An Axiomatic Approach to Hyperconnectivity

Michael H.F. Wilkinson

Institute for Mathematics and Computing Science, University of Groningen
P.O. Box 407, 9700 AK Groningen, The Netherlands

Abstract. In this paper the notion of hyperconnectivity, first put forward by Serra as an extension of the notion of connectivity is explored theoretically. Hyperconnectivity operators, which are the hyperconnected equivalents of connectivity openings are defined, which supports both hyperconnected reconstruction and attribute filters. The new axiomatics yield insight into the relationship between hyperconnectivity and structural morphology. The latter turns out to be a special case of the former, which means a continuum of filters between connected and structural exists, all of which falls into the category of hyperconnected filters.

1 Introduction

Connected filters are object-based morphological filters which allow edge preserving filtering based on a range of criteria [1, 2, 3]. Fig. 1 shows the difference between the structural opening and the opening by reconstruction [4]. In some cases, however, such strict edge preservation is not desirable, because thin structures can link up different entities in an image. For example, the thin stripes on the clothes in Fig. 1 link up the face area to other structures nearby. To circumvent some problems with the strictness of the edge preserving nature of these filters, and their inability to handle overlapping objects as separate entities, several solutions have been put forward [5, 6, 7]. One of these is hyperconnectivity, first proposed by Serra [6] and extended in [8]. Recently, hyperconnectivity has moved from a theoretical concept to a practical one, in particular in fuzzy connectivity [9], in fast reconstruction using reconstruction criteria [10], and in hyperconnected attribute filtering using k-flat zones (overlapping connected regions of with grey level total variations no more than k grey levels) [11]. The latter are useful for separation of galaxies from stars in astronomical imaging (see Fig. 2).

In this paper, a new axiomatics for hyperconnectivity is derived. We will first deal with some theoretical preliminaries. After this, connectivity and connectivity openings are treated. Then we replace the hyperconnectivity openings proposed in [8] by operators which return sets of hyperconnected components. It is then shown that hyperconnected counterparts of the connected attribute filters introduced by Breen and Jones [1] can only be constructed using the new framework. Finally, it is shown that any structural morphology can be seen as a special case of hyperconnected filters. This means that a large family of filters exist between the extremes of edge preserving connected filters, and structural filters. A variant of the work in [5] is shown to be part of that family.

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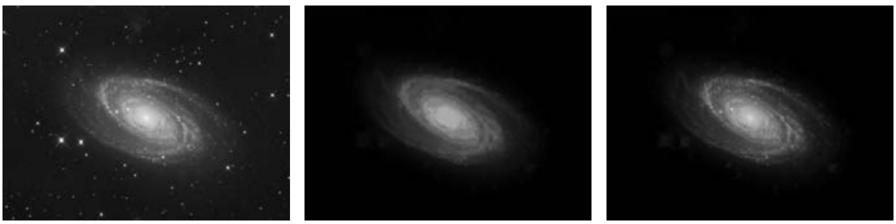
(a) (b) (c) (d)

Fig. 1. Structural, connected, and hyperconnected filters:(a) original image f (b) opening with Euclidean disc of diameter 21 $g = \gamma_{21} f$; (c) connected reconstruction of f by g (d) hyperconnected reconstruction of f by g according to (32)



(a) (b) (c)

Fig. 2. Separating galaxies from stars: (a) spiral galaxy M81, original image, courtesy Giovanni Benintende; (b) stars suppressed by an area attribute filter with 2000 $\leq$ area $\leq$ 240000; (c) k-flat hyperconnected variant of (b), showing improved suppression of stellar, and better retention of galactic detail

2 Theory

Let E denote some *finite*, universal, non-empty set, and $\mathcal{P}(E)$ the set of all subsets of E. $\mathcal{P}(E)$is also finite. A cover $\mathcal{A} = \{A_i\}$ of E is a subset of $\mathcal{P}(E)$ such that $\cup_i A_i = E$. A partition $\mathcal{A} = \{A_i\}$ of E is a cover such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, and all A_i are non-empty. Covers of any $X \subseteq E$ can be defined likewise. Because covers and partitions are sets of subsets of E they are elements of $\mathcal{P}(\mathcal{P}(E))$. To avoid confusion, $\emptyset$ denotes the least element of $\mathcal{P}(E)$, and $\emptyset_{\mathcal{P}(E)}$ denotes least element of $\mathcal{P}(\mathcal{P}(E))$.

A cover $\mathcal{A}$, or indeed any element of $\mathcal{P}(\mathcal{P}(E))$ will be called *redundant* if there exists at least one pair of elements $A_i, A_j \in \mathcal{A}$ such that $A_i \subset A_j$. Obviously, partitions are non-redundant covers. We denote the set of all non-redundant subsets of $\mathcal{P}(E)$ as $\mathcal{N}(\mathcal{P}(E))$.

Any redundant cover can be reduced to a non-redundant cover by means of a *binary reduction operator* $\Phi_{\subset}$. This reduces any redundant subset $\mathcal{A} \subseteq \mathcal{P}(E)$ to the largest, non-redundant subset of $\mathcal{A}$.

Definition 1. *The binary reduction operator* $\Phi_\subset : \mathcal{P}(\mathcal{P}(E)) \to \mathcal{N}(\mathcal{P}(E))$ *is defined as*

$$\Phi_\subset(\mathcal{A}) = \mathcal{A} \setminus \{A_i \in \mathcal{A} \mid \exists A_j \in \mathcal{A} : A_i \subset A_j\}. \tag{1}$$

It is important to observe that if E is not finite, $\Phi_\subset(\mathcal{A})$ might be empty. Let $E = [0,1]$ and $\mathcal{A} = \{[0, 1-\frac{1}{n}] : n \in \mathbb{N}\}$. It can easily be verified that $\Phi_\subset(\mathcal{A}) = \emptyset$ in this case. This problem does not arise in finite, discrete images used in practice. Obviously, $\Phi_\subset$ has the following property

Proposition 1. *For any* $\mathcal{A} \in \mathcal{P}(\mathcal{P}(E))$

$$\bigcup \mathcal{A} = \bigcup \Phi_\subset(\mathcal{A}) \tag{2}$$

Proof. Because $\Phi_\subset(\mathcal{A}) \subseteq \mathcal{A}$ by definition, we only need to show that all elements of $\bigcup \mathcal{A}$ are contained in $\bigcup \Phi_\subset(\mathcal{A})$. Consider a point $x \in \bigcup \mathcal{A}$. This means that there is some $A_i \in \mathcal{A}$ such that $x \in A_i$. If $A_i \in \Phi_\subset(\mathcal{A})$, x is obviously contained in $\bigcup \Phi_\subset(\mathcal{A})$. If $A_i \notin \Phi_\subset(\mathcal{A})$, there must exist an $A_j \in \Phi_\subset(\mathcal{A})$ such that $A_i \subset A_j$, and x is also contained in $\bigcup \Phi_\subset(\mathcal{A})$.

We can define a partial order on $\mathcal{N}(\mathcal{P}(E))$ as

$$\mathcal{A} \preccurlyeq \mathcal{B} \quad \equiv \quad \forall A_i \in \mathcal{A} \; \exists \; B_j \in \mathcal{B} : A_i \subseteq B_j. \tag{3}$$

This is the same partial order as used for partitions in [12]. Suppose we have some elements $\mathcal{C}_i$ of $\mathcal{N}(\mathcal{P}(E))$, with $i \in I$, and I some index set, under $\preccurlyeq$ the infimum is

$$\bigwedge_{i \in I} \mathcal{C}_i = \Phi_\subset\left(\left\{\bigcap_{i \in I} D_i | D_i \in \mathcal{C}_i\right\}\right), \tag{4}$$

i.e. we first compute all sets which are intersections of one element from each of the sets $\mathcal{C}_i$. These are the maximal sets which are subset of some set in *each* of the $\mathcal{C}_i$. In general this set is redundant, so we map it back to $\mathcal{N}(\mathcal{P}(E))$ using $\Phi_\subset$. If the $\mathcal{C}_i$ are partitions, (4) is equal to the infimum of partitions in [12]. The supremum is given by

$$\bigvee_{i \in I} \mathcal{C}_i = \Phi_\subset\left(\bigcup_{i \in I} \mathcal{C}_i\right), \tag{5}$$

i.e. we create a new cover by first combining all elements of all $\mathcal{C}_i$, and then removing any redundant ones. For any $D \in C_i$ there exist an element $S \in \bigvee_{i \in I} \mathcal{C}_i$ such that $D \subseteq S$. Conversely, because for any $S \in \bigvee_{i \in I} \mathcal{C}_i$ there exists a C_i such that $S \in C_i$. Therefore, we cannot replace any $S \in \bigvee_{i \in I} \mathcal{C}_i$ by some smaller set, without violating $C_i \preccurlyeq \bigvee_{i \in I} \mathcal{C}_i$. Therefore (5) defines a supremum under $\preccurlyeq$. Within $\mathcal{N}(\mathcal{P}(E))$ the least element under $\preccurlyeq$ is $\emptyset_{\mathcal{P}(E)}$ and the maximal element is $\{E\}$. If $\mathcal{A}_1 \preccurlyeq \mathcal{A}_2$ for two partitions or covers we state that $\mathcal{A}_1$ is finer than $\mathcal{A}_2$, or, equivalently, $\mathcal{A}_2$ is coarser than $\mathcal{A}_1$.

Note that $\preccurlyeq$ is not a partial order on $\mathcal{P}(\mathcal{P}(E))$. Suppose I have some redundant $\mathcal{A} \in \mathcal{P}(\mathcal{P}(E))$, i.e., $A_i \subset A_j$ for some $A_i, A_j \in \mathcal{A}$. We then have

$$\mathcal{A} \preccurlyeq \mathcal{A} \setminus \{A_i\} \;\; \wedge \;\; \mathcal{A} \setminus \{A_i\} \preccurlyeq \mathcal{A} \tag{6}$$

but

$$\mathcal{A} \setminus \{A_i\} \neq \mathcal{A}. \tag{7}$$

2.1 Connectivity

Connectivity such as is used in morphological filtering is defined through the notion of connectivity classes or *connections* [13, 14, 6].

Definition 2. *A connection $\mathcal{C} \subseteq \mathcal{P}(E)$ is a set of sets with the following two properties:*

1. *$\emptyset \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$ for all $x \in E$*
2. *for each family $\{C_i\} \subset \mathcal{C}$, $\cap C_i \neq \emptyset$ implies $\cup C_i \in \mathcal{C}$.*

Any set $C \in \mathcal{C}$ is said to be connected. Using such a notion of connectivity, any set $X \in \mathcal{P}(E)$ can be partitioned into connected components. These are the connected subsets of X of maximal extent, i.e. if $C \subseteq X$ and $C \in \mathcal{C}$ and there exists no set $D \in \mathcal{C}$ such that $C \subset D \subseteq X$, then C is a connected component of X. Let $\mathcal{C}_X$ be defined as

$$\mathcal{C}_X = \{C \in \mathcal{C} \mid C \subseteq X\}, \tag{8}$$

in other words $\mathcal{C}_X$ is the set of all connected subsets of X. $\mathcal{C}_X$ is obviously a cover of X because for every $x \in X$ $\{x\} \in \mathcal{C}_X$. Therefore every $x \in X$ is represented in the union of all elements of $\mathcal{C}_X$. The set of all connected components $\mathcal{C}^*_X$ of X is simply

$$\mathcal{C}^*_X = \Phi_{\subset}(\mathcal{C}_X). \tag{9}$$

It is well known that this constitutes a partition of X because any $C, D \in \mathcal{C}^*_X$ are either disjoint or equal.

Connected components can be accessed through *connectivity openings* [6]:

Definition 3. *The binary connectivity opening Γ_x of X at a point $x \in E$ is given by*

$$\Gamma_x(X) = \begin{cases} \bigcup\{C_i \in \mathcal{C} \mid x \in C_i \ \wedge \ C_i \subseteq X\} & \textit{if } x \in X \\ \emptyset & \textit{otherwise.} \end{cases} \tag{10}$$

In this definition the notion of maximum extent is derived by taking the union of all connected subsets of X containing x. It can readily be shown that this is equivalent to

$$\Gamma_x(X) = \begin{cases} C_i \in \mathcal{C}^*_X : x \in C_i & \text{if } x \in X \\ \emptyset & \text{otherwise.} \end{cases} \tag{11}$$

This equivalence stems from the fact that connected subsets of X which contain x have a non-empty intersection, and that their union is therefore connected.

An important theorem links connectivity openings to connections [6].

Theorem 1. *The datum of a connection $\mathcal{C}$ in $\mathcal{P}(E)$ is equivalent to the family $\{\Gamma_x, x \in E\}$ of openings on x such that:*

1. *Γ_x is an algebraic opening marked by $x \in E$*
2. *for all $x \in E$, we have $\Gamma_x(\{x\}) = \{x\}$*
3. *for all $X \in \mathcal{P}(E)$ and all $x \in E$, we have that $x \notin X \ \Rightarrow \ \Gamma_x(X) = \emptyset$.*
4. *for all $X \in \mathcal{P}(E)$, $x, y \in E$, if $\Gamma_x(X) \cap \Gamma_y(X) \neq \emptyset \ \Rightarrow \ \Gamma_x(X) = \Gamma_y(X)$, i.e. $\Gamma_x(X)$ and $\Gamma_y(X)$ are equal or disjoint.*

2.2 Hyperconnectivity

Hyperconnectivity is a generalization of connectivity, which generalizes the second condition of Definition 2 [6]. Instead of using a non-empty intersection, we can use any *overlap criterion* $\perp$ which is defined as follows.

Definition 4. *An overlap criterion in* $\mathcal{P}(E)$ *is a mapping* $\perp : \mathcal{P}(\mathcal{P}(E)) \to \{0,1\}$ *such that* $\perp$ *is decreasing, i.e., for any* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(E)$

$$\mathcal{A} \subseteq \mathcal{B} \quad \Rightarrow \quad \perp(\mathcal{B}) \leq \perp(\mathcal{A}). \tag{12}$$

Any $\mathcal{A} \subseteq \mathcal{P}(E)$ for which $\perp(\mathcal{A}) = 1$ is said to be *overlapping*, otherwise $\mathcal{A}$ is non-overlapping. We can now define a *hyperconnectivity class* or *hyperconnection* as follows.

Definition 5. *A hyperconnection* $\mathcal{H} \subseteq \mathcal{P}(E)$ *is a set of sets with the following two properties:*

1. $\emptyset \in \mathcal{H}$ *and* $\{x\} \in \mathcal{H}$ *for all* $x \in E$
2. *for each family* $\{H_i\} \subset \mathcal{H}$, $\perp(\{H_i\}) = 1$ *implies* $\bigcup_i H_i \in \mathcal{H}$,

with $\perp$ *an overlap criterion such that* $\perp(\{H_i\}) \Rightarrow \cap_i H_i \neq \emptyset$.

Any set $H \in \mathcal{H}$ is said to be hyperconnected. Note that inserting the overlap criterion

$$\perp_\cap(\{H_i\}) = \begin{cases} 1 & \text{if } \bigcap_i H_i \neq \emptyset \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

into Definition 5 just yields a connection, showing that a connection is a special case of hyperconnection [6].

As can be seen from Definition 5, $\perp_\cap$ is the least strict overlap criterion to be used in a hyperconnection, i.e., $\perp(\{H_i\}) \leq \perp_\cap(\{H_i\})$ in general. For example we might require that the intersection contains a ball B_r of some diameter r for which $B_r \subseteq \bigcap_i H_i$. This leads to a "viscous" hyperconnectivity [10], which has been used to implement hyperconnected reconstruction shown in Fig. 1(d).

Like the notion of *connected* components for connection, we need to define the notion of *hyperconnected component*, which are hyperconnected subsets of X of maximal extent. In complete analogy with connected components we can first define the set $\mathcal{H}_X$ of all hyperconnected subsets of $X \in \mathcal{P}(E)$:

$$\mathcal{H}_X = \{H \in \mathcal{H} \mid H \subseteq X\}, \tag{14}$$

which is a cover of X for the same reasons as for $\mathcal{C}_X$. The set of hyperconnected components $\mathcal{H}_X^*$ is defined equivalently

$$\mathcal{H}_X^* = \Phi_\subset(\mathcal{H}_X). \tag{15}$$

Note that $\mathcal{H}_X^*$ is not necessarily a partition of X, because two hyperconnected components H_j, H_k may have a non-zero intersection, but $H_j \cup H_k$ need not be a member of $\mathcal{H}_X$ if $\perp(\{H_j, H_k\}) = 0$.

Braga-Neto and Goutsias [8] define a hyperconnectivity opening H_x as follows

Definition 6. *The binary hyperconnectivity opening H_x of X at point $x \in E$ is given by*

$$H_x(X) = \begin{cases} \bigcup\{H_i \in \mathcal{H}_X \mid x \in H_i\} & \text{if } x \in X \\ \emptyset & \text{otherwise.} \end{cases} \tag{16}$$

Unlike the connectivity opening Γ_x, which always returns a connected set, the hyperconnectivity opening H_x does not necessarily return a hyperconnected set, as pointed out by Braga-Neto and Goutsias in [8]. In this paper I propose a different approach.

Instead of the hyperconnectivity opening, we introduce the *hyperconnectivity operator* $\Upsilon_x : \mathcal{P}(E) \rightarrow \mathcal{P}(\mathcal{P}(E))$ which returns a set of hyperconnected sets. In the case of the connectivity opening in definition 3, we capture the notion of maximal extent by taking the union of all connected sets within X which contain the point x. This is not possible in the hyperconnected case, where we use the more explicit formulation using set inclusion used in the definition of hyperconnected components.

Definition 7. *The hyperconnectivity operator $\Upsilon_x : \mathcal{P}(E) \rightarrow \mathcal{P}(\mathcal{P}(E))$ associated with hyperconnection $\mathcal{H}$ is defined as*

$$\Upsilon_x(X) = \begin{cases} \Phi_{\subset}(\{H_i \in \mathcal{H}_X \mid x \in H_i\}), & \text{if } x \in X \\ \{\emptyset\} & \text{otherwise,} \end{cases} \tag{17}$$

In other words, Υ_x extracts the set of hyperconnected components of X containing x. It is obvious that the relationship between Υ_x and H_x is a simple one:

$$H_x(X) = \bigcup_{H_i \in \Upsilon_x(X)} H_i. \tag{18}$$

Fig. 3 illustrates the difference between the two operators.

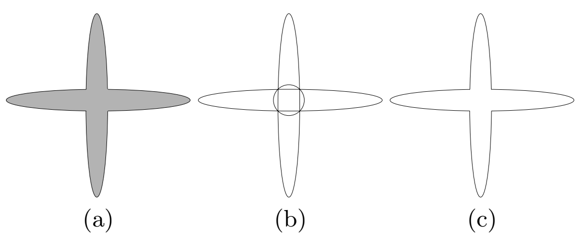


Fig. 3. Hyperconnectivity opening vs. hyperconnectivity operator: (a) binary image X; (b) outlines of hyperconnected components H_1, H_2, H_3 for some hypothetical hyperconnection $\mathcal{H}$; (c) outline of union of these hyperconnected components. Hyperconnectivity opening $H_x(X)$ returns the set outlined in (c) for any x in the intersection $\bigcap_{i=1}^{3} H_i$, whereas hyperconnectivity operator $\Upsilon_x(X)$ returns one or more of the sets outlined in (b).

We now define the properties a family of mappings $\Upsilon_x : \mathcal{P}(E) \to \mathcal{P}(\mathcal{P}(E))$ requires to define a hyperconnection. A few properties are "inherited" from connectivity openings:

1. $\Upsilon_x(H_i) = \{H_i\}$ for all $H_i \in \Upsilon_x(X)$ for all $X \in \mathcal{P}(E)$ and all $x \in E$;
2. $H_i \subseteq X$ for all $H_i \in \Upsilon_x(X)$ for all $X \in \mathcal{P}(E)$ and all $x \in X$;
3. for any $X, Y \in \mathcal{P}(E)$ we have $X \subseteq Y \;\Rightarrow\; \Upsilon_x(X) \preccurlyeq \Upsilon_x(Y)$ for all $x \in X$;
4. for all $x \in E$ we have $\Upsilon_x(\{x\}) = \{\{x\}\}$
5. for all $X \in \mathcal{P}(E)$, and all $x \in E$ we have $x \notin X \;\Rightarrow\; \Upsilon_x(X) = \{\emptyset\}$;
6. for any $H_i \in \Upsilon_x(X)$, $y \in H_i$ implies $H_i \in \Upsilon_y(X)$;
7. for all $x \in E$ and all $X \in \mathcal{P}(E)$, and any $H_i, H_j \in \Upsilon_x(X)$ we have $H_i \neq H_j \;\Rightarrow\; \perp(\{H_i, H_j\}) = 0$.

The first property ensures each $H_i \in \Upsilon_x(X)$ is hyperconnected according to the associated hyperconnection $\mathcal{H}$, and it contains x, it is the largest hyperconnected set contained in itself. Therefore, by definition 7, it is the only set $\Upsilon_x(H_i)$ should return. The second property ensures any hyperconnected component of X is a subset of X.

The third property is increasingness in the sense of (3), which can be shown as follows. Let $X \subseteq Y$. In this case any $H_i \in \Upsilon_x(X)$ is a subset of Y, through property 2. This means that either $H_i \in \Upsilon_x(Y)$, or there exists an $H_j \in \Upsilon_x(Y)$ such that $H_i \subset H_j$. Because all sets in $\Upsilon_x(X)$ have a set in $\Upsilon_x(Y)$ which is a superset or equal, the union of all members of $\Upsilon_x(X)$ is a subset of the union of all sets in $\Upsilon_x(Y)$.

The fourth property ensures that all singletons are members of $\mathcal{H}$, and the fifth that each hyperconnected component is marked *only* by its members.

The sixth property can be derived as follows. Because $H_i \in \Upsilon_x(X)$, there exists no hyperconnected set $H_j \subseteq X$, such that $H_i \subset H_j$. If $y \in H_i$ but $H_i \notin \Upsilon_y(X)$, this would imply that there is some $H_j \subseteq X$, such that $H_i \subset H_j$, leading to contradiction. This also ensures that each hyperconnected component is marked by *all* its members.

The seventh property is related, and states that no two different sets $H_i, H_j \in \Upsilon_x(X)$ can overlap in the sense of $\perp$. If they did, $H_i \cup H_j \in \mathcal{H}$ and $x \in H_i \cup H_j$. This means there exists a hyperconnected superset of both H_i and H_j containing x, and they should therefore not be members of $\Upsilon_x(X)$.

2.3 Relationship with Connectivity Openings

We will now investigate how the properties of hyperconnectivity operators relate to those of connectivity openings. Let $\#\Upsilon_x(X)$ denote the cardinality of $\Upsilon_x(X)$.

Proposition 2. *A hyperconnection $\mathcal{H}$ is a connection if and only if*

$$\#\Upsilon_x(X) = 1 \quad \textit{for all } x \in E \textit{ and all } X \in \mathcal{P}(E), \tag{19}$$

with Υ_x the hyperconnectivity operator associated with $\mathcal{H}$. In this case $H_x(X) = \bigcup_{H_i \in \Upsilon_x(X)} H_i$ is a connectivity opening.

Proof. If $\#\Upsilon_x(X) > 1$ for some $x \in E$ and some $X \in \mathcal{P}(E)$, $\mathcal{H}$ cannot be a connection because there are at least two hyperconnected components of X to which x belongs. Therefore, there are at least two sets $H_1, H_2 \in \mathcal{H}$ with non-empty intersection, but for which $H_1 \cup H_2 \notin \mathcal{H}$. This violates property 3 of Definition 2, and $\mathcal{H}$ is not a connection.

If $\#\Upsilon_x(X) = 1$ for all $x \in E$ and all $X \in \mathcal{P}(E)$ then the hyperconnected opening H_x is just a way of extracting the single element from $\Upsilon_x(X)$, i.e. $H_x(X) \in \Upsilon_x(X)$, implying $H_x(X) \in \mathcal{H}$ for all $x \in E$ and all $X \in \mathcal{P}(E)$. It has been shown that H_x is an algebraic opening [8], proving the first requirement of Theorem 1.

The second requirement of Theorem 1 follows from property 4, which states that $\Upsilon_x(\{x\}) = \{\{x\}\}$ for all hyperconnectivity operators, and therefore $H_x(\{x\}) = \{x\}$ for all $X \in \mathcal{P}(E)$. The third requirement derives from property 5, i.e. $\Upsilon_x(X) = \{\emptyset\}$ if $x \notin X$, which implies $H_x(X) = \emptyset$ for all $x \notin X$, for all $X \in \mathcal{P}(E)$.

The fourth requirement of Theorem 1 derives from property 6 above. If $y \in H_x(X)$ it follows from property 6 that $H_x(X) \in \Upsilon_y(X)$, and because $\#\Upsilon_y(X) = 1$, it follows that $H_x(X) = H_y(X)$. If $y \notin H_x(X)$, suppose that there exists some $z \in H_x(X) \cap H_y(X)$. For the previously given reasons, this implies $H_z(X) = H_x(X) = H_y(X)$, and therefore $y \in H_x(X)$, leading to contradiction. Therefore $y \notin H_x(X)$ implies $H_x(X) \cap H_y(X) = \emptyset$. Thus H_x is a connectivity opening.

Because $H_x \in \mathcal{H}$ for all $x \in E$ and $X \in \mathcal{P}(E)$, $\mathcal{H}$ is a connectivity class associated with the family of connectivity openings $\{H_x, x \in E\}$, proving Proposition 2.

2.4 Hyperconnected Filters

We will now turn to hyperconnected attribute filters, which were not considered by either Serra or Braga-Neto and Goutsias. Hyperconnected attribute filters can be defined in much the same way as connected attribute filters. We do this using a *trivial filter* $\Psi_\Lambda(H)$ which returns H if the criterion $\Lambda(H) = 1$ and $\emptyset$ otherwise. Let $\Psi_\Lambda(\mathcal{H}^*_X)$ be shorthand for the subset of all $H_i \in \mathcal{H}^*_X$ for which $\Lambda(H_j) = 1$.

Definition 8. *A hyperconnected attribute filter $\Psi^\Lambda : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ based on criterion $\Lambda : \mathcal{H} \rightarrow \{0, 1\}$ is defined as*

$$\Psi^\Lambda(X) = \bigcup_{x \in X} \bigcup_{H_i \in \Upsilon_x(X)} \Psi_\Lambda(H_i) = \bigcup_{H_j \in \mathcal{H}^*_X} \Psi_\Lambda(H_j) = \bigcup_{H_k \in \Psi_\Lambda(\mathcal{H}^*_X)} H_k, \tag{20}$$

We can define an alternative attribute filter Ψ^Λ_H using hyperconnectivity openings H_x as

$$\Psi^\Lambda_H = \bigcup_{x \in X} \Psi_\Lambda(H_x(X)) = \bigcup_{x \in X} \Psi_\Lambda\left(\bigcup \Upsilon_x(X)\right) \neq \bigcup_{x \in X} \bigcup_{H_i \in \Upsilon_x(X)} \Psi_\Lambda(H_i). \tag{21}$$

Here we see a clear distinction between the framework using hyperconnected openings H_x versus the proposed framework using operators Υ_x, because Ψ_Λ does

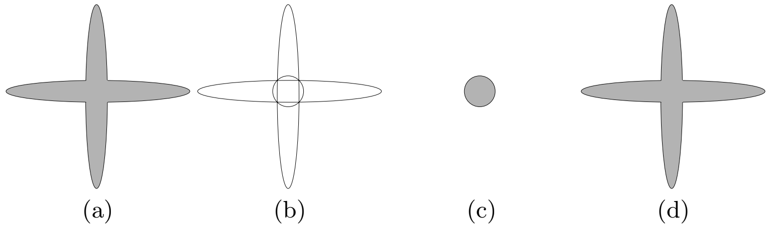


Fig. 4. Hyperconnected attribute filter with criterion Λ according to (22): (a) original images; (b) outlines of hyperconnected components; (c) union of trivial thinnings applied to hyperconnected components; (d) trivial thinning applied to union of hyperconnected components

not necessarily commute with set union. Consider the non-increasing criterion for 2-D images

$$\Lambda(H) = \begin{cases} 1 & \text{if } \Delta_x(H) = \Delta_y(H) \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

in which $\Delta_x(H)$ and $\Delta_y(H)$ are the maximal extents in x and y direction respectively. This requires that the minimum enclosing, axis-aligned rectangle is a square. Fig. 4 demonstrates the different outcomes of attribute filtering using hyperconnectivity openings and hyperconnectivity operators. The small circle in the centre is not seen as a separate entity by the H_x, whereas the "cross" preserved by Ψ_H^Λ is not hyperconnected.

3 Relationship to Structural Filters

In this section we will show the relationship with structural morphology. Let $S \subseteq E$ be an arbitrary structuring element centred at the origin $\mathbf{0}$, and $\mathcal{S}$ be the set of singletons in E, i.e.

$$\mathcal{S} = \{\{x\} | x \in E\}. \tag{23}$$

Furthermore, consider a finite chain $\mathcal{A} \subseteq \mathcal{P}(E)$, i.e. a totally ordered ordered family of sets under $\subseteq$ such that for an appropriate index set I, $A_i \subseteq A_j$ for any $i \leq j$. Obviously, if $\mathcal{A}$ is a chain, so is any subset of $\mathcal{A}$. Furthermore,

$$\bigcup_{i \in I} A_i = A_{\max I}, \tag{24}$$

provided E is finite. We can now show that the following set

$$\mathcal{H}_S = \{\emptyset\} \cup \mathcal{S}, \cup \{\{x\} \oplus S, x \in E\}, \tag{25}$$

is a hyperconnection, if provided with the overlap criterion

$$\perp_0(\mathcal{A}) = \begin{cases} 1 & \text{if } \mathcal{A} \text{ is a finite chain} \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

In other words, $\mathcal{H}_S$ consist of the empty set, all singletons, and all translates of S. The overlap criterion states that only chains of hyperconnected components overlap. It is easily seen that a hyperconnected area opening using $\mathcal{H}_S$ with an area threshold between 1 and the area of S is just the structural opening with S. Thus, *any* structural opening using any structuring element can be represented as a hyperconnected area opening. By duality, the same holds for closings.

If we combine this result with the well-established result from Serra [6] that connected filters are a special case of hyperconnected filters, we see that hyperconnected filters form a family of filters in between the two extremes. An example of such a filter is inspired by [5, 15], but now based on hyperconnected filters. Let B be a ball centred on the origin, and $\mathcal{C}$ some connection on E. Consider

$$\mathcal{H}_B = \{\emptyset\} \cup \mathcal{S} \cup \{H \in \mathcal{P}(E) \mid \exists C \in \mathcal{C} : H = \delta_B C\}, \tag{27}$$

which is just the set of all dilates by B of all connected sets, augmented with the empty set and all singletons. This set is a hyperconnection with overlap criterion

$$\perp_B(\{A_i\}) = \bigcup_i (\epsilon_B A_i) \neq \emptyset. \tag{28}$$

This overlap criterion is true if and only if the intersection of all sets A_i eroded by B is non-empty. Equivalently, the intersection of A_i must contain at least one translate of B. In this hyperconnectivity, any image is constructed from a series of hyperconnected components which all lie within $\gamma_B f$ and a series of singletons which lie in $f - \gamma_B f$. Reconstruction from markers becomes

$$\rho_{\mathcal{H}_B}(f|g) = \delta_B \rho(\epsilon_B f|\epsilon_B g). \tag{29}$$

Thus, we erode the image and the marker, and then reconstruct all those parts of the eroded image which are marked by the eroded marker. This means those parts of f which overlap with g in the sense of $\perp_B$. After this, we dilate the result to reconstitute the hyperconnected components retained in the reconstruction. If marker g is obtained by an opening with some ball B_r, we can move (more-or-less) continuously from a structural opening, when $B_r \subseteq B$, through a "viscous" hyperconnected reconstruction ($B_0 \subset B \subset B_r$) to connected reconstruction when $B = B_0$, as in [5].

A drawback of this approach is that the end result of this is a subset of $\gamma_B f$ except when singletons are included in the result. This could seriously reduce the edge-preserving qualities of this filter. We can partly amend this by performing a geodesic dilation within f, similar to [5]. The geodesic dilation by a unit ball $\bar{\delta}^1_X$ within X is defined as

$$\bar{\delta}^1_X Y = X \cap \delta^1 Y. \tag{30}$$

with δ^1 the dilation by a unit ball.

$$\mathcal{H}^X_B = \{\emptyset\} \cup \mathcal{S} \cup \{H \in \mathcal{P}(E) \mid \exists C \in \mathcal{C} : H = \bar{\delta}^1_X \delta_B C\}. \tag{31}$$

This is a hyperconnection under the the overlap criterion from (28). In this case we simply perform a geodesic dilation by a unit ball after the processing, i.e.:

$$\rho_{\mathcal{H}^f_B}(f|g) = \bar{\delta}^1_f \delta_B \rho(\epsilon_B f|\epsilon_B g), \tag{32}$$

Fig. 5. Viscous hyperconnections: (a) reconstruction of Fig 4(a) by Fig 4(b) according to (29); (b) same according to (32); (c) difference (contrast stretched)

as put forward in [10]. The difference between reconstruction according to (29) and (32) is quite small, as shown in Fig. 5.

4 Conclusion

In this paper new axiomatics for hyperconnected filters have been introduced. It has been shown that this is needed to define hyperconnected attribute filters. Before these are of any practical use, efficient algorithms for these filters must be devised. Currently work is in progress to extend the work in [10] to attribute filters in general. A drawback of the formulation chosen is that it applies to finite images, and work is in progress to obtain a more general result. An important conclusion is the relationship to structural filters. This means that there is a (semi-)continuum of operators stretching from the edge-preserving connected filters to structural filters, all of which are hyperconnected. The relationship to path openings [16] and attribute-space connectivity [7] is explored in the next paper in this volume [17].

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