpha \in]0,1[.$$

The theorem below shows that the preference ordering on set of states which satisfies the proposed axioms can always be represented by a utility function.

Theorem 1 (Representation Theorem ([57], 1944)). *A preference ordering relation $\preceq f$ on $\mathbf{P}(X)$ satisfies axioms NM1, NM2, NM3 and NM4 if and only if, there is a real-valued function U: $\mathbf{P}(X)\to\mathbb{R}$ such that*

(i) U represents $\preceq$, i.e., for $p,q\in\mathbf{P}(X)$ holds $p \preceq fq$ if and only if

$$U(p) \preceq U(q);$$

(ii) U is affine, i.e., for every $p,q\in\mathbf{P}(X)$ and every $\alpha\in]0,1[$ we have

$$U(\alpha p+(1-\alpha)q) = \alpha U(p)+(1-\alpha)U(q).$$

Moreover, U is unique up to a linear transformation.

As consequence of these axioms there were deduced the following three important theorems [23, 24].

Theorem 2 (The expected utility theorem.). *Given a pair of options, each with a range of possible outcomes and associated probabilities of occurrence, that is, two lotteries, the preferred choice is the option (or lottery) that has the highest expected utility.*

Theorem 3 (The substitution theorem.). *A decision maker is indifferent between a lottery L and a certainty outcome whose utility is equal to the expected utility of the lottery.*

The person who has a transitive preference relation usually is called rational, in the opposite case he is irrational. The famous Arrow's Impossibility Theorem [3, 23] states that a group consisting only of rational individuals need not exhibit transitive preferences.

Theorem 4 (Arrow's Impossibility Theorem.). *Groups consisting of rational people are not necessarily rational.*

3 Generalization of the Probabilistic Approach

Pseudo-analysis is based on the semiring structure on the real interval $[a,b]\subseteq[-\infty,\infty]$, see [38, 40]. For some engineering applications see [4, 43]. In this paper we restrict ourselves on the special case, operations on the interval $[0,1]$ (see [28]) and therefore on special non-additive measures on so called pseudo-additive (decomposable) measures (see [28, 38, 40]).

Definition 1. A triangular conorm (t-conorm for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $S:[0,1]^2\to[0,1]$ such that for all $x,y,z\in[0,1]$ the following four axioms are satisfied:

(S1) *Commutativity* $S(x,y)=S(y,x)$,

(S2) *Associativity* $S(x,S(y,z))=S(S(x,y),z)$,

(S3) *Monotonicity* $S(x,y)\le S(x,z)$ whenever $y\le z$,

(S4) *Boundary Condition* $S(x,0)=x$.

If S is a t-conorm, then its dual t-norm $T:[0,1]^2\to[0,1]$ is given by

$$T(x,y)=1-S(1-x,1-y).$$

Definition 2. A t-norm T is restricted distributive over a t-conorm S if for all $x,y,z\in[0,1]$ we have

$$(RD)\quad T(x,S(y,z))=S(T(x,y),T(x,z)),$$

whenever $S(y,z)<1$.

The complete characterization of the pair (S,T) satisfying condition (RD) is given in [28].

A mapping $m: 2^X \to [0,1]$ is called a pseudo-additive measure (S-measure), if $m(\emptyset)=0, m(X)=1$ and if for all $A,B \in 2^X$ with $A \cap B = \emptyset$ we have $m(A \cup B) = S(m(A), m(B))$, see [10, 28, 38]. Important example is the maxitive measure, i.e., max-measure, where $m(A) = \sup_{x \in A} \pi(x)$.

We present now the possibilistic approach to the utility theory [13]. The belief state about which situation in X is the actual one is supposed to be represented by a possibility distribution π. A possibility distribution π defined on X takes its values on a valuation scale V, where V is supposed to be linearly ordered. V is assumed to be bounded and we take $\sup(V)=1$ and $\inf(V)=0$. Define $\mathbf{Pi}(X)$ as set of consistent possibility distributions over X, i.e.,

$$\mathbf{Pi}(X) = \{\pi : X \to V \mid \exists x \in X \pi(x) = 1\}.$$

The possibilistic mixture is an operation defined on $\mathbf{Pi}(X)$ which combines two possibility distributions π and π' into a new one, denoted $P(\pi,\pi';\alpha,\beta)$, with $\alpha,\beta \in V$ and $\max(\alpha,\beta)=1$, given by

$$P(\pi,\pi';\alpha,\beta) = \max(\min(\alpha,\pi),\min(\beta,\pi')).$$

Let $\sqsubseteq$ be a binary relation over $P(\pi,\pi';\alpha,\beta)$. Hence, we can write $\pi \sqsubseteq \pi'$ to indicate that possibilistic lottery π' is "preferred to or equivalent to" lottery π.

The proposed axiom systems for the possibilistic optimistic utility is

Pos 1 $\mathbf{Pi}(X)$ is equipped with a complete preordering structure $\sqsubseteq$.

Pos 2 (Continuity) For every $\pi \in \mathbf{Pi}(X)$ there exist $\lambda \in V$ such that $\pi \sim P(\overline{\pi},\underline{\pi};\lambda,1)$, where $\overline{\pi}$ and $\underline{\pi}$ are a maximal and a minimal element of $\mathbf{Pi}(X)$ w.r.t. $\sqsubseteq$, respectively.

Pos 3 (Independence) $\pi \sim \pi'$ implies $P(\pi,\pi'';\lambda,\mu) \sim P(\pi',\pi'';\lambda,\mu)$, for every $\pi'' \in \mathbf{Pi}(X)$ and every $\lambda,\mu \in V$.

Pos 4 (Uncertainty prone): $\pi \le \pi'$ implies $\pi \sqsubseteq \pi'$.

The set of axioms Pos1, Pos2, Pos3 and Pos4 characterize the preference ordering induced by an optimistic utility.

Theorem 5 (Representation Theorem ([13], 1998)). *A preference ordering relation* $\sqsubseteq f$ *on* $\mathbf{Pi}(X)$ *satisfies axioms Pos1, Pos2, Pos3 and Pos4 if and only if, there exist*

(i) *a linearly ordered utility scale* U, *with* $\inf(U)=0$ *and* $\sup(U)=1$*;*

(ii) *a preference function* $u : X \to U$ *such that* $u^{-1}(1) \neq \varnothing \neq u^{-1}(0)$, *and*

(iii) *an onto ordered preserving function* $h : V \to U$ *such that*

h(0)=0, h(1)=1,

in such a way that it holds: $\pi \sqsubseteq \pi'$ *if and only if* $\pi \lhd_u \pi`$, *where* $\lhd_u$ *is the ordering on* $\mathbf{Pi}(X)$ *induced by the qualitative utility*

$$QU(\pi) = \max_{x \in X} \min\Big(h\big(\pi(x)\big), u(x)\Big).$$

We present now the hybrid probabilistic-possibilistic utility theory [14, 15]. In order to generalize stated sets of axioms for utility theory, we denote $X = \{x_1, x_2, ..., x_n\}$ set of outcomes, $\Delta(X)$ the set of all S-measures defined on X.

Definition 3. A hybrid mixture operation which combines two S-measures m and m' into a new one, denoted $M(m,m';\alpha,\beta)$, for $a \in [0,1]$ and that (α,β) belongs to with $\Phi_{S,a} = \Big\{\alpha,\beta \Big| \alpha,\beta \in \left]0,1\right[, \alpha+\beta = 1+a$ or $\min(\alpha,\beta), \leq a, \max((\alpha,\beta)=1 \Big\}$ is given by

$$M(m,m';\alpha,\beta) = S(T(\alpha,m), T(\beta,m')),$$

where (S,T) is a pair of continuous t-conorm and t-norm, respectively, which satisfy the property of restricted distributivity (RD).

We propose the following set of axioms for a preference relation $\preceq_h$ defined over $\Delta(X)$ to represent optimistic utility

H1 $\Delta(X)$ is equipped with a complete preordering structure $\preceq_h$ (i.e., $\preceq_h$ is reflexive, transitive and complete).

H2 (Continuity) If $m \preceq_h m' \preceq_h m''$ then we have

(i) for $m, m', m'' > a$ there exists $\alpha \in \left]a,1\right[$ such that

$$m' \sim_h M(m, m''; 1 + a - \alpha, \alpha);$$

(ii) there exists $\alpha \in \left]0,a\right]$ such that $m' \sim_h M(m,m'';1,\alpha)$.

H3 (Independence) For all $m, m', m'' \in \Delta(X)$ and for all $\alpha, \beta \in \Phi_{S,a}$ we have that $m' \preceq_h m''$ is equivalent with $M(m', m; \alpha, \beta) \preceq_h M(m'', m; \alpha, \beta)$.

H4 (Uncertainty prone)

(i) if $m, m' > a$ then $m \preceq_h m'$ implies $m \preceq_h M(m, m'; \alpha, 1 + a - \alpha) \preceq_h m'$ for $\alpha \in]a, 1[$;

(ii) otherwise $m < m'$ implies $m \preceq_h m'$.

Now, we define a function of optimistic utility for all $m \in \Delta(X)$ in the following way

$$U(m) = S_{x_i \in X} T(m(x_i), u(x_i)),$$

where $u : X \to U$ is a preference function that assigns to each consequence of X a preference level of U, such that $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$.

Remark 1. It is interesting to note that U preserves the hybrid mixture in the sense that

$$\begin{aligned} U(M(m,m';\alpha,\beta)) &= S(T(\alpha, U(m)), T(\beta, U(m'))) \\ &= M(U(m), U(m'); \alpha, \beta). \end{aligned}$$

Theorem 6 (Representation Theorem - Optimistic Utility, [44]). *Let* $\Delta(X)$ *be the set of all* S *-measures defined on* 2^X, *and* $\preceq_h$ *a binary preference relation on* $\Delta(X)$. *Then the relation* $\preceq_h$ *satisfies the set of axioms H1, H2, H3, H4 if and only if there exist*

(i) *a linearly ordered utility scale* U, *with* $\inf(U) = 0$ *and* $\sup(U) = 1$;

(ii) *a preference function* $u : X \to [0, 1]$,

such that $m \preceq_h m'$ *if and only if* $mf \sqsubseteq_h m'$, *where* $f \sqsubseteq_h$ *is the ordering in* $\Delta(X)$ *induced by the optimistic utility function given by*

$$U(m) = S\,(T(m(x), u(x)),$$

where (S, T) *is a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (RD).*

4 Fuzzy Systems

A decision is made under the risk if the only available knowledge related the outcome states is the probability distribution. This can be used in the optimization of the utility function. If the knowledge about the probabilities of the outcome is unknown, then the decision have to be made under uncertainty. In engineering design one of the most critical problems is the preliminary design decision when the design is imprecise and most costly [2, 37, 59]. In such situations, fuzzy decision making can be used to handle this vagueness. This fuzziness can be modeled in different ways: fuzzy sets (membership function) [37] (The Method of Imprecision), [62], fuzzy measures (Choquet and Sugeno integrals) [21, 38, 58]. There are other design methodologies as optimization tools (linear, nonlinear, integer programming, multi-objective optimization - e.g., with weighted sum technique [36]), probability methods [54].

An overall evaluation of design alternatives have two parts: their partial evaluation and the importance of the criteria taken into account. The first step consists in the determination of fuzzy sets representing partial evaluation of the alternatives. Since there are many different judgments with respect to the expression of the suitability of variety of alternatives there is need for some methods for this purpose. The Analytical Hierarchy Process [48] and some other matrix methods as [45, 46] are very convenient tools for that purpose. At the second step all partial information is aggregated into a final rating. In engineering design the preliminary design decisions are very important although the design description is still imprecise. Fuzzy design methods are convenient for representing and manipulating design imprecision [25, 37, 62]. By the Method of Imprecision [37, 50] constraints can be imprecise permitting to choose preferences over a range of values. This method was specially developed for engineering design and implies that the trade-off combination functions (aggregation operators) have to satisfy the boundary conditions, monotonicity, continuity, annihilation and idempotency. Then it follows by [37, 51] that any weigthed quasi-linear mean that satisfies the annihilation property is design-appropriate. A weighted aggregation function which continuous, strictly monotonic, idempotent and bisymmetrical has the representation ([1, 17])

$$\mathbf{M}^{f}_{\omega_1,\ldots,\omega_n}(x_1,\ldots,x_n) = f^{-1}\left(\frac{\omega_1 f(x_1) + \cdots + \omega_n f(x_n)}{\omega_1 + \cdots + \omega_n}\right),$$

where f is a strictly monotone continuous function. By this representation it is possible to construct special convenient families of aggregation functions ([17, 21, 30, 37]). Drakopoulos [9] proved that probabilities have a higher representational power than fuzzy sets (with respect to max-min) and possibilities (special fuzzy measure) for finite domain, but at the cost of higher computational complexity and reduced computational efficiency (they have equal representational power when their domains are infinite).

5 Aggregation in Engineering Design

The aggregation of incoming data plays a key role in applications of several intelligent systems. The aggregation functions (operators) form a fundamental part of multi-criteria decision making, engineering design, expert systems, pattern recognition, neural networks, fuzzy controllers, genetic algorithms, etc. ([17, 20, 28, 37, 61]).

We restrict ourselves to the inputs and outputs from the unit interval $[0,1]$. Note that the case of any other closed interval is the question of rescaling only.

Definition 4. An *aggregation function* $\mathbf{A}$ is a non-decreasing mapping

$$\mathbf{A} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$$

fulfilling the following conditions

(i) $0 \le x_i \le y_i ?1, i = 1,\le, n$ imply $\mathbf{A}(x_1,...,x_n) \le \mathbf{A}(y_1,...,y_n)$;

(ii) $\mathbf{A}(x) = x$ for all $x \in [0,1]$;

(iii) $\mathbf{A}(0,...,0) = 0$ and $\mathbf{A}(1,...,1) = 1$.

Property (i) in Definition 4 is *the monotonicity* and properties (ii) and (iii) are *the boundary conditions*. Each aggregation function $\mathbf{A}$ can be represented as a system $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of n-ary operators $\mathbf{A}_n, n \in \mathbb{N}$, on the unit interval, where $\mathbf{A}_1$ is the identity operator on $[0,1]$ and each $\mathbf{A}_n, n \ge 2$, is non-decreasing and $\mathbf{A}_n(0,...,0) = 0, \mathbf{A}_n(1,...,1) = 1$.

Depending on the field of application, several additional properties can be required and/or examined, such as commutativity, associativity, continuity, idempotency, compensation, cancellativity, etc. Note, for example, that the associativity of an aggregation function $\mathbf{A}$ means that the binary function $\mathbf{A}_2$ is associative and its corresponding n-ary extensions (for $n > 2$) are just the relevant n-ary operators $\mathbf{A}_n$. Therefore, an associative aggregation function $\mathbf{A}$ is fully determined by $\mathbf{A}_2$. If $\mathbf{A}$ is an aggregation function, then the operator $\mathbf{DA} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ defined by

$$\mathbf{DA}(x_1,...,x_n) = 1 - \mathbf{A}(1 - x_1,...,1 - x_n)$$

is called the *dual operator* of $\mathbf{A}$. $\mathbf{DA}$ is also an aggregation function.

Fuzzy design methods are convenient for representing and manipulating design imprecision [37, 62]. The Method of Imprecision [37] was specially developed for engineering design and implies that the trade-off combination functions (aggregation operators) have to satisfy the boundary conditions, monotonicity, continuity, annihilation, and idempotency, where annihilation means that if one argument (when the preference for any one attribute of the design sinks to zero) of the aggregation operator is zero then the value of the aggregation operator (the overall preference of the design) is zero. We remark that if the weights $\{\omega_i\}$ in a weighted aggregation function are given with respect to a ratio scale, then ω_i are not uniquely determined, since any other system of weights $\{\omega_i'\}$ with $\omega_i' = C\omega_i$ for a positive rational number C is convenient, e.g., $\omega_i' = \frac{\omega_i}{\sum_i \omega_i}$. Specially important cases are $f(x)=x, f(x)=\log x$ and $f(x)=x^s$. The last case is interesting since it generates a parameterized family of aggregation functions. For $s>0$, the annihilation property fails, but it can be handled in the practical engineering design by assuming that preferences smaller than some small ε are not relevant for the designer [51]. There are also design-appropriate aggregation functions which are not weighted quasi-arithmetical means, that is, they are not strictly monotone.

Starting from a given t-norm and/or t-conorm, several useful operations on $[0,1]$ can be introduced. The conditions (i) - (iii) required for an aggregation operator $\mathbf{A}$ are the genuine properties of triangular norms and conorms.

From the practical application point of view, there are suggestions to use the special aggregation functions, so-called *compensatory operators,* in order to model intersection and union in many-valued logic. The main goal of compensatory operators is to model an aggregation of incoming values. If two values are aggregate by a t-norm, then there is no compensation between low and high values. On the other hand, a t-conorm based aggregation provides the full compensation. None of the above cases covers the real decision making. To avoid such inaccuracies, in [63] suggested two kinds of so-called compensatory operators, see [33]. The first of them was γ-operator, $\Gamma : [0,1] \to [0,1],\ \gamma \in [0,1],\ n \geqslant 2$

$$\Gamma_\gamma(x_1,\dots,x_n)=\left(\prod_{i=1}^{n}x_i\right)^{1-\gamma}\left(1-\prod_{i=1}^{n}(1-x_i)\right)^{\gamma}.$$

Here parameter γ indicates the degree of compensation. Note that γ-operators are a special class of exponential compensatory operators [28]. For a given t-norm T t-conorm S (not necessarily dual to T) and parameter γ indicating the degree of compensation, the exponential compensatory operator $E_{T,S,\gamma} : [0,1]^n \to [0,1],\quad n \geq 2,$ is defined by

$$E_{T,S,\gamma}(x_1,\dots,x_n)=(T(x_1,\dots,x_n))^{1-\gamma}(S(x_1,\dots,x_n))^{\gamma}.$$

It is obvious that γ-operator is based an $T_{\mathbf{P}}$ $S_{\mathbf{P}}$, $\Gamma_\gamma = E_{T_{\mathbf{P}},S_{\mathbf{P}},\gamma}$. Further note that $E_{T,S,\gamma}$ is a logarithmic convex combination of T and S and up to the case when $\gamma\in\{0,1\}$ it is non-associative. Another class of compensatory operators proposed by [63, 64] are so-called convex-linear compensatory operators.

It was proposed an *associative class of compensatory operators* in [27]. The degree of compensation is ruled by two parameters, namely by the neutral element e and the compensation factor k. Let T be a given strict t-norm with additive generator $f, f(\frac{1}{2})=1,$ and let S be a given strict t-conorm with an additive generator $g, g(\frac{1}{2})=1$. For a given $e\in\,]0,1[,\ k\in\,]0,\infty[$, we define an associative compensatory operator

$$C(T,S,e,k) = C : [0,1]^2 \setminus \{(0,1),(1,0)\} \to [0,1]$$

by

$$C(x,y) = h^{-1}(h(x) + h(y)),$$

where $h : [0,1] \to [-\infty,\infty]$ is a strictly increasing bijection such that

$$h(x) = \begin{cases} kf(\frac{x}{e}) & \text{if } x \in [0,e] \\ g(\frac{x-e}{1-e}) & \text{if } x \in]e,1]. \end{cases}$$

Engineering decision making need more general mathematical models, which involve also non-additive measures. Previously used additive probability measures could not model some situations, e.g., the Ellsberg Paradox, see [21]. For the non-additive set function (measure) m defined on a σ-algebra $\mathcal{A}$ of subsets of a set X (for finite X it is usually taken $\mathcal{A} = 2^X$, the family of all subsets), the difference $m(A \cup B) - m(B)$ depends on B and can be interpreted as the effect of A joining B, [21, 38, 52, 53, 58]. A monotone set function m with $m(\emptyset) = 0$ is usually called fuzzy measure. More than the contribution of the extension principle of the fuzzy sets [55], fuzzy connectives [16, 21, 27, 63] and fuzzy measures are important in the problem of the modeling of the behavior of decision makers. Utility theory [17, 21, 47] deals with preference relations describing the decision behavior, and as the basis of decision theory, is well axiomatically based on the fuzzy measures and Choquet integral [17, 20, 22, 38]. The Choquet integral approach is generalized in many directions ([28, 34, 38]). As the mapping, the fuzzy integral is defined by a set of 2^n (for n elements basic set X) parameters and a t-conorm system. The word "identification" has the origin in the system theory and is preferred to the word "learning", though the algorithms for finding the appropriate fuzzy measure could be the learning samples minimizing certain criterion. Unknown measure to be identified can be regarded as the part of the parameter identification [19, 20, 21].

Conclusion

We have given a short overview of some basic facts from the theory of pseudo-analysis, mostly related to pseudo-operations. As a generalization of von Neumann and Morgestern utility theory, using pseudo-analysis, there are presented approach based on possibility theory, and as a common generalization through hybridization of the both approaches is given. We modeled uncertainty in engineering design with fuzzy systems, which involves more general real operations: aggregation functions [20]. We remark that S-measures and corresponding integrals have the advantage that for n elements of the basic set X they require only n parameters.

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