Cycle-Free Finite Automata in Partial Iterative Semirings

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Abstract. We consider partial Conway semirings and partial iteration semirings, both introduced by Bloom, Ésik, Kuich [2]. We develop a theory of cycle-free elements in partial iterative semirings that allows us to define cycle-free finite automata in partial iterative semirings and to prove a Kleene Theorem. We apply these results to power series over a graded monoid with discounting.

1 Introduction

Cycle-free power series $r \in S\langle\langle \Sigma^* \rangle\rangle$, where S is a semiring and Σ is an alphabet, are defined by the condition that (r, ε), the coefficient of r at the empty word ε, is nilpotent. Transferring this notion via its transition matrix to a finite automaton assures that the behavior of a cycle-free finite automaton is well defined. This fact makes it possible to generalize classical finite automata with ε-moves to weighted cycle-free finite automata (see Kuich, Salomaa [11], Ésik, Kuich [9]).

In this paper, we take an additional step of generalization. We consider cycle-free elements in a partial iterative semiring and consider cycle-free finite automata. This generalization preserves all the nice results of weighted cycle-free finite automata and allows us to prove the usual Kleene Theorem stating the coincidence of the sets of recognizable and rational elements.

This paper consists of this and three more sections. In Section 2 we consider partial iterative semirings and partial Conway semirings, both introduced by Bloom, Ésik, Kuich [9]. Moreover, we define cycle-free elements in partial iterative semirings and prove several identities involving these cycle-free elements. In Section 3 we introduce cycle-free finite automata in partial iterative semirings,

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define recognizable and rational elements and prove a Kleene Theorem: an element is recognizable iff it is rational. In Section 4 we apply the results to power series over a finitely generated graded monoid with discounting.

2 Cycle-Free Elements in Partial Iterative Semirings

Suppose that S is a semiring and I is an ideal of S, so that $0 \in I$, $I + I \subseteq I$ and $IS \cup SI \subseteq I$. According to Bloom, Ésik, Kuich [2], S is a *partial iterative semiring over* I if for all $a \in I$ and $b \in S$ the equation $x = ax + b$ has a unique solution in S. We denote this unique solution by a^*b.

Example. This is a running example for the whole paper. Let S be a semiring and Σ an alphabet, and consider the power series semiring $S\langle\langle \Sigma^* \rangle\rangle$. A power series $r \in S\langle\langle \Sigma^* \rangle\rangle$ is called *proper* if $(r, \varepsilon) = 0$. Clearly, the collection of proper power series forms an ideal $I = \{r \in S\langle\langle \Sigma^* \rangle\rangle \mid (r, \varepsilon) = 0\}$. By Theorem 5.1 of Droste, Kuich [4], $S\langle\langle \Sigma^* \rangle\rangle$ is a partial iterative semiring over the ideal I, where the * of a proper power series r is defined by $r^* = \sum_{j \geq 0} r^j$. □

In the rest of this section we suppose that S is a partial iterative semiring over I. Moreover, we let J denote the set of all $a \in S$ such that $a^k \in I$ for some $k \geq 1$. Note that if $a^k \in I$ then $a^m \in I$ for all $m \geq k$. When a^k is in I, we say that a is *cycle free with index* k. We clearly have $I \subseteq J$.

Proposition 1. *If $a \in I$ and $b \in J$ then $a + b \in J$. Moreover, if $a, b \in S$ with $ab \in J$ then $ba \in J$.*

Proof. If $a \in I$ and $b \in J$ with $b^k \in I$, then $(a + b)^k$ is a sum of terms which are k-fold products over $\{a, b\}$. Each such product is in I since it is either b^k or contains a as a factor. Since I is closed under sum, it follows that $(a + b)^k$ is in I and thus $a + b$ is in J.

Suppose now that $a, b \in S$ with $(ab)^k \in I$ for some $k \geq 1$. Then $(ba)^{k+1} = b(ab)^k b \in I$, proving that $ba \in J$. □

The following fact was shown in Bloom, Ésik, Kuich [2].

Proposition 2. *Suppose that $a \in J$ and $b \in S$. Then the equation $x = ax + b$ has a unique solution. Moreover, its unique solution is a^*b, where a^* is the unique solution of the equation $x = ax + 1$.*

Thus, we have a partial *-operation $S \to S$ defined on the set J of cycle-free elements.

Proposition 3. *Suppose that $a, b \in J$ and $c \in S$. If $ac = cb$, then $a^*c = cb^*$. In particular, $a(a^m)^* = (a^m)^* a$, for all $a \in J$ and $m \geq 1$.*

Proof. We have $acb^* + c = cbb^* + c = c(bb^* + 1) = cb^*$, so that $a^*c = cb^*$ by uniqueness. □

Proposition 4. *Suppose that $a, b \in S$ such that $ab \in J$. Then $ba \in J$, moreover, $(ab)^*a = a(ba)^*$ and $a(ba)^*b + 1 = (ab)^*$.*

Proof. Since $(ab)a = a(ba)$ and $ab, ba \in J$, we can apply Proposition 3 to get $(ab)^*a = a(ba)^*$. Using this, $a(ba)^*b + 1 = ab(ab)^* + 1 = (ab)^*$. □

Proposition 5. *Suppose that $a, b \in S$ such that $a, a+b$ and a^*b are all in J. Then $(a+b)^* = (a^*b)^*a^*$.*

Proof. We show that $(a^*b)^*a^*$ is a solution to the equation $x = (a+b)x + 1$:

$$\begin{aligned}(a+b)(a^*b)^*a^* + 1 &= a(a^*b)a^* + b(a^*b)a^* + 1\\ &= aa^*(ba^*)^* + (ba^*)^*\\ &= (aa^* + 1)(ba^*)^*\\ &= a^*(ba^*)^*\\ &= (a^*b)^*a^*.\end{aligned}$$

□

Corollary 1. *If $a \in J$ and $b \in I$ then $(a+b)^* = (a^*b)^*a^*$.*

Proposition 6. *If $a \in J$ then $a^m \in J$ for all $m \geq 1$ and $a^* = (a^m)^*(a^{m-1} + \ldots + 1) = (a^{m-1} + \ldots + 1)(a^m)^*$.*

Proof. The fact that $(a^m)^*(a^{m-1} + \ldots + 1) = (a^{m-1} + \ldots + 1)(a^m)^*$ follows from Proposition 3. The fact that $a^* = (a^{m-1} + \ldots + 1)(a^m)^*$ follows by noting that

$$\begin{aligned}a(a^{m-1} + \ldots + 1)(a^m)^* + 1 &= a^m(a^m)^* + 1 + (a^{m-1} + \ldots + a)(a^m)^*\\ &= (a^m)^* + (a^{m-1} + \ldots + a)(a^m)^*\\ &= (a^{m-1} + \ldots + 1)(a^m)^*.\end{aligned}$$

□

The following fact is from Bloom, Ésik, Kuich [2].

Proposition 7. *If S is a partial iterative semiring over I, then $S^{n\times n}$ is a partial iterative semiring over $I^{n\times n}$.*

Below we will consider fixed point equations $X = AX + B$, where $A \in S^{n\times n}$ and $B \in S^{n\times m}$. We will assume that A and B are partitioned as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e \\ f \end{pmatrix}$$

where $a \in S^{n_1\times n_2}$, $b \in S^{n_1\times n_2}$, $c \in S^{n_2\times n_1}$, $d \in S^{n_2\times n_2}$, $e \in S^{n_1\times m}$, $f \in S^{n_2\times m}$.

Corollary 2. *If A is cycle-free so that $A^k \in I^{n\times n}$ for some k, then the equation $X = AX + B$ has a unique solution.*

Again, this unique solution is A^*B, where A^* is the unique solution to the equation $X = AX + E_n$, where E_n denotes the unit matrix in $S^{n\times n}$.

Proposition 8. *Let $A \in S^{n\times n}$ be cycle-free and assume that $a, a + bd^*c, d, d + ca^*b$ are all cycle-free. Then*

$$A^* = \begin{pmatrix} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{pmatrix}. \qquad (1)$$

Proof. Consider the system of fixed point equations

$$x = ax + by + e \tag{2}$$

$$y = cx + dy + f \tag{3}$$

where x ranges over $S^{n_1 \times m}$ and y ranges over $S^{n_2 \times m}$. We show that it has a unique solution

$$x = (a + bd^*c)^*(e + bd^*f) \tag{4}$$

$$y = (d + ca^*b)^*(f + ca^*e) \tag{5}$$

Since a is cycle free, from (2) we have $x = a^*by + a^*e$. Substituting this for x in (3) gives

$$y = (d + ca^*b)y + ca^*e + f$$

Since $d + ca^*b$ is cycle-free, this gives (5). The proof of (4) is similar. □

Proposition 9. *Let $A \in S^{n \times n}$ and assume that a and d are cycle-free and $b \in I^{n_1 \times n_2}$ or $c \in I^{n_2 \times n_1}$. Then A is cycle-free and (1) holds.*

Proof. We only prove the case where $c \in I^{n_2 \times n_1}$. The proof of the other case is similar. It is clear that for each $j \geq 1$,

$$A^j = \begin{pmatrix} a^j + x & y + z \\ u & d^j + v \end{pmatrix},$$

where the entries of x, y, u, v are all in I since they are finite sums of j-fold products containing at least one occurrence of the factor c. Moreover, z is a sum of j-fold products over $\{a, b, d\}$ having a single factor equal to b. Since a and d are cycle-free, it follows that for large enough j each such product is also a matrix with entries in I, so that each entry of z is in I. We have thus proved that when j is sufficiently large, then $A^j \in I^{n \times n}$ so that A^* is defined. Also, for each $j \geq 1$, $(a + bd^*c)^j = a^j + x$ and $(d + ca^*b)^j = d^j + y$ where x, y are matrices with entries in I. Since a and d are cycle-free, it follows again that when j is sufficiently large, then the entries of $(a + bd^*c)^j$ and $(d + ca^*b)^j$ are all in I, so that $a + bd^*c$ and $d + ca^*b$ are cycle-free and $(a + bd^*c)^*$ and $(d + ca^*b)^*$ exist. Thus, the assumptions of Proposition 8 are satisfied and our proposition is proved. □

Corollary 3. *Let $A \in S^{n \times n}$ and assume that a and d are cycle-free and $c = 0$. Then A is cycle-free and*

$$A^* = \begin{pmatrix} a^* & a^*bd^* \\ 0 & d^* \end{pmatrix}.$$

In Bloom, Ésik, Kuich [2], a *partial Conway semiring* is defined as a semiring S equipped with a distinguished ideal I and a partial operation $^* : S \to S$ defined on I which satisfies the *sum *-identiy*

$$(a + b)^* = (a^*b)^*a^*$$

for all $a, b \in I$ and *product *-identity*

$$(ab)^* = 1 + a(ba)^*b$$

for all $a, b \in S$ with $a \in I$ or $b \in I$. By Propositions 4 and 5 we have that each partial iterative semiring is a partial Conway semiring. It is known that when S is a partial Conway semiring with distinguished ideal I, then for each n, $S^{n\times n}$ is also a partial Conway semiring equipped with the ideal $I^{n\times n}$. Moreover, (1) holds for all decompositions of a matrix $A \in I^{n\times n}$. A *Conway semiring* (see Conway [3] and Bloom, Ésik [1]) is a partial Conway semiring S whose distinguished ideal is S, so that the *-operation is completely defined.

3 Cycle-Free Finite Automata

In this section we establish a Kleene Theorem in partial iterative semirings. To this end, we define a general notion of cycle-free finite automaton in partial iterative semirings. Defining the set of recognizable elements to be the set of behaviors of cycle-free finite automata, and the set of rational elements to be the least partial iterative semiring generated by some particular elements, the Kleene Theorem states that an element is recognizable iff it is rational.

In this section, S is a partial iterative semiring over the ideal I of S, Σ is a subset of I, and S_0 is a subsemiring of S. Moreover, $S_0\Sigma$ denotes the set of all finite linear combinations over Σ with coefficients in S_0, and $S_0 + S_0\Sigma$ denotes the set of sums of elements of S_0 with elements of $S_0\Sigma$. (See Bloom, Ésik, Kuich [2], Section 6.)

A *finite automaton in S and I over* (S_0, Σ) $\mathbf{A} = (\alpha, A, \beta)$ is given by

(i) a *transition matrix* $A \in (S_0 + S_0\Sigma)^{n\times n}$,
(ii) an *initial vector* $\alpha \in S_0^{1\times n}$,
(iii) a *final vector* $\beta \in S_0^{n\times 1}$.

The integer $n \geq 1$ is called the *dimension* of $\mathbf{A}$. Briefly, we call $\mathbf{A}$ finite automaton if S, I, S_0, Σ are understood.

The finite automaton $\mathbf{A} = (\alpha, A, \beta)$ is called *cycle-free* if A is cycle-free over $I^{n\times n}$. The *behavior* $|\mathbf{A}|$ of such a cycle-free finite automaton $\mathbf{A}$ is given by

$$|\mathbf{A}| = \alpha A^* \beta\,.$$

We say that $a \in S$ is *recognizable* if a is the behavior of some cycle-free finite automaton in S and I over (S_0, Σ). We let $\mathbf{Rec}_{S,I}(S_0, \Sigma)$ denote the set of all elements of S which are recognizable.

We say that $a \in S$ is *rational* if it is contained in the partial iterative semiring $\mathbf{Rat}_{S,I}(S_0, \Sigma)$ over $\mathbf{Rat}_{S,I}(S_0, \Sigma) \cap I$ generated by $S_0 \cup \Sigma$; i. e., if it is contained in the least set containing $S_0 \cup \Sigma$ and closed under the *rational operations* $+, \cdot, ^*$, where * is applied only to elements of I.

Observe that $\mathbf{Rat}_{S,I}(S_0, \Sigma)$ may be defined in an equivalent way as follows, due to Proposition 6: $\mathbf{Rat}_{S,I}(S_0, \Sigma)$ is the least set containing $S_0 \cup \Sigma$ which is closed under the operations $+, \cdot, ^*$, where * is applied only to cycle-free elements.

We will show that under a certain additional condition on S_0, $\mathbf{Rec}_{S,I}(S_0, \Sigma) = \mathbf{Rat}_{S,I}(S_0, \Sigma)$.

Example. We let S_0 be the subsemiring $S\langle\{\varepsilon\}\rangle = \{a\varepsilon \mid a \in S\}$ of $S\langle\langle\Sigma^*\rangle\rangle$. Then the finite automata in Subsection 2.1 of Ésik, Kuich [9] are essentially the finite automata in $S\langle\langle \Sigma^* \rangle\rangle$ and I over $(S\langle\{\varepsilon\}\rangle, \Sigma)$, where I is the ideal of proper series. (See Theorem 2.1 of Ésik, Kuich [9].)

The sets $S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$ and $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$ in Ésik, Kuich [9] are then the specializations of the sets of recognizable and rational elements of $S\langle\langle \Sigma^* \rangle\rangle$, respectively; i. e., $S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle = \mathbf{Rec}_{S\langle\langle \Sigma^* \rangle\rangle, I}(S\langle\{\varepsilon\}\rangle, \Sigma)$ and $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle = \mathbf{Rat}_{S\langle\langle \Sigma^* \rangle\rangle, I}(S\langle\{\varepsilon\}\rangle, \Sigma)$. Then $\mathbf{Rec}_{S,I}(S_0, \Sigma) = \mathbf{Rat}_{S,I}(S_0, \Sigma)$ is the Kleene-Schützenberger Theorem, usually written as $S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle = S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$, and the theory of cycle-free finite automata developed in this section is a generalization of Subsection 2.1 of Ésik, Kuich [9]. $\square$

Two cycle-free finite automata $\mathbf{A}$ and $\mathbf{A}'$ are *equivalent* if $|\mathbf{A}| = |\mathbf{A}'|$. A finite automaton $\mathbf{A} = (\alpha, A, \beta)$ of dimension n is called *normalized* if $n \geq 2$ and

(i) $\alpha_1 = 1$, $\alpha_i = 0$, for all $2 \leq i \leq n$;
(ii) $\beta_n = 1$, $\beta_i = 0$, for all $1 \leq i \leq n-1$;
(iii) $A_{i,1} = A_{n,i} = 0$, for all $1 \leq i \leq n$.

(See also Ésik, Kuich [9], below Theorem 2.9.)

Proposition 10. *Each cycle-free finite automaton is equivalent to a normalized cycle-free finite automaton.*

Proof. Let $\mathbf{A} = (\alpha, A, \beta)$ be a cycle-free finite automaton of dimension n. Define the finite automaton

$$\mathbf{A}' = ((1\ 0\ 0), \begin{pmatrix} 0 & \alpha & 0 \\ 0 & A & \beta \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$$

of dimension $n+2$. Then $\mathbf{A}'$ is normalized. Applying Corollary 3 twice on the transition matrix of $\mathbf{A}'$ proves that $\mathbf{A}'$ is cycle-free and

$$|\mathbf{A}'| = (\begin{pmatrix} 0 & \alpha & 0 \\ 0 & A & \beta \\ 0 & 0 & 0 \end{pmatrix}^*)_{1,n+2} = \alpha A^* \beta = |\mathbf{A}|\,.$$

$\square$

We now show that, under an additional condition on S_0, each cycle-free finite automaton is equivalent to one where the entries of the transition matrix are in the ideal I. (See condition (23) in Section 6 of Bloom, Ésik, Kuich [2].)

Definition 1. *Suppose that S is a partial iterative semiring over the ideal I, S_0 is a subsemiring of S. We say (S, S_0, I)* ***is cycle-free*** *if for all $a \in S_0$ and all $b \in I$, if*

$$a + b \in I$$

then $a = 0$.

Thus, when (S, S_0, I) is cycle-free, we understand that S, S_0, I satisfy the assumptions of Definition 1.

Proposition 11. *Suppose* (S, S_0, I) *is cycle-free and* $\Sigma \subseteq I$*. Then each cycle-free finite automaton in* S *and* I *over* (S_0, Σ) *is equivalent to a cycle-free automaton* $\mathbf{A}' = (\alpha', A', \beta')$ *in* S *and* I *over* (S_0, Σ)*, where* $A' \in (S_0\Sigma)^{n\times n}$*, and* $\alpha'_1 = 1$*,* $\alpha'_i = 0$ *for all* $2 \le i \le n$*.*

Proof. For each cycle-free finite automaton there exists, by Proposition 10, an equivalent normalized cycle-free automaton $\mathbf{A} = (\alpha, A, \beta)$. The definition of the transition matrix A implies that it can be written (not necessarily in a unique way) in the form $A = A_0 + A_1$, where $A_0 \in S_0^{n\times n}$ and $A_1 \in (S_0\Sigma)^{n\times n}$. Assume that A is cycle-free of index k. Then $A^k = A_0^k + B \in I^{n\times n}$, where $A_0^k \in S_0^{n\times n}$ and $B \in I^{n\times n}$. By the additional condition on S_0 we obtain $A_0^k = 0$ and $A_0^* = A_0^{k-1} + \ldots + E \in S_0^{n\times n}$. Hence $A_0^*A_1 \in (S_0\Sigma)^{n\times n}$ and $A_0^*\beta \in S_0^{n\times n}$.

We now define the finite automaton $\mathbf{A}'$ by $A' = A_0^*A_1$, $\alpha' = \alpha$, $\beta' = A_0^*\beta$ and show the equivalence of $\mathbf{A}$ and $\mathbf{A}'$:

$$|\mathbf{A}'| = \alpha(A_0^*A_1)^*A_0^*\beta = \alpha(A_0 + A_1)^*\beta = \alpha A^*\beta = |\mathbf{A}|\,.$$

Here we have applied Corollary 1 in the second equality. □

We now define, for given finite automata $\mathbf{A} = (\alpha, A, \beta)$ and $\mathbf{A}' = (\alpha', A', \beta')$ of dimensions n and n', respectively, the finite automata $\mathbf{A} + \mathbf{A}'$ and $\mathbf{A} \cdot \mathbf{A}'$ of dimension $n + n'$:

$$\mathbf{A} + \mathbf{A}' = \left((\alpha\ \alpha'), \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}, \begin{pmatrix} \beta \\ \beta' \end{pmatrix}\right),$$

$$\mathbf{A} \cdot \mathbf{A}' = \left((\alpha\ 0), \begin{pmatrix} A & \beta\alpha' \\ 0 & A' \end{pmatrix}, \begin{pmatrix} 0 \\ \beta' \end{pmatrix}\right).$$

Since the entries of $\beta\alpha'$ are in S_0, the entries of the transition matrices of $\mathbf{A}+\mathbf{A}'$ and $\mathbf{A} \cdot \mathbf{A}'$ are in $S_0 + S_0\Sigma$. If $\mathbf{A}$ and $\mathbf{A}'$ are cycle-free then, by Corollary 3, the transition matrices of $\mathbf{A} + \mathbf{A}'$ and $\mathbf{A} \cdot \mathbf{A}'$ are cycle-free. Hence, $\mathbf{A} + \mathbf{A}'$ and $\mathbf{A} \cdot \mathbf{A}'$ are then again cycle-free finite automata.

Proposition 12. *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be cycle-free finite automata. Then* $\mathbf{A}+\mathbf{A}'$ *and* $\mathbf{A} \cdot \mathbf{A}'$ *are again cycle-free finite automata and*

$$|\mathbf{A} + \mathbf{A}'| = |\mathbf{A}| + |\mathbf{A}'| \qquad \textit{and} \qquad |\mathbf{A} \cdot \mathbf{A}'| = |\mathbf{A}||\mathbf{A}'|\,.$$

Proof. For the proof of the equalities we apply Corollary 3:

$$|\mathbf{A} + \mathbf{A}'| = (\alpha\ \alpha') \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}^* \begin{pmatrix} \beta \\ \beta' \end{pmatrix} =$$

$$(\alpha\ \alpha') \begin{pmatrix} A^* & 0 \\ 0 & A'^* \end{pmatrix} \begin{pmatrix} \beta \\ \beta' \end{pmatrix} = \alpha A^*\beta + \alpha' A'^*\beta' = |\mathbf{A}| + |\mathbf{A}'|\,;$$

$$|\mathbf{A} \cdot \mathbf{A}'| = (\alpha\ 0) \begin{pmatrix} A & \beta\alpha' \\ 0 & A' \end{pmatrix}^* \begin{pmatrix} 0 \\ \beta' \end{pmatrix} =$$

$$(\alpha\ 0) \begin{pmatrix} A^* & A^*\beta\alpha' A'^* \\ 0 & A'^* \end{pmatrix} \begin{pmatrix} 0 \\ \beta' \end{pmatrix} = \alpha A^*\beta\alpha' A'^*\beta' = |\mathbf{A}||\mathbf{A}'|\,.$$

□

Proposition 13. *Let $a \in S_0 + S_0\Sigma$. Then $a \in \mathbf{Rec}_{S,I}(S_0, \Sigma)$.*

Proof. Consider the following finite automaton $\mathbf{A}_a$, $a \in S_0 + S_0\Sigma$, of dimension 2:

$$\mathbf{A}_a = ((1\ 0), \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) .$$

Clearly, $\mathbf{A}_a$ is cycle-free of index 2 and we obtain

$$|\mathbf{A}_a| = (1\ 0) \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a .$$

□

Corollary 4. $\mathbf{Rec}_{S,I}(S_0, \Sigma)$ *is a subsemiring of S containing $S_0 \cup \Sigma$.*

We define, for a given finite automaton $\mathbf{A} = (\alpha, A, \beta)$ the finite automaton $\mathbf{A}^+ = (\alpha, A + \beta\alpha, \beta)$. Since the entries of $\beta\alpha$ are in S_0, the entries of the transition matrix of $\mathbf{A}^+$ are in $S_0 + S_0\Sigma$.

Proposition 14. *Suppose that (S, S_0, I) is cycle-free and $\Sigma \subseteq I$. Then, for $a \in \mathbf{Rec}_{S,I}(S_0, \Sigma) \cap I$, $a^* \in \mathbf{Rec}_{S,I}(S_0, \Sigma)$.*

Proof. Let $a \in \mathbf{Rec}_{S,I}(S_0, \Sigma) \cap I$. Then, by Proposition 11, there exists a finite automaton $\mathbf{A} = (\alpha, A, \beta)$ with $A \in (S_0\Sigma)^{n\times n}$, $\alpha \in S_0^{1\times n}$ and $\beta \in S_0^{n\times 1}$ such that $a = |\mathbf{A}|$. Since $a = \alpha A^*\beta = \alpha\beta + \alpha AA^*\beta$, where $\alpha\beta \in S_0$ and $\alpha AA^*\beta \in I$, we infer by the additional condition on S_0 that $\alpha\beta = 0$.

Considering the transition matrix of the finite automaton $\mathbf{A}^+$, we obtain $(A + \beta\alpha)^2 = A^2 + A\beta\alpha + \beta\alpha A + \beta\alpha\beta\alpha = A^2 + A\beta\alpha + \beta\alpha A \in I^{n\times n}$. Hence, the transition matrix of $\mathbf{A}^+$ is cycle-free of index 2. Observe that $A^*\beta\alpha = \beta\alpha + AA^*\beta\alpha$ is cycle-free for a similar reason; thus we can in the following computation apply Proposition 5 in the second equality and Proposition 4 in the third equality and obtain

$$|\mathbf{A}^+| = \alpha(A + \beta\alpha)^*\beta = \alpha(A^*\beta\alpha)^*A^*\beta = (\alpha A^*\beta)(\alpha A^*\beta)^* = |\mathbf{A}||\mathbf{A}|^* .$$

Hence, $aa^* \in \mathbf{Rec}_{S,I}(S_0, \Sigma)$.

Consider now the cycle-free finite automaton $\mathbf{A}_1 + \mathbf{A}^+$. It has the behavior $1 + |\mathbf{A}||\mathbf{A}|^* = |\mathbf{A}|^*$. Hence, $a^* \in \mathbf{Rec}_{S,I}(S_0, \Sigma)$. □

Corollary 5. *Suppose that (S, S_0, I) is cycle-free and $\Sigma \subseteq I$. Then $a^* \in \mathbf{Rec}_{S,I}(S_0, \Sigma)$ if $a \in \mathbf{Rec}_{S,I}(S_0, \Sigma)$ is cycle-free.*

Corollary 6. *Suppose that (S, S_0, I) is cycle-free and $\Sigma \subseteq I$. Then $\mathbf{Rec}_{S,I}(S_0, \Sigma)$ is a partial iterative subsemiring of S (and hence, a partial Conway subsemiring of S) containing $S_0 \cup \Sigma$ over the ideal $\mathbf{Rec}_{S,I}(S_0, \Sigma) \cap I$ of $\mathbf{Rec}_{S,I}(S_0, \Sigma)$.*

Corollary 4 and Propositions 13, 14 show that, under an additional condition on S_0, $\mathbf{Rat}_{S,I}(S_0, \Sigma) \subseteq \mathbf{Rec}_{S,I}(S_0, \Sigma)$. We now prove the converse.

Proposition 15. $\mathbf{Rec}_{S,I}(S_0, \Sigma) \subseteq \mathbf{Rat}_{S,I}(S_0, \Sigma)$.

Proof. Let $\mathbf{A} = (\alpha, A, \beta)$ be a cycle-free finite automaton, where A is cycle-free of index k. Then $|\mathbf{A}| = \alpha A^* \beta = \alpha (A^k)^* (A^{k-1} + \ldots + E)\beta = \alpha(A^{k-1} + \ldots + E)\beta + \alpha A^k (A^k)^* (A^{k-1} + \ldots + E)\beta$. By a proof analogous to that of Lemma 6.8 of Bloom, Ésik, Kuich [2], the entries of $A^k(A^k)^*$ are in $\mathbf{Rat}_{S,I}(S_0, \Sigma)$. Since the entries of $\alpha, \beta, A^{k-1}, \ldots, E$ are also in $\mathbf{Rat}_{S,I}(S_0, \Sigma)$, the behavior $|\mathbf{A}|$ is in $\mathbf{Rat}_{S,I}(S_0, \Sigma)$. □

Corollary 7. *Suppose that (S, S_0, I) is cycle-free and $\Sigma \subseteq I$. Then*

$$\mathbf{Rec}_{S,I}(S_0, \Sigma) = \mathbf{Rat}_{S,I}(S_0, \Sigma).$$

Corollary 8. *Let (S, S_0, I) be cycle-free, and suppose that $\Sigma \subseteq I$.Then $\mathbf{Rec}_{S,I}(S_0, \Sigma)$ is the least partial iterative subsemiring of S (and hence, the least partial Conway subsemiring of S) containing $S_0 \cup \Sigma$ over the ideal $\mathbf{Rec}_{S,I}(S_0, \Sigma) \cap I$ of $\mathbf{Rec}_{S,I}(S_0, \Sigma)$.*

Corollary 4.11 and Corollary 6.13 of Bloom, Ésik, Kuich [2] show that under the conditions of Corollary 8, our set $\mathbf{Rec}_{S,I}(S_0, \Sigma)$ coincides with the set $\mathbf{Rec}_S(S_0, \Sigma)$ of Bloom, Ésik, Kuich [2].

4 Cycle-Free Finite Automata with Discounting

In this section we apply our results to a generalization of the usual power series semiring: to power series semirings over a graded monoid with discounting. We reprove a result of Droste, Sakarovitch, Vogler [6].

A monoid $\langle M, \cdot, e\rangle$ is called *graded* if it is equipped with a length function $| \, | : M \to \mathbb{N}$ that is an additive morphism. (See Sakarovitch [12,13].)

For a semiring S, we denote by $\mathrm{End}(S)$ the monoid of all endomorphisms of S, with composition as monoid operation and the identity morphism as unit.

For the rest of this section, let $\langle M, \cdot, e\rangle$ be a finitely generated graded monoid with length function $| \, |$, let $\langle S, +, \cdot, 0, 1\rangle$ be a semiring and let $\phi : M \to \mathrm{End}(S)$ be a monoid morphism.

A *formal power series over M and S* is a mapping $r : M \to S$, written as $r = \sum_{m \in M} (r, m)m$, where $(r, m) = r(m)$ is the *coefficient* of m. The set of all these power series is denoted by S^M. Let $r, s \in S^M$. *Addition* of r, s is defined pointwise by letting $(r + s, m) = (r, m) + (s, m)$ for all $m \in M$. *Multiplication* of r, s is defined by the *ϕ-Cauchy product* $r \cdot_\phi s$ of r and s by letting

$$(r \cdot_\phi s, m) = \sum_{m=uv} (r, u)\phi(u)(s, v) \qquad \text{for all } m \in M.$$

The usual definitions on power series over Σ^* and S, Σ an alphabet, can be easily transferred to power series in S^M.

Theorem 1 **(**Droste, Kuske [5], Droste, Sakarovitch, Vogler [6]**).** *The algebra $S_\phi\langle\langle M\rangle\rangle = \langle S^M, +, \cdot_\phi, 0, e\rangle$ is a semiring. Moreover, the algebra $S_\phi\langle M\rangle$ of polynomials is a subsemiring of $S_\phi\langle\langle M\rangle\rangle$.*

In the sequel we write $S_\phi\langle\langle M\rangle\rangle$ for the set S^M of formal power series over M and S.

Theorem 2. *Let S be a partial iterative semiring over the ideal I'. Then $S_\phi\langle\langle M\rangle\rangle$ is a partial iterative semiring over the ideal $\{r \in S_\phi\langle\langle M\rangle\rangle \mid (r,e) \in I'\}$.*

Proof. Consider the equation $y = ry + s$, $r, s \in S_\phi\langle\langle M\rangle\rangle$ with $(r,e) \in I'$. Let $r^* = \sum_{j\geq 0} r^j$. Here $r^0 = 1$ and $r^{j+1} = r \cdot_\phi r^j = r^j \cdot_\phi r$, $j \geq 0$. Clearly, $\{r^j \mid j \geq 0\}$ is locally finite and hence, r^* is well defined.

By an argument similar to that of Theorem 5.6 of Kuich [10], r^* satisfies

$$(r^*, e) = (r,e)^*\,,$$
$$(r^*, m) = \textstyle\sum_{uv=m,\ u\neq e}(r^*, e)(r,u) \cdot_\phi (r^*, v)\,.$$

Let $t \in S_\phi\langle\langle M\rangle\rangle$ be any solution of $y = ry + s$. Then, for all $m \in M$,

$$(t,m) = \sum_{uv=m} (r,u) \cdot_\phi (t,v) + (s,m)\,.$$

We claim that $(t,m) = (r^* \cdot_\phi s, m)$ for all $m \in M$ and prove it by induction on $|m|$.

Let $m = e$. Then $(t,e) = (r,e)(t,e) + (s,e)$. Hence, $(t,e) = (r,e)^*(s,e) = (r^* \cdot_\phi s, e)$.

Let now $|m| > 1$. Then

$$(t,m) = (r,e)(t,m) + \sum_{uv=m,\ u\neq e} (r,u) \cdot_\phi (r^* \cdot_\phi s, v) + (s,m)$$

implies

$$(t,m) = (r^*, e)\textstyle\sum_{uv_1v_2=m,\ u\neq e}(r,u) \cdot_\phi (r^*, v_1) \cdot_\phi (s, v_2) + (r^*, e)(s,m) =$$
$$\textstyle\sum_{u_1v_2=m,\ u_1\neq e}(r^*, u_1) \cdot_\phi (s, v_2) + (r^*, e)(s,m) = (r^* \cdot_\phi s, m)\,.$$

Hence, $r^* \cdot_\phi s$ is the unique solution of $y = ry + s$. □

In the sequel, $S_\phi\langle\{e\}\rangle$ denotes the subsemiring $\{ae \mid a \in S\}$ of $S_\phi\langle\langle M\rangle\rangle$ and I an ideal of $S_\phi\langle\langle M\rangle\rangle$. A *finite automaton in $S_\phi\langle\langle M\rangle\rangle$ and I over $(S_\phi\langle\{e\}\rangle, M)$*

$$\mathbf{A} = (\alpha, A, \beta)$$

is given by

(i) a *transition matrix* $A \in (S_\phi\langle M\rangle)^{n\times n}$,
(ii) an *initial vector* $\alpha \in (S_\phi\langle\{e\}\rangle)^{1\times n}$,
(iii) a *final vector* $\beta \in (S_\phi\langle\{e\}\rangle)^{n\times 1}$.

This definition is a specialization of the definition of finite automaton in Section 3. The finite automaton $\mathbf{A} = (\alpha, A, \beta)$ is called *proper* or *cycle-free* if A is proper or cycle-free, respectively. The *behavior* $|\mathbf{A}|$ of a cycle-free finite automaton $\mathbf{A}$ is given by

$$|\mathbf{A}| = \alpha \cdot_\phi A^* \cdot_\phi \beta\,.$$

Let now $S^{\mathrm{rec}}_{I,\phi}\langle\langle M\rangle\rangle$ and $S^{\mathrm{rat}}_{I,\phi}\langle\langle M\rangle\rangle$ denote the sets $\mathbf{Rec}_{S_\phi\langle\langle M\rangle\rangle,I}(S_\phi\langle\{e\}\rangle, M)$ and $\mathbf{Rat}_{S_\phi\langle\langle M\rangle\rangle,I}(S_\phi\langle\{e\}\rangle, M)$, respectively. (Here the definition of **Rec** and **Rat** is adjusted from Σ^* to M.)

Corollary 8 implies the next theorem.

Theorem 3. *Let S be a partial iterative semiring over the ideal I' and $I = \{r \in S_\phi\langle\langle M\rangle\rangle \mid (r, e) \in I'\}$. Suppose $(S_\phi\langle\langle M\rangle\rangle, S, I)$ is cycle-free. Then*

$$S^{\mathrm{rec}}_{I,\phi}\langle\langle M\rangle\rangle = S^{\mathrm{rat}}_{I,\phi}\langle\langle M\rangle\rangle$$

is the least partial iterative subsemiring of $S_\phi\langle\langle M\rangle\rangle$ (and hence, the least Conway subsemiring of $S_\phi\langle\langle M\rangle\rangle$) containing $S_\phi\langle\{e\}\rangle \cup M$ over the ideal $S^{\mathrm{rec}}_{I,\phi}\langle\langle M\rangle\rangle \cap I$.

This theorem generalizes the Kleene-Schützenberger Theorem of Schützenberger [14].

The finite S-automata over M in Droste, Sakarovitch, Vogler [6] are nothing other than our finite automata in $S_\phi\langle\langle M\rangle\rangle$ and $I = \{r \in S_\phi\langle\langle M\rangle\rangle \mid (r, e) = 0\}$ over $(S_\phi\langle\{e\}\rangle, M - \{e\})$ with proper transition matrix.

Corollary 9 (Droste, Sakarovitch, Vogler [6])**.** *Let $I = \{r \in S_\phi\langle\langle M\rangle\rangle \mid (r, e) = 0\}$. Then $S^{\mathrm{rec}}_{I,\phi}\langle\langle M\rangle\rangle = S^{\mathrm{rat}}_{I,\phi}\langle\langle M\rangle\rangle$ is the least partial iterative subsemiring of $S_\phi\langle\langle M\rangle\rangle$ (and hence, the least Conway subsemiring of $S_\phi\langle\langle M\rangle\rangle$) containing $S_\phi\langle\{e\}\rangle \cup M$ over the ideal $S^{\mathrm{rec}}_{I,\phi}\langle\langle M\rangle\rangle \cap I$.*

We now assume, for the rest of this section, that S is a partial Conway semiring.

Theorem 4. *If S is a Conway semiring then so is $S_\phi\langle\langle M\rangle\rangle$.*

Proof. In the definition of r^*, $r \in S_\phi\langle\langle M\rangle\rangle$, and in the proof of Corollary 2.4 of Kuich [10] replace $\varphi^{|w|}$ by $\phi(w)$, $w \in \Sigma^*$ by $w \in M$, and ε by e. □

In the next theorem, we assume S is a Conway semiring, and $S_\phi\langle\langle M\rangle\rangle$ is a partial Conway semiring over the ideal $S_\phi\langle\langle M\rangle\rangle$ and apply Corollary 6.12 of Bloom, Ésik, Kuich [2] or Theorem 3.2 of Ésik, Kuich [7].

Corollary 10. *Let S be a Conway semiring. Then*

$$S^{\mathrm{rec}}_{S_\phi\langle\langle M\rangle\rangle,\phi}\langle\langle M\rangle\rangle = S^{\mathrm{rat}}_{S_\phi\langle\langle M\rangle\rangle,\phi}\langle\langle M\rangle\rangle$$

is the least Conway subsemiring of $S_\phi\langle\langle M\rangle\rangle$ which contains $S\langle\{e\}\rangle \cup M$.

Theorem 5. *If S is a partial Conway semiring over the ideal I' then $S_\phi\langle\langle M\rangle\rangle$ is a partial Conway semiring over the ideal $I = \{r \in S_\phi\langle\langle M\rangle\rangle \mid (r, e) \in I'\}$.*

Proof. In a first step, change the proof of Corollary 2.4 of Kuich [10] according to the proof of Theorem 4. Now inspect this proof and assume that the power series r and s are in I. We have to check, whether the * of all power series, taken in the proof of Theorem 4, does exist; i. e., we have to check that the *-operation is applied only to power series t where $(t, e) \in I'$. Inspection shows that this is the case and the * of all used power series is defined. □

Corollary 6.13 of Bloom, Ésik, Kuich [2] implies our next result.

Corollary 11. *Let S be a partial Conway semiring with distinguished ideal I' and $I = \{r \in S_\phi\langle\langle M\rangle\rangle \mid (r,e) \in I'\}$. Suppose $(S_\phi\langle\langle M\rangle\rangle, S, I)$ is cycle-free. Then*

$$S^{\mathrm{rec}}_{I,\phi}\langle\langle M\rangle\rangle = S^{\mathrm{rat}}_{I,\phi}\langle\langle M\rangle\rangle$$

is the least partial Conway subsemiring of $S_\phi\langle\langle M\rangle\rangle$ containing $S_\phi\langle\{e\}\rangle \cup M$ with distinguished ideal $S^{\mathrm{rec}}_{I,\phi}\langle\langle M\rangle\rangle \cap I$.

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