C.3 Spatial Econometric Methods for Modeling Origin-Destination Flows

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C.3.1 Introduction

Spatial econometric theory and practice have been dominated by a focus on object data. In economic analysis these objects correspond to economic agents with discrete locations in geographic space, such as addresses, census tracts and regions. In contrast spatial interaction or flow data pertain to measurements each of which is associated with a link or pair of origin-destination locations that represent points or areas in space. While there is a voluminous literature on the specification and estimation of models for cross-sectional object data (see, Chapter C.1 in this volume), less attention has been paid to sample data consisting of origin-destination pairs that form the basic units of analysis in spatial interaction models.

Spatial interaction models represent a class of methods which are used for modeling origin-destination flow data. The interest in such models is motivated by the need to understand and explain flows of tangible entities such as persons and commodities or intangible ones such as capital, information or knowledge across geographic space. By adopting a spatial interaction modeling perspective attention is focused on interaction patterns at the aggregate rather than the individual level.

The basis of modeling is the use of a discrete zone system. Discrete zone systems can obviously take many different forms, both in relation to the level of resolution and the shape of zones. The subdivision of the geography into zones introduces spatial aggregation problems. Such problems come from the fact that substantially different conclusions can be obtained from the same dataset and the same spatial interaction model, but at another spatial aggregation level (see, for example, Batty and Sidkar 1982). Spatial aggregation problems involve both a scale issue and a zoning issue. The tidiest, and often most convenient system to use would be a square grid. But quite often one is forced to use administratively defined regions, such as NUTS-2 regions in Europe, counties in a country or the wards of a city.

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The subject of spatial interaction modeling has a long and distinguished history that has led to the emergence of three major schools of analytical thought: the macroscopic school based upon a statistical equilibrium approach (see Wilson 1967; Roy 2004), the microscopic school based on a choice-theoretic approach (see Smith 1975; Sen and Smith 1995), and the geocomputational school based upon the neural network approach that processes spatial interaction models as universal function approximators (see Fischer 2002; Fischer and Reismann 2002). In these schools there is a deep-seated view that spatial interaction implies movement of entities, and that this has little to do with spatial association (Getis 1991).

Spatial interaction models typically rely on three types of factors to explain mean interaction frequencies between origins and destinations of interaction: (i) origin-specific attributes that characterize the ability of the origins to produce or generate flows, (ii) destination-specific attributes that represent the attractiveness of destinations, and (iii) origin-destination variables that characterize the way spatial separation of origins from destinations constrains or impedes the interaction. They implicitly assume that using spatial separation variables such as distance will eradicate the spatial dependence among the sample of spatial flows.

However, research dating back to the 1970s, noted that spatial dependence or autocorrelation might be intermingled in spatial interaction model specifications. This idea was first put forth in a theoretical context by Curry (1972), with some subsequent debate in Curry et al. (1975). Griffith and Jones (1980) documented the presence of spatial dependence in conventional spatial interaction models. Despite this, most practitioners assume independence among observations and few have used spatial lags of the dependent variable or disturbances in spatial interaction models. Exceptions are Bolduc et al. (1992), and Fischer and Griffith (2008) who rely on spatial lags of the disturbances, and LeSage and Pace (2008) who use lags of the dependent variable.

The focus of this chapter is on problems that plague empirical implementation of conventional regression-based spatial interaction models and econometric extensions that have recently appeared in the literature. These new models replace the conventional assumption of independence between origin-destination flows with formal approaches that allow for spatial dependence in flow magnitudes. We follow LeSage and Pace (2008) and extend the generic version of the spatial interaction model to include spatial lags of the dependent variable.

C.3.2 The analytical framework

Spatial interaction data represent phenomena that may be described in their most general terms as interactions between populations of actors and opportunities distributed over some relevant geographic space. Such interactions may involve movements of individuals from one location to another, such as daily traffic flows in which case the relevant actors are individual travellers (commuters, shoppers, etc.) and the relevant opportunities are their destinations (jobs, stores, etc.). Similarly, one may consider annual migration flows, where the relevant actors are migrants (individuals, family units, firms, etc.) and the relevant opportunities are their possible new locations. Interactions may also involve flows of information such as telephone calls or electronic messages. Here the callers or message senders may be the relevant actors, and the possible receivers of calls or electronic messages may be considered as the relevant opportunities (Sen and Smith 1995). With this range of examples in mind, the purpose of this section is to outline a framework in which all such spatial interaction behaviour can be studied.

The classical spatial interaction model

Suppose we have a spatial system consisting of n discrete zone (locations, regions) where i ($i = 1, \ldots, n$) denotes the origin and j ($j = 1, \ldots, n$) the destination of interaction. Let $m(i, j)$ denote observations on random variables, say $M(i, j)$, each of which corresponds to a movement of tangible or intangible entities from i to j. The $M(i, j)$ are assumed to be independent random variables. They are sampled from a specified probability distribution that is dependent upon some mean, say $\mu\ (i, j)$. Let us assume that no a priori information is given about the origin and destination totals of the observed flow matrix. Then the mean interaction frequencies between origin i and destination j may be modeled by

$$\mu(i, j) = C\ A(i)\ B(j)\ S(i, j) \tag{C.3.1}$$

where $\mu(i, j) = E[M(i, j)]$ is the expected flow, C denotes a constant term, the quantities $A(i)$ and $B(j)$ are called origin and destination factors or variables respectively, and $S(.)$ is some unspecified distance deterrence function (see Fischer and Griffith 2008). Note if the outflow totals for each origin zone and/or the inflow totals into each destination zone are known a priori, then model (C.3.1) would need to be modified to incorporate the explicitly required constraints to match exact totals. Imposing origin and/or destination constraints leads to so-called *production-constrained*, *attraction-constrained* and *production-attraction-constrained* spatial interaction models that may be convincingly justified using entropy maximizing methods (see Fotheringham and O'Kelly 1989; Bailey and Gatrell 1995 for a discussion).

Equation (C.3.1) is a very general version of the classical (unconstrained) spatial interaction model. The exact functional form of the three terms $A(.)$, $B(.)$ and $S(.)$ on the right hand side of Eq. (C.3.1) is subject to varying degrees of conjecture. There is wide agreement that the origin and destination factors are generally best given by power functions

$$A(i) = (A_i)^{\beta} \tag{C.3.2a}$$

$$B(j) = (B_j)^{\gamma} \tag{C.3.2b}$$

where A_i represents some appropriate variable measuring the propulsiveness of origin i, and B_j some appropriate variable measuring the attractiveness of destination j in a specific spatial interaction context. The product $A(i)\,B(j)$ can be interpreted simply as the number of distinct (i, j)-interactions that are possible. Thus, for origin-destination pairs (i, j) with the same level of separation, it follows from Eq. (C.3.1) that mean interaction levels are proportional to the number of possible interactions between such (i, j)-pairs. The exponents, β and γ, indicate the origin and destination effects respectively, and are treated as statistical parameters to be estimated.

If more than one origin and one destination variable are relevant in a specific context the above specification may be extended to

$$A(i) = \prod_{q \in Q} (A_{iq})^{\beta_q} \tag{C.3.3a}$$

$$B(j) = \prod_{r \in R} (B_{jr})^{\gamma_r} \tag{C.3.3b}$$

where A_{iq} $(q \in Q)$ and B_{jr} $(r \in R)$ represent sets of relevant (positive) origin-specific and destination-specific variables, respectively. The exponents $(\beta_q : q \in Q)$ and $(\gamma_r : r \in R)$ are parameters to be estimated. See Fotheringham and O'Kelly (1989) for a range of explicit variable specifications.

The distance deterrence function $S(i, j)$ constitutes the very core of spatial interaction models. Hence, a number of alternative specifications have been proposed in the literature (for a discussion see Sen and Smith 1995). One prominent example is the following power function specification given by

$$S(i, j) = [D(i, j)]^{\theta} \tag{C.3.4}$$

for any positive scalar distance measure, $D(i, j)$, and negative distance sensitivity parameter θ that has to be estimated. Another popular specification is the exponential function $S(i, j) = \exp[-\theta\, D(i, j)]$, where θ has to be an univariate parameter with specific value depending on the choice of units for distance (see Sen and Smith 1995).

The deterrence function reflects the way in which spatial separation or distance constrains or impedes movement across space. In general we will refer to this as distance between an origin *i* and a destination *j*, and denote it as $D(i, j)$. At relatively large scales of geographical inquiry this might be simply the great circle distance separating an origin from a destination zone measured in terms of the distance between their respective centroids. In other cases, it might be transportation or travel time, cost of transportation, perceived travel time or any other sensible measure such as political distance, language distance or cultural distance measured in terms of nominal or categorical attributes. To allow for the possibility of multiple measures of spatial separation, the power function specification in Eq. (C.3.4) can be extended to the following class of multivariate power deterrence functions

$$S(i,j) = \prod_{k \in K} [{}^{k} D(i,j)]^{\theta_k} \tag{C.3.5}$$

with corresponding distance sensitivity vector $\theta = (\theta_k : k \in K)$.

From the positivity of the functions $A(.)$, $B(.)$ and $S(.)$, it follows that the spatial interaction model (C.3.1) with the specifications (C.3.3) and (C.3.4) can be expressed equivalently as a log-additive model of the form

$$y(i,j) = c + \sum_{q \in Q} \beta_q \, a_q(i) + \sum_{r \in R} \gamma_r \, b_r(j) + \theta \, d(i,j) \tag{C.3.6}$$

where $y(i,j) = \log \mu(i,j)$, $c = \log C$, $a_q(i) = \log A_{iq}$, $b_r(j) = \log B_{jr}$, and $d(i,j) = \log D(i,j)$. In the sequel we will illustrate how these $n^2 (= N)$ equations can be written more compactly using vector and matrix notation.

The spatial interaction model in matrix notation

Let $\boldsymbol{Y}$ denote an n-by-n square matrix of origin-destination flows from each of the n origin zones to each of the n destination zones as shown in Eq. (C.3.7) where the n columns represent different origins and the n rows different destinations. The elements on the main diagonal of the matrix represent intrazonal flows, and we use $N = n^2$ for notational simplicity.

$$\boldsymbol{Y} = \begin{bmatrix} y(1,1) & \cdots & y(1,i) & \cdots & y(1,n) \\ \vdots & & \vdots & & \vdots \\ y(j,1) & \cdots & y(j,i) & \cdots & y(j,n) \\ \vdots & & \vdots & & \vdots \\ y(n,1) & \cdots & y(n,i) & \cdots & y(n,n) \end{bmatrix} \tag{C.3.7}$$

LeSage and Pace's (2008) introduction of notational conventions allow use of *origin-centric* or *destination-centric* flow matrices. An *origin-centric* ordering of the flow matrix $\boldsymbol{Y}$ is shown in Table C.3.1, where the dyad label denotes the overall index from 1, …, N for the ordering. The first n elements in the stacked vector $\boldsymbol{y}$ reflect flows from origin zone $i=1$ to all n destinations and the last n elements flows from origin zone $i=n$ to destinations 1, …, n. This case often arises in practice when intraregional flows cannot be measured or are difficult to measure.

Table C.3.1. Data organization convention

Dyad label	ID origin	ID destination	Flows	Origin variables	Destination variables	Distance variable
1	1	1	$y(1,1)$	$a_1(1)\ldots a_Q(1)$	$b_1(1)\ldots b_R(1)$	$d(1,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
n	1	n	$y(1,n)$	$a_1(1)\ldots a_Q(1)$	$b_1(n)\ldots b_R(n)$	$d(1,n)$
$n+1$	2	1	$y(2,1)$	$a_1(2)\ldots a_Q(1)$	$b_1(1)\ldots b_R(1)$	$d(2,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
$2n$	2	n	$y(2,n)$	$a_1(2)\ldots a_Q(2)$	$b_1(n)\ldots b_R(n)$	$d(2,n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
$N-n+1$	n	1	$y(n,1)$	$a_1(n)\ldots a_Q(n)$	$b_1(1)\ldots b_R(1)$	$d(n,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
N	n	n	$y(n,n)$	$a_1(n)\ldots a_Q(n)$	$b_1(n)\ldots b_R(n)$	$d(n,n)$

The least-squares regression approach widely used in practice to explain variation in origin-destination flows relies on two sets of explanatory variable matrices. One is an N-by-Q matrix of Q origin-specific variables for the n regions that we label $\boldsymbol{X}_o$. This matrix reflects an n-by-q matrix of explanatory variables $\boldsymbol{X}_q$ (q = 1, …, Q) that is *repeated* n times using $\boldsymbol{X}_o = \boldsymbol{X} \otimes \boldsymbol{\iota}_n$, where $\boldsymbol{\iota}_n$ is an n-by-1 vector of ones. The matrix Kronecker product $(\otimes)$ works to multiply the right-hand argument $\boldsymbol{\iota}_n$ times each element in the matrix $\boldsymbol{X}$, which strategically repeats the explanatory variables so they are associated with observations treated as origins. Specifically, the matrix product would repeat the origin characteristics of the first zone to form the first n rows, the origin characteristics of the second zone n times for the next n rows and so on (see Table C.3.1), resulting in the N-by-Q matrix $\boldsymbol{X}_o$. LeSage and Pace (2008) point out that if we organized the matrix of flows $\boldsymbol{Y}$

using a *destination-centric* ordering based on $\boldsymbol{Y}^{\mathrm{T}}$, then the matrix of origin-specific explanatory variables would consist of $\boldsymbol{X}_o = \boldsymbol{\iota}_n \otimes \boldsymbol{X}$.

The second matrix is an N-by-R matrix $\boldsymbol{X}_d = \boldsymbol{\iota}_n \otimes \boldsymbol{X}_r \; (r = 1, ..., R)$ that represents the R destination characteristics of the n regions. The Kronecker product works to repeat the matrix $\boldsymbol{X}_r$ n times to produce an N-by-R matrix representing destination characteristics (see Table C.3.1) that we label $\boldsymbol{X}_d$.

In addition to explanatory variables consisting of origin and destination characteristics, a vector of distances between each origin-destination dyad is included in the regression model. This vector is formed using the n-by-n distance matrix $\boldsymbol{D}$ containing distances between each origin and destination zone. The N-by-1 vector of distances is formed using $\boldsymbol{d} = \mathrm{vec}(\boldsymbol{D})$, where vec is an operator that converts a matrix to a vector by stacking the columns of the matrix, as shown in Table C.3.1.

This results in a regression model of the type shown in Eq. (C.3.8) that represents the *log-additive power deterrence function spatial interaction model* in matrix notation

$$\boldsymbol{y} = \alpha \, \boldsymbol{\iota}_n + \boldsymbol{X}_o \, \boldsymbol{\beta} + \boldsymbol{X}_d \, \boldsymbol{\gamma} + \theta \, \boldsymbol{d} + \boldsymbol{\varepsilon} \tag{C.3.8}$$

where

$\boldsymbol{y}$	N-by-*1* vector of origin-destination flows,
$\boldsymbol{X}_o$	N-by-Q matrix of Q origin-specific variables that characterize the ability of the origin zones to produce flows,
$\boldsymbol{\beta}$	the associated Q-by-1 parameter vector that reflects the origin effects,
$\boldsymbol{X}_d$	N-by-R matrix of R destination-specific variables that represent the attractiveness of the destination zones,
$\boldsymbol{\gamma}$	the associated R-by-1 parameter vector that reflects the destination effects,
$\boldsymbol{d}$	N-by-1 vector of distances between origin and destination zones,
θ	scalar distance sensitivity parameter that comes from the power deterrence function and reflects the distance effects,
$\boldsymbol{\iota}_n$	N-by-1 vector of ones,
α	constant term parameter on $\boldsymbol{\iota}_n$,
$\boldsymbol{\varepsilon}$	N-by-1 vector of disturbances with $\boldsymbol{\varepsilon} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}_N)$.

This spatial interaction model is based on the independence assumption for the case of a square matrix where each origin zone is also a destination zone and where no a priori information is given on the row and/or column totals of the interaction data matrix. In the sequel we will refer to this model as the *independence* (log-normal) *model*.

C.3.3 Problems that plague empirical use of conventional spatial interaction models

There are several problems that arise in applied practice when estimating the conventional spatial interaction model given by Eq. (C.3.8). We enumerate each of these problems in the following section and discuss solutions that have been proposed in the literature. These solutions often rely on elaborations of the basic model specification given in Eq. (C.3.8).

Efficient computation

One problem that can arise in cases where the sample of regions n is large involves computational memory. For the case of the U.S. counties, for example, we have $n > 3,000$ leading to N-by-Q and N-by-R matrices for the explanatory variables involving $N = n^2 > 9,000,000$. LeSage and Pace (2008) propose a solution for the case where $Q = R = k$ and we rely on the same n-by-k explanatory variables matrix $\boldsymbol{X}$ for both origin and destination characteristics. They point out that repeating the same sample of n-by-k explanatory variable information is not necessary if we take a moment matrix approach to the estimation problem.

If we let $\boldsymbol{Z} = (\boldsymbol{\iota}_N \quad \boldsymbol{X}_d \quad \boldsymbol{X}_o \quad \boldsymbol{d})$, we can form the moment matrix $\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{Z}$ shown in Eq. (C.3.9), with the symbol $\mathbf{0}_k$ denoting a 1-by-k vector of zeros, and tr representing the *trace* operator

$$\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{Z} = \begin{pmatrix} N & \mathbf{0}_k & \mathbf{0}_k & \mathbf{0} \\ \mathbf{0}_k^{\mathrm{T}} & n\,\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} & \mathbf{0}_k^{\mathrm{T}}\,\mathbf{0}_k & \boldsymbol{X}^{\mathrm{T}}\boldsymbol{D}\,\boldsymbol{\iota}_n \\ \mathbf{0}_k^{\mathrm{T}} & \mathbf{0}_k^{\mathrm{T}}\,\mathbf{0}_k & n\,\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} & \boldsymbol{X}^{\mathrm{T}}\boldsymbol{D}\,\boldsymbol{\iota}_n \\ \mathbf{0} & \boldsymbol{\iota}_n\,\boldsymbol{D}^{\mathrm{T}}\boldsymbol{X} & \boldsymbol{\iota}_n\,\boldsymbol{D}^{\mathrm{T}}\boldsymbol{X} & tr(\boldsymbol{D}^2) \end{pmatrix} \tag{C.3.9}$$

where we assume that the matrix $\boldsymbol{X}$ and vector $\boldsymbol{d}$ are in deviation from means form. This leads to many of the entries in Eq. (C.3.9) taking values of zero.

For the case of the $\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{y}$ required to produce least-squares estimates for the parameters, $\boldsymbol{\delta} = (\boldsymbol{Z}^{\mathrm{T}}\,\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\mathrm{T}}\,\boldsymbol{y}$, we have

$$\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{y} = \begin{pmatrix} \boldsymbol{\iota}_n\boldsymbol{Y}\boldsymbol{\iota}_n \\ \boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}\boldsymbol{\iota}_n \\ \boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{\iota}_n \\ tr(\boldsymbol{D}\boldsymbol{Y}) \end{pmatrix}. \tag{C.3.10}$$

Kronecker products prove extremely useful in working with origin-destination flows, as we will see. However, there are limitations associated with this approach that were not fully elaborated by LeSage and Pace (2008). One limitation is that the system of flows is a *closed system* with the same number of origins (n) as destinations (n). This will be required when we discuss modeling spatial dependence by constructing spatial lags of the dependent variable or disturbance terms. For example, if we were modeling shopping trips from various residential locations to a *single* store, this limitation would come into play.

Another limitation pertains to moment-based expressions in Eqs. (C.3.9) and (C.3.10) for working with large problems. These require that the same matrix $\boldsymbol{X}$ is used to form both the origin and destination characteristics matrices so that $\boldsymbol{X}_d = \boldsymbol{\iota}_n \otimes \boldsymbol{X}$ and $\boldsymbol{X}_o = \boldsymbol{X} \otimes \boldsymbol{\iota}_n$. This is equivalent to imposing the restriction that $Q = R$ in Table C.3.1. The moment-based expressions in Eqs. (C.3.9) to (C.3.10) also assume the matrix $\boldsymbol{X}$ is in deviation from means form, but LeSage and Pace (2009a) provide moment expressions that relax this requirement.

If these limitations are consistent with the problem at hand, the moment-based approach to estimation of the model parameters saves a great deal of computer memory. This is accomplished by working with n-by-n matrices rather than n^2-by-$(2k + 2)$, where we have k explanatory variables for regions treated as origins, k for the destination regions in addition to the intercept term and distance vector.

Spatial dependence in origin-destination flows

As already indicated, numerous applied work has pointed to the presence of spatial dependence in the least-squares disturbances from models involving origin-destination data samples (Porojan 2001; Lee and Pace 2005; Fischer and Griffith 2008).

One way to incorporate spatial dependence into a log-normal spatial interaction model of the form (C.3.8) is to specify a spatial process that governs the spatial interaction variable $\boldsymbol{y}$. This approach leads to a family of models depending on restrictions imposed on the spatial origin-destination filter specification set forth in LeSage and Pace (2009a). Specifically, this type of model specification takes the form

$$\boldsymbol{y} = \rho_o \boldsymbol{W}_o \boldsymbol{y} + \rho_d \boldsymbol{W}_d \boldsymbol{y} + \rho_w \boldsymbol{W}_w \boldsymbol{y} + \alpha \boldsymbol{\iota}_n + \boldsymbol{X}_o \boldsymbol{\beta} + \boldsymbol{X}_d \boldsymbol{\gamma} + \theta \boldsymbol{d} + \boldsymbol{\varepsilon} \qquad \text{(C.3.11a)}$$

$$\boldsymbol{\varepsilon} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}_N) \qquad \text{(C.3.11b)}$$

where the spatial weight matrix $\boldsymbol{W}_o = \boldsymbol{W} \otimes \boldsymbol{I}_n$ is used to form a spatial lag vector $\boldsymbol{W}_o \boldsymbol{y}$ that captures *origin-based dependence* arising from flows (observation dyads) that neighbor the origins. The n-by-n spatial weight matrix $\boldsymbol{W}$ is a nonnegative sparse matrix with diagonal elements set to zero to prevent an observation from being defined as a neighbor to itself. Non-zero values for element pairs (i, j) denote that zone i is *a neighbor* to zone j. Neighbors could be defined using contiguity or other measures of spatial proximity such as cardinal distance (for example, kilometers) and ordinal distance (for example, the five closest neighbors). The spatial weight matrix is typically standardized to have row sums of unity, and this is required to produce linear combinations of flows from neighboring regions in the model given by Eq. (C.3.11).

Given an origin-centric organization of the sample data, the spatial weight matrix $\boldsymbol{W}_o = \boldsymbol{W} \otimes \boldsymbol{I}_n$ will form an *N*-by-1 vector containing a linear combination of flows from regions neighboring each observation (dyad) treated as an origin. In the case where neighbors are weighted equally, we would have an average of the neighboring region flows. Similarly, a spatial lag of the dependent variable formed using the weight matrix $\boldsymbol{W}_d = \boldsymbol{I}_n \otimes \boldsymbol{W}$ to produce an *N*-by-1 vector $\boldsymbol{W}_d\,\boldsymbol{y}$ captures *destination-based dependence* using an average (or linear combination) of flows associated with observations (dyads) that neighbor the destination regions. Finally, a spatial weight matrix, $\boldsymbol{W}_w = \boldsymbol{W} \otimes \boldsymbol{W}$ can be used to form a spatial lag vector that captures *origin-to-destination based dependence* using a linear combination of neighbors to both the origin and destination regions.

This model specification can also be written as

$$(\boldsymbol{I}_n - \rho_o \boldsymbol{W}_o)(\boldsymbol{I}_n - \rho_d \boldsymbol{W}_d)\,\boldsymbol{y} = \boldsymbol{Z}\boldsymbol{\delta} + \boldsymbol{\varepsilon} \tag{C.3.12a}$$

$$(\boldsymbol{I}_n - \rho_o \boldsymbol{W}_o - \rho_d \boldsymbol{W}_d + \rho_o\,\rho_d\,\boldsymbol{W}_o \boldsymbol{W}_d)\,\boldsymbol{y} = \boldsymbol{Z}\boldsymbol{\delta} + \boldsymbol{\varepsilon} \tag{C.3.12b}$$

$$\{\boldsymbol{I}_n - \rho_o[\boldsymbol{W} \otimes \boldsymbol{I}_n] - \rho_d[\boldsymbol{I}_n \otimes \boldsymbol{W}] + \rho_o\,\rho_d[\boldsymbol{W} \otimes \boldsymbol{W}]\} = \boldsymbol{Z}\boldsymbol{\delta} + \boldsymbol{\varepsilon} \tag{C.3.12c}$$

where the matrix cross-product term, $\rho_o\,\rho_d\,\boldsymbol{W}_o\boldsymbol{W}_d \equiv \rho_w\,\boldsymbol{W}_w$ motivates the term reflecting *origin-to-destination based dependence*. LeSage and Pace (2008) note that this specification implies that $\rho_w = -\rho_o\,\rho_d$, but these restrictions need to be applied during estimation. There is a need to impose restrictions on the values of the scalar dependence parameters ρ_d, ρ_o, ρ_w to ensure stationarity in the case where ρ_w is free of the restriction. LeSage and Pace (2008) discuss maximum likelihood estimation of this specification, and LeSage and Pace (2009a) set forth a Bayesian heteroscedastic variant of the model along with Markov Chain Monte Carlo (MCMC) estimation methods.

This variant allows for non-constant variance in the disturbances by introducing a set of N scalar variance parameters. Specifically, $\boldsymbol{\varepsilon} \sim \mathcal{N}(\boldsymbol{0}_N, \boldsymbol{\Sigma})$, where the *N*-by-*N* *diagonal matrix* $\boldsymbol{\Sigma}$ contains variance scalar parameters to be estimated on the diagonal and zeros elsewhere.

A virtue of the model in Eq. (C.3.11) is that changes in the value of an explanatory variable associated with a single region will potentially impact flows to all other regions. For example, a *ceteris paribus* change in observation i of the explanatory variables matrix $\boldsymbol{X}$ for variable X_r implies that region i will be viewed differently as both an origin and destination. Given the structure of the matrices $\boldsymbol{X}_o, \boldsymbol{X}_d$ changes in observation i imply changes in $2n$ observations from the explanatory variables matrices. This is true for the *independence model* as well as the spatial model. In the case of the *independence model* such a ceteris paribus change will lead to changes in the flows associated with the same $2n$ observations and no others. Intuitively, if, for example, the labor market opportunities in a single region i decrease, this region will look less attractive as a destination when considered by workers residing in the own and other $n-1$ regions in a migration application context, for example. This should lead to a decrease in migration *pull* from within and outside region i, the impact of changing the n-elements in $\boldsymbol{X}_d$ and associated parameter. Region i will exert more *push* leading to an increase in out-migration to the other $n-1$ regions (as well as a decrease in within-region migration). This impact is reflected by the n-elements in $\boldsymbol{X}_o$ and associated parameter. In the independence model, changes in the explanatory variables associated with the $2n$ observations can only impact changes in flows in the *same* $2n$ observations (by definition).

Turning to the spatial model that includes spatial lags of the dependent variable, these $2n$ changes will lead to changes in flows involving more than the $2n$ observations whose explanatory variables have changed. The additional impacts arising from changes in a single region's characteristics represent *spatial spillover effects*. Intuitively, a decrease in labor market opportunities for region i will indirectly impact the attractiveness of a region that neighbors i, say region j. Region j will become less attractive as a destination for migrants given the decrease in labor market opportunities in neighboring region i. Residents of region j who work in region i and suffer from the labor market downturn in this neighboring region might also find out-migration more attractive. In-migrants to region j may consider labor market opportunities not only in region j but also in neighboring regions such as i. The partial derivative impacts on observations y_i arising from changes in the explanatory variables associated with observations j are zero (by definition) in the independence model, but not in the spatial model containing lags of the dependent variable (see LeSage and Pace 2009a for a discussion of this). Correct calculation and interpretation of the partial derivative impacts associated with the spatial lag model allow one to quantify the spatial spillover impacts.

LeSage and Polasek (2008) provide a minor modification to the model that can be used in the case of commodity flows. In an application involving truck and train commodity flows between 40 Austrian regions, they provide a procedure that adjusts the spatial weight matrix to account for the presence or absence of interregional transport connectivity. Since the mountainous terrain of Austria precludes the presence of major rail and highway infrastructure in all regions, they use this priori non-sample knowledge regarding the transportation network structure connecting regions to produce a modified spatial weight structure. Bayesian model comparison methods indicate that these adjustments to the spatial weight matrix result in an improved model.

Another approach to dealing with spatial dependence in origin-destination flows is to specify a spatial process for the disturbance terms, structured to follow a (first-order) spatial autoregressive process (see Fischer and Griffith 2008). This specification could be estimated using maximum likelihood methods. In this framework, the spatial dependence resides in the disturbance process $\boldsymbol{\varepsilon}$, as in the case of serial correlation in time series regression models. Griffith (2007) also takes this specification approach that focuses on dependence in the disturbances but relies on a *spatial filtering* estimation methodology.

Specifically, the most general variant of this type of model specification takes the form

$$\boldsymbol{y} = \alpha\,\boldsymbol{\iota}_n + \boldsymbol{X}_o\,\boldsymbol{\beta} + \boldsymbol{X}_d\,\boldsymbol{\gamma} + \theta\,\boldsymbol{d} + \boldsymbol{u} \tag{C.3.13a}$$

$$\boldsymbol{u} = \rho_o\,\boldsymbol{W}_o\,\boldsymbol{u} + \rho_d\,\boldsymbol{W}_d\,\boldsymbol{u} + \rho_w\,\boldsymbol{W}_w\,\boldsymbol{u} + \boldsymbol{\varepsilon} \tag{C.3.13b}$$

$$\boldsymbol{\varepsilon} \sim \mathcal{N}(0,\, \sigma^2 \boldsymbol{I}_N) \tag{C.3.13c}$$

where the definitions for the spatial lags involving the disturbance terms in Eq. (C.3.13), $\boldsymbol{W}_0\,\boldsymbol{u}$, $\boldsymbol{W}_d\,\boldsymbol{u}$ and $\boldsymbol{W}_w\,\boldsymbol{u}$, are analogous to those for the spatial lags of the dependent variable in Eq. (C.3.12).

Simpler models can be constructed by imposing restrictions on the general specification in Eq. (C.3.13). For example, we could specify the disturbances using

$$\boldsymbol{u} = \rho\,\widetilde{\boldsymbol{W}}\,\boldsymbol{u} + \varepsilon \tag{C.3.14a}$$

$$\boldsymbol{\varepsilon} \sim \mathcal{N}(0,\, \sigma^2 \boldsymbol{I}_N) \tag{C.3.14b}$$

which merges origin- and destination-based dependence to produce a single (row-normalized) spatial weight matrix $\widetilde{\boldsymbol{W}}$ consisting of the sum of $\boldsymbol{W}_o$ and $\boldsymbol{W}_d$ which is row-normalized to produce a single vector $\widetilde{\boldsymbol{W}}\boldsymbol{u}$ reflecting a spatial lag of the disturbances. This specification also restricts the origin-to-destination based dependence in the disturbances to be zero, since ρ_w is implicitly set to zero.

The virtue of a simpler model such as this is that conventional software for estimating spatial error models could be used to produce an estimate for the parameter ρ along with the remaining model parameters α, $\boldsymbol{\beta}$, γ and θ. It may or may not be apparent that estimating the more general models that involve more than a single spatial dependence parameter requires customized algorithms of the type set forth in LeSage and Pace (2008). These are needed to maximize a log-likelihood that is concentrated with respect to the parameters α, $\boldsymbol{\beta}$, γ, θ and σ^2 resulting in an optimization problem involving the three dependence parameters ρ_d, ρ_o, ρ_w. Of note is the fact that an extended version of the moment-based expressions involving the matrix $\boldsymbol{Z}$ from Eq. (C.3.9) and Eq. (C.3.10) can be used for both maximum likelihood and Bayesian MCMC estimation (see LeSage and Pace 2009a for details).

One point to note regarding modeling spatial dependence in the model disturbances is that the coefficient estimates α, $\boldsymbol{\beta}$, γ, θ will be asymptotically equal to those from least-squares estimation. However, there may be an efficiency gain that arises from modeling dependence in the disturbances. Another point is that the partial derivative impacts associated with this model are the same as those from the *independence model*. That is, no spatial spillover impacts arise in this type of model so that ceteris paribus changes in region i 's explanatory variable only result in changes in the $2n$ regions associated with the n^2 dyad relationships involving region i.

A third approach to modeling spatial dependence is motivated by the use of *fixed effects* parameters for origin and destination regions in non-spatial versions of the gravity model in the empirical trade literature (Feenstra 2002). Assuming the origin-centric data organization set forth in Table C.3.1, a fixed effects model would take the form in Eq. (C.3.15). The N-by-n matrix $\boldsymbol{\Delta}_o$ contains elements that equal one if region I is the origin region and zero otherwise, and $\boldsymbol{\theta}_o$ is an n-by-1 vector of associated fixed effects estimates for regions treated as origins. Similarly, the N-by-n matrix $\boldsymbol{\Delta}_d$ contains elements that equal one if region j is the destination region and zero otherwise leading to an n-by-1 vector $\boldsymbol{\theta}_d$ of fixed effects estimates for regions treated as destinations

$$\boldsymbol{y} = \alpha + \boldsymbol{\beta}_o\boldsymbol{X}_o + \boldsymbol{\beta}_d\boldsymbol{X}_d + \gamma\boldsymbol{d} + \boldsymbol{\Delta}_o\boldsymbol{\theta}_o + \boldsymbol{\Delta}_d\boldsymbol{\theta}_d + \boldsymbol{\varepsilon}. \tag{C.3.15}$$

LeSage and Llano (2007) extend this model to the case of spatially structured random effects. This involves introduction of latent effects parameters that are structured to follow a spatial autoregressive process. This is accomplished using a Bayesian prior that the origin and destination effects parameters are similar for neighboring regions.

In the context of commodity flows between Spanish regions, the model takes the form

$$\boldsymbol{y} = \boldsymbol{Z}\delta + \boldsymbol{\Delta}_d\boldsymbol{\theta}_d + \boldsymbol{\Delta}_o\boldsymbol{\theta}_o + \boldsymbol{\varepsilon} \tag{C.3.16a}$$

$$\boldsymbol{\theta}_d = \rho_d\boldsymbol{W}\boldsymbol{\theta}_d + \boldsymbol{u}_d \tag{C.3.16b}$$

$$\boldsymbol{\theta}_o = \rho_o\boldsymbol{W}\boldsymbol{\theta}_o + \boldsymbol{u}_0 \tag{C.3.16c}$$

$$\boldsymbol{u}_d \sim \mathcal{N}(0,\, \sigma_d^2\boldsymbol{I}_n) \tag{C.3.16d}$$

$$\boldsymbol{u}_o \sim \mathcal{N}(0,\, \sigma_o^2\boldsymbol{I}_n). \tag{C.3.16e}$$

Given our origin-centric orientation of the flow matrix (columns as origins and rows as destinations), the matrices $\boldsymbol{\Delta}_d = \boldsymbol{I}_n \otimes \boldsymbol{\iota}_n$ and $\boldsymbol{\Delta}_o = \boldsymbol{\iota}_n \otimes \boldsymbol{I}_n$ produce N-by-n matrices. It should be noted that estimates for these two sets of random effects parameters are identified, since a set of n sample data observations are aggregated through the matrices $\boldsymbol{\Delta}_d$ and $\boldsymbol{\Delta}_o$ to produce each estimate in $\boldsymbol{\theta}_d$ and $\boldsymbol{\theta}_o$.

The spatial autoregressive prior structure placed on the destination effects parameters $\boldsymbol{\theta}_d$ (conditional on the parameters ρ_d and σ_d^2) is shown in Eq. (C.3.17) and that for the spatially structured origin effects parameters $\boldsymbol{\theta}_o$ in Eq. (C.3.18), where we use the symbol $\pi(.)$ to denote a prior distribution:

$$\pi(\theta_d \mid \rho_d, \sigma_d^2) \sim (\sigma_d^2)^{n/2} \mid \boldsymbol{B}_d \mid \exp\left(-\frac{1}{2\sigma_d^2}\boldsymbol{\theta}_d^{\mathrm{T}}\boldsymbol{B}_d^{\mathrm{T}}\boldsymbol{B}_d\boldsymbol{\theta}_d\right) \tag{C.3.17}$$

$$\pi(\theta_o \mid \rho_o, \sigma_o^2) \sim (\sigma_o^2)^{n/2} \mid \boldsymbol{B}_o \mid \exp\left(-\frac{1}{2\sigma_0^2}\boldsymbol{\theta}_o^{\mathrm{T}}\boldsymbol{B}_o^{\mathrm{T}}\boldsymbol{B}_o\boldsymbol{\theta}_o\right) \tag{C.3.18}$$

$$\boldsymbol{B}_d = (\boldsymbol{I}_n - \rho_d\boldsymbol{W}) \tag{C.3.19}$$

$$\boldsymbol{B}_o = (\boldsymbol{I}_n - \rho_o\boldsymbol{W}). \tag{C.3.20}$$

Estimation of the spatially structured effects parameters requires that we estimate the dependence parameters ρ_d, ρ_o and associated variances σ_d^2, σ_o^2. LeSage and Llano (2007) provide details regarding using of Markov Chain Monte Carlo methods for estimation of this model.

This model does not allow directly for spatial spillover effects. It does, however, provide a *spatially structured effect* adjustment for each origin and destination region. These act in the same fashion as non-spatial effects parameters producing an intercept shift adjustment that would be added to the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ when considering the partial derivative impacts arising from ceteris paribus changes in region i's explanatory variable. Another point about the spatially structured prior is that if the scalar spatial dependence parameters (ρ_o, ρ_d) are not significantly different from zero, the spatial structure of the effects vectors disappears, leaving us with normally distributed random effects parameters for the origins and destinations similar to the conventional effects models described in Feenstra (2002).

Large diagonal flow matrix elements

Another problem that arises in empirical work is the fact that the diagonal elements of the flow matrix $\boldsymbol{Y}$ representing intraregional flows are often quite large relative to the off-diagonal elements reflecting interregional flows. Since the objective of spatial interaction modeling is typically a model that attempts to explain variation in interregional rather than intraregional flows, practitioners often view intraregional flows as a nuisance, and introduce dummy variables for these observations (see, for example, Koch et al. 2007). For the case of the independence model this approach is fine, but it can have deleterious impacts on models involving spatial lags of the dependent variable. To see this, consider the case of a simple model involving

$$\boldsymbol{y} = \rho \, \widetilde{\boldsymbol{W}} \; \boldsymbol{y} + \boldsymbol{Z} \, \delta + \boldsymbol{\varepsilon} \tag{C.3.21a}$$

$$\boldsymbol{\varepsilon} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}_N) \tag{C.3.21b}$$

where $\widetilde{\boldsymbol{W}}$ is a row-normalized version of the sum of the spatial weight matrices $\boldsymbol{W}_o$, $\boldsymbol{W}_d$, $\boldsymbol{W}_w$. The n zero elements associated with the diagonal of the vectorized flow matrix $\boldsymbol{y} =$ vec($\boldsymbol{Y}$) in the N-by-1 vector of flows will have the impact of producing outliers in the spatial lags when these observations are involved in the linear combination used to form $\widetilde{\boldsymbol{W}} \, \boldsymbol{y}$.

To avoid this problem, LeSage and Pace (2008) suggest a procedure that embeds a separate model for the intraregional flows into the spatial interaction model. This is accomplished by adjusting the explanatory variables matrices $\boldsymbol{X}_o, \boldsymbol{X}_d$ and the intercept vector ι_n to have zero values for the n observations associated with the main diagonal elements (intraregional flows) of the flow matrix $\boldsymbol{Y}$. We use $\tilde{\boldsymbol{X}}_o, \tilde{\boldsymbol{X}}_d$ to denote these adjusted matrices. A new matrix that we label $\boldsymbol{X}_i$ is introduced containing the n observations associated with intraregional flows set to zero in the matrices $\boldsymbol{X}_o, \boldsymbol{X}_d$, and zeros in the other $N-n$ observations. That is,

$\tilde{\boldsymbol{X}}_o = \boldsymbol{X}_o - \boldsymbol{X}_i$, and $\tilde{\boldsymbol{X}}_d = \boldsymbol{X}_d - \boldsymbol{X}_i$. In addition, a new intercept vector $\boldsymbol{\iota}_i$ is introduced that contains ones in the n positions so that $\tilde{\boldsymbol{\iota}}_N = \boldsymbol{\iota}_N - \boldsymbol{\iota}_i$. The adjusted *independence model* now takes the form

$$\boldsymbol{y} = \alpha \tilde{\boldsymbol{\iota}}_N + \alpha_i \boldsymbol{\iota}_i + (\boldsymbol{X}_o - \boldsymbol{X}_i)\boldsymbol{\beta} + (\boldsymbol{X}_d - \boldsymbol{X}_i)\boldsymbol{\gamma} + \boldsymbol{X}_i \psi + \theta \boldsymbol{d} + \boldsymbol{\varepsilon} \tag{C.3.22a}$$

$$\boldsymbol{y} = \alpha \tilde{\boldsymbol{\iota}}_N + \alpha_i \boldsymbol{\iota}_i + \tilde{\boldsymbol{X}}_o \boldsymbol{\beta} + \tilde{\boldsymbol{X}}_d \boldsymbol{\gamma} + \boldsymbol{X}_i \psi + \theta \boldsymbol{d} + \boldsymbol{\varepsilon} \tag{C.3.22b}$$

where a corresponding adjustment can be used for the case of the spatial lag model in Eq. (C.3.11) or the spatial error model in Eq. (C.3.13). This model uses the (orthogonal) intercept term $\boldsymbol{\iota}_i$ and explanatory variables $\boldsymbol{X}_i$ (and associated Ψ) to capture variation in the vector of flows $\boldsymbol{y}$ across dyads representing intraregional flows and the adjusted variables: $\tilde{\boldsymbol{\iota}}_N, \tilde{\boldsymbol{X}}_d, \tilde{\boldsymbol{X}}_o$ to model variation in interregional flows.

Of course, it is not necessary to rely on the same set of explanatory variables for $\boldsymbol{X}_o, \boldsymbol{X}_d, \boldsymbol{X}_i$, but this will simplify computation via the moment matrices for models involving large samples n as discussed earlier. LeSage and Pace (2009a) provide expressions for the moment matrices that arise for these adjustments to the model.

As an example, consider that variation in intraregional flows might be explained by variables such as the *area of the regions* or in the case of a migration flow model the *population of the regions*. We would expect that regions having larger population and area should exhibit more intraregional migration. This subset of two explanatory variables could then be used to form the matrix $\boldsymbol{X}_i$, with corresponding adjustments to these two variables undertaken for the matrices $\boldsymbol{X}_o, \boldsymbol{X}_d$ to produce $\tilde{\boldsymbol{X}}_o$, $\tilde{\boldsymbol{X}}_d$. Inference regarding the parameter ψ for these two variables would not be of primary interest (since associated with the intraregional control variables) whereas the focus of the model is on the parameters $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and θ.

The advantage of this approach is that non-zero intraregional flows can be included in the matrix $\boldsymbol{Y}$ used to form the dependent variable vector $\boldsymbol{y}$ and the spatial lags $\boldsymbol{W}_o\, \boldsymbol{y}$, $\boldsymbol{W}_d\, \boldsymbol{y}$, $\boldsymbol{W}_w\, \boldsymbol{y}$. Variation in the flows associated with the large diagonal elements is captured by the embedded model variables $\boldsymbol{\iota}_i$ and $\boldsymbol{X}_i$ allowing the coefficient estimates associated with the adjusted explanatory variables $\tilde{\boldsymbol{X}}_o$, $\tilde{\boldsymbol{X}}_d$ to more accurately characterize variation in interregional flows.

As an illustration of the differences that arise from these adjustments to the model, we use a sample of 1998 commodity flows between the 48 lower U.S. states plus the District of Columbia leading to a sample size of $n = 49$ and $N =$ 2,401. The commodity flows were taken from the Federal Highway Administration Freight Analysis Framework State to State Commodity flow Database. As explanatory variables we use the (logged) area of each state and the 1998 Gross State Product (*gsp*). The model was based on a single spatial weight matrix constructed using a row-normalized matrix consisting of $\boldsymbol{W}_d + \boldsymbol{W}_o + \boldsymbol{W}_w$, where the n-by-n matrix $\boldsymbol{W}$ was based on six nearest neighbors. Following convention, the commodity flows were transformed using logs as were the explanatory variables representing area and *gsp*.

Table C.3.2 shows the coefficient estimates labelled $\hat{\boldsymbol{\beta}}_1$ for the adjusted model along with those from the unadjusted model labeled $\hat{\boldsymbol{\beta}}_0$. In the table, we use the symbol *I_gsp* and *I_area* to denote the variables contained in the matrix $\boldsymbol{X}_i$ in the adjusted model expression given by Eq. (C.3.22). A *t*-test for significant differences between the coefficients $(\hat{\boldsymbol{\beta}}_0 - \hat{\boldsymbol{\beta}}_1)$ common to the two models is presented in Table C.3.3. From the table reporting test results for differences in the two sets of estimates we see evidence of differences that are significant at the 99 percent level in the coefficients on distance and the spatial lag of the dependent variable. There is also a difference between the *origin area* explanatory variable that is significant at the 90 percent level. It is also worth noting that twice the difference in the log-likelihood function values from the two models is 249, which suggests a significant difference between the models. This would be an informal indication since the two models cannot be viewed as formally nested.

Table C.3.2. Unadjusted and adjusted model estimates

Variables	Unadjusted model		Adjusted model	
	Coefficient $(\hat{\boldsymbol{\beta}}_0)$	*t*-statistic	Coefficient $(\hat{\boldsymbol{\beta}}_1)$	*t*-statistic
Constants				
$\boldsymbol{\iota}_N$ / $\tilde{\boldsymbol{\iota}}_N$	−19.2770	−38.9	−19.9888	−41.1
$\boldsymbol{\iota}_i$	–	–	−5.2012	−2.2
Origin variables				
O_gsp / $\tilde{O}$ *_gsp*	0.3397	15.7	0.3520	17.0
O_area / $\tilde{O}$ *_area*	0.5679	27.1	0.4961	23.6
Destination variables				
D_gsp / $\tilde{D}$ *_gsp*	0.7374	30.7	0.7021	30.8
D_area / $\tilde{D}$ *_area*	0.2806	17.2	0.2608	16.5
I_gsp	–	–	0.6169	4.3
I_area	–	–	0.3738	3.5
Distance	−0.5123	−22.2	−0.3101	−13.1
ρ	0.5219	23.5	0.6429	31.6
σ^2	1.1549		1.0337	
Log-likelihood	−2762.7		−2638.2	

We can also use this model and sample data to illustrate how problems arise when setting the intraregional flows to zero values. For this illustration a spatial weight matrix based on row-normalized $\boldsymbol{W}_d + \boldsymbol{W}_o$ was used, and the unadjusted model was estimated for values of the dependent variable representing intraregional flows flows set to zero as well as the full set of non-zero flows.

Table C.3.3. Test for significant differences between the unadjusted and adjusted model estimates

Variables	$(\hat{\beta}_0 - \hat{\beta}_1)$	*t*-statistic	*t*-probability
Constant	0.7118	0.7264	0.4677
Origin variables			
O_gsp	–0.0123	–0.2905	0.7715
O_area	0.0718	1.7139	0.0867
Destination variables			
D_gsp	0.0352	0.7529	0.4516
D_area	0.0198	0.6195	0.5357
Distance	–0.2022	–4.3319	0.0000
ρ	–0.1210	–2.8511	0.0044

The results from this illustration are presented in Table C.3.4 where we see a serious degradation in the log-likelihood function value for the zero-flows model and a dramatic six-fold rise in the noise variance estimate σ^2. A number of problematical coefficient estimates arise, for example the coefficient on distance is negative but not significantly different from zero, contrary to the conventional result. The magnitude of the spatial dependence parameter ρ decreased dramatically, consistent with our admonition that setting the main diagonal elements of the flow matrix to zero will have an adverse impact on the spatial nature of the sample flow data. Finally, given the reported *t*-statistics, we can infer that the coefficient estimates on the origin and destination *gsp* variables are significantly different in the two regressions.

Table C.3.4. Zero intraregional flows versus non-zero intraregional flows

Variables	Zero diagonal flows		Non-zero diagonal flows	
	Coefficient $(\hat{\beta}_0)$	*t*-statistic	Coefficient $(\hat{\beta}_1)$	*t*-statistic
Constant	2.1675	2.30	–16.1351	–33.55
Origin variables				
O_gsp	0.3801	7.73	0.2805	13.92
O_area	0.5573	13.42	0.4552	22.72
Destination variables				
D_gsp / $\tilde{D}$ *_gsp*	0.8504	15.35	0.5969	25.77
D_area / $\tilde{D}$ *_area*	0.1801	5.01	0.2341	15.31
Distance	–0.0230	–0.75	–0.4113	–19.15
ρ	0.2979	6.80	0.6449	33.71
σ^2	5.8627		0.9911	
Log-likelihood	–4,707.2		–2,612.1	

The zero flows problem

Another problem that arises involves the presence of a large number of zero flows[1]. This problem arises when analyzing sample data collected using a fine spatial scale. As an example, population migration flows between the largest 50 U.S. metropolitan areas over the period 1995-2000 resulted in only 3.76 percent of the OD-pairs contained zero flows, whereas 9.38 percent of the OD-pairs were zero for the largest 100 metropolitan areas and for the largest 300 metropolitan areas, 32.89 percent of the OD pairs exhibited zero flows.

The presence of a large number of zero flows invalidates use of least-squares regression as a method for estimating the *independence model* and maximum likelihood methods for spatial variants of the interaction model. This is because zero values for a large proportion of the dependent variable invalidate the normality assumption required for inference in the regression model and validity of the maximum likelihood method. Despite this, a number of applications can be found where the dependent variable is modified using log $(1 + \boldsymbol{y})$ to accommodate the log transformation. This, however, ignores the mixed discrete/continuous nature of the flow distribution. Intuitively, this type of practice should lead to downward bias in the coefficient estimates for the model.

If we can view flows as arising from say positive utility in the case of migration flows or positive profits when considering commodity flows, then the presence of zero flows might be indicative of negative utility or profits. This type of argument is often used to motivate sample censoring models such as in the Tobit regression model. In a non-spatial application to international trade flows, Ranjan and Tobias (2007) treat zero flows using a threshold Tobit model. Their argument is that zero trade flows are indicative of situations where the transportation and other costs associated with trade exceed a threshold making trade unprofitable. A similar argument could be applied to migration flows. Non-zero flows could be viewed as an indication that the origin versus destination characteristics are such that at least one migrant perceives positive utility arising from movement between the origin-destination dyad. In contrast, zero observed migration flows could be interpreted to mean that no individual views destination utility to be greater than utility at the origin for these OD dyads, leading to net negative utility from migration. We note that similar arguments regarding utility from program participation have been used to motivate sample truncation leading to the use of Tobit regression models when evaluating the level of program participation by individuals.

LeSage and Pace (2009a) set forth estimation methods for Tobit models where a spatial lag of the dependent variable is involved. This requires Bayesian MCMC estimation where a set of parameters representing negative utility are introduced for the zero-valued dependent variable observations. Some important caveats are associated with this approach to dealing with zero-valued flows. One is that Tobit

[1] Note that zero counts present no serious problem in Poisson regression, but must be handled in the log-normal spatial interaction model case.

models assume the dependent variable follows a truncated normal distribution. This assumption seems reasonable when we are faced with a sample of flows containing less than 50 to 70 percent zero or censored values. However, in situations where we are faced with a very large proportion of zero values, the assumption of a truncated normal distribution seems less plausible.

In the context of modeling knowledge flows between European Union regions, LeSage et al. (2007) note that a large proportion of zero knowledge flows between the sample of European regions should be viewed as indicative that knowledge flows are perhaps a rare event. This view is more consistent with a Poisson distribution for the dependent variable. We will have more to say about this later.

To demonstrate how spatial autoregressive Tobit models can be used to address the issue of zero observations we generated a sample of 2,401 OD flow observations using the explanatory variables *area* and *gsp* from our previous example involving state level commodity flows involving the 48 lower U.S. states and the District of Columbia. A Queen-based spatial contiguity weight matrix was used for $\boldsymbol{W}$ and a single matrix $\tilde{\boldsymbol{W}}$ was generated using a row-normalized version of $\boldsymbol{W}_d + \boldsymbol{W}_o$. The true parameter values for $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ were set to one and minus one for the *gsp* and *area* variables respectively. Use of both positive and negative coefficient values ensures that the generated flows will include negative values. The parameter θ for distance was set to minus one and that for the intercept to 20. A value of $\rho = 0.65$ was used. This procedure for producing data-generated flows resulted in 1,020 negative flows out of 2,401 observations, or slightly more than 42 percent sample censoring. We should view the dependent variable generated in this fashion as profitability associated with interregional commodity flows, so the magnitude of commodity flows is proportional to profitability. Consistent with this view, we set negative values of the dependent variable to zero, reflecting the absence of commodity flows between dyads where negative profits existed.

Estimates from the set of continuous values for the flows/profitability were constructed using maximum likelihood estimation of the spatial autoregressive model in Eq. (C.3.11). These estimates should of course be close to the true values used to generate the sample data. *A second set of estimates* were based on the sample with zero values assigned for negative values of the generated dependent variable, to explore the impact of ignoring zero flow values and proceeding with conventional maximum likelihood estimation of the spatial autoregressive model. Here we would expect to see downward bias in the coefficient estimates due to the sample truncation.

A third set of spatial autoregressive Tobit model estimates were based on the sample with zero values assigned for negative values of the dependent variable. Ideally, the spatial Tobit model parameters should be close to the true parameter values used to generate the sample of flows, if we have been successful in our spatial econometric treatment of zero valued flows as representing sample truncation. MCMC estimation methods described in LeSage and Pace (2009a) were used to produce estimates for the spatial autoregressive Tobit model.

Results from this illustration are reported in Table C.3.5, where we see coefficient estimates labeled *Uncensored sample* close to the true values used to generate the flow vector $\boldsymbol{y}$. These were based on the sample flow vector that did not impose sample truncation on the negative values of the dependent variable. The estimates labeled *Non-Tobit censored* are those based on ignoring the existence of zero valued flows. The Bayesian spatial autoregressive Tobit model estimates are reported in the columns labeled *Tobit censored*, where the posterior mean reported in the table is based on a sample of 1,000 MCMC draws. The posterior mean was divided by the posterior standard deviation to produce a pseudo t-statistic for comparability with these measures of dispersion for the maximum likelihood estimates.

From the table we see that ignoring zero valued flows produces a dramatic downward bias in the coefficient estimates. Most of the estimates are around 50 to 60 percent lower than the true parameters used to generate the sample $\boldsymbol{y}$-vector. In contrast, the spatial autoregressive Tobit estimates produced coefficients very close to the true parameters as well as the benchmark estimates based on the *uncensored sample*. A point worth noting is that use of the spatial autoregressive Tobit model will lead to larger dispersion in the estimates, which from a Bayesian viewpoint reflects greater uncertainty in the posterior means.

Table C.3.5. Spatial Tobit experimental results

Variables	True	Uncensored sample		Non-Tobit censored		Tobit censored	
		Coefficient	t-statistic [a]	Coefficient	t-statistic [a]	Coefficient	t-statistic [a]
Constant	20	19.2933	31.4	15.7547	24.9	19.5794	29.9
Origin variables							
O_gsp	1	1.0309	42.5	0.4746	21.2	1.0519	32.5
O_area	–1	–1.0055	–45.0	–0.6128	–29.3	–1.0169	–45.4
Destination variables							
D_gsp	1	0.9833	41.1	0.4564	20.5	0.9940	31.8
D_area	–1	–0.9691	–44.3	–0.5985	–29.0	–0.9849	–43.2
Distance	–1	–0.9861	–42.8	–0.6016	–27.9	–1.0075	–41.7
ρ	0.65	0.6569	81.9	0.7719	90.8	0.6475	75.1
σ^2	1	0.9654		0.9853		0.9786	

Notes: [a] Pseudo t-statistic, posterior mean divided by posterior standard deviation

Some caveats regarding this approach to dealing with zero-valued flows are in order. As already mentioned, this approach is most likely applicable for situations where there is not an excessive amount of zero values. The ability of this approach to produce quality estimates depends on the ability of the spatial Tobit procedure to produce good estimates for the latent parameters introduced in the model (see LeSage and Pace 2009a for a detailed discussion of this). As economists are fond of saying, there is no such thing as a free lunch. This applies to the spatial Tobit model where the cost of censoring is increased uncertainty regarding the posterior estimates. Intuitively, as the proportion of the sample that is censored increases, so does our uncertainty in the estimation outcomes. A final point is that this same approach can be used to deal with zero flow values for the spatially structured effects model set forth in Eq. (C.3.16). LeSage and Pace (2009a) discuss this and LeSage et al. (2008) provide details including an applied example using commuting flows in Toulouse. This involves introducing latent parameters for the zero-valued flows and estimating these using Bayesian MCMC procedures.

As already mentioned, cases where the proportion of zero-valued flows is very large are not amenable to the Tobit model approach. LeSage et al. (2007) provide an extension of the model given by Eq. (C.3.16) that can be used to accommodate this situation. They rely on a variant of the model in Eq. (C.3.16) where the flows are assumed to follow a Poisson distribution, and treat interregional patent citations from a sample of European Union regions as representing knowledge flows. The counts of patents originating in region i that were cited by regions $j = 1, \ldots, n$ are used to form a *knowledge flows* matrix. Since cross-region patent citations are both counts and rare events, a Poisson distribution seems much more plausible than the normal distribution assumption made for the Tobit model.

The extension of the spatially structured effects model relies on work by Frühwirth-Schnatter and Wagner (2008) who argue that (non-spatial) Poisson regression models (including those with random-effects) can be treated as a partially Gaussian regression model by conditioning on two strategically chosen sequences of artificially missing data. These sequences are similar in spirit to the latent parameters approach described above for estimating the spatial autoregressive Tobit model (LeSage and Pace 2009a). After conditioning on both of these latent sequences, Frühwirth-Schnatter and Wagner (2008) show that the resulting model can be estimated using an MCMC procedure.

The one drawback to the approach pointed out by LeSage et al. (2007) is that one must sample two sets of latent parameters equal to $y_{ij} + 1$, where y_{ij} denotes the count for observation i. This can lead to very long sequences of artifically missing data that need to be manipulated during MCMC estimation thousands of times. The authors report that for a sample of $n = 188$ regions 23,718 zero values and 199,817 non-zero values, a total of 133,535 latent observations were needed to sample each of the two latent variable vectors. The estimation procedure took over two days to produce estimates for the moderately sized sample based on n = 188.

For the spatially structured random effects model from Eqs. (C.3.17) to (C.3.20), let $\boldsymbol{y} = (y_1, \ldots, y_N)$ denote our sample of $N = n^2$ counts for dyads of flows between regions. The assumption regarding y_i is that $y_i \mid \lambda_i$ follows a Poisson, $P(\lambda_i)$ distribution, where λ_i depends on (standardized) covariates Z_i reflecting the ith row of the explanatory variables matrix $\mathbf{Z}$, with $i = 1, \ldots, N$. The Poisson variant of this model can be expressed as

$$y_i \mid \lambda_i \sim P(\lambda_i) \qquad \text{(C.3.23a)}$$

$$\lambda_i = \exp(z_i\,\delta + \delta_{di}\,\theta_d + \delta_{oi}\,\theta_o) \qquad \text{(C.3.23b)}$$

where δ_{di} represents the ith row from the matrix Δ_d in Eq. (C.3.16) that identifies region i as a destination region and δ_{oi} identifies origin regions using rows from the matrix Δ_o of Eq. (C.3.16). The insight of Frühwirth-Schnatter and Wagner (2008) was that conditional on the sequences of artifically missing data MCMC samples can be constructed from the posterior distribution of the parameters using draws from a series of distributions that take known forms.

C.3.4 Concluding remarks

In addition to the challenges discussed above that face practitioners interested in empirical implementation of spatial interaction models, there is a need to provide a theoretical justification for the use of spatial lags of the dependent variable (or disturbances) in spatial interaction models. The description provided here motivates the need for these models based on empirically observed spatial dependence in flows.

LeSage and Pace (2008) provide a purely econometric motivation for inclusion of spatial lags of the dependent variable based on missing variables, and LeSage and Pace (2009a) provide a number of additional econometric motivations for use of spatial autoregressive regions models in applied settings not specific to modeling origin-destination flows. Many of these empirical motivations could be extended to the case of flow modeling.

However, a theoretical basis would give the strongest justification for use of these models. Koch et al. (2007) provide a starting point for the special case of international trade flows by extending the work of Anderson and van Wincoop (2004). They rely on a monopolistic competition model in conjunction with a CES (constant elasticity of substitution) utility function to derive a gravity equation for trade flows that contains spatial lags of the dependent variable. A study of theoretical work in the trade literature (Anderson and van Wincoop 2004; Koch et al. 2007) suggests that spatial interaction models may suffer from their focus on bilateral flows between origin-destination dyads. The conclusion drawn from recent theoretical developments in the trade literature is that *bilateral relationships* may not readily extend to a *multilateral world*. Simple relationships based on dyads ignore indirect interactions that link all trading partners. The theoretical work of Koch et al. (2007) leading to a spatial interaction model for trade flows that includes spatial lags of the dependent variable has some important implications for spatial interaction modeling in more general circumstances. One implication is that introducing spatial dependence leads to a situation where dyad relationships are no longer of central importance. In the context of trade flows *and* spatial dependence, price differences between bilateral partners spillover to produce an implicit dependence that quickly encompasses *all* other trading partners. Specifically, the authors argue that when goods are gross substitutes, trade flows from any origin to any destination may depend on the entire distribution of bilateral trade barriers, which reflect prices of substitute goods.

As already motivated, use of spatial regression models that include spatial lags of the dependent variable leads to an implication consistent with the work of Koch et al. (2007). Returning to our example of a ceteris paribus change in labor market opportunities for a single region i, the spatial spillover impacts that arise for these models have the potential to reflect dependence on the entire distribution of regional labor market opportunities available in all regions.

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