Chapter 7
Optimal Ellipsoidal Estimates of Uncertain Systems: An Overview and New Results

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Abstract The method of ellipsoids for the guaranteed state estimation of uncertain dynamical systems is associated with optimal two-sided ellipsoidal bounds for reachable sets of the systems. Being based on the set-membership approach to uncertainties, the method can be regarded as a natural counterpart to well-known stochastic, or probabilistic, techniques. Basic concepts and results of the method are outlined, and certain results are presented. Various possible applications to problems in control, estimation, and observation are considered.

7.1 Introduction

Dynamical systems subjected to unknown but bounded perturbations appear in numerous applications. The set-membership approach that is a natural counterpart to the well-known stochastic, or probabilistic, one makes it possible to obtain guaranteed estimates on reachable sets and thus to evaluate the family of all possible trajectories of the perturbed system.

In the framework of the set-membership approach, the ellipsoidal estimation seems to be the most efficient technique. Among its advantages are the explicit form of approximations, invariance with respect to linear transformations, possibility of optimization, etc. The earlier results on the ellipsoidal estimation were presented in [33]. The concept of optimality for two-sided (inner and outer) approximating ellipsoids was first introduced in [1] and generalized, extended, and summarized in books [2, 3, 5].

In this paper, basic concepts and results of the method of optimal ellipsoids are outlined, and certain recent results are presented.

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Dynamical systems subjected to bounded controls and/or perturbations are considered. For these systems, nonlinear differential equations are obtained that describe the evolution of the optimal ellipsoids representing two-sided (inner and outer) estimates for reachable sets. The approximating ellipsoids depend on the choice of the optimality criterion (e.g., volume of ellipsoids, sum of their squared axes), and on the notion of local/global optimality.

Various useful properties of the optimal approximating ellipsoids have been established. Asymptotic behavior of the ellipsoids near the initial point and at infinity have been studied. As a rule, the nonlinear equations for these ellipsoids are to be integrated numerically. However, certain explicit analytical solutions have been obtained.

Outer and inner ellipsoidal approximations can be used for various applications in control and estimation, including two-sided approximations for optimal control and differential games, analysis of practical stability and parameter excitation, state estimation in the presence of observation errors, control in the presence of uncertain perturbations, etc.

7.2 Reachable Sets

Consider a dynamical system subjected to control or disturbance and described by a system of ordinary differential equations

$$\dot{x} = f(x, u, t), \quad t \geq s \tag{7.1}$$

Here, $x = (x_1, \ldots, x_n)$ is the vector of state, t is time, s is the initial time instant, $u = (n_1, \ldots, u_m)$ is the vector of control or disturbance, and f is a given function. At each time instant, vector $u(t)$ should belong to the given set $U(t)$, so that

$$u(t) \in U(t) \subset R^m, \quad t \geq s\ . \tag{7.2}$$

The initial state belongs to a given initial set:

$$x(s) \in M \subset R^n\ . \tag{7.3}$$

An alternative description of the system (7.1)–(7.3) is given by the differential inclusion:

$$\dot{x} \in f(x, U(t), t), \quad x(s) \in M\ .$$

For systems under consideration, the notion of a reachable set is introduced.

The *reachable*, or *attainable*, set $D(t, s, M)$ of system (7.1) for $t \geq s$ is defined as the set of all end points $x(t)$ at the instant t of all state trajectories $x(\cdot)$ compatible

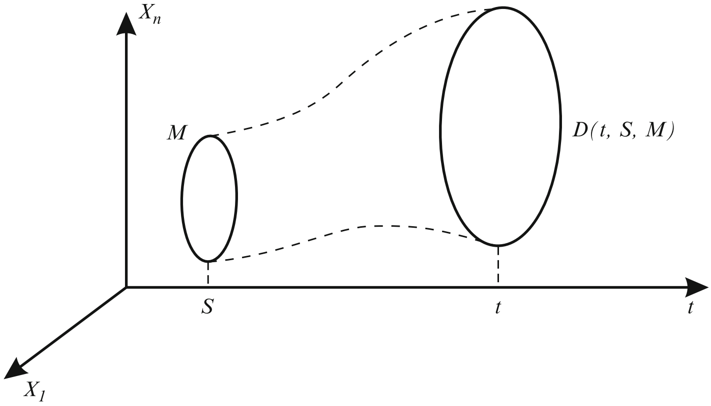

Fig. 7.1 Reachable set

with (7.1)–(7.3). The reachable set has the following evolutionary property

$$D(t, s, M) = D(t, \tau, D(\tau, s, M)) \tag{7.4}$$

that holds for all $\tau \in [s, t]$. Reachable sets are important for systems subjected to control or disturbance because they give a description of all possible states of the system at any given time t (Fig. 7.1).

In various applications, it is often required to verify whether a given state x^* is reachable at the given instant of time t^*, i.e., if the inclusion

$$x^* \in D(t^*, s, M) \tag{7.5}$$

holds. However, the practical determination of reachable sets in the n-dimensional space presents usually serious computational difficulties. In a number of cases, it is sufficient to obtain simple and efficient two-sided (inner and outer) bounds $D^-(t)$ and $D^+(t)$ on reachable sets such that the following inclusions

$$D^-(t) \subset D(t, s, M) \subset D^+(t) \tag{7.6}$$

are true for $t \geq s$. If the bounds D^- and D^+ are known, then the inclusions

$$x^* \in D^-(t^*), \quad x^* \in D^+(t^*) \tag{7.7}$$

can serve, respectively, as a sufficient and necessary conditions for the inclusion (7.5).

If we deal with a control system and u in (7.1) is a control to be chosen, then the inner bound $D^-(t)$ is important. The first inclusion (7.7) implies that the state x^* is reachable at the instant t^*, and there exists an admissible control $u(t)$ bringing the system to this state.

For an uncertain system, $u(t)$ is a disturbance, and the outer bound $D^+(t)$ provides a guaranteed estimate for all possible trajectories: the system cannot reach any state outside the set $D^+(t^*)$ at the instant t^*.

In the well-known stochastic (probabilistic) approach to uncertain systems, the Gaussian distribution plays a specific role and results in the simplest mathematical description. To some extent, ellipsoidal sets for the set-membership approach to uncertainties are analogous to the Gaussian distributions. In fact, the sets of constant probability for the n-dimensional Gaussian distribution are the surfaces of n-dimensional ellipsoids. Moreover, there is a certain similarity between stochastic systems subjected to the white noise and dynamical systems subjected to uncertain disturbances bounded by ellipsoids.

Denote by $E(a, Q)$ the following n-dimensional ellipsoid

$$E(a, Q) = \left\{x : (Q^{-1}(x - a), (x - a)) \le 1\right\}, \tag{7.8}$$

where $a \in R^n$ is its center, Q is a symmetric positive definite $n \times n$ matrix, and $(.,.)$ denotes the scalar product of vectors.

Ellipsoids have a number of advantages as approximating sets. They provide a satisfactory approximation for a wide class of convex sets [1–3]. For any n-dimensional convex set D, there exists an ellipsoid $E(a, Q)$ such that

$$E(a, Q) \subset D \subset E(a, n^2 Q).$$

If the convex set is symmetric with respect to some point, then there exists an ellipsoid $E(a, Q)$ for which the inclusions.

$$E(a, Q) \subset D \subset E(a, nQ)$$

hold. These inclusions provide two-sided estimates for possible approximations of any convex set by means of ellipsoids.

The class of ellipsoids is invariant with respect to affine transformations. The class of parallelepipeds has the same invariance property but ellipsoids are defined by less number of parameters: $n + n(n + 1)/2$ for ellipsoids, $n(n + 1)$ for parallelepipeds. Note that rectangular parallelepipeds are defined by the same number of parameters as ellipsoids but they are not invariant with respect to affine transformations.

The last but not the least advantage of ellipsoids is the following one. Simple explicit formulas were obtained [1–3] for the basic algebraic operations for ellipsoidal sets, namely, for the multiplication by a constant, the sum, and intersection of ellipsoids. These operations are, in a certain sense, optimal and provide a basis for the ellipsoidal estimation of dynamical systems subjected to controls and/or uncertain disturbances.

The ellipsoidal state estimation occurs to be the most effective technique in the framework of the set-membership approach. In what follows, we will remind the basic concepts and results of this technique.

7.3 Ellipsoidal Bounds

Let us specify the general system (7.1) and consider a linear system of ordinary differential equations

$$\dot{x} = A(t)x + B(t)u + f(t), \quad t \geq s. \tag{7.9}$$

Here, $x \in R^n$ is the n-vector of state, $u \in R^m$ is the m-vector of control or unknown disturbances, A is an $n \times n$ matrix, B is an $n \times m$ matrix, and f is an n-vector. The matrices $A(t)$ and $B(t)$ as well as the vector $f(t)$ are given functions of time t for $t \geq s$.

Suppose the set $U(t)$ that bounds vector $u(t)$ in (7.2) is an ellipsoid. We have

$$u(t) \in E(0, G(t)), \quad t \geq s, \tag{7.10}$$

where $G(t)$ is a positive definite $m \times m$ matrix specified for $t \geq s$.

The initial set M in (7.3) is also supposed to be an ellipsoid. We have the following bound on the initial state:

$$x(s) \in M = E(a_0, Q_0), \tag{7.11}$$

where a_0 is a given n-vector, and Q_0 is a given positive definite $n \times n$ matrix.

Our goal is to obtain two-sided bounds (7.6) on reachable sets, where the bounding sets $D^-(t)$ and $D^+(t)$ are n-dimensional ellipsoids. In other words, we look for two families of n-dimensional ellipsoids:

$$E^-(t) = E(a^-(t), Q^-(t)), \quad E^+(t) = E(a^+(t), Q^+(t)) \tag{7.12}$$

defined, according to notation (7.8), by their centers $a^-(t), a^+(t)$ and positive definite $n \times n$ matrices $Q^-(t), Q^+(t)$ for $t \geq s$ and such that the following inclusions hold:

$$E^-(t) \subset D(t, s, M) \subset E^+(t), \quad t \geq s. \tag{7.13}$$

In addition, we require that the families of ellipsoids $E^-(t)$ and $E^+(t)$ possess the properties of subreachability and superreachability, respectively.

The families of ellipsoids $E^{-}(t)$ and $E^{+}(t)$ defined by (7.11) are called *subreachable* and *superreachable*, respectively, if

$$E^{-}(t) \subset D(t, \tau, E^{-}(\tau)), \quad E^{+}(t) \supset D(t, \tau, E^{+}(\tau))$$

for all $\tau \in [s, t]$. These properties are similar to the evolutionary property (7.4) of reachable sets. It occurs that subreachable and superreachable ellipsoids have certain advantages: they can be determined efficiently in a straightforward way and provide a sufficiently good approximations of reachable sets.

7.4 Optimality

To make the approximating ellipsoids (7.12) closer to reachable sets, it is quite natural to impose certain optimality conditions upon these ellipsoids. The optimality criteria should, to some extent, reflect the "size" of ellipsoids, and the inner ellipsoid E^{-} should be "larger", whereas the outer ellipsoid E^{+}"smaller", in the sense of the chosen criterion.

Let us characterize an ellipsoid $E(a, Q)$ by a scalar optimality criterion J which is a given function $L(Q)$ of the matrix Q, i.e., $J = L(Q)$. The function $L(Q)$ is defined for all symmetric positive definite matrices Q, is smooth and monotone. The monotonicity means that $L(Q_1) \geq L(Q_2)$, if $Q_1 - Q_2$ is a nonnegative definite matrix.

Consider some important particular cases of the general criterion $L(Q)$:

1. The volume of an ellipsoid [1–3] is given by

$$J = c_n (\det Q)^{1/2}, \tag{7.14}$$

   where c_n is a constant depending on n.
2. The sum of the squared semiaxes of an ellipsoid is equal to

$$J = \mathrm{Tr} Q.$$

3. A linear optimality criterion [3, 5]

$$J = \mathrm{Tr}(CQ), \tag{7.15}$$

   where C is a symmetric nonnegative definite $n \times n$ matrix, is a generalization of the previous case.
4. The following criterion

$$J = L(Q) = (Qv, v), \tag{7.16}$$

where v is a given nonzero n-vector, is a particular case of (7.15) with

$$C = v * v, \quad C_{ij} = v_i v_j, \quad i, j = 1, \ldots, n. \tag{7.17}$$

Here, the symbol $*$ denotes the dyadic product of vectors.

Criterion (7.16) has a clear geometric interpretation: it is related to the projection $\Pi_v(E)$ of the ellipsoid onto the direction of the vector v as follows:

$$\Pi_v(E) = 2(Qv, v)^{1/2}/|v|. \tag{7.18}$$

To prove (7.18), note that the projection $\Pi_v(E)$ is related to the support function $H_E(v)$ of the ellipsoid as follows:

$$\Pi_v(E) = [H_E(v) + H_E(-v)]/|v|. \tag{7.19}$$

Since the support function of the ellipsoid is given by

$$H_E(v) = (a, v) + (Qv, v)^{1/2}, \tag{7.20}$$

equation (7.18) follows immediately from (7.19) and (7.20).

By virtue of (7.18), the minimization of criterion (7.16) is equivalent to the minimization of the projection of the ellipsoid onto the direction of vector v. Other examples of optimality criteria are given in [3].

Below, we consider locally and globally optimal ellipsoids [3, 26].

A smooth family of ellipsoids $E^*(a(t), Q(t))$ is called *locally optimal*, if it is superreachable/subreachable and

$$dL(Q(\tau))/d\tau|_{\tau=t} \to \min/\max,$$

where the minimum/maximum is taken over all smooth families of superreachable/subreachable ellipsoids $E^{\pm}(t)$ such that $E^{\pm}(t) = E^*(t)$.

A smooth family of superreachable/subreachable ellipsoids is called *globally optimal* for a given $t = T$, if the minimum/maximum of $L(Q(T))$ over all superreachable/subreachable ellipsoids is attained on this family.

All definitions and results related to the optimal ellipsoids are true also for the case, where the criterion depends also on time t, so that $J = L(Q, t)$. For example, matrix C in (7.15) and vector v in (7.16) can depend on t: $C = C(t)$, $v = v(t)$.

Note that the volume of an ellipsoid as an optimality criterion (7.14) has a property that singles it out among all other criteria. The optimality of a given ellipsoid in the sense of volume remains intact under any affine transformation in R^n. Thus, the optimality in the sense of volume seems more "basic" property of an approximating ellipsoid that its optimality in the sense of other criteria.

7.5 Equations of Ellipsoids

We consider linear system (7.9) where $u(t)$ is subjected to constraint (7.10) and initial conditions are specified by (7.11).

The center $a^+(t)$ and the matrix $Q^+(t)$ of the outer locally optimal ellipsoid satisfy the following differential equations and initial conditions [3, 5]:

$$\dot{a}^+ = A(t)a^+ + f(t), \quad a^+(s) = a_0, \tag{7.21}$$
$$\dot{Q}^+ = A(t)Q^+ + Q^+A^T(t) + hQ^+ + h^{-1}K, \quad Q^+(s) = Q_0. \tag{7.22}$$

Here, ^{T} denotes the transposed matrix, $K(t)$ is expressed via given matrices by the formula

$$K(t) = B(t)G(t)B^T(t), \tag{7.23}$$

and the following notation is used

$$h = \left[\mathrm{Tr}\left(\frac{\partial L}{\partial Q^+}K\right)/\mathrm{Tr}\left(\frac{\partial L}{\partial Q^+}Q^+\right)\right]^{1/2}. \tag{7.24}$$

Here, Tr is the trace of a matrix, and $\partial L/\partial Q^+$ is a symmetric matrix of partial derivatives $\partial L(Q)/\partial Q^+_{ij}$, $i, j = 1, \ldots n$.

Note that (7.21) for the vector $a(t)$ is linear and does not depend on the chosen optimality criterion $L(Q)$. By contrast, (7.22) for the matrix $Q^+(t)$ is nonlinear and depends on $L(Q)$ via (7.24) for h. For ellipsoids optimal in the sense of volume [see (7.14)], expression (7.24) becomes

$$h = \left\{n^{-1}\mathrm{Tr}\left[(Q^+)^{-1}K\right]\right\}^{1/2}. \tag{7.25}$$

For the linear criterion (7.15), we have

$$h = [\mathrm{Tr}(CK)/\mathrm{Tr}(CQ^+)]^{1/2}. \tag{7.26}$$

Further simplifications are possible for the criterion (7.16). Substituting C from (7.17) into (7.26), we obtain

$$h = [(Kv, v)/(Q^+v, v)]^{1/2}. \tag{7.27}$$

Equations for inner approximating ellipsoids locally optimal in the sense of volume [3, 5] are as follows:

$$\begin{aligned} \dot{a}^- &= A(t)a^- + f(t), \quad a^-(s) = a_0, \\ \dot{Q}^- &= A(t)Q^- + Q^-A^T(t) + 2K^{1/2}(K^{-1/2}Q^-K^{-1/2})^{1/2}K^{1/2}, \\ Q^-(s) &= Q_0. \end{aligned} \tag{7.28}$$

Here, matrix $K(t)$ is again defined by (7.23). Note that equations for the centers of the inner and outer approximating ellipsoids (7.21) and (7.28) coincide, hence, $a^-(t) \equiv a^+(t)$ for $t \geq s$.

It occurs that (7.28) are true for inner approximating ellipsoids optimal in the sense of all criteria $L(Q)$ satisfying the conditions imposed in the beginning of Sect. 7.4. Thus, these equations have a universal nature.

Let us consider now globally optimal outer ellipsoids. The centers of these ellipsoids coincide with those of locally optimal ones and satisfy the initial value problem (7.21). The matrix $Q^+(t)$ of globally optimal ellipsoids satisfies equation and initial condition (7.22), where matrix K is defined by (7.23), whereas the scalar h is, instead of (7.24), given by the expression

$$h = [\mathrm{Tr}(PK)/\mathrm{Tr}(PQ^+)]^{1/2}. \tag{7.29}$$

Here, $P(t)$ is a symmetric positive definite matrix satisfying the following linear differential equation

$$\dot{P} = -PA(t) - A^T(t)P \tag{7.30}$$

and initial condition at $t = T$:

$$P(T) = [\partial L(Q^+)/\partial Q^+]_{t=T}. \tag{7.31}$$

Hence, we have a two-point boundary value problem for the pair of matrices Q^+ and P described by (7.22), (7.23), (7.29)–(7.31). For the linear criterion (7.15), the initial condition (7.31) is reduced to

$$P(T) = C(T). \tag{7.32}$$

Therefore, for outer ellipsoids globally optimal in the sense of criterion (7.15), the boundary value problem for matrices Q^+ and P becomes decoupled and reduces to two initial value problems: a linear one for $P(t)$ defined by (7.30) and (7.32), and a nonlinear one for $Q^+(t)$ defined by (7.22) and (7.29). First, the problem for $P(t)$ should be solved (from $t = T$ to $t = s$), and then the problem for $Q^+(t)$ (from $t = s$ to $t = T$).

Further simplifications are possible for criterion (7.16). For globally optimal outer ellipsoids, we have, on the strength of (7.17) and (7.32):

$$P(T) = v * v,$$

where v is a given constant n-vector. Let us introduce the adjoint n-vector $\psi(t)$ satisfying the initial value problem:

$$\dot{\psi} = -A^T(t)\psi, \quad \psi(T) = v. \tag{7.33}$$

Then we have [6, 7]

$$P(t) = \psi(t) * \psi(t).$$

Thus, in order to find the matrix $Q^+(t)$ of a globally optimal ellipsoid in the case of criterion (7.16), one is to solve first the linear n-dimensional initial value problem (7.33) for $\psi(t)$ (instead of $n(n+1)/2$-dimensional problem for $P(t)$ defined by (7.30) and (7.32)), then substitute

$$h = [(K\psi, \psi)/(Q\psi, \psi)]^{1/2}$$

into (7.22) and solve the resultant initial value problem for Q^+.

Thus, linear optimality criterion (7.15) and, especially, its particular case (7.16) lead to a considerable simplification of equations for optimal outer ellipsoids.

7.6 Transformation of the Equations

In what follows, we restrict ourselves mostly with outer approximating ellipsoids and omit the superscript $^+$, so that we denote below: $a^+(t) = a(t)$, $Q^+(t) = Q(t)$.

Equation (7.22) for locally optimal ellipsoids depends on two given matrices: $A(t)$ and $K(t)$, where the symmetric matrix $K(t)$ is given by (7.23). By the change of variables

$$Q = VQ_*V^T, \tag{7.34}$$

where $V(t)$ is an invertible $n \times n$ matrix to be specified below, and Q_* is a new matrix variable, it is possible to simplify (7.22) so that it will contain only one given matrix instead of two.

1. Let us define $\Phi(t)$ as the fundamental matrix of system (7.7):

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(s) = I.$$

Here, I is the $n \times n$ identity matrix.
If we set

$$V(t) = \Phi(t), \tag{7.35}$$

then transformation (7.34) reduces (7.22) to the form

$$\dot{Q}_* = h_*Q_* + h_*^{-1}K_*(t), \quad Q_*(s) = Q_0, \tag{7.36}$$
$$K_*(t) = \Phi^{-1}(t)K(t)[\Phi^{-1}(t)]^T.$$

Here, h_* is given by (7.24) where Q should be replaced by $\Phi Q_* \Phi^T$. Explicit expressions [3, 5, 7] for h_* are presented below for criteria (7.14)–(7.16), respectively:

$$\begin{aligned} h_* &= [n^{-1}\mathrm{Tr}(Q_*^{-1}K_*)]^{1/2}, \\ h_* &= [\mathrm{Tr}(C_*K_*)/\mathrm{Tr}(C_*Q_*)]^{1/2}, \quad C_* = \Phi^T C \Phi, \\ h_* &= [(K_*v_*, v_*)/(Q_*v_*, v_*)]^{1/2}, \quad v_* = \Phi^T v. \end{aligned} \tag{7.37}$$

2. Let matrix $K(t)$ from (7.23) be positive definite. By taking

$$V(t) = [K(t)]^{1/2} \tag{7.38}$$

in (7.34), we convert (7.22) to the form

$$\begin{aligned} \dot{Q}_* &= A_*Q_* + Q_*A_*^T + h_*Q_* + h_*^{-1}I, \\ Q_*(s) &= K^{-1/2}(s)Q_0K^{-1/2}(s). \end{aligned} \tag{7.39}$$

Here, matrix $A_*(t)$ is given by

$$A_*(t) = K^{-1/2}(AK^{1/2} - dK^{1/2}/dt), \tag{7.40}$$

and explicit formulas [7] for h_* are given below for criteria (7.14)–(7.16), respectively:

$$\begin{aligned} h_* &= (n^{-1}\mathrm{Tr}Q_*^{-1})^{1/2}, \\ h_* &= [\mathrm{Tr}C_*/\mathrm{Tr}(C_*Q_*)]^{1/2}, \quad C_* = K^{1/2}CK^{1/2}, \\ h_* &= [(v_*, v_*)/\mathrm{Tr}(Q_*v_*, v_*)]^{1/2}, \quad v_* = K^{1/2}v. \end{aligned}$$

Thus, by choosing matrix V in (7.34) according to (7.35), we reduce (7.22) for matrix Q to the form (7.36) that corresponds to the case where $A = 0$. On the other hand, by taking matrix V in accordance with (7.38), we come to (7.39) where K is replaced by the unity matrix I. Therefore, we can restrict ourselves with considering (7.22) only in the cases where either $A = 0$ or $K = I$.

Similar simplifications take place also for (7.28) for inner ellipsoids.

Let us consider now equations for globally optimal outer ellipsoids [7] and use the change of variables (7.34) for Q and

$$P = (V^{-1})^T P_* V^{-1} \tag{7.41}$$

for P, where P_* is a new variable. Let us restrict ourselves with the case of linear criteria (7.15) and (7.16). If V is defined by (7.35), we obtain from (7.30) and (7.32) for criterion (7.15):

$$P_*(t) = \Phi^T(T)C(T)\Phi(T) = \text{const.}$$

As a result, the equation for matrix $Q_*(t)$ coincides with (7.36), where

$$h_* = [\mathrm{Tr}(P_* K_*)/\mathrm{Tr}(P_* Q_*)]^{1/2}.$$

For criterion (7.16), h_* is determined by (7.37) with $v_* = \Phi^T(T)v$.

By defining V in (7.34) and (7.41) in accordance with (7.38), we again obtain for criterion (7.15), equations (7.39) for Q_*, equations

$$\dot{P}_* = -P_* A_* - A_*^T P_*, \quad P_*(T) = K^{1/2}(T)C(T)K^{1/2}(T)$$

for P_*, and the same expression (7.40) for A_*.

For criterion (7.16), (7.39) for Q_* and (7.40) for A_* still hold. Here, h_* is defined by

$$h_* = [(\psi_*, \psi_*)/(Q_* \psi_*, \psi_*)]^{1/2},$$

where the adjoint vector $\psi_*(t)$ satisfies the following initial value problem that replaces (7.33):

$$\dot{\psi}_* = -A_*^T \psi_*, \quad \psi_*(T) = K^{1/2}(T)v.$$

7.7 Properties of Optimal Ellipsoids

Outer approximating ellipsoids $E(a(t), Q(t))$ optimal in the sense of criterion (7.16) have the following properties:

1. Globally optimal ellipsoids touch reachable sets $D(t, s, M)$ for all $t \in [s, T]$ at points $x(t)$, where the normal to the boundary of these sets is parallel to the vector $\psi(t)$ defined by (7.33) [6]. In other words, these ellipsoids are tight in the sense of [16].
2. Globally optimal ellipsoids are also locally optimal for the vector $v(t) = \psi(t)$.
3. Locally optimal ellipsoids for the vector $v(t)$ defined by the initial value problem:

$$v(t) = \psi(t), \quad \dot{\psi} = -A^T(t)\psi, \quad \psi(s) = v^0, \tag{7.42}$$

where v^0 is an arbitrary vector, are also globally optimal for any terminal time instant $T \geq s$ and for the criterion $J = (Qv(T), v(T))$.

To construct these locally (and also globally) optimal ellipsoids, one is to solve the linear initial value problem (7.21) for $a^+(t)$ and also initial value problem consisting of (7.22) for Q^+ and (7.42) for $\psi(t)$. Here, the initial vector v^0 can be chosen arbitrarily, and different vectors v^0 correspond to different approximating ellipsoids touching reachable sets at different points.

Various properties of nonlinear equations (7.22) and (7.28) governing the evolution of locally optimal ellipsoids have been studied [1–3, 5–7, 26–28].

As a rule, the nonlinear differential equations for ellipsoids are to be integrated numerically. However, a number of explicit analytical solutions have been obtained both for locally [1–3, 5] and globally [6, 26–28] optimal ellipsoids.

Asymptotic behavior of the solution of (7.22) and (7.28) have been analyzed in the vicinity of the initial point $t = s$, if $Q_0 = 0$ [2, 3]. This case corresponds to the situation, where the initial set M in (7.11) is a given point $x(s) = a_0$. In this important case, (7.22) and (7.28) have a singularity [see also (7.24)–(7.27)], and the obtained asymptotic expansions of the solution are needed to start the numerical integration of equations near the initial point $t = s$ for the case where $Q_0 = 0$.

Also, asymptotic behavior for solutions of (7.22) and (7.28) at infinity ($t \to \infty$) have been analyzed [2, 3, 6, 26–28].

7.8 Generalizations

The method of ellipsoids has been extended to the case, where the parameters of the linear system (7.9) are uncertain and/or subjected to unknown but bounded perturbations. Consider the following system

$$\dot{x} = [A_0(t) + A_1(t)]x + f(t), \tag{7.43}$$

where $x \in R^n$ is the state, matrix $A_0(t)$ and n-vector $f(t)$ are given functions of time, whereas the matrix $A_1(t)$ is unknown, and its elements $a_{ij}^1(t)$ are bounded:

$$|a_{ij}^1(t)| \le b_{ij}, \quad i, j = 1, \dots, n, \quad t \ge s. \tag{7.44}$$

Here, b_{ij} are given nonnegative numbers. The system described by (7.43) and (7.44) models the situation, where some parameters of the system are uncertain (fixed but unknown) or changeable, for example, in the case of parametric excitation.

Outer ellipsoidal estimates $E(a(t), Q(t))$ on the reachable sets of the system described by (7.43) and (7.44) have been obtained [4].

Suppose the initial data are given by (7.11). The equation for the center $a(t)$ of the approximating ellipsoid is still the same as (7.21), where $A(t)$ is replaced by $A_0(t)$, so that we have

$$\dot{a} = A_0(t)a + f(t), \quad a(s) = a_0.$$

Nonlinear matrix equation for $Q(t)$ differs from (7.22) and has the form

$$\begin{aligned} \dot{Q} &= A_0 Q + Q A_0^T + hQ + h^{-1} R(a, Q), \\ h &= [n^{-1} \mathrm{Tr}(Q^{-1} R)]^{1/2}, \quad R = \mathrm{diag}(R_1^2, \dots, R_n^2), \\ R_i &= \sum_{j=1}^{n} b_{ij} |a_j| + \left(\max_{\sigma} \sum Q_{jk} b_{ij} b_{ik} \sigma_{ij} \sigma_{ik} \right)^{1/2}. \end{aligned} \tag{7.45}$$

Here, the maximum is taken over all $\sigma_{ij} = \pm 1$, $i, j = 1, \dots, n$. Note that, in contrast to (7.22), the right-hand side of (7.45) for Q depends on vector a.

The method of ellipsoids can be extended also to nonlinear systems [2, 3]. The main idea is to construct a linear comparison system described by (7.9), (7.10), and (7.11), so that all possible motions of the original nonlinear system are within the reachable sets of the linear one. For example, consider a nonlinear system

$$\dot{x} = A(t)x + \varphi(u,t), \quad |\varphi(u,t)| \le \varphi_0(t), \quad t \ge s, \tag{7.46}$$

where u is the disturbance, and the absolute value of the nonlinearity $\varphi(u,t)$ is bounded by $\varphi_0(t)$ for all admissible u and $t \ge s : |\varphi(u,t)| \le \varphi_0(t)$.

Then, the following linear system

$$\dot{x} = A(t)x + w, \quad w \in E(0, \varphi_0^2(t)I) \tag{7.47}$$

can serve as a comparison system for (7.46). Here, w is a disturbance bounded by a ball.

An outer approximating ellipsoid $E(a(t), Q(t))$ for linear system (7.47) will provide an outer bound also for reachable sets of the original nonlinear system (7.46).

Similarly, inner ellipsoidal bounds for reachable sets of nonlinear systems can be obtained [2, 3].

The class of approximating sets can be extended: besides ellipsoids, also intersections and unions of several ellipsoids can be considered [2, 3]. Thus, the approximation of reachable sets can be improved significantly.

7.9 Applications

Two-sided ellipsoidal approximations of reachable sets can be used for the solution and approximation of various problems in control and state estimation. Here, we will only briefly mention these applications; see [3, 5] for details.

7.9.1 Two-Sided Estimates in Optimal Control

Consider the following optimal control problem for system (7.9) under the initial condition $x(s) = x^0$.

Find the control subject to constraint (7.10) that provides the minimum of the given terminal functional $J = F(x(T))$ at $t = T > s$.

Let us obtain two-sided approximating ellipsoids (locally or globally optimal) and evaluate the following minima of the function $F(x)$ over the ellipsoids

$$F^{\pm} = \min F(x), \quad x \in E(a^{\pm}(T), Q^{\pm}(T)).$$

Then the two-sided bounds $F^+ \leq \min J \leq F^-$ are true for the required minimum of the functional J. To obtain these bounds, one needs, first, to find the inner and outer ellipsoids, and, second, to solve the problems of nonlinear programming, namely, to minimize the function $F(x)$ over the ellipsoidal sets. Note that the first part of this procedure does not depend on the function $F(x)$; if we change this function, we are to change only the second part of the procedure.

7.9.2 Two-Sided Bounds on Time for the Time-Optimal Problem

Let the time-optimal problem ($T \to \min$) is set up for the system (7.9) under the initial condition $x(s) = x_0$. The terminal state is fixed: $x(T) = x^*$. Let us find the minimal time instants T^+ and T^- when the ellipsoids $E(a^\pm(t), Q^\pm(t))$ contain the point x^*:

$$T^\pm = \min t, \quad x^* \in E(a^\pm(t), Q^\pm(t)), \quad t \geq s.$$

Then the two-sided bounds $T^+ \leq T \leq T^-$ for T are true.

7.9.3 Suboptimal Control

Consider again a linear system described by (7.9) and (7.10) under the initial condition $x(s) = x^0$. Suppose that the given terminal state $x(T) = x^*$ at some instant T belongs to the inner ellipsoid $E(a^-(T), Q^-(T))$. Then there exists an admissible control $u(t)$ bringing the system to this terminal state at $t = T$. This control can be constructed efficiently [12, 13] and is defined by the following procedure. First, find the solution of the initial value problems (7.28) for functions $a^-(t)$ and $Q^-(t)$ that determine the inner approximating ellipsoid.

Denote

$$R(t) = K^{1/2}(K^{-1/2}Q^-K^{-1/2})^{1/2}K^{1/2}$$

and solve the auxiliary initial value problem

$$\begin{aligned} \dot{\psi} &= -A^T\psi - (Q^-)^{-1}R\psi \quad t \in [s, T], \\ \psi(T) &= [Q^-(T)]^{-1}[x^* - a^-(T)]. \end{aligned}$$

Then the admissible control $u(t)$ bringing system (7.9) from the initial state $x(s) = x^0$ to the prescribed terminal state $x(T) = x^*$ is given by the expression

$$u(t) = R(t)\psi(t), \quad t \in [s, T].$$

This control can be called suboptimal since, if the terminal time T is equal to the upper bound T^- for the time-optimal problem introduced in Sect. 7.9.2, this control brings our system to the prescribed terminal state at $t = T = T^-$.

7.9.4 Differential Games

Consider now a differential game of two players X and Y described by equations similar to (7.9) and (7.10):

$$\begin{aligned} X: \quad & \dot{x} = A_x(t)x + B_x(t)u + f_x(t), \\ & u \in E(0, G_x(t)), \quad x(s) = x^0; \\ Y: \quad & \dot{y} = A_y(t)y + B_y(t)v + f_y(t), \\ & v \in E(0, G_y(t)), \quad y(s) = y^0. \end{aligned}$$

Here, x and y are the state vectors of the players X and Y, respectively, u and v are their controls, and x^0 and y^0 are the respective initial states. The cost functional J is a given scalar function of the terminal states of the players at the prescribed time instant T:

$$J = \Phi(x(T), y(T)), \quad T > s.$$

Player X seeks to minimize J while player Y, which can be also identified with uncertain disturbances, opposes Y and tends to maximize J. The following inequalities

$$\max_{y \in D_y} \min_{x \in D_x} \Phi(x, y) \leq J^* \leq \min_{x \in D_x} \max_{y \in D_y} \Phi(x, y),$$

where D_x and D_y are the reachable sets of players X and Y, respectively, at the instant T, provide evident two-sided bounds on the optimal value J^* of the functional J that corresponds to optimal strategies of both players.

Using ellipsoidal bounds

$$E_x^{\pm} = E(a_x^{\pm}(t), Q_x^{\pm}(t)), \quad E_y^{\pm} = E(a_y^{\pm}(t), Q_y^{\pm}(t))$$

on reachable sets for players X and Y, we obtain the following two-sided estimates on J^*:

$$\Phi_1 \leq J^* \leq \Phi_2, \quad \Phi_1 = \max_{y \in E_y^-} \min_{x \in E_x^+} \Phi(x, y), \quad \Phi_2 = \min_{x \in E_x^-} \max_{y \in E_y^+} \Phi(x, y). \tag{7.48}$$

Suppose the pairs of points x_1^*, y_1^* and x_2^*, y_2^* are found where the maximin Φ_1 and minimax Φ_2 from (7.48) are attained, respectively. Using results of Sect. 7.9.3, we can find the control $u(t)$ bringing player X to the state $x_2^* \in E_x^-$ at $t = T$, and also the control $v(t)$ bringing player Y to the state $y_1^* \in E_y^-$ at $t = T$. If player X applies the open-loop control $u(t)$, then the value of functional J does not exceed Φ_2 under any admissible control of player Y. On the other hand, if player Y applies the open-loop control $v(t)$, than the value of functional J is not less than Φ_1 under any admissible control of player X. Thus, using approximating ellipsoids, we can obtain two-sided bounds on the value of the cost functional and determine open-loop controls of the players that ensure these bounds.

Approximating ellipsoids can be also used for obtaining two-sided bounds in games of pursuit-evasion. In this context, the rule of extremal aiming [14, 15] is used. Numerical example is presented in [11].

7.9.5 Control of Uncertain Systems

Consider a system subjected to both the control $u(t)$ and disturbance $v(t)$:

$$\dot{x} = A(t)x + B(t)u + C(t)v + f(t), \quad t \geq a. \tag{7.49}$$

Knowing the bound $v(t) \in E(0, G(t))$ on the disturbance, we can obtain the outer bound $x(t) \in E(a(t), Q(t))$ on all trajectories of system (7.49) subjected to the given control $u(t)$. Then, equations (7.21) for $a(t)$ and (7.22) for $Q(t)$ become

$$\dot{a} = Aa + Bu + f, \quad \dot{Q} = AQ + QA^T + hQ + h^{-1}K, \quad K = BGB^T$$

and can be considered as a control system for the whole set of possible trajectories. Various control problems can be set up for this system, and different control methods can be applied to these problems.

7.9.6 Other Applications

In case where $u(t)$ is a bounded disturbance in (7.9), the outer approximating ellipsoid provides an estimate on possible deviations of the trajectory caused by the disturbance. Such estimates are sometimes associated with the notion of "practical stability". For example, possible deviation of the trajectory of a moving body in the presence of wind disturbances can be evaluated.

Various applications of ellipsoidal bounds to parameter estimation are considered in [30, 31]. Aerospace applications of approximating ellipsoids are discussed in [25, 32].

7.9.7 State Estimation in the Presence of Observation Errors

Consider again the system subjected to uncertain disturbances and described by (7.9), (7.10), and (7.11). Suppose that, at the given time instants t_i, the results of observations

$$y_i = H_i x(t_i) + \xi_i, \quad t_i \geq s, \quad i = 0, 1, \ldots, \tag{7.50}$$

become available. Here, y_i are m-vectors of observation results, H_i are given $m \times n$ matrices, and ξ_i are m-dimensional observation errors subject to constraints $\xi_i \in E(0, L_i)$, where L_i are given symmetric positive definite $m \times m$ matrices. Thus, at the instants t_i, the state $x(t_i)$ of the system belongs to the intersection of two ellipsoidal sets:

$$\begin{aligned} &x(t) \in E(t_i) \cap \tilde{E}_i, \\ &\tilde{E}_i = \{x : (L_i^{-1}[H_i x(t_i) - y_i], [H_i x(t_i) - y_i]) \leq 1\}. \end{aligned} \tag{7.51}$$

Here, $E(t_i)$ is the outer ellipsoidal state estimate of the system based on all information available for $t < t_i$, and $\tilde{E}_i$ is the ellipsoid corresponding to the observation (7.50) at $t = t_i$.

To design the process of the ellipsoidal state estimation, we are to construct the ellipsoid $E(t_i + 0)$ that contains the intersection of ellipsoids (7.51). Then we can use $E(t_i + 0)$ as the initial ellipsoid for the next time interval (t_i, t_{i+1}), see Fig. 7.2. It is desirable to find $E(t_i + 0)$ that minimizes certain optimality criterion (see Sect. 7.4). Using optimal and suboptimal outer ellipsoidal bounds for the intersection of two ellipsoids [2, 3], the recursive procedure for the ellipsoidal state estimation in the presence of observation errors has been developed. The state estimation procedure

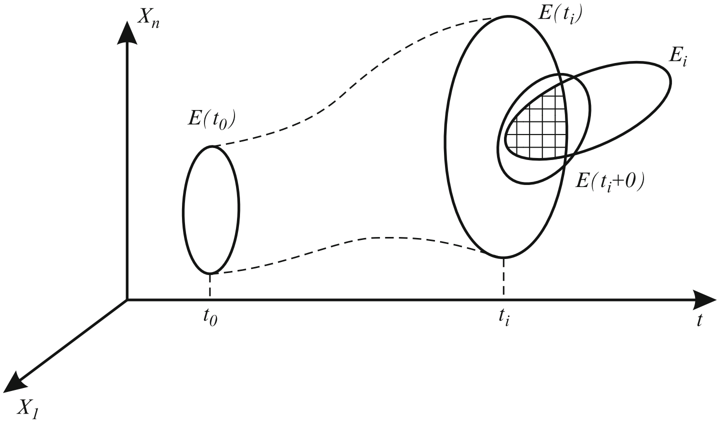

Fig. 7.2 Ellipsoidal state estimation

has been extended also to the case of continuous observations. Thus, the guaranteed analogue of the well-known Kalman filtering has been elaborated [2, 3, 5].

Applications of ellipsoidal technique to state estimation are considered in [17, 18, 25, 30, 31].

7.10 Ellipsoidal vs. Interval Analysis

There exists a vast literature on the interval analysis that is widely used in the computational mathematics [8,9,20,22]. In the interval analysis, to obtain the guaranteed error estimates, one operates with intervals $[x^-, x^+]$, instead of precise value x that is unknown but bounded, $x \in [x^-, x^+]$. When this approach is applied to vectors $x = (x_1, \dots, x_n)$, one is to deal with boxes

$$B = \{x : \ x_i^- \leq x \leq x_i^+, \quad i = 1, \dots, n\}, \tag{7.52}$$

i.e., rectangular parallelepipeds with sides parallel to coordinate axes.

The interval approach and its generalizations, where uncertainty domains are bounded by various polytopes, are also applied in the control and identification, see [10, 19, 21, 23, 24, 34].

Let us discuss and compare the method of ellipsoids with the interval methods.

One can consider several levels of generalization of interval methods. The sets of uncertainty can be bounded by: (a) boxes (7.52); (b) rectangular parallelepipeds; (c) arbitrary parallelepipeds; (d) polytopes. The class (d) of arbitrary polytopes seems to be too wide. As already mentioned in Sect. 7.2, the class (c), like the class of ellipsoids, is invariant with respect to linear transformations, but it requires almost two times more parameters for its description than the class of ellipsoids. The classes (a) and (b) are not invariant with respect to linear transformations.

In this context, the class of boxes (7.52) seems to be more attractive among classes (a)–(d) because it has the simplest description and requires only $2n$ parameters.

However, this class can lead to an undesirable but essential error increase.

Consider a simplest vectorial operation $y = Ax$, where x is defined by a box (7.52) and A is a given $n \times n$ matrix. As a result of this operation, box (7.52) is transformed to a parallelepiped P. To obtain a box B' for vector y, one needs to take a box that contains P, $B' \supset P$, and thus to expand the set of possible vectors y. In other words, the error estimate will become wider. In [29], the results of operation $y = Ax$ have been analyzed both for ellipsoidal and interval uncertainty bounds. It has been shown that the ellipsoidal bounds are frequently superior to interval ones and lead to tighter error estimates than those given by the interval analysis.

Thus, it seems that the method of ellipsoids can be of use for error estimation in numerical analysis.

7.11 Conclusions

The method of ellipsoids seems to be an efficient technique for the analysis of dynamical systems subjected to uncertain perturbations and observation errors. By means of this approach, exact and approximate solutions as well as reliable two-sided bounds for a number of basic problems in control and estimation can be obtained.

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References

1. Chernousko, F.L.: Optimal guaranteed estimates of indeterminacies using ellipsoids, Parts 1–3. Izv. Akad. Nauk SSSR, Tekh. Kibern. **3**, 3–11; **4**, 3–11; **5**, 5–11 (1980)
2. Chernousko, F.L.: Estimation of Phase State for Dynamical Systems. Nauka, Moscow (1988)
3. Chernousko, F.L.: State Estimation for Dynamic Systems. CRC, Boca Raton (1994)
4. Chernousko, F.L.: Ellipsoidal approximation of attainability sets of linear system with indeterminate matrix. J. Appl. Math. Mech. **60**(6), 921–931 (1996)
5. Chernousko, F.L.: What is ellipsoidal modelling and how to use it for control and state estimation? In: Elishakoff, I. (ed.) Whys and Hows in Uncertainty Modelling, pp. 127–188. Springer, Vienna (1999)
6. Chernousko, F.L., Ovseevich, A.I.: Some properties of optimal ellipsoids approximating reachable sets. Dokl. Math. **67**(1), 123–126 (2003)
7. Chernousko, F.L., Ovseevich, A.I.: Properties of the optimal ellipsoids approximating the reachable sets of uncertain systems. J. Optim. Theor. Appl. **120**(2), 223–246 (2004)
8. Goetz, A., Herzberger, J.: Introduction to Interval Analysis. Academic, New York (1983)
9. Jaulin, L., Kieffer, M., Didrit, O., Walter, E.: Applied Interval Analysis. Springer, London (2001)
10. Kieffer, M., Walter, E.: Interval analysis for guaranteed non-linear parameter and state estimation. Math. Comput. Model. Dyn. Syst. **11**(2), 171–181 (2005)
11. Klepfish, B.R.: Method of obtaining the two-side estimate on the time of pursuit. Izv. Akad. Nauk SSSR, Tekh. Kibern. **4**, 156–160 (1984)
12. Komarov, V.A.: Estimates on reachable sets and construction of admissible controls for linear systems. Dokl. Akad. Nauk SSSR **268**, 537–541 (1982)
13. Komarov, V.A.: Estimates on reachable sets for linear systems. Izv. Akad. Nauk SSSR, Ser. Mat. **48**, 865–879 (1984)
14. Krasovskii, N.N.: Game Problems of Meeting of Motions. Nauka, Moscow (1970)
15. Krasovskii, N.N., Subbotin, A.I.: Positional Differential Games. Springer, Berlin (1988)
16. Kurzhanski, A.B., Varaiya, P.: Ellipsoidal techniques for reachability analysis. Lecture Notes in Computer Science vol. 1790, pp. 202–214. Springer, Berlin (2000)
17. Maksarov, D.G., Norton, J.P.: State bounding with ellipsoidal set description of the uncertainty. Int. J. Control **65**, 847–866 (1996)
18. Maksarov, D.G., Norton, J.P.: Computationally efficient algorithms for state estimation with ellipsoidal approximations. Int. J. Adapt. Control Signal Process. **16**, 411–434 (2002)
19. Milanese, M., Norton, J., Piet-Lahanier, H., Walter, E. (eds.): Bounding Approaches to System Identification. Plenum, New York (1996)
20. Moore, R.E.: Methods and Applications of Interval Analysis. SIAM, Philadelphia (1979)

21. Nazin, S.A., Polyak, B.T.: Interval parameter estimation under model uncertainty. Math. Comput. Model. Dyn. Syst. **11**(2), 225–237 (2005)
22. Neumaier, A.: Interval Methods for Systems of Equations. Cambridge University Press, Cambridge (1990)
23. Norton, J.P. (ed.): Special issue on bounded-error estimation, 1. Int. J. Adapt. Control Signal Process. **8**(1) (1994)
24. Norton, J.P. (ed.): Special issue on bounded-error estimation, 2. Int. J. Adapt. Control Signal Process. **9**(2) (1995)
25. Norton, J.P.: Results to aid applications of ellipsoidal state bounds. Math. Comput. Model. Dyn. Syst. **11**(2), 209–224 (2005)
26. Ovseevich, A.I.: Limit behavior of attainable and superattainable sets. In: Modelling, Estimation, and Control of Systems with Uncertainty, pp. 324–333. Birkhäuser, Boston (1991)
27. Ovseevich, A.I.: On equations of ellipsoids approximating attainable sets. J. Optim. Theor. Appl. **95**(3), 659–676 (1997)
28. Ovseevich, A.I., Chernousko, F.L. : Methods of ellipsoidal estimation for linear control systems. In: Proc. 17th World Congress, The International Federation of Automatic Control, pp. 15345–15348. Seoul, Korea (2008)
29. Ovseevich, A.I., Taraban'ko, Yu.V., Chernousko, F.L.: A comparison of interval and ellipsoidal error bounds for vector operations. Dokl. Math. **71**(1), 127–130 (2005)
30. Polyak, B.T., Nazin, S.A., Durieu, C., Walter, E.: Ellipsoidal parameter or state estimation under model uncertainty. Automatica **40**, 1171–1179 (2004)
31. Pronzato, L., Walter, E.: Minimum-volume ellipsoids containing compact sets: application to parameter bounding. Automatica **30**, 1731–1739 (1994)
32. Rokityanskiy, D.Ya., Veres, S.M.: Application of ellipsoidal estimation to satellite control design. Math. Comput. Model. Dyn. Syst. **11**(2), 239–249 (2005)
33. Schweppe, F.C.: Uncertain Dynamic Systems. Prentice-Hall, Englewood Cliffs, NJ (1973)
34. Walter, E. (ed.): Special issue on parameter identification with error bound. Math. Comput. Simul. **32** (1990)