Chapter 8
Expected Total Cost Minimum Design of Plane Frames by Means of Stochastic Linear Programming Methods

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Abstract Yield stresses, allowable stresses, moment capacities (plastic moments with respect to compression, tension and rotation), applied loadings, cost factors, manufacturing errors, etc., are not given fixed quantities in structural analysis and optimal design problems, but must be modeled as random variables with a certain joint probability distribution. Problems from plastic analysis and optimal plastic design are based on the convex yield (feasibility) criterion and the linear equilibrium equation for the stress (state) vector.

After the formulation of the basic mechanical conditions including the relevant material strength parameters and load components as well as the involved design variables (as, e.g., sizing variables) for plane frames, several approximations are considered: (1) approximation of the convex yield (feasible) domain by means of convex polyhedrons (piecewise linearization of the yield domain); (2) treatment of symmetric and non symmetric yield stresses with respect to compression and tension; (3) approximation of the yield condition by using given reference capacities.

As a result, for the survival of plane frames a certain system of necessary and/or sufficient linear equalities and inequalities is obtained. Evaluating the recourse costs, i.e., the costs for violations of the survival condition by means of piecewise linear convex cost functions, a linear program is derived for the minimization of the total costs including weight-, volume- or more general initial (construction) costs. Appropriate cost factors are given. Considering then the expected total costs from construction as well as from possible structural damages or failures, a stochastic linear optimization problem (SLP) is obtained. Finally, discretization of the probability distribution of the random parameter vector yields a (large scale) linear program (LP) having a special structure. For LP's of this type numerous, very efficient LP-solvers are available – also for the solution of very large scale problems.

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8.1 Introduction

8.1.1 Plastic Analysis of Structures

Many materials, e.g., most of metals, have distinct, plastic properties, i.e., they are ductile, see, e.g., [10,23]. Even after the stress intensity attains the yield point stress, such materials can deform considerably without breaking. This implies that if the stress intensity at a certain point of a hyperstatic structure reaches the critical (yield) value, the structure does not necessarily fail or deform excessively. Instead, a certain amount of stress redistribution takes place and some further load increments can be supported. Structural failure does not occur before a kinematic mechanism of unconstrained plastic flow develops. Thus, the actual load-carrying capacity of a structure is higher (in some cases quite considerably) than that derived from classical elastic analysis. A crucial question for the engineer designing structures like buildings, bridges, etc., or structural components is to which extent a plastic deformation is permissible without leading to a failure of the structure, the component, resp., with respect to the expected load and material strength conditions. Applying standard methods, the load carrying capacity is determined using a certain code with general rules for safety evaluations. The use of such general rules may be very expensive in the safety evaluation and design of a structure. On the other hand, safety assessment and design based on stochastic optimization techniques, taking into account the available knowledge about random parameter variations, reduce the expected total project costs (primary costs, e.g., costs of construction, plus recourse cots, e.g., strengthening costs) considerably. Consequently, this way one obtains more robust (safe) information about the maximum load factors, hence, the carrying capacity, as well as about robust optimal designs. A further big advantage of stochastic optimization methods is the possibility of updates of the maximum load factors and robust optimal designs based on inspection, sampling and other posterior information about the probability distribution of the random parameters. For elastic-perfectly plastic materials, the ultimate load condition corresponding to complete collapse of the structure can be obtained through application of a pair of dual theorems [14,18,23]:

(ST) Static Theorem (lower bound or safe theorem) If any stress distribution throughout the structure can be found which is everywhere in equilibrium internally and balances the external loads, and at the same time does not violate the yield condition, those external loads will be carried safely by the structure.

(KT) Kinematic Theorem (upper bound or unsafe theorem)
Collapse occurs if a collapse mechanism, fulfilling the compatibility condition, exists such that the work done by the external loads is larger than the corresponding internal plastic work.
Limit analysis is concerned [5,8,10,14,16,18,19,22,24,33,34,36,37,40,41] with establishing the strength of a structure, i.e., its capacity for the supporting of loads. Hence, using the plastic ductility of structural materials in

improving the design of structures, limit analysis is not concerned with deformation: it can not therefore provide the load carrying capacity for a structure with elements that have a limited ductility or deformability, nor for a structure which becomes unstable because of the displacements induced by plastic deformation, see [10, 13, 18, 24].

8.1.2 Limit (Collapse) Load Analysis of Structures as a Linear Programming Problem

Assuming that the material behaves as an elastic-perfectly plastic material [17, 23, 32] a conservative estimate of the collapse load factor λ_T is based [5–8, 10, 13, 14, 22, 33, 40] on the following linear program:

$$\text{maximize } \lambda \tag{8.1a}$$

s.t.

$$F^L \le F \le F^U \tag{8.1b}$$

$$CF = \lambda R_0. \tag{8.1c}$$

Here, (8.1c) is the equilibrium equation of a statically indeterminate loaded structure involving an $m \times n$ matrix $C = (c_{ij}), m < n$, of given coefficients $c_{ij}, 1 \le i \le m, 1 \le j \le n$, depending on the undeformed geometry of the structure having n_0 members (elements). After taking into account the given support (boundary) conditions, we may suppose that rank $C = m$. Furthermore, R_0 is an external load m-vector, and F denotes the n-vector of internal forces and bending-moments in the relevant points (sections, nodes or elements) of lower and upper bounds F^L, F^U.

For a plane or spatial truss [25, 38] we have that $n = n_0$, the matrix C contains the direction cosines of the members, and F involves only the normal (axial) forces moreover,

$$F_j^L := \sigma_{yj}^L A_j, F_j^U := \sigma_{yj}^U A_j, j = 1, \dots, n(= n_0), \tag{8.2}$$

where A_j is the (given) cross-sectional area, and $\sigma_{yj}^L, \sigma_{yj}^U$, respectively, denotes the yield stress in compression (negative values) and tension (positive values) of the j-th member of the truss. In case of a plane frame, F is composed of subvectors [38],

$$F^{(k)} = \begin{pmatrix} F_1^{(k)} \\ F_2^{(k)} \\ F_3^{(k)} \end{pmatrix} = \begin{pmatrix} t_k \\ m_k^+ \\ m_k^- \end{pmatrix}, \tag{8.3a}$$

where $F_1^{(k)} = t_k$ denotes the normal (axial) force, and $F_2^{(k)} = m_k^+, F_3^{(k)} = m_k^-$ are the bending-moments at the positive ("right"), negative ("left") end of the k-th member with respect to a certain chosen orientation of the members. In this case

F^L, F^U contain – for each member k – the subvectors

$$F^{(k)L} = \begin{pmatrix} \sigma^L_{yk} A_k \\ -M_{kpl} \\ -M_{kpl} \end{pmatrix}, F^{(k)U} = \begin{pmatrix} \sigma^U_{yk} A_k \\ M_{kpl} \\ M_{kpl} \end{pmatrix}, \tag{8.3b}$$

resp., where $M_{kpl}, k = 1, \dots, n_0$, denotes the plastic moments (moment capacities) [17, 32] given by

$$M_{kpl} = \sigma^U_{yk} W_{kpl}, \tag{8.3c}$$

and $W_{kpl} = W_{kpl}(A_k)$ is the plastic section modulus of the cross-section of the k-th member (beam) with respect to the local z-axis. In order to omit instabilities, such as buckling, σ^L_{yk} can be selected by

$$\sigma^L_{yk} := -\kappa_k \sigma^U_{yk} \tag{8.3d}$$

with a certain reduction factor κ_k.

For a *space frame* [25,38], corresponding to the k-th member (beam), F contains the subvector

$$F^{(k)} := (t_k, m_{kT}, m^+_{k\bar{y}}, m^+_{k\bar{z}}, m^-_{k\bar{y}}, m^-_{k\bar{z}})^T, \tag{8.4a}$$

where t_k is the normal (axial) force, m_{kT} the twisting moment, and $m^+_{k\bar{y}}, m^+_{k\bar{z}}$, $m^-_{k\bar{y}}, m^-_{k\bar{z}}$ denote four bending moments with respect to the local $\bar{y}, z$-axis at the positive, negative end of the beam, respectively. Finally, the bounds F^L, F^U for F are given by

$$F^{(k)L} = \left(\sigma^L_{yk} A_k, -M^{\bar{p}}_{kpl}, -M^{\bar{y}}_{kpl}, -M^{\bar{z}}_{kpl}, -M^{\bar{y}}_{kpl}, -M^{z}_{kpl}\right)^T \tag{8.4b}$$

$$F^{(k)U} = \left(\sigma^U_{yk} A_k, M^{\bar{p}}_{kpl}, M^{\bar{y}}_{kpl}, M^{\bar{z}}_{kpl}, M^{\bar{y}}_{kpl}, M^{\bar{z}}_{kpl}\right)^T, \tag{8.4c}$$

where [17, 32]

$$M^{\bar{p}}_{kpl} := \tau_{yk} W^{\bar{p}}_{kpl}, M^{\bar{y}}_{kpl} := \sigma^U_{yk} W^{\bar{y}}_{kpl}, M^{\bar{z}}_{kpl} := \sigma^U_{yk} W^{\bar{z}}_{kpl}, \tag{8.4d}$$

are the plastic moments of the cross-section of the k-th element with respect to the local twisting axis, the local $\bar{y}-, \bar{z}$-axis, respectively. In (8.4d), $W^{\bar{p}}_{kpl} = W^{\bar{p}}_{kpl}(x)$ and $W^{\bar{y}}_{kpl} = W^{\bar{y}}_{kpl}(x), W^{\bar{z}}_{kpl} = W^{\bar{z}}_{kpl}(x)$, resp., denote the polar, axial modulus of the cross-sectional area of the k-th beam and τ_{yk} denotes the yield stress with respect to torsion; we suppose that $\tau_{yk} = \kappa_{\tau k} \sigma^U_{yk}$ with a reduction factor $\kappa_{\tau k}$. Moreover, x denotes the r-vector of design variables, see Sect. 8.1.1 for more details.

Remark 8.1. Possible plastic hinges [17, 24, 32] are taken into account by inserting appropriate eccentricities $e_{kl} > 0, e_{kr} > 0, k = 1, \dots, n_0$, with $e_{kl}, e_{kr} << L_k$, where L_k is the length of the k-th beam.

Remark 8.2. Working with more general yield polygons [1, 40, 42], the stress condition (8.1b) is replaced by the more general system of inequalities

$$H(F_d^U)^{-1}F \leq h. \tag{8.5a}$$

Here, (H, h) is a given $\nu \times (n+1)$ matrix, and $F_d^U := (F_j^U \delta_{ij})$ denotes the $n \times n$ diagonal matrix of principal axial and bending plastic capacities

$$F_j^U := \sigma_{yk_j}^U A_{kj}, F_j^U := \sigma_{yk_j}^U W_{k_j pl}^{\kappa j}, \tag{8.5b}$$

where $kj, \kappa j$ are indices as arising in (8.3b)–(8.4d). The more general case (8.5a) can be treated by similar methods as the case (8.1b) which is considered here.

8.1.3 Plastic and Elastic Design of Structures

In the plastic design of trusses and frames [22,26,27,29,34,36] having n_0 members, the n-vectors F^L, F^U of lower and upper bounds

$$F^L = F^L(\sigma_y^L, \sigma_y^U, x), F^U = F^U(\sigma_y^L, \sigma_y^U, x), \tag{8.6}$$

for the n-vector F of internal member forces and bending moments $F_j, j = 1, \ldots, n$, are determined [13, 22] by the yield stresses, i.e., compressive limiting stresses (negative values) $\sigma_y^L = \left(\sigma_{y1}^L, \ldots, \sigma_{yn_0}^L\right)^T$, the tensile yield stresses $\sigma_y^U = \left(\sigma_{y1}^U, \ldots, \sigma_{yn_0}^U\right)^T$, and the r-vector

$$x = (x_1, x_2, \ldots, x_r)^T \tag{8.7}$$

of design variables of the structure. In case of trusses we have that, cf. (8.2),

$$\begin{aligned} F^L &= \sigma_{yd}^L A(x) = A(x)_d \sigma_y^L, \\ F^U &= \sigma_{yd}^U A(x) = A(x)_d \sigma_y^U, \end{aligned} \tag{8.8}$$

where $n = n_0$, and $\sigma_{yd}^L, \sigma_{yd}^U$ denote the $n \times n$ diagonal matrices having the diagonal elements $\sigma_{yj}^L, \sigma_{yj}^U$, respectively, $j = 1, \ldots, n$, moreover,

$$A(x) = \Big[A_1(x), \ldots, A_n(x)\Big]^T \tag{8.9}$$

is the n-vector of cross-sectional area $A_j = A_j(x), j = 1, \ldots, n$, depending on the r-vector x of design variables $x_\kappa, \kappa = 1, \ldots, r$, and $A(x)_d$ denotes the $n \times n$ diagonal matrix having the diagonal elements $A_j = A_j(x), 1 < j < n$.

Corresponding to (8.1c), here the equilibrium equation reads

$$CF = R_u, \tag{8.10}$$

where R_u describes [22] the ultimate load [representing constant external loads or self-weight expressed in linear terms of $A(x)$].

The *plastic design* of structures can be represented then [1,2] by the optimization problem

$$\min G(x), \tag{8.11a}$$

s.t.

$$F^L(\sigma_y^L, \sigma_y^U, x) \le F \le F^U(\sigma_y^L, \sigma_y^U, x) \tag{8.11b}$$

$$CF = R_u \tag{8.11c}$$

$$x \in D, \tag{8.11d}$$

where $G = G(x)$ is a certain objective function, e.g., the volume or weight of the structure, and $C \subset \mathbb{R}^+$ denotes the convex set of admissible design vecotrs x.

Remark 8.3. As mentioned in Remark 8.2, working with more general yield polygons, (8.11b) is replaced by the condition

$$H[F^U(\sigma_y^U, x)_d]^{-1} F \le h. \tag{8.11e}$$

For the *elastic design* we must replace the yield stresses σ_y^L, σ_y^U by the allowable stresses σ_a^L, σ_a^U and instead of ultimate loads we consider service loads R_s. Hence, instead of (8.11a–d) we have the related program

$$\min G(x), \tag{8.12a}$$

s.t.

$$F^L(\sigma_a^L, \sigma_a^U, x) \le F \le F^U(\sigma_a^L, \sigma_a^U, x), \tag{8.12b}$$

$$CF = R_s, \tag{8.12c}$$

$$x^L \le x \le x^U, \tag{8.12d}$$

where x^L, x^U still denote lower and upper bounds for x.

8.2 Plane Frames

For each bar $i = 1, \ldots, B$ of a plane frame with member load vector $F_i = (t_i, m_i^+, m_i^-)^T$ we consider [37,41] the load at the negative, positive end

$$F_i^- := (t_i, m_i^-)^T, F_i^+ := (t_i, m_i^+)^T, \tag{8.13}$$

respectively.

Furthermore, for each bar/beam with rigid joints we have several plastic capacities: The plastic capacity N_{ipl}^L of the bar with respect to axial compression, hence, the maximum axial force under compression is given by

$$N_{ipl}^L = |\sigma_{yi}^L| \cdot A_i, \tag{8.14a}$$

where $\sigma_{yi}^L < 0$ denotes the (negative) yield stress with respect to compression and A_i is the cross sectional area of the ith element. Correspondingly, the plastic capacity with respect to (axial) tension reads:

$$N_{ipl}^U = \sigma_{yi}^U \cdot A_i, \tag{8.14b}$$

where $\sigma_{yi}^U > 0$ is the yield stress with respect to tension. Besides the plastic capacities with respect to the normal force, we have the moment capacity

$$M_{ipl} = \sigma_{yi}^U \cdot W_{ipl} \tag{8.14c}$$

with respect to the bending moments at the ends of the bar i.

Remark 8.4. Note that all plastic capacities have nonnegative values.

Using the plastic capacities (8.14a–c), the load vectors F_i^+, F_i^+ given by (8.13) can be replaced by dimensionless quantities

$$F_i^{L-} := \left(\frac{t_i}{N_{ipl}^L}, \frac{m_i^-}{M_{ipl}}\right)^T, F_i^{U-} := \left(\frac{t_i}{N_{ipl}^U}, \frac{m_i^-}{M_{ipl}}\right)^T \tag{8.15a,b}$$

$$F_i^{L+} := \left(\frac{t_i}{N_{ipl}^L}, \frac{m_i^+}{M_{ipl}}\right)^T, F_i^{U+} := \left(\frac{t_i}{N_{ipl}^U}, \frac{m_i^+}{M_{ipl}}\right)^T \tag{8.15c,d}$$

for the negative, positive end, resp., of the ith bar.

Remark 8.5 (Symmetric yield stresses under compression and tension).
In the important special case that the absolute values of the yield stresses under compression (<0) and tension (>0) are equal, hence,

$$\sigma_{yi}^L = -\sigma_{yi}^U \tag{8.16a}$$

$$N_{ipl}^L = N_{ipl}^U =: N_{ipl}. \tag{8.16b}$$

The limit between the elastic and plastic state of the elements is described by the feasibility or yield condition: At the negative end we have the condition

$$F_i^{L-} \in K_i, F_i^{U-} \in K_i \tag{8.17a,b}$$

and at the positive end the condition reads

$$F_i^{L+} \in K_i, F_i^{U+} \in K_i. \tag{8.17c,d}$$

Here, K_i, $K_i \subset \mathbb{R}^2$, denotes the feasible domain of bar "i" having the following properties:

- K_i is a closed, convex subset of $\mathbb{R}^2$.
- The origin 0 of $\mathbb{R}^2$ is an interior point of K_i.
- The interior $\mathring{K}_i$ of K_i represents the elastic states.
- At the boundary ∂K_i yielding of the material starts.

Considering, e.g., bars with rectangular cross sectional areas and symmetric yield stresses, cf. Remark 8.5, K_i is given by $K_i = K_{0,sym}$, where [19, 21]

$$K_{0,\text{sym}} = \{(x, y)^T : x^2 + |y| \leq 1\} \tag{8.18}$$

where $x = \frac{N}{N_{pl}}$ and $y = \frac{M}{M_{pl}}$, (see Fig. 8.1). Note that the symbols $x, y, (x, y)$, resp., denote in this Sect. 8.2 just real variables, a point in the real plane.

In case (8.18) and supposing symmetric yield stresses, the yield condition (8.17a–d) reads

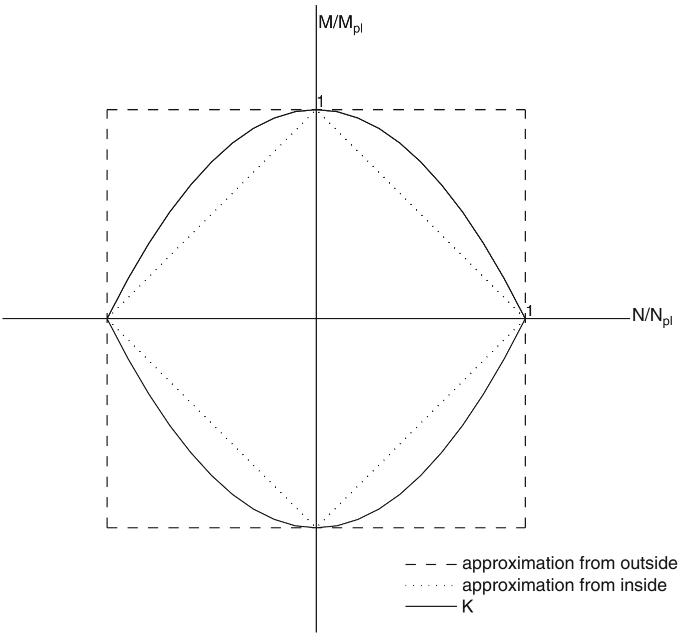

Fig. 8.1 Domain $K_{0,\text{sym}}$ with possible approximations

$$\left(\frac{t_i}{N_{ipl}}\right)^2 + \left|\frac{m_i^-}{M_{ipl}}\right| \leq 1, \tag{8.19a}$$

$$\left(\frac{t_i}{N_{ipl}}\right)^2 + \left|\frac{m_i^+}{M_{ipl}}\right| \leq 1. \tag{8.19b}$$

Remark 8.6. Because of the connection between the normal force t_i and the bending moments, (8.19a,b) is also called "M–N-interaction".

If the M–N-interaction is not taken into account, $K_{0,\mathrm{sym}}$ is approximated from outside, see Fig. 8.1, by

$$K^u_{0,\mathrm{sym}} := \{(x,y)^T : |x|, |y| \leq 1\}. \tag{8.20}$$

Hence, (8.17a–d) are replaced, cf. (8.19a,b), by the simpler conditions

$$|t_i| \leq N_{ipl} \tag{8.21a}$$

$$|m_i^-| \leq M_{ipl} \tag{8.21b}$$

$$|m_i^+| \leq M_{ipl}. \tag{8.21c}$$

Since the symmetry condition (8.16a) does not hold in general, some modifications of the basic conditions (8.19a,b) are needed. In the non symmetric case $K_{0,xm}$ must be replaced by the intersection

$$K_0 = K_0^U \cap K_0^L \tag{8.22}$$

of two convex sets K_0^U and K_0^L. For a rectangular cross-sectional area we have

$$K_0^U = \left\{(x,y)^T : x \leq \sqrt{1-|y|},\ |y| \leq 1\right\} \tag{8.23a}$$

for tension and

$$K_0^L = \left\{(x,y)^T : -x \leq \sqrt{1-|y|},\ |y| \leq 1\right\} \tag{8.23b}$$

compression, where again $x = \frac{N}{N_{pl}}$ and $y = \frac{M}{M_{pl}}$ (see Fig. 8.2).

In case of tension, see (8.17b,d), from (8.23a) we obtain then the feasibility condition

$$\frac{t_i}{N^U_{ipl}} \leq \sqrt{1 - \left|\frac{m_i^-}{M_{ipl}}\right|} \tag{8.24a}$$

$$\frac{t_i}{N^U_{ipl}} \leq \sqrt{1 - \left|\frac{m_i^+}{M_{ipl}}\right|} \tag{8.24b}$$

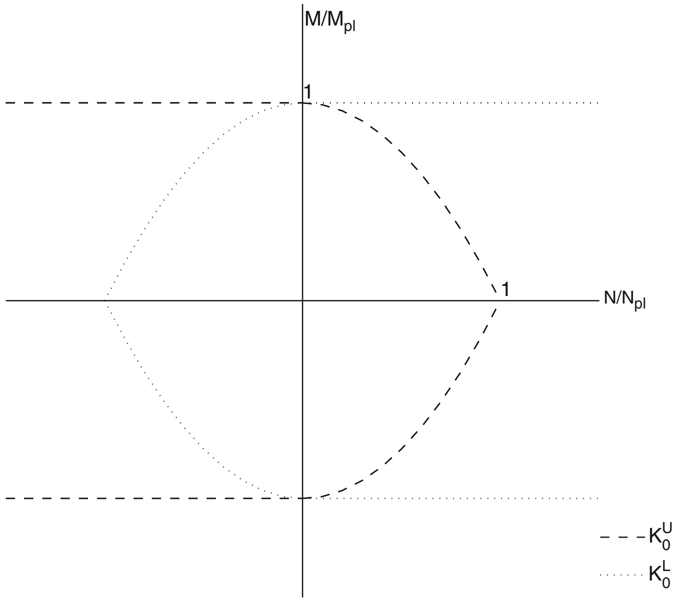

Fig. 8.2 Feasible domain as intersection of K_0^U and K_0^L

$$\left|\frac{m_i^-}{M_{ipl}}\right| \le 1 \tag{8.24c}$$

$$\left|\frac{m_i^+}{M_{ipl}}\right| \le 1. \tag{8.24d}$$

For compression, with (8.17a,c) and (8.23b) we get the feasibility condition

$$-\frac{t_i}{N_{ipl}^L} \le \sqrt{1 - \left|\frac{m_i^-}{M_{ipl}}\right|} \tag{8.24e}$$

$$-\frac{t_i}{N_{ipl}^L} \le \sqrt{1 - \left|\frac{m_i^+}{M_{ipl}}\right|} \tag{8.24f}$$

$$\left|\frac{m_i^-}{M_{ipl}}\right| \le 1 \tag{8.24g}$$

$$\left|\frac{m_i^+}{M_{ipl}}\right| \le 1. \tag{8.24h}$$

From (8.24a), (8.24e) we get

$$- N^L_{ipl}\sqrt{1-\left|\frac{m^-_i}{M_{ipl}}\right|} \le t_i \le N^U_{ipl}\sqrt{1-\left|\frac{m^-_i}{M_{ipl}}\right|}. \tag{8.25a}$$

and (8.24b), (8.24f) yield

$$- N^L_{ipl}\sqrt{1-\left|\frac{m^+_i}{M_{ipl}}\right|} \le t_i \le N^U_{ipl}\sqrt{1-\left|\frac{m^+_i}{M_{ipl}}\right|}. \tag{8.25b}$$

Furthermore, (8.24c), (8.24g) and (8.24d), (8.24h) yield

$$|m^-_i| \le M_{ipl} \tag{8.25c}$$

$$|m^+_i| \le M_{ipl}. \tag{8.25d}$$

For computational purposes, piecewise linearizations are applied [30] to the nonlinear conditions (8.25a,b), see also [3, 15]. A basic approximation of K^L_0 and K^U_0 is given by

$$K^{Uu}_0 := \{(x,y)^T : x \le 1, |y| \le 1\} =]\infty, 1] \times [-1, 1] \tag{8.26a}$$

and

$$K^{Lu}_0 := \{(x,y)^T : x \ge -1, |y| \le 1\} = [-1, \infty] \times [-1, 1] \tag{8.26b}$$

with $x = \frac{N}{N_{pl}}$ and $y = \frac{M}{M_{pl}}$ (see Fig. 8.3).

Since in this approximation the M–N-interaction is not taken into account, condition (8.17a–d) is reduced to

$$- N^L_{ipl} \le t_i \le N^U_{ipl} \tag{8.27a}$$

$$|m^-_i| \le M_{ipl} \tag{8.27b}$$

$$|m^+_i| \le M_{ipl}. \tag{8.27c}$$

8.2.1 *Yield Condition in Case of $M - N$-Interaction*

8.2.1.1 Symmetric Yield Stresses

Consider first the case

$$\sigma^U_{yi} = -\sigma^L_{yi} =: \sigma_{yi},\ i = 1, \dots, B. \tag{8.28a}$$

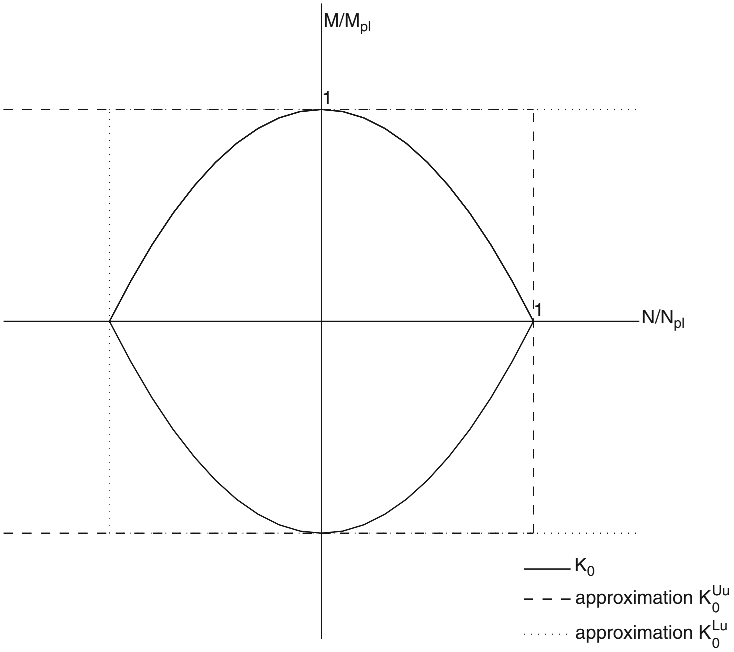

Fig. 8.3 Approximation of K_0 by K_0^{Lu} and K_0^{Uu}

Then,

$$N_{ipl}^L := |\sigma_{yi}^L| A_i = \sigma_{yi}^U A_i =: N_{ipl}^U, \tag{8.28b}$$

hence,

$$N_{ipl} := N_{ipl}^L = N_{ipl}^U = \sigma_{yi} A_i. \tag{8.28c}$$

Moreover,

$$M_{ipl} = \sigma_{yi}^U W_{ipl} = \sigma_{yi} W_{ipl} = \sigma_{yi} A_i \bar{y}_{ic}, \; i = 1, \dots, B, \tag{8.28d}$$

where $\overline{y}_{ic}$ denotes the arithmetic mean of the centroids of the two half areas of the cross-sectional area A_i of bar i.

Depending on the geometric from of the cross-sectional areal (rectangle, circle, etc.), for the element load vectors

$$F_i = \begin{pmatrix} t_i \\ m_i^+ \\ m_i^- \end{pmatrix}, i = 1, \dots, B, \tag{8.29}$$

of the bars we have the yield condition:

$$\left|\frac{t_i}{N_{ipl}}\right|^\alpha + \left|\frac{m_i^-}{M_{ipl}}\right| \le 1 \quad \text{(negative end)} \tag{8.30a}$$

$$\left|\frac{t_i}{N_{ipl}}\right|^\alpha + \left|\frac{m_i^+}{M_{ipl}}\right| \le 1 \quad \text{(positive end).} \tag{8.30b}$$

Here, $\alpha > 1$ is a constant depending on the type of the cross-sectional area of the ith bar. Defining the convex set

$$K_0^\alpha := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x|^\alpha + |y| \le 1 \right\}, \tag{8.31}$$

for (8.30a,b) we have also the representation

$$\begin{pmatrix} \frac{t_i}{N_{ipl}} \\ \frac{m_i^-}{M_{ipl}} \end{pmatrix} \in K_0^\alpha \quad \text{(negative end)} \tag{8.32a}$$

$$\begin{pmatrix} \frac{t_i}{N_{ipl}} \\ \frac{m_i^+}{M_{ipl}} \end{pmatrix} \in K_0^\alpha \quad \text{(positive end).} \tag{8.32b}$$

Piecewise Linearization of K_0^α

Due to the symmetry of K_0^α with respect to the transformation

$$x \to -x, y \to -y,$$

K_0^α is piecewise linearized as follows.

Starting from a boundary point of K_0^α, hence,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \partial K_0^\alpha \quad \text{with} \quad u_1 \ge 0,\ u_2 \ge 0, \tag{8.33a}$$

we consider the gradient of the boundary curve

$$f(x, y) := |x|^\alpha + |y| - 1 = 0$$

of K_0^α in the four points

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}. \tag{8.33b}$$

We have

$$\nabla f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1^{\alpha-1} \\ 1 \end{pmatrix} \tag{8.34a}$$

$$\nabla f \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\alpha(-(-u_1))^{\alpha-1} \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha u_1^{\alpha-1} \\ 1 \end{pmatrix} \tag{8.34b}$$

$$\nabla f \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} -\alpha(-(-u_1))^{\alpha-1} \\ -1 \end{pmatrix} = \begin{pmatrix} -\alpha u_1^{\alpha-1} \\ -1 \end{pmatrix} \tag{8.34c}$$

$$\nabla f \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1^{\alpha-1} \\ -1 \end{pmatrix}, \tag{8.34d}$$

where

$$\nabla f \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = -\nabla f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{8.35a}$$

$$\nabla f \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} = -\nabla f \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}. \tag{8.35b}$$

Furthermore, in the two special points

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

of ∂K_0^α we have, cf. (8.34a), (8.34d), resp., the gradients

$$\nabla f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.36a}$$

$$\nabla f \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{8.36b}$$

Though $f(x, y) = |x|^\alpha + |y| - 1$ is not differentiable at

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

we define

$$\nabla f \begin{pmatrix} 1 \\ 0 \end{pmatrix} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.36c}$$

$$\nabla f \begin{pmatrix} -1 \\ 0 \end{pmatrix} := \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{8.36d}$$

Using a boundary point $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of K_0^α with $u_1, u_2 > 0$, the feasible domain K_0^α can be approximated from outside by the convex polyhedron defined as follows.

From the gradients (8.36a–d) we obtain next to the already known conditions (no M–N-interaction):

$$\nabla f \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} x \\ y-1 \end{pmatrix} \le 0$$

$$\nabla f \begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} x \\ y+1 \end{pmatrix} \le 0$$

$$\nabla f \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x-1 \\ y \end{pmatrix} \le 0$$

$$\nabla f \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x+1 \\ y \end{pmatrix} \le 0.$$

This yields

$$\begin{aligned} y-1 &\le 0 \\ -1(y+1) &\le 0 \\ x-1 &\le 0 \\ -1(x+1) &\le 0 \end{aligned}$$

or

$$|x| \le 1 \tag{8.37a}$$

$$|y| \le 1. \tag{8.37b}$$

Moreover, with the gradients (8.34a–d), cf. (8.35a,b), we get the additional conditions

$$\nabla f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = \begin{pmatrix} \alpha u_1^{\alpha-1} \\ 1 \end{pmatrix}^T \begin{pmatrix} x-u_1 \\ y-u_2 \end{pmatrix} \le 0 \qquad \text{(1st quadrant)} \tag{8.38a}$$

$$\nabla f \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} \right) = -\begin{pmatrix} \alpha u_1^{\alpha-1} \\ 1 \end{pmatrix}^T \begin{pmatrix} x+u_1 \\ y+u_2 \end{pmatrix} \le 0 \qquad \text{(3rd quadrant)} \tag{8.38b}$$

$$\nabla f\begin{pmatrix}-u_1\\ u_2\end{pmatrix}^T\left(\begin{pmatrix}x\\ y\end{pmatrix}-\begin{pmatrix}-u_1\\ u_2\end{pmatrix}\right)$$
$$=\begin{pmatrix}-\alpha u_1^{\alpha-1}\\ 1\end{pmatrix}^T\begin{pmatrix}x+u_1\\ y-u_2\end{pmatrix}\le 0 \qquad \text{(2nd quadrant)} \tag{8.38c}$$
$$\nabla f\begin{pmatrix}u_1\\ -u_2\end{pmatrix}^T\left(\begin{pmatrix}x\\ y\end{pmatrix}-\begin{pmatrix}u_1\\ -u_2\end{pmatrix}\right)$$
$$=-\begin{pmatrix}-\alpha u_1^{\alpha-1}\\ 1\end{pmatrix}^T\begin{pmatrix}x-u_1\\ y+u_2\end{pmatrix}\le 0 \quad \text{(4th quadrant).} \tag{8.38d}$$

This means

$$\alpha u_1^{\alpha-1}x-\alpha u_1^{\alpha}+y-u_2=\alpha u_1^{\alpha-1}x+y-\left(\alpha u_1^{\alpha}+u_2\right)\le 0 \tag{8.39a}$$
$$-\left(\alpha u_1^{\alpha-1}x+\alpha u_1^{\alpha}+y+u_2\right)=-\left(\alpha u_1^{\alpha-1}x+y+\left(\alpha u_1^{\alpha}+u_2\right)-\right)\le 0 \tag{8.39b}$$
$$-\alpha u_1^{\alpha-1}x-\alpha u_1^{\alpha}+y-u_2=-\alpha-u_1^{\alpha-1}x+y-\left(\alpha u_1^{\alpha}+u_2\right)\le 0 \tag{8.39c}$$
$$-\alpha u_1^{\alpha-1}x-\alpha u_1^{\alpha}-y-u_2=\alpha-u_1^{\alpha-1}x-y-\left(\alpha u_1^{\alpha}+u_2\right)\le 0. \tag{8.39d}$$

With

$$\alpha u_1^{\alpha}+u_2=\nabla f\begin{pmatrix}u_1\\ u_2\end{pmatrix}^T\begin{pmatrix}u_1\\ u_2\end{pmatrix}=:\beta(u_1,u_2) \tag{8.40}$$

we get the equivalent constraints

$$\begin{aligned}\alpha u_1^{\alpha-1}x+y-\beta(u_1,u_2)&\le 0\\ \alpha u_1^{\alpha-1}x+y+\beta(u_1,u_2)&\ge 0\\ -\alpha u_1^{\alpha-1}x+y-\beta(u_1,u_2)&\le 0\\ \alpha u_1^{\alpha-1}x-y-\beta(u_1,u_2)&\le 0.\end{aligned}$$

This yields the double inequalities

$$|\alpha u_1^{\alpha-1}x+y|\le\beta(u_1,u_2) \tag{8.41a}$$
$$|\alpha u_1^{\alpha-1}x-y|\le\beta(u_1,u_2). \tag{8.41b}$$

Thus, a point $u=\begin{pmatrix}u_1\\ u_2\end{pmatrix}\in\partial K_0^{\alpha},\ u_1>0, u_2>0$, generates therefore the inequalities

$$-1\le x\le 1 \tag{8.42a}$$
$$-1\le y\le 1 \tag{8.42b}$$
$$-\beta(u_1,u_2)\le\alpha u_1^{\alpha-1}x+y\le\beta(u_1,u_2) \tag{8.42c}$$
$$-\beta(u_1,u_2)\le\alpha u_1^{\alpha-1}x-y\le\beta(u_1,u_2). \tag{8.42d}$$

Obviously, each further point $\hat{u} \in \partial K_0^\alpha$ with $\hat{u}_1 > 0, \hat{u}_2 > 0$ yields additional inequalities of the type (8.42c,d).

Condition (8.42a–d) can be represented in the following vectorial form:

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} \le I \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8.43a}$$

$$-\beta(u_1, u_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \le H(u_1, u_2) \begin{pmatrix} x \\ y \end{pmatrix} \le \beta(u_1, u_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{8.43b}$$

with the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; H(u_1, u_2) = \begin{pmatrix} \alpha u_1^{\alpha-1} & 1 \\ \alpha u_1^{\alpha-1} & -1 \end{pmatrix}. \tag{8.44}$$

Choosing a further boundary point $\hat{u}$ of K_0^α with $\hat{u}_1 > 0, \hat{u}_2 > 0$, we get additional conditions of the type (8.43b).

Using (8.42a–d), for the original yield condition (8.32a,b) we get then the approximative feasibility condition:

1. Negative end of the bar

$$-N_{ipl} \le t_i \le N_{ipl} \tag{8.45a}$$

$$-M_{ipl} \le m_i^- \le M_{ipl} \tag{8.45b}$$

$$-\beta(u_1, u_2) \le \alpha u_1^{\alpha-1} \frac{t_i}{N_{ipl}} + \frac{m_i^-}{M_{ipl}} \le \beta(u_1, u_2) \tag{8.45c}$$

$$-\beta(u_1, u_2) \le \alpha u_1^{\alpha-1} \frac{t_i}{N_{ipl}} - \frac{m_i^-}{M_{ipl}} \le \beta(u_1, u_2). \tag{8.45d}$$

2. Positive end of the bar

$$-N_{ipl} \le t_i \le N_{ipl} \tag{8.45e}$$

$$-M_{ipl} \le m_i^+ \le M_{ipl} \tag{8.45f}$$

$$-\beta(u_1, u_2) \le \alpha u_1^{\alpha-1} \frac{t_i}{N_{ipl}} + \frac{m_i^+}{M_{ipl}} \le \beta(u_1, u_2) \tag{8.45g}$$

$$-\beta(u_1, u_2) \le \alpha u_1^{\alpha-1} \frac{t_i}{N_{ipl}} - \frac{m_i^+}{M_{ipl}} \le \beta(u_1, u_2). \tag{8.45h}$$

Defining

$$\Gamma^{(i)} := \begin{pmatrix} \frac{1}{N_{ipl}} & 0 & 0 \\ 0 & \frac{1}{M_{ipl}} & 0 \\ 0 & 0 & \frac{1}{M_{ipl}} \end{pmatrix}, \; F_i = \begin{pmatrix} t_i \\ m_i^+ \\ m_i^- \end{pmatrix}, \tag{8.46}$$

conditions (8.45a–h) can be represented also by

$$-\begin{pmatrix}1\\1\\1\end{pmatrix} \le \Gamma^{(i)} F_i \le \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{8.47a}$$

$$-\beta(u_1,u_2)\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \le \begin{pmatrix}\alpha u_1^{\alpha-1} & 1 & 0\\ \alpha u_1^{\alpha-1} & -1 & 0\\ \alpha u_1^{\alpha-1} & 0 & 1\\ \alpha u_1^{\alpha-1} & 0 & -1\end{pmatrix} \Gamma_i F_i \le \beta(u_1,u_2)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}. \tag{8.47b}$$

Multiplying (8.45a,c,d,g,h) with N_{ipl}, due to

$$\frac{N_{ipl}}{M_{ipl}} = \frac{\sigma_{yi} A_i}{\sigma_{yi} W_{ipl}} = \frac{\sigma_{yi} A_i}{\sigma_{yi} A_i \bar{y}_{ic}} = \frac{1}{\bar{y}_{ic}}, \tag{8.48}$$

for (8.45a,c,d,g,h) we also have

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1} t_i + \frac{m_i^-}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl} \tag{8.49a}$$

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1} t_i - \frac{m_i^-}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl} \tag{8.49b}$$

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1} t_i + \frac{m_i^+}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl} \tag{8.49c}$$

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1} t_i - \frac{m_i^+}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl}. \tag{8.49d}$$

8.2.2 Approximation of the Yield Condition by Using Reference Capacities

According to (8.31), (8.32a,b) for each bar $i = 1,\dots,B$ we have the condition

$$\begin{pmatrix}\frac{t}{N_{pl}}\\ \frac{m}{M_{pl}}\end{pmatrix} \in K_0^\alpha = \left\{\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 : |x|^\alpha + |y| \le 1\right\}$$

with $(t,m) = (t_i, m_i^\pm),\ (N_{pl}, M_{pl}) = (N_{ipl}, M_{ipl})$.

Selecting fixed reference capacities

$$N_{i0} > 0, M_{i0} > 0, i = 1,\dots,B,$$

related to the plastic capacities N_{ipl}, M_{ipl}, we get

$$\left|\frac{t_i}{N_{ipl}}\right|^{\alpha} + \left|\frac{m_i^{\pm}}{M_{ipl}}\right| = \left|\frac{t_i}{N_{i0}} \cdot \frac{1}{\frac{N_{ipl}}{N_{i0}}}\right|^{\alpha} + \left|\frac{m_i^{\pm}}{M_{i0}} \cdot \frac{1}{\frac{M_{ipl}}{M_{i0}}}\right|.$$

Putting

$$\rho_i = \rho_i(a(\omega), x) := \min\left\{\frac{N_{ipl}}{N_{i0}}, \frac{M_{ipl}}{M_{i0}}\right\}, \tag{8.50}$$

we have

$$\frac{\rho_i}{\frac{N_{ipl}}{N_{i0}}} \leq 1, \frac{\rho_i}{\frac{M_{ipl}}{M_{i0}}} \leq 1$$

and therefore

$$\left|\frac{t_i}{N_{ipl}}\right|^{\alpha} + \left|\frac{m_i^{\pm}}{M_{ipl}}\right| = \left|\frac{t_i}{\rho_i N_{i0}}\right|^{\alpha} \cdot \left|\frac{\rho_i}{\frac{N_{ipl}}{N_{i0}}}\right|^{\alpha} + \left|\frac{m_i^{\pm}}{\rho_i M_{i0}}\right| \cdot \left|\frac{\rho_i}{\frac{M_{ipl}}{M_{i0}}}\right| \leq \left|\frac{t_i}{\rho_i N_{i0}}\right|^{\alpha} + \left|\frac{m_i^{\pm}}{\rho_i M_{i0}}\right|. \tag{8.51}$$

Thus, the yield condition (8.30a,b) or (8.32a,b) is guaranteed by

$$\left|\frac{t_i}{\rho_i N_{i0}}\right|^{\alpha} + \left|\frac{m_i^{\pm}}{\rho_i M_{i0}}\right| \leq 1$$

or

$$\begin{pmatrix} \frac{t_i}{\rho_i N_{i0}} \\ \frac{m_i^{\pm}}{\rho_i M_{i0}} \end{pmatrix} \in K_0^{\alpha}. \tag{8.52}$$

Applying the piecewise linearization described in Sect. 8.2.1 to condition (8.52), we obtain, cf. (8.45a–h), the approximation stated below. Of course, conditions (8.45a,b,e,f) are not influenced by this procedure. Thus, we find

$$-N_{ipl} \leq t_i \leq N_{ipl} \tag{8.53a}$$

$$-M_{ipl} \leq m_i^- \leq M_{ipl} \tag{8.53b}$$

$$-M_{ipl} \leq m_i^+ \leq M_{ipl} \tag{8.53c}$$

$$-\beta(u_1, u_2) \leq \alpha u_1^{\alpha-1} \frac{t_i}{\rho_i N_{i0}} + \frac{m_i^-}{\rho_i M_{i0}} \leq \beta(u_1, u_2) \tag{8.53d}$$

$$-\beta(u_1, u_2) \leq \alpha u_1^{\alpha-1} \frac{t_i}{\rho_i N_{i0}} - \frac{m_i^-}{\rho_i M_{i0}} \leq \beta(u_1, u_2) \tag{8.53e}$$

$$-\beta(u_1, u_2) \leq \alpha u_1^{\alpha-1} \frac{t_i}{\rho_i N_{i0}} + \frac{m_i^+}{\rho_i M_{i0}} \leq \beta(u_1, u_2) \tag{8.53f}$$

$$-\beta(u_1, u_2) \leq \alpha u_1^{\alpha-1} \frac{t_i}{\rho_i N_{i0}} - \frac{m_i^+}{\rho_i M_{i0}} \leq \beta(u_1, u_2). \tag{8.53g}$$

Remark 8.7. Multiplying with ρ_i we get quotients $\frac{t_i}{N_{i0}}, \frac{m_i^{\pm}}{M_{i0}}$ with fixed denominators.

Hence, multiplying (8.53d–g) with ρ_i, we get the equivalent system

$$-N_{ipl} \le t_i \le N_{ipl} \tag{8.54a}$$
$$-M_{ipl} \le m_i^- \le M_{ipl} \tag{8.54b}$$
$$-M_{ipl} \le m_i^+ \le M_{ipl} \tag{8.54c}$$
$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}} \le \beta(u_1,u_2)\rho_i \tag{8.54d}$$
$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}} \le \beta(u_1,u_2)\rho_i \tag{8.54e}$$
$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}} \le \beta(u_1,u_2)\rho_i \tag{8.54f}$$
$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}} \le \beta(u_1,u_2)\rho_i. \tag{8.54g}$$

Obviously, (8.54a–g) can be represented also in the following form:

$$|t_i| \le N_{ipl} \tag{8.55a}$$
$$|m_i^-| \le M_{ipl} \tag{8.55b}$$
$$|m_i^+| \le M_{ipl} \tag{8.55c}$$
$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i \tag{8.55d}$$
$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i \tag{8.55e}$$
$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i \tag{8.55f}$$
$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i. \tag{8.55g}$$

By means of definition (8.50) of ρ_i, system (8.55a–g) reads

$$|t_i| \le N_{ipl} \tag{8.56a}$$
$$|m_i^-| \le M_{ipl} \tag{8.56b}$$
$$|m_i^+| \le M_{ipl} \tag{8.56c}$$
$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{N_{ipl}}{N_{i0}} \tag{8.56d}$$
$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{M_{ipl}}{M_{i0}} \tag{8.56e}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{N_{ipl}}{N_{i0}} \tag{8.56f}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{M_{ipl}}{M_{i0}} \tag{8.56g}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{N_{ipl}}{N_{i0}} \tag{8.56h}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{M_{ipl}}{M_{i0}} \tag{8.56i}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{N_{ipl}}{N_{i0}} \tag{8.56j}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{M_{ipl}}{M_{i0}} \tag{8.56k}$$

Corresponding to Remark 8.7, the variables

$$t_i, m_i^+, m_i^-, A_i \text{ or } x$$

enters linearly. Increasing the accuracy of approximation by taking a further point $\hat{u} = (\hat{u}_1, \hat{u}_2)$ with the related points $(\hat{u}_1, -\hat{u}_2), (-\hat{u}_1, \hat{u}_2), (-\hat{u}_1, -\hat{u}_2)$, we obtain further inequalities of the type (8.55d–g), (8.56d–g) respectively.

8.3 Stochastic Optimization

Due to (8.1c), (8.10), (8.11c), (8.12c), the $3B$ – vector

$$F = (F_1^T, \ldots, F_B^T)^T \tag{8.57a}$$

of all interior loads fulfills the equilibrium condition

$$CF = R \tag{8.57b}$$

with the external load vector R and the equilibrium matrix C.

In the following we collect all random model parameters [31, 35, 39], such as external load factors, material strength parameters, cost factors, etc., into the random ν-vector

$$a = a(\omega),\ \omega \in (\Omega, \mathcal{A}, \mathcal{P}), \tag{8.58a}$$

on a certain probability space $(\Omega, \mathcal{A}, P)$.

Thus, since in some cases the vector R of external loads depend also on the design r-vector x, we get

$$R = R\left(a(\omega), x\right). \tag{8.58b}$$

Of course, the plastic capacities depend also on the vectors x and $a(\omega)$, hence,

$$N_{ipl}^{\Gamma} = N_{ipl}^{\Gamma}\left(a(\omega), x\right), \ \Gamma = L, U \tag{8.58c}$$

$$M_{ipl} = M_{ipl}\left(a(\omega), x\right). \tag{8.58d}$$

We assume that the probability distribution and/or the needed moments of the random parameter vector $a = a(\omega)$ are known [2, 28, 31, 42].

The remaining deterministic constraints for the design vector x are represented by

$$x \in \mathcal{D} \tag{8.59}$$

with a certain convex subset D of $\mathbb{R}^r$.

8.3.1 Violation of the Yield Condition

Consider in the following an interior load distribution F fulfilling the equilibrium condition (8.58b).

According to the analysis given in Sect. 8.2, after piecewise linearization, the yield condition for the ith bar can be represented by an inequality of the type

$$H\Gamma^{(i)}\Big(a(\omega), x\Big) \le h^{(i)}\Big(a(\omega), x\Big), i = 1, \ldots, B, \tag{8.60}$$

with matrices $H, \Gamma^{(i)} = \Gamma^{(i)}\Big(a(\omega), x\Big)$ and a vector $h^{(i)} = h^{(i)}\Big(a(\omega), x\Big)$ as described in Sect. 8.2.

In order to take into account violations of condition (8.60), we consider the equalities

$$H\Gamma^{(i)}F_i + z_i = h^{(i)}, i = 1, \ldots, B. \tag{8.61}$$

If

$$z_i \ge 0, \text{ for all } i = 1, \ldots, B, \tag{8.62a}$$

then (8.60) holds, and the yield condition is then fulfilled too, or holds with a prescribed accuracy.

However, in case

$$z_i \not\ge 0 \text{ for some bars } i \in \{1, \ldots, B\}, \tag{8.62b}$$

the survival condition is violated at some points of the structure. Hence, structural failures may occur. The resulting costs Q of failure, damage and reconstructure of the frame is a function of the vectors z_i, $i = 1, \ldots, B$, defined by (8.61). Thus, we have

$$Q = Q(z) = Q(z_1, \ldots, z_B), \tag{8.63a}$$

where

$$z := (z_1^T, z_2^T, \ldots, z_B^T)^T, \tag{8.63b}$$

$$z_i := h^{(i)} - H\Gamma^{(i)} F_i,\ i = 1, \ldots, B. \tag{8.63c}$$

8.3.2 Cost Function

Due to the survival condition (8.62a), we may consider cost functions Q such that

$$Q(z) = 0,\ \text{if } z \geq 0, \tag{8.64}$$

hence, no (recourse) costs arise if the yield condition (8.60) holds.

In many cases the recourse or failure costs of the structure are defined by the sum

$$Q(z) = \sum_{i=1}^{B} Q_i(z_i), \tag{8.65}$$

of the element failure costs $Q_i = Q_i(z_i), i = 1, \ldots, B$.

Using the representation

$$z_i = y_i^+ - y_i^-, \quad y_i^+, y_i^- \geq 0, \tag{8.66a}$$

the member cost functions $Q_i = Q_i(z_i)$ are often defined [20, 31] by considering the linear function

$$q_i^{-T} y_i^- + q_i^{+T} y_i^+ \tag{8.66b}$$

with certain vectors q_i^+, q_i^- of cost coefficients for the evaluation of the condition $z_i \geq 0, z_i \not\geq 0$, respectively.

The cost function $Q_i = Q_i(z_i)$ is then defined by the minimization problem

$$\min\ q_i^{-T} y_i^- + q_i^{+T} y_i^+ \tag{8.67a}$$

$$\text{s.t.}\ \ y_i^+ - y_i^- = z_i \tag{8.67b}$$

$$y_i^-, y_i^+ \geq 0. \tag{8.67c}$$

If $z_i := (z_{i1}, \ldots, z_{i\mu})^T, y_i^{\pm} := (y_{i1}^{\pm}, \ldots, y_{i\mu}^{\pm})^T, i = 1, \ldots, B$ then (8.67a–c) can also be represented by

$$\min \sum_{l=1}^{\mu} (q_{il}^- y_{il}^- + q_{il}^+ y_{il}^+) \tag{8.68a}$$

$$\text{s.t.}\ \ y_{il}^+ - y_{il}^- = z_{il}, \quad l = 1, \ldots, \mu \tag{8.68b}$$

$$y_{il}^-, y_{il}^+ \geq 0, \quad l = 1, \ldots, \mu. \tag{8.68c}$$

Since the pairs of variables $(y_{il}^-, y_{il}^+), l = 1, \ldots, \mu$, are not connected with each other by the constraints, and the objective function is separabel with respect to these pairs of variables, (8.68a–c) can be decomposed into μ separated minimization problems

$$\min\; q_{il}^- y_{il}^- + q_{il}^+ y_{il}^+ \tag{8.69a}$$

$$\text{s.t.}\;\; y_{il}^+ - y_{il}^- = z_{il} \tag{8.69b}$$

$$y_{il}^-, y_{il}^+ \geq 0, \tag{8.69c}$$

for the pairs of variables (y_{il}^-, y_{il}^+), $l = 1, \ldots, \mu$.

Under the condition

$$q_{il}^- + q_{il}^+ \geq 0, l = 1, \ldots, \mu, \tag{8.70}$$

the following result holds:

Lemma 8.1. *Suppose that (8.70) holds. Then the minimum value function $Q_{il} = Q_{il}(z_{il})$ of (8.69a–c) is a piecewise linear, convex function given by*

$$Q_{il}(z_{il}) := \max\{q_{il}^+ z_{il}, -q_{il}^- z_{il}\}. \tag{8.71}$$

Hence, the member cost functions $Q_i = Q_i(z_i)$ reads

$$Q_i(z_i) = \sum_{l=1}^{\mu} Q_{il}(z_{il}) = \sum_{l=1}^{\mu} \max\{q_{il}^+ z_{il}, -q_{il}^- z_{il}\}, \tag{8.72a}$$

and the total cost function $Q = Q(z)$ is given by

$$Q(z) = \sum_{i=1}^{B} Q_i(z_i) = \sum_{i=1}^{B} \sum_{l=1}^{\mu} \max\{q_{il}^+ z_{il}, -q_{il}^- z_{il}\}. \tag{8.72b}$$

8.3.3 Choice of the Cost Factors

Under elastic conditions the change $\Delta\sigma$ of the total stress σ and the change ΔL of the element length L are related by

$$\Delta L = \frac{L}{E} \Delta\sigma, \tag{8.73a}$$

where E denotes the modulus of elasticity.

Assuming that the neutral axis is equal to the axis of symmetry of the element, for the total stress $\Delta\sigma$ in the upper ("+") lower ("−") fibre of the boundary we have

$$\Delta\sigma = \frac{\Delta t}{A} \pm \frac{\Delta m}{W}, \tag{8.73b}$$

where W denotes the axial modulus of the cross-sectional area of the element (beam).

Representing the change ΔV of volume of an element by

$$\Delta V = A \cdot \Delta L, \tag{8.74a}$$

then

$$\begin{aligned} \Delta V &= A \cdot \Delta L = A \cdot \frac{L}{E}\Delta\sigma = A \cdot \frac{L}{E}\left(\frac{\Delta t}{A} \pm \frac{\Delta m}{W}\right) \\ &= \frac{L}{E}\Delta t \pm \frac{L}{E}\frac{A}{W}\Delta m = \frac{L}{E}\Delta t \pm \frac{L}{E}\frac{1}{\frac{W}{A}}\Delta m = \frac{L}{E}\Delta t \pm \frac{L}{E} \cdot \frac{1}{\bar{y}_c}\Delta m, \end{aligned} \tag{8.74b}$$

where $\bar{y}_c$ is the cross-sectional parameter as defined in (8.28d).

Consequently, due to (8.73a,b), for the evaluation of violations Δt of the axial force constraint we may use a cost factor of the type

$$\Gamma_K := \frac{L}{E}, \tag{8.75a}$$

and an appropriate cost factor for the evaluation of violations of moment constraints reads

$$\Gamma_M = \frac{L}{E} \cdot \frac{1}{\bar{y}_c}. \tag{8.75b}$$

8.3.4 Total Costs

Denoting by

$$G_0 = G_0\Big(a(\omega), x\Big) \tag{8.76a}$$

the primary costs, such as weighted negative load factors, material costs, costs of construction, etc., the total costs including failure or recourse costs are given by

$$G = G_0\big(a(\omega), x\big) + Q\Big(z\big(a(\omega), x, F(\omega)\big)\Big). \tag{8.76b}$$

Hence, the total costs $G = G\big(a(\omega), x, F(\omega)\big)$ depend on the vector $x = (x_1, \ldots, x_r)^T$ of design variables, the random vector $a(\omega) = \big(a_1(\omega), \ldots, a_\nu(\omega)\big)^T$ of model parameters and the random vector $F = F(\omega)$ of all internal loadings.

Minimizing the expected total costs, we get the following stochastic optimization problem (SOP) of recourse type [20]

$$\min\ E\Big(G_0\big(a(\omega), x\big) + Q\big(z\,(a(\omega), x, F(\omega))\big)\Big) \tag{8.77a}$$

$$\text{s.t.}\ \ H\Gamma^{(i)}\Big(a(\omega), x\Big)F_i(\omega) + z_i(\omega) = h^{(i)}\Big(a(\omega), x\Big) \quad \text{a.s.,}$$
$$i = 1, \ldots, B \tag{8.77b}$$

$$CF(\omega) = R\Big(a(\omega), x\Big) \quad \text{a.s.} \tag{8.77c}$$

$$x \in D. \tag{8.77d}$$

Using representation (8.65), (8.67a–c) of the recourse or failure cost function $Q = Q(z)$, problem (8.77a–d) takes also the following equivalent from

$$\min\ E\left(G_0\Big(a(\omega), x\Big) + \sum_{i=1}^{B}\big(q_i^-(\omega)^T y_i^-(\omega) + q_i^+(\omega)^T y_i^+(\omega)\big)\right) \tag{8.78a}$$

$$\text{s.t.}\ \ H\Gamma^{(i)}\Big(a(\omega), x\Big)F_i(\omega) + y_i^+(\omega) - y_i^-(\omega) = h^{(i)}\Big(a(\omega), x\Big) \quad \text{a.s.,}$$
$$i = 1, \ldots, B \tag{8.78b}$$

$$CF(\omega) = R\,(a(\omega), x) \quad \text{a.s.} \tag{8.78c}$$

$$x \in D,\ y_i^+(\omega), y_i^-(\omega) \geq 0 \quad \text{a.s.,} \quad i = 1, \ldots, B. \tag{8.78d}$$

Remark 8.8. Stochastic optimization problems of the type (8.78a–d) are called “two-stage stochastic programs” or “stochastic problems with recourse”.

In many cases the primary cost function $G_0 = G_0\Big(a(\omega), x\Big)$ represents the volume or weight of the structural, hence,

$$\begin{aligned} G_0\Big(a(\omega), x\Big) &:= \sum_{i=1}^{B} \Gamma_i(\omega) V_i(x) \\ &= \sum_{i=1}^{B} \Gamma_i(\omega) L_i A_i(x), \end{aligned} \tag{8.79}$$

with certain (random) weight factors $\Gamma_i = \Gamma_i(\omega)$.

In case

$$A_i(x) := u_i x_i \tag{8.80}$$

with fixed sizing parameters $u_i, i = 1, \ldots, B$, we get

$$\begin{aligned} G_0\,(a(\omega), x) &= \sum_{i=1}^{B} \Gamma_i(\omega) L_i u_i(x) = \sum_{i=1}^{B} \Gamma_i(\omega) L_i h_i x_i \\ &= c\,(a(\omega))^T\, x, \end{aligned} \tag{8.81a}$$

where

$$c\,(a(\omega)) := (\Gamma_1(\omega)L_1u_1,\ldots,\Gamma_B(\omega)L_Bu_B)^T\,. \tag{8.81b}$$

Thus, in case (8.80), $G_0 = G_0\Big(a(\omega),x\Big)$ is a linear function of x.

8.3.5 Discretization Methods

The expectation in the objective function of the stochastic optimization problem (8.78a–d) must be computed numerically. One of the main methods is based on the discretization of the probability distribution $P_{a(\cdot)}$ of the random parameter ν-vector $a = a(\omega)$, hence,

$$P_{a(\cdot)} \approx \mu := \sum_{k=1}^{s} \alpha_k \epsilon_{a^{(k)}} \tag{8.82a}$$

with

$$\alpha_k \geq 0,\ k = 1,\ldots,s,\ \sum_{k=1}^{s} \alpha_k = 1. \tag{8.82b}$$

Corresponding to the realizations $a^{(k)}, k = 1,\ldots,s$, of the discrete approximate (8.82a,b), we have the realizations $y_i^{-(k)}, y_i^{+(k)}$ and $F^{(k)}$, $F_i^{(k)}, k = 1,\ldots,s$, of the random vectors $y_i^-(\omega), y_i^+(\omega), i = 1,\ldots,B$, and $F(\omega)$. then

$$\begin{aligned} &E\left(G_0\big(a(\omega),x\big) + \sum_{i=1}^{B}\Big(q_i^{-T}y_i^-(\omega) + q_i^{+T}y_i^+(\omega)\Big)\right) \\ &\approx \sum_{k=1}^{s} \alpha_k \left(G_0(a^{(k)},x) + \sum_{i=1}^{B}(q_i^{-T}y_i^{-(k)} + q_i^{+T}y_i^{+(k)}\right). \end{aligned} \tag{8.83a}$$

Furthermore, the equilibrium equation (8.77c) is approximated by

$$CF^{(k)} = R(a^{(k)},x),\ k = 1,\ldots,s, \tag{8.83b}$$

where $F^{(k)} := \left(F_1^{(k)^T},\ldots,F_B^{(k)^T}\right)^T$, and we have, cf. (8.78d), the nonnegativity constraints

$$y_i^{+(k)},\ y_i^{-(k)} \geq 0,\ k = 1,\ldots,s,\ i = 1,\ldots,B. \tag{8.83c}$$

Thus, (SOP) (8.78a–d) is reduced to the parameter optimization problem

$$\min\ \overline{G}_0(a^{(k)},x) + \sum_{i=1}^{B} \alpha_k \left(q_i^{-T}y_i^{-(k)} + q_i^{+T}y_i^{+(k)}\right) \tag{8.84a}$$

$$\text{s.t.} \quad H\Gamma^{(i)}(a^{(k)}, x)F_i^{(k)} + y_i^{+(k)} - y_i^{-(k)} = h^{(i)}(a^{(k)}, x),$$
$$i = 1, \ldots, B, \ k = 1, \ldots, s \tag{8.84b}$$
$$CF^{(k)} = R(a^{(k)}, x), \ k = 1, \ldots, s \tag{8.84c}$$
$$x \in D, \ y_i^{+(k)}, \ y_i^{-(k)} \geq 0, \ k = 1, \ldots, s, \ i = 1, \ldots, B. \tag{8.84d}$$

A further important class of methods for computing expectations and probabilities, hence, multiple integrals, occurring in stochastic optimization problems, reliability analysis and reliability-based optimal design (RBO), cf. [9, 11, 30, 31], are simulation methods, such as Monte Carlo Simulation (MCS) procedures. Simulation techniques are used especially in cases with only few information about the analytical properties of the underlying technical device, e.g., in case of analytically almost unavailable limit state functions. In principle, MSC is a very simple technique which is widely applicable on the one hand, but may have a very low efficiency of estimation on the other hand. Hence, several improvements were considered in the last time, such as Advanced Monte Carlo Simulation techniques: Variance reduction methods reducing the sampling error, based, e.g., on importance sampling methods, direction sampling, subset simulation, etc., see, e.g., [4]. Further improvements can be obtained by combining these simulation/estimation techniques with Response Surface Methods (RSM) for estimating unknown functions using regression techniques and advanced nonlinear programming procedures (NLP), cf. [12].

8.3.6 Complete Recourse

According to Sect. 8.3.1, the evaluation of the violation of the yield condition (8.60) is based on (8.61), hence

$$H\Gamma^{(i)}F_i + z_i = h^{(i)}, i = 1, \ldots, B.$$

In generalization of the so-called "simple recourse" case (8.67a–c), in the "complete recourse" case the deviation

$$z_i = h^{(i)} - H\Gamma^{(i)}F_i$$

is evaluated by means of the minimum value $Q_i = Q_i(z_i)$ of the linear program, cf. (8.68a–c)

$$\min q^{(i)T} y^{(i)} \tag{8.85a}$$
$$\text{s.t.} \quad M^{(i)} y^{(i)} = z_i \tag{8.85b}$$
$$y^{(i)} \geq 0. \tag{8.85c}$$

Here, $q^{(i)}$ is a given cost vector and $M^{(i)}$ denotes the so-called recourse matrix [20].

We assume that the linear equation (8.85b) has a solution $y^{(i)} \geq 0$ for each vector z_i. This property is called "complete recourse".

In the present case the stochastic optimization problem (8.77a–c) reads

$$\min E\left(G_0\Big(a(\omega), x\Big) + \sum_{i=1}^{B} q^{(i)T} y^{(i)}(\omega)\right) \tag{8.86a}$$

$$\text{s.t.}\quad H\Gamma^{(i)}\Big(a(\omega), x\Big) F_i(\omega) + M^{(i)} y^{(i)}(\omega) = h^{(i)}\Big(a(\omega), x\Big) \text{ a.s.,} \quad i = 1, \ldots, B \tag{8.86b}$$

$$CF(\omega) = R\left(a(\omega), x\right) \quad \text{a.s.} \tag{8.86c}$$

$$x \in D,\ y^{(i)}(\omega) \geq 0 \quad \text{a.s.,} \quad i = 1, \ldots, B. \tag{8.86d}$$

As described in Sect. 8.3.5, problem (8.86a–d) can be solved numerically by means of discretization methods and application of linear/nonlinear programming techniques.

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