Chapter 8
Integral Extensions

The concept of an integral ring extension is a generalization of the concept of an algebraic field extension. In the first section of this chapter, we develop the algebraic theory of integral extensions, and introduce the concept of a normal ring. Section 8.2 studies the morphism $\text{Spec}(S) \to \text{Spec}(R)$ induced from an integral extension $R \subseteq S$. In Section 8.3, we turn our attention to affine algebras again. We prove the Noether normalization theorem, and use it to prove, among other results, that all maximal ideals of an affine domain have equal height.

8.1 Integral Closure

In the previous section we have considered ring homomorphisms $\varphi \colon R \to S$. We will now assume that φ is injective, so we view R as a subring of S or (equivalently) S as a ring extension of R.

Definition 8.1. *Let S be a ring and $R \subseteq S$ a subring.*

(a) Let $s \in S$. A monic polynomial

$$g = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \in R[x]$$

with $g(s) = 0$ is called an **integral equation** *for s over R.*

(b) An element $s \in S$ is called **integral** *over R if there exists an integral equation for s over R. (The difference between this definition and that of "algebraic" is that here we insist that the polynomial equation for s be monic.)*

(c) S is called **integral** *over R if all elements from S are integral over R. In this case we call S an* **integral extension** *of R.*

Example 8.2. (1) $\sqrt{2} \in \mathbb{R}$ is integral over $\mathbb{Z}$. The ring $\mathbb{Z}[\sqrt{2}]$ is an integral extension of $\mathbb{Z}$.

G. Kemper, *A Course in Commutative Algebra*, Graduate Texts in Mathematics 256, DOI 10.1007/978-3-642-03545-6_9,

(2) $1/\sqrt{2} \in \mathbb{R}$ is not integral over $\mathbb{Z}$ (although it is algebraic). To see this, assume

$$\frac{1}{\sqrt{2}^n} + a_1 \frac{1}{\sqrt{2}^{n-1}} + \cdots + a_{n-1} \frac{1}{\sqrt{2}} + a_n = 0$$

with $a_i \in \mathbb{Z}$. Observe that 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$. Multiplying the above equation by $\sqrt{2}^n$ and picking out the summands that lie in $\mathbb{Q}$ yields

$$1 + 2a_2 + 4a_4 + \cdots = 0,$$

a contradiction.

(3) $s = \frac{1+\sqrt{5}}{2} \in \mathbb{R}$ is integral over $\mathbb{Z}$, since $s^2 - s - 1 = 0$. Therefore s is also integral over $R := \mathbb{Z}\left[\sqrt{5}\right] \subset \mathbb{R}$ (the subalgebra generated by $\sqrt{5}$). What is remarkable about this is that there exists an algebraic equation for s over R of degree 1 (so $s \in \operatorname{Quot}(R)$), but the smallest *integral* equation has degree 2. ◁

We wish to prove that products and sums of integral elements are again integral. The proof is quite similar to the standard proof of the analogous result in field theory, and requires the following lemma.

Lemma 8.3 (Integral elements and finite modules)**.** *Let S be a ring, $R \subseteq S$ a subring, and $s \in S$. Then the following statements are equivalent:*

(a) The element s is integral over R.
(b) The subalgebra $R[s] \subseteq S$ generated by s is finitely generated as an R-module.
(c) There exists an $R[s]$-module M with $\operatorname{Ann}(M) = \{0\}$ *such that M is finitely generated as an R-module.*

Proof. Assume that s is integral over R, so we have an integral equation $x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in R[x]$ for s. We claim that $R[s]$ is generated by the s^i, $i \in \{0, \ldots, n-1\}$, i.e.,

$$R[s] = \left(1, s, \ldots, s^{n-1}\right)_R = \sum_{i=0}^{n-1} R s^i =: N.$$

Indeed, for $k \geq n$, we have $s^k = -\left(a_1 s^{k-1} + \cdots + a_n s^{k-n}\right)$, so it follows by induction that all s^k lie in N. So (a) implies (b). Moreover, it is clear that (b) implies (c): Take $M = R[s]$, then $1 \in M$, so $\operatorname{Ann}(M) = \{0\}$.

Now assume (c). We have $M = (m_1, \ldots, m_r)_R$, so for each $i \in \{1, \ldots, r\}$ there exist $a_{i,j} \in R$ with $s \cdot m_i = \sum_{j=1}^r a_{i,j} m_j$. By Lemma 7.2 this implies

$$\det\left(\delta_{i,j} s - a_{i,j}\right)_{1 \leq i,j \leq r} \in \operatorname{Ann}(M),$$

so by hypothesis the determinant is zero. Therefore $\det\left(\delta_{i,j} x - a_{i,j}\right)_{1 \leq i,j \leq r} \in R[x]$ is an integral equation for s. □

The following theorem is in perfect analogy to the result that a finitely generated field extension is finite if and only if it is algebraic. It also implies that sums and products of integral elements are again integral.

Theorem 8.4 (Generated by integral elements implies integral)**.** *Let S be a ring and $R \subseteq S$ a subring such that $S = R[a_1, \dots, a_n]$ is finitely generated as an R-algebra. Then the following statements are equivalent:*

(a) All a_i are integral over R.
(b) S is integral over R.
(c) S is finitely generated as an R-module.

Proof. Clearly (b) implies (a). We use induction on n to show that (a) implies (c). We may assume $n > 0$. By induction, $S' := R[a_1, \dots, a_{n-1}]$ is finitely generated as an R-module, so $S' = (m_1, \dots, m_r)_R = \sum_{i=1}^r Rm_i$ with $m_i \in S'$. We also have that a_n is integral over S', so Lemma 8.3 yields $S'[a_n] = \sum_{j=1}^l S'n_j$ with $n_j \in S$. Putting things together, we obtain

$$S = S'[a_n] = \sum_{j=1}^{l} \sum_{i=1}^{r} Rm_i n_j,$$

so (c) holds.

Finally, (c) implies (b) by Lemma 8.3 (take $M = S$ in Lemma 8.3(c)). □

Corollary 8.5 (Integral elements form a subalgebra)**.** *Let S be a ring and $R \subseteq S$ a subring. Then the set*

$$S' := \{s \in S \mid s \textit{ is integral over } R\} \subseteq S$$

is an R-subalgebra.

Proof. Clearly all elements from R lie in S'. So all we need to show is that if $a, b \in S'$, then also $a + b \in S'$ and $a \cdot b \in S'$. But this follows since $R[a, b]$ is integral over R by Theorem 8.4. □

We obtain a further consequence of Lemma 8.3 and Theorem 8.4.

Corollary 8.6 (Towers of integral extensions)**.** *Let T be a ring and $R \subseteq S \subseteq T$ subrings. If T is integral over S and S is integral over R, then T is integral over R.*

Proof. For every $t \in T$ we have an integral equation

$$t^n + s_1 t^{n-1} + \cdots + s_{n-1} t + s_n = 0$$

with $s_i \in S$. So t is integral over $S' := R[s_1, \dots, s_n] \subseteq S$. By Lemma 8.3, $S'[t]$ is finitely generated as an S'-module, and by Theorem 8.4, S' is finitely generated as an R-module. It follows that $S'[t]$ is finitely generated as an R-module, so applying Lemma 8.3 again shows that t is integral over R. □

Corollary 8.5 prompts the following definition.

Definition 8.7.

(a) Let S be a ring and $R \subseteq S$ a subring. Then the set S' of all elements from S that are integral over R is called the **integral closure** *of R in S. If $S' = R$, we say that R is* **integrally closed** *in S.*

(b) An integral domain R is called **normal** *if it is integrally closed in its field of fractions* $\mathrm{Quot}(R)$*. One can extend this definition to rings that need not be integral domains by calling a ring normal if it is integrally closed in its total ring of fractions. In this book, normality is understood in the above narrower sense.*

(c) If R is an integral domain, the **normalization** *of R, often written as $\widetilde{R}$, is the integral closure of R in its field of fractions* $\mathrm{Quot}(R)$*. Observe that $\widetilde{R}$ is normal by Corollary 8.6.*

(d) An irreducible affine variety X over a field K is called **normal** *if the coordinate ring $K[X]$ is normal.*

Before giving some examples, we prove an elementary result.

Proposition 8.8. *Every factorial ring is normal.*

Proof. Let R be a factorial ring, and let $a/b \in \mathrm{Quot}(R)$ be integral over R with $a, b \in R$ coprime. So we have

$$\frac{a^n}{b^n} + a_1 \frac{a^{n-1}}{b^{n-1}} + \cdots + a_{n-1}\frac{a}{b} + a_n = 0$$

with $a_i \in R$. Multiplying this by b^n shows that b divides a^n, so every prime factor of b divides a. By the coprimality, b has no prime factors, so it is invertible in R. Therefore $a/b \in R$. □

Example 8.9. (1) By Proposition 8.8, $\mathbb{Z}$ is normal, and so is every polynomial ring $K[x_1, \ldots, x_n]$ over a field.

(2) By Example 8.2(3), $R := \mathbb{Z}\left[\sqrt{5}\right]$ is not normal. In fact, the normalization is

$$\widetilde{R} = \mathbb{Z}\left[\left(1 + \sqrt{5}\right)/2\right] =: S.$$

To see this, let $a + b\sqrt{5} \in \mathbb{Q}\left[\sqrt{5}\right] = \mathrm{Quot}(S)$ (with $a, b \in \mathbb{Q}$) be integral over S. Since S is integral over $\mathbb{Z}$ by Theorem 8.4, $a + b\sqrt{5}$ is integral over $\mathbb{Z}$ by Corollary 8.6, and so is $a - b\sqrt{5}$ (satisfying the same integral equation over $\mathbb{Z}$). So the sum $2a$ and the product $a^2 - 5b^2$ of these two elements are also integral over $\mathbb{Z}$. Since $\mathbb{Z}$ is integrally closed, it follows that $2a \in \mathbb{Z}$ and $a^2 - 5b^2 \in \mathbb{Z}$. Now it is easy to see that this implies $a + b\sqrt{5} \in S$.

It may be interesting to note that the ring S is actually factorial.

(3) A rather different case is $R = \mathbb{Z}\left[\sqrt{-5}\right] \subseteq \mathbb{C}$. For an element $a + b\sqrt{-5} \in \mathbb{Q}\left[\sqrt{-5}\right]$, we obtain the conditions $2a \in \mathbb{Z}$ and $a^2 + 5b^2 \in \mathbb{Z}$ for integrality

over $\mathbb{Z}$. It is easy to see that this implies $a, b \in \mathbb{Z}$, so R is normal. However, R is not factorial, as the nonunique factorization

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \tag{8.1}$$

shows. In fact, one needs to show that the factors in (8.1) really are irreducible, and that the factorizations are essentially distinct, i.e., not the same up to the order of the factors and up to invertible elements. For $z = a + b\sqrt{-5} \in R$, write $N(z) := a^2 + 5b^2 = z \cdot \overline{z}$ (z times its complex conjugate) for the so-called **norm** of z. Assume that $2 = z_1 z_2$ with $z_i \in R$. Since the norm is multiplicative, it follows that $4 = N(z_1) \cdot N(z_2)$. But 2 does not occur as a norm of an element of R, so z_1 or z_2 has norm 1. But this means $z_1 = \pm 1$ or $z_2 = \pm 1$, so z_1 or z_2 is invertible. Since every invertible element of R has norm 1, 2 itself is not invertible, so 2 is irreducible in R. Since 3 is not a norm, either, it follows by the same argument that 3 and $1 \pm \sqrt{-5}$ are irreducible, too. Finally, none of the quotients $(1 \pm \sqrt{-5})/2$ and $(1 \pm \sqrt{-5})/3$ lie in R, so the factorizations in (8.1) are essentially different.

This example shows that the converse of Proposition 8.8 does not hold.

(4) Let K be an algebraically closed field. An example from geometry is the singular cubic curve

$$X = \mathcal{V}_{K^2}\left(y^2 - x^2(x+1)\right)$$

over a field K, which is shown in Fig. 8.1, and which has a (visible) singular point at the origin. The coordinate ring of X is

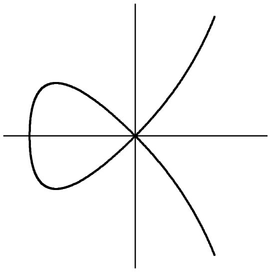

Fig. 8.1. A singular cubic curve

$$A := K[X] = K[x, y] \,/\left(y^2 - x^2(x+1)\right) =: K[\overline{x}, \overline{y}].$$

We have

$$(\overline{y}/\overline{x})^2 - \overline{x} - 1 = 0,$$

so $\overline{y}/\overline{x} \in \text{Quot}(A)$ is integral over A. The above equation also tells us that $\overline{x}$ and $\overline{y} = (\overline{y}/\overline{x}) \cdot \overline{x}$ lie in $K[\overline{y}/\overline{x}]$, so $A \subseteq K[\overline{y}/\overline{x}] \subseteq \widetilde{A}$. Since $K[\overline{y}/\overline{x}]$ is normal by Example 8.9(1), we obtain

$$\widetilde{A} = K[\overline{y}/\overline{x}].$$

It is interesting to consider the morphism of varieties induced by the embedding $A \hookrightarrow \widetilde{A}$. This is given by

$$K^1 \to X, \ \zeta \mapsto (\zeta^2 - 1, \zeta^3 - \zeta).$$

Observe that K^1 has no singular points, and that every nonsingular point of X has precisely one preimage in K^1, whereas the unique singular point of X has two preimages. So the normalization amounts to a desingularization here. As we will see later, these observations are no coincidence. In fact, we will prove in Section 14.1 that normality and nonsingularity coincide in dimension 1. This is one (but not the only) reason why normal rings are interesting. ◁

As the following proposition shows, normality is a *local property*, meaning that it holds globally if and only if it holds locally everywhere.

Proposition 8.10 (Normal rings and localization)**.** *For an integral domain R, the following statements are equivalent:*

(a) R is normal.
(b) For every multiplicative subset $U \subset R$ with $0 \notin U$, the localization $U^{-1}R$ is normal.
(c) For every maximal ideal $\mathfrak{m} \in \text{Spec}_{\text{max}}(R)$, the localization $R_{\mathfrak{m}}$ is normal.

Proof. Let $K = \text{Quot}(R)$ be the field of fractions. Assume that R is normal, and let $U \subset R$ be a multiplicative subset with $0 \notin U$. We have $U^{-1}R \subseteq K$ and $\text{Quot}(U^{-1}R) = K$. To show that $U^{-1}R$ is normal, let $a \in K$ be integral over $U^{-1}R$. Then there exist $u \in U$ and $a_1, \ldots, a_n \in R$ such that

$$a^n + \frac{a_1}{u}a^{n-1} + \cdots + \frac{a_{n-1}}{u}a + \frac{a_n}{u} = 0.$$

Multiplying this by u^n yields an integral equation for ua over R. So by assumption, $ua \in R$, so $a \in U^{-1}R$. We have shown that the statement (a) implies (b). Clearly (b) implies (c).

Now assume that (c) holds, and let $a \in K$ be integral over R. Consider the ideal $I := \{b \in R \mid ba \in R\} \subseteq R$. For every $\mathfrak{m} \in \text{Spec}_{\text{max}}(R)$, a is integral over $R_{\mathfrak{m}}$, so $a \in R_{\mathfrak{m}}$ by assumption. It follows that there exists $b \in I \setminus \mathfrak{m}$. This means that I is not contained in any maximal ideal. But if $I \subsetneq R$, Zorn's lemma would yield the existence of a maximal ideal containing I. So $1 \in I$, and $a \in R$ follows. So we have shown that (c) implies (a). □

Proposition 8.10 implies that an irreducible affine variety X is normal if and only if for every point $x \in X$ the local ring $K[X]_x$ is normal. Normality also behaves well with respect to passing from R to the polynomial ring $R[x]$, as Exercise 8.7 shows.

We finish the section with a lemma that will be used in Chapter 12. If R is an integral domain, then an element $s \in \operatorname{Quot}(R)$ is said to be **almost integral** (over R) if there exists a nonzero $c \in R$ such that $cs^n \in R$ for all nonnegative integers n.

Lemma 8.11 (Almost integral elements)**.** *In the above setting, if s is integral, then it is almost integral. If R is Noetherian, the converse holds.*

Proof. By Lemma 8.3, s is integral if and only if $R[s] \subseteq \operatorname{Quot}(R)$ is finitely generated as an R-module. In this case there exists $c \in R \setminus \{0\}$ such that $cf \in R$ for all $f \in R[s]$. In particular, $cs^n \in R$ for all n.

Conversely, if s is almost integral, then $R[s]$ is contained in $c^{-1}R \subseteq \operatorname{Quot}(R)$, which is finitely generated (by c^{-1}) as an R-module. If R is Noetherian, it follows with Theorem 2.10 that the same holds for $R[s]$. □

8.2 Lying Over, Going Up, and Going Down

If $R \subseteq S$ is an extension of rings, we have a map $f\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$, $Q \mapsto R \cap Q$, induced from the inclusion. We know from Exercise 4.2 that this map is dominant. The following theorem shows that if S is integral over R, then f is, in fact, surjective, and its fibers are finite if S is finitely generated as an R-algebra.

Theorem 8.12 (Lying over and going up)**.** *Let $R \subseteq S$ be an integral extension of rings, $P \in \operatorname{Spec}(R)$ a prime ideal, and $I \subseteq S$ an ideal with $R \cap I \subseteq P$. (Notice that the zero ideal always satisfies the condition on I.) Set*

$$\mathcal{M} := \{Q \in \operatorname{Spec}(S) \mid R \cap Q = P \text{ and } I \subseteq Q\}.$$

Then the following hold:

(a) $\mathcal{M}$ is nonempty.
(b) There exist no $Q, Q' \in \mathcal{M}$ with $Q \subsetneqq Q'$.
(c) If S is finitely generated as an R-algebra, then $\mathcal{M}$ is finite.

The keywords "lying over" and "going up," with which we advertised Theorem 8.12, refer to the following: A prime ideal $Q \in \operatorname{Spec}(S)$ with $R \cap Q = P$ is said to *lie over* P. If additionally I is contained in Q, we say that we are *going up* from I. The situation is illustrated in Fig. 8.2.

Proof of Theorem 8.12. With $S' := S/I$, $R' := R/(R \cap I)$, and $P' := P/(R \cap I)$, we have an integral extension $R' \subseteq S'$, and Lemma 1.22 yields an inclusion-preserving bijection

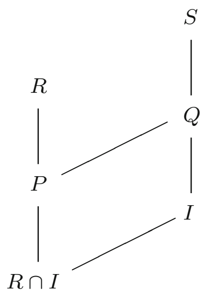

Fig. 8.2. Lying over and going up

$$\mathcal{M} \to \{Q' \in \operatorname{Spec}(S') \mid R' \cap Q' = P'\}.$$

Substituting all objects by their primed versions, we may therefore assume that $I = \{0\}$. By Proposition 7.11, we have to show that the fiber ring $S_{[P]}$ is not the zero ring (implying (a)), has Krull dimension 0 (implying (b)), and has a finite spectrum if S is finitely generated (implying (c)).

By way of contradiction, assume that $S_{[P]} = \{0\}$. By the definition of $S_{[P]}$, this is equivalent to the existence of $u \in R \setminus P$ with $u \in (P)_S$. Forming the localization $S_P := (R \setminus P)^{-1}S$, we obtain $1 \in (P_P)_{S_P}$, so

$$1 = \sum_{i=1}^{n} s_i a_i \quad \text{with} \quad s_i \in S_P,\ a_i \in P_P.$$

Form $\widetilde{S} := R_P[s_1, \ldots, s_n] \subseteq S_P$. Then the above equation implies $(P_P)_{\widetilde{S}} = \widetilde{S}$, which we may write as $P_P\widetilde{S} = \widetilde{S}$. Since $\widetilde{S}$ is an integral extension of R_P, it is finitely generated as an R_P-module by Theorem 8.4. Applying Nakayama's lemma (Theorem 7.3) yields $\widetilde{S} = \{0\}$. Since R_P is embedded into $\widetilde{S}$, this contradicts the fact that local rings are never zero. So we conclude that $S_{[P]}$ is nonzero.

The homomorphism

$$K := \operatorname{Quot}(R/P) \to S_{[P]}, \ \frac{a + P}{b + P} \mapsto \frac{a + (P)_S}{b + (P)_S},$$

makes $S_{[P]}$ into a K-algebra. The hypothesis that S is integral over R translates into the fact that $S_{[P]}$ is algebraic over K. So if $Q \in \operatorname{Spec}\left(S_{[P]}\right)$, then the quotient ring $S_{[P]}/Q$ is algebraic over K as well, and Lemma 1.1(a) yields that $S_{[P]}/Q$ is a field. This shows that $\dim\left(S_{[P]}\right) = 0$.

Finally, if S is finitely generated as an R-algebra, then $S_{[P]}$ is an affine K-algebra, so Theorem 5.11 yields that $\operatorname{Spec}_{\max}\left(S_{[P]}\right)$ is finite. Since $S_{[P]}$ has dimension 0, $\operatorname{Spec}\left(S_{[P]}\right) = \operatorname{Spec}_{\max}\left(S_{[P]}\right)$, so we are done. □

Let $R \subseteq S$ be an integral extension of rings. If $P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$ is a chain of prime ideals $P_i \in \text{Spec}(R)$, we can use Theorem 8.12 to construct a chain $Q_0 \subseteq \cdots \subseteq Q_n$ of prime ideals in $\text{Spec}(S)$ with $R \cap Q_i = P_i$. In particular, all inclusions of the Q_i are proper. So $\dim(S) \geq n$, which implies

$$\dim(R) \leq \dim(S). \tag{8.2}$$

On the other hand, if $Q \in \text{Spec}(S)$ is a prime ideal and $Q_0 \subsetneqq Q_1 \subsetneqq \cdots \subsetneqq Q_n \subseteq Q$ is a chain of prime ideals in $\text{Spec}(S)$, then $P_i := R \cap Q_i$ yields a chain in $\text{Spec}(S)$, and it follows from Theorem 8.12(b) that the inclusions of the P_i are proper. So with $P := R \cap Q$ we obtain

$$\text{ht}(Q) \leq \text{ht}(P). \tag{8.3}$$

This implies

$$\dim(S) \leq \dim(R). \tag{8.4}$$

By putting (8.2) and (8.4) together, we obtain the following corollary.

Corollary 8.13. *Let $R \subseteq S$ be an integral extension of rings. Then*

$$\dim(R) = \dim(S).$$

We now pose the question whether the reverse inequality of (8.3) also holds, i.e., whether (8.3) is in fact an equality. For proving this, we need to start with a chain of prime ideals in $\text{Spec}(R)$ that are all contained in P, and construct an equally long chain of prime ideals in $\text{Spec}(S)$ that are all contained in Q. The way to do this is to work our way downwards from Q. But what we need for being able to do this is the going down property, which was discussed in Section 7.2 (see on page 85). We have proved the following:

Corollary 8.14. *Let $R \subseteq S$ be an integral extension of rings such that going down holds for the inclusion $R \hookrightarrow S$. If $Q \in \text{Spec}(S)$ and $P := R \cap Q$, then*

$$\text{ht}(P) = \text{ht}(Q).$$

Unfortunately, going down does not always hold for integral ring extensions, as Exercise 8.9 shows. We have proved that a sufficient condition for going down is freeness (see Lemma 7.16). However, freeness is rarely found for integral extensions. We will exhibit another sufficient condition for going down (see Theorem 8.17). For proving this, we need two lemmas. The effort is worth it, since the reverse inequality of (8.3) is of crucial importance for proving some important results about affine algebras, such as Theorem 8.22 and its corollaries. The first lemma is a result from field theory. The proof uses some standard results from field theory, which we will quote from Lang [33].

Lemma 8.15 (Elements fixed by field automorphisms)**.** *Let N be a field of characteristic $p \geq 0$ and let $K \subseteq N$ be a subfield such that N is finite and* *normal over K (see Lang [33, Chapter VII, Theorem 3.3] for the definition of a normal field extension). Let $G := \mathrm{Aut}_K(N)$ be the group of automorphisms of N fixing K elementwise. Then for every $\alpha \in N^G$ in the fixed field of G, there exists $n \in \mathbb{N}_0$ such that $\alpha^{p^n} \in K$. If N is separable over K, then $n = 0$, so $\alpha \in K$.*

Proof. In the separable case, the lemma follows directly from Galois theory. The proof we give works for the separable case, too.

Let $g = \mathrm{irr}(\alpha, K) \in K[x]$ be the minimal polynomial of α over K. Let $\overline{N}$ be the algebraic closure of N, and let $\beta \in \overline{N}$ be a zero of g. Since $K[\alpha] \cong K[x]/(g) \cong K[\beta]$ with an isomorphism sending α to β, we have a homomorphism $\sigma \colon K[\alpha] \to \overline{N}$ of K-algebras with $\sigma(\alpha) = \beta$. By Lang [33, Chapter VII, Theorem 2.8], this extends to a homomorphism $\sigma \colon N \to \overline{N}$. The normality of N implies $\sigma \in G$ (see Lang [33, Chapter VII, Theorem 3.3]). Since $\sigma(\alpha) = \beta$, the hypothesis of the lemma implies $\beta = \alpha$. So α is the only zero of g, and we obtain $g = (x - \alpha)^m$ with $m \in \mathbb{N}$. Write $m = k \cdot p^n$ with $p \nmid k$. If N is separable over K, then g has to be separable, so $m = 1$ and $n = 0$. We have

$$g = (x - \alpha)^m = (x^{p^n} - \alpha^{p^n})^k = x^{kp^n} - k \cdot \alpha^{p^n} \cdot x^{(k-1)p^n} + (\text{lower terms}),$$

so $g \in K[x]$ implies $a^{p^n} \in K$. □

Lemma 8.16. *Let N be a field and $K \subseteq N$ a subfield such that N is finite and normal over K. Let $R \subseteq K$ be a subring that is integrally closed in K, and let $S \subseteq N$ be the integral closure of R in N. Then for two prime ideals $Q, \widetilde{Q} \in \mathrm{Spec}(S)$ with $R \cap Q = R \cap \widetilde{Q}$, there exists $\sigma \in G := \mathrm{Aut}_K(N)$ with $\widetilde{Q} = \sigma(Q)$.*

Proof. Let $a \in \widetilde{Q}$. Then the product $\prod_{\sigma \in G} \sigma(a)$ lies in N^G, so by Lemma 8.15 there exists $n \in \mathbb{N}_0$ with

$$b := \prod_{\sigma \in G} \sigma(a)^{p^n} \in K, \tag{8.5}$$

where $p = \mathrm{char}(K)$ and $n = 0$ if $p = 0$. Since a is integral over R and all $\sigma \in G$ fix R elementwise, all $\sigma(a)$ are integral over R as well. So b is integral over R, too, and (8.5) implies $b \in R$. Moreover, b is an S-multiple of a, so $b \in R \cap \widetilde{Q} = R \cap Q \subseteq Q$. Since Q is a prime ideal, it follows from (8.5) that there exists $\sigma \in G$ with $\sigma(a) \in Q$. Since this holds for all $a \in \widetilde{Q}$, we conclude that

$$\widetilde{Q} \subseteq \bigcup_{\sigma \in G} \sigma(Q).$$

By the prime avoidance lemma (Lemma 7.7), this implies that there exists $\sigma \in G$ with $\widetilde{Q} \subseteq \sigma(Q)$. Since σ fixes R elementwise, we have $R \cap \sigma(Q) = R \cap Q = R \cap \widetilde{Q}$, so by Theorem 8.12(b), the inclusion $\widetilde{Q} \subseteq \sigma(Q)$ cannot be strict. □

Theorem 8.17 (Going down for integral extensions of normal rings)**.** *Let S be a ring and $R \subseteq S$ a subring such that*

(1) S is an integral domain,
(2) R is normal,
(3) S is integral over R, and
(4) S is finitely generated as an R-algebra.

Then going down holds for the inclusion $R \hookrightarrow S$. In particular, the conclusion of Corollary 8.14 holds.

Proof. The proof is not difficult but a bit involved. Fig. 8.3 shows what is going on. Given prime ideals $P \in \operatorname{Spec}(R)$ and $Q' \in \operatorname{Spec}(S)$ with $P \subseteq Q'$, we need to produce $Q \in \operatorname{Spec}(S)$ with $R \cap Q = P$ and $Q \subseteq Q'$. The field of fractions $L := \operatorname{Quot}(S)$ is a finite field extension of $K := \operatorname{Quot}(R)$. By Lang [33, Chapter VII, Theorem 3.3], there exists a finite normal field extension N of K such that $L \subseteq N$. Let $T \subseteq N$ be the integral closure of R in N, so $S \subseteq T$. By Theorem 8.12, there exist $\widetilde{Z}, Z' \in \operatorname{Spec}(T)$ such that $R \cap \widetilde{Z} = P$ and $S \cap Z' = Q'$. We cannot assume that $\widetilde{Z}$ is contained in Z'. However, applying Theorem 8.12 again, we see that there exists $\widetilde{Z}' \in \operatorname{Spec}(T)$ such that $R \cap \widetilde{Z}' = R \cap Q'$ and $\widetilde{Z} \subseteq \widetilde{Z}'$. We have

$$R \cap Z' = R \cap S \cap Z' = R \cap Q' = R \cap \widetilde{Z}'.$$

So by Lemma 8.16 there exists $\sigma \in \operatorname{Aut}_K(N)$ with $Z' = \sigma(\widetilde{Z}')$. Set $Z := \sigma(\widetilde{Z})$ and $Q := S \cap Z \in \operatorname{Spec}(S)$. Then

$$R \cap Q = R \cap Z = R \cap \sigma(\widetilde{Z}) = R \cap \widetilde{Z} = P$$

and

$$Q = S \cap \sigma(\widetilde{Z}) \subseteq S \cap \sigma(\widetilde{Z}') = S \cap Z' = Q'.$$

This finishes the proof. □

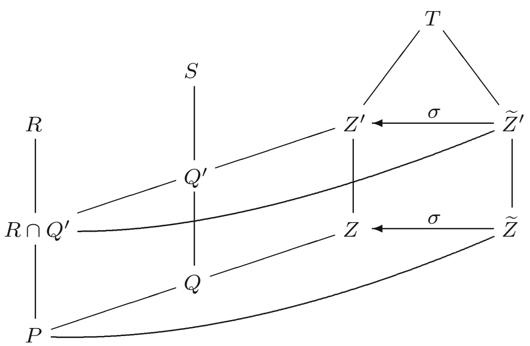

Fig. 8.3. Going down: given P and Q', construct Q

We finish this section by drawing some conclusions about geometric properties of normalization.

Proposition 8.18 (Geometric properties of normalization)**.** *Let R be an integral domain with normalization $\widetilde{R}$, and consider the morphism $f\colon \operatorname{Spec}(\widetilde{R}) \to \operatorname{Spec}(R)$ induced from the inclusion $R \subseteq \widetilde{R}$. Then*

(a) $\dim(\widetilde{R}) = \dim(R)$.
(b) The morphism f is surjective.
(c) Let $P \in \operatorname{Spec}(R)$ be such that R_P is normal. Then the fiber $f^{-1}(\{P\})$ consists of one point.

Proof. Parts (a) and (b) follow from Corollary 8.13 and Theorem 8.12(a).

To prove (c), take $P \in \operatorname{Spec}(R)$ with R_P normal. Both R_P and $\widetilde{R}$ are contained in $\operatorname{Quot}(R)$. With $U := R \setminus P$ we have $U^{-1}\widetilde{R} \subseteq \operatorname{Quot}(R) = \operatorname{Quot}(R_P)$, and $U^{-1}\widetilde{R}$ is integral over R_P, so $U^{-1}\widetilde{R} = R_P$ by the normality of R_P. Let $Q \in \operatorname{Spec}(\widetilde{R})$ be in the fiber of P, so $R \cap Q = P$. By Theorem 6.5 it follows that $U^{-1}Q \in \operatorname{Spec}(U^{-1}\widetilde{R}) = \operatorname{Spec}(R_P)$, and $\widetilde{R} \cap U^{-1}Q = Q$, so $R \cap U^{-1}Q = P$. But Theorem 6.5 also says that P_P is the only prime ideal in R_P whose intersection with R is P, so $U^{-1}Q = P_P$. It follows that $Q = \widetilde{R} \cap P_P$, showing uniqueness. □

8.3 Noether Normalization

We now turn our attention to the special case of affine algebras. Let A be an affine K-algebra with $\dim(A) = n$. By Theorem 5.9, there exist algebraically independent elements $a_1, \ldots, a_n \in A$ such that A is algebraic over the subalgebra $K[a_1, \ldots, a_n]$. As we will see in the following theorem, more can be said.

Theorem 8.19 (Noether normalization)**.** *Let $A \neq \{0\}$ be an affine K-algebra. Then there exist algebraically independent elements $c_1, \ldots, c_n \in A$ (with $n \in \mathbb{N}_0$) such that A is integral over the subalgebra $C := K[c_1, \ldots, c_n]$. In particular, A is finitely generated as a C-module, and C is isomorphic to a polynomial ring (with $C = K$ if $n = 0$).*

If $c_1, \ldots, c_n \in A$ are algebraically independent and A is integral over $K[c_1, \ldots, c_n]$, then $n = \dim(A)$.

Proof. Write A as a quotient ring of a polynomial ring: $A = K[x_1, \ldots, x_m]/I$. We use induction on m for proving the first statement. There is nothing to show for $m = 0$. If $I = \{0\}$, we can set $c_i = x_i + I$, and again there is nothing to show. If $I \neq \{0\}$, choose $f \in I \setminus \{0\}$. We can write f as

$$f = \sum_{(i_1, \ldots, i_m) \in S} \alpha_{i_1, \ldots, i_m} \cdot x_1^{i_1} \cdots x_m^{i_m}$$

with $\emptyset \neq S \subset \mathbb{N}_0^m$ a finite subset and $\alpha_{i_1,\ldots,i_m} \in K \setminus \{0\}$. Choose $d > \deg(f)$ (in fact, it suffices to choose d bigger than all x_i-degrees of f). Then the function $s\colon S \to \mathbb{N}_0,\ (i_1,\ldots,i_m) \mapsto \sum_{j=1}^m i_j \cdot d^{j-1}$ is injective. For $i = 2,\ldots,m$ set $y_i := x_i - x_1^{d^{i-1}}$. Then

$$\begin{aligned} f &= f\left(x_1, y_2 + x_1^d, \ldots, y_m + x_1^{d^{m-1}}\right) \\ &= \sum_{(i_1,\ldots,i_m)\in S} \alpha_{i_1,\ldots,i_m} \left(x_1^{s(i_1,\ldots,i_m)} + g_{i_1,\ldots,i_m}(x_1, y_2, \ldots, y_m)\right) \end{aligned}$$

with $g_{i_1,\ldots,i_m}$ polynomials satisfying $\deg_{x_1}(g_{i_1,\ldots,i_m}) < s(i_1,\ldots,i_m)$. We have exactly one $(i_1,\ldots,i_m) \in S$ such that $k := s(i_1,\ldots,i_m)$ becomes maximal. Since $A \neq \{0\}$, f is not constant, so $k > 0$. We obtain

$$f = \alpha_{i_1,\ldots,i_m} \cdot x_1^k + h(x_1, y_2, \ldots, y_m)$$

with $\deg_{x_1}(h) < k$, so

$$x_1^k + \alpha_{i_1,\ldots,i_m}^{-1} h(x_1, y_2, \ldots, y_m) \in I.$$

Set $B := K[y_2 + I, \ldots, y_m + I] \subseteq A$. Then $A = B[x_1 + I]$, and the above equation and Theorem 8.4 show that A is integral over B. By induction, there exist algebraically independent $c_1, \ldots, c_n \in B$ such that B is integral over $K[c_1, \ldots, c_n]$, and the same follows for A by Corollary 8.6.

The statement $n = \dim(A)$ follows from Corollaries 5.7 and 8.13. □

The above proof can be turned into an algorithm for computing $c_1, \ldots, c_n$. This algorithm uses Gröbner bases and is dealt with in Exercise 9.12.

Remark 8.20. In Exercise 8.10, the following stronger (but slightly less general) version of Noether normalization is shown: If the field K is infinite and $A = K[a_1, \ldots, a_m]$, then the elements $c_1, \ldots, c_n$ satisfying Theorem 8.19 can be chosen as linear combinations

$$c_i = a_i + \sum_{j=n+1}^{m} \gamma_{i,j} \cdot a_j \quad (\gamma_{i,j} \in K)$$

of the "original" generators a_i. ◁

It is not hard to give geometric interpretations of Noether normalization. In fact, Theorem 8.19 tells us that for an affine variety X of dimension n over a field K, there exists a morphism

$$f\colon X \to K^n$$

induced by the inclusion $C \subseteq K[X]$, and by Theorem 8.12, f is surjective and has finite fibers. So Noether normalization tells us that every affine variety may be interpreted as a "finite covering" of some K^n. A slightly different interpretation is that Noether normalization provides a new coordinate system such that the first n coordinates can be set to arbitrary values, which will be attained by finitely many points from the variety. So the first n coordinates act as "independent parameters." With both interpretations, it makes intuitive sense that X should have dimension n, which is a further indication that our definition of dimension is a good one. In Exercise 8.11, a further interpretation of Noether normalization as a "global system of parameters" is given.

Example 8.21. Consider the affine variety $X = \mathcal{V}_{K^2}(x_1x_2 - 1)$, which is a hyperbola as shown in Fig. 8.4. We write $\overline{x}_i$ for the image of x_i in the coordinate ring $K[X] = K[x_1, x_2]/(x_1x_2 - 1) = K[\overline{x}_1, \overline{x}_2]$. Notice that $K[X]$ is not integral over $K[\overline{x}_1]$ or over $K[\overline{x}_2]$. Motivated by Remark 8.20, we try $c = \overline{x}_1 - \overline{x}_2$ and find

$$0 = \overline{x}_1\overline{x}_2 - 1 = \overline{x}_1^2 - \overline{x}_1 c - 1,$$

so $K[X]$ is integral over $C := K[c]$. The morphism induced by the embedding $C \hookrightarrow K[X]$ is $f\colon X \to K^1$, $(\xi_1, \xi_2) \mapsto \xi_1 - \xi_2$. It is surjective, and all $\eta \in K$ with $\eta^2 \neq -4$ have two preimages, as indicated by the arrows in Fig. 8.4. ◁

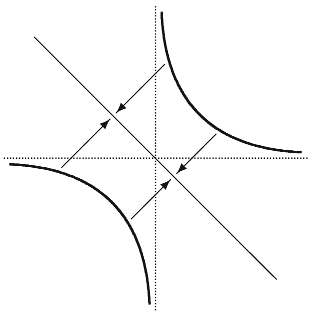

Fig. 8.4. A hyperbola and Noether normalization

We now turn our attention to chains of prime ideals in an affine algebra. Generally, in a set $\mathcal{M}$ whose elements are sets, a **maximal chain** is a subset $\mathcal{C} \subseteq \mathcal{M}$ that is totally ordered by inclusion "$\subseteq$" such that $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{M}$ with $\mathcal{D}$ totally ordered implies $\mathcal{C} = \mathcal{D}$. In particular, a chain

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

of prime ideals $P_i \in \operatorname{Spec}(R)$ in some ring is maximal if no further prime ideal can be added into the chain by insertion or by appending at either end. In general rings, it is not true that all maximal chains of prime ideals have equal length. Examples for this are affine algebras that are not equidimensional, or, more subtly, the ring studied in Exercise 8.12. However, the following theorem says that this is the case for affine domains.

Theorem 8.22 (Chains of prime ideals in an affine algebra)**.** *Let A be an affine algebra and let*

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n \tag{8.6}$$

be a maximal chain of prime ideals $P_i \in \operatorname{Spec}(A)$. Then

$$n = \dim(A/P_0).$$

In particular, if A is equidimensional (which is always the case if A is an affine domain), then every maximal chain of prime ideals of A has length equal to $\dim(A)$*.*

Proof. We use induction on n. Substituting A by A/P_0, we may assume that A is an affine domain and $P_0 = \{0\}$. If $n = 0$, then P_0 is a maximal ideal, so A is a field and we are done. So we may assume $n > 0$. Applying Lemma 1.22 yields a maximal chain $P_1/P_1 \subsetneqq P_2/P_1 \subsetneqq \cdots \subsetneqq P_n/P_1$ of prime ideals in A/P_1. Using induction, we obtain $n - 1 = \dim(A/P_1)$. So we need to show that $\dim(A/P_1) = \dim(A) - 1$.

Using Noether normalization (Theorem 8.19), we obtain $C \subseteq A$ with A integral over C and C isomorphic to a polynomial ring. By the maximality of (8.6), we have $\operatorname{ht}(P_1) = 1$. By Proposition 8.8, C is normal, so all hypotheses of Theorem 8.17 are satisfied. We obtain $\operatorname{ht}(C \cap P_1) = 1$. By the implication (b) $\Rightarrow$ (a) of Theorem 5.13, this implies $\dim(C/(C \cap P_1)) = \dim(C) - 1$. Since A/P_1 is integral over $C/(C \cap P_1)$, Corollary 8.13 yields

$$\dim(A/P_1) = \dim(C/(C \cap P_1)) = \dim(C) - 1 = \dim(A) - 1.$$

This finishes the proof. □

A ring R is called **catenary** if for two prime ideals $P \subseteq Q$ in $\operatorname{Spec}(R)$, all maximal chains of prime ideals between P and Q have the same length. So Theorem 8.22 implies that all affine algebras are catenary. It is not easy to find examples of noncatenary rings (see Nagata [41, Appendix, Example E2], Matsumura [37, Example 14E], or Hutchins [28, Example 27]). We get two immediate consequences of Theorem 8.22. The first one says that in affine domains, the height of an ideal and the dimension of the quotient ring behave complementarily.

Corollary 8.23 (Dimension and height)**.** *Let A be an affine domain or, more generally, an equidimensional affine algebra. If $I \subseteq A$ is an ideal, then*

$$\operatorname{ht}(I) = \dim(A) - \dim(A/I).$$

Proof. If I is a prime ideal, there exists a maximal chain $\mathcal{C} \subseteq \operatorname{Spec}(A)$ with $I \in \mathcal{C}$, so the result follows from Theorem 8.22, Lemma 1.22, and Definition 6.10(a). For $I = A$, it follows from Definition 6.10(b). For all other I, Definition 6.10(b) and the fact that

$$\dim(A/I) = \max\left\{\dim(A/P) \mid P \in \mathcal{V}_{\operatorname{Spec}(A)}(I)\right\}$$

allow reduction to the case that I is a prime ideal. □

The following corollary is about the height of maximal ideals in affine algebras. In the case of a maximal ideal $\mathfrak{m} \in \operatorname{Spec}_{\max}(K[X])$ belonging to a point $x \in X$ of an affine variety, it says that the height of $\mathfrak{m}$ is equal to the largest dimension of an irreducible component of X containing x.

Corollary 8.24 (Height of maximal ideals)**.** *Let A be an affine algebra with minimal prime ideals $P_1, \ldots, P_n$. (There are finitely many P_i by Corollaries 2.12 and 3.14*(a)*.) If* $\mathfrak{m} \in \operatorname{Spec}_{\max}(A)$ *is a maximal ideal, then*

$$\operatorname{ht}(\mathfrak{m}) = \max\left\{\dim(A/P_i) \mid P_i \subseteq \mathfrak{m}\right\}.$$

In particular, if A is an affine domain or, more generally, equidimensional, then all maximal ideals have $\operatorname{ht}(\mathfrak{m}) = \dim(A)$.

Proof. This is an immediate consequence of Theorem 8.22. □

To get a better appreciation of the last three results, it is important to see an example of a Noetherian domain (= a Noetherian integral domain) for which they fail. Such an example is given in Exercise 8.12.

The following result restates the principal ideal theorem (Theorem 7.5) for the special case of affine domains. Corollary 8.23 allows us to convert the statement from Theorem 7.5 on height into a statement on dimension. The theorem exemplifies the common paradigm that "imposing n further equations makes the dimension of the solution set go down by at most n."

Theorem 8.25 (Principal ideal theorem for affine domains)**.** *Let A be an affine domain or, more generally, an equidimensional affine algebra, and let $I = (a_1, \ldots, a_n) \subseteq A$ be an ideal generated by n elements. Then every prime ideal* $P \in \operatorname{Spec}(A)$ *that is minimal over I satisfies*

$$\dim(A/P) \geq \dim(A) - n.$$

In particular, if $I \neq A$, then

$$\dim(A/I) \geq \dim(A) - n,$$

and if equality holds, then A/I is equidimensional.

Proof. By Theorem 7.5, every $P \in \operatorname{Spec}(A)$ that is minimal over I satisfies $\operatorname{ht}(P) \leq n$, so

$$\dim(A/P) \geq \dim(A) - n$$

by Corollary 8.23. The other claims follows directly from this. □

If $f_1, \ldots, f_n \in K[x_1, \ldots, x_m]$ are polynomials over an algebraically closed field, then by Theorem 8.25, the affine variety in $X = \mathcal{V}_{K^m}(f_1, \ldots, f_n)$ is empty or has dimension at least $m - n$. If the dimension is equal to $m - n$, then X is called a **complete intersection** ("intersection" referring to the intersection of the hypersurfaces given by the f_i). So the second assertion of Theorem 8.25 tells us that complete intersections are equidimensional. By a slight abuse of terminology, an affine K-algebra A is also called a complete intersection if $A \cong K[x_1, \ldots, x_m]/(f_1, \ldots, f_n)$ with $\dim(A) = m - n \geq 0$.

Geometrically, the first part of Theorem 8.25 gives a dimension bound for the intersection of affine varieties $X, Y \subseteq K^m$, where X is equidimensional and Y is given by n equations. A generalization is contained in Exercise 8.14.

We will close this chapter by proving that the normalization of an affine domain is again an affine domain, and applying this result to affine varieties. Although this material is interesting, it will be used in this book only in Chapter 14 to prove two results: the existence of a desingularization of an affine curve, and the fact that the integral closure of $\mathbb{Z}$ in a number field is Noetherian (which follows from Lemma 8.27). So readers may choose to skip the rest of this chapter.

Theorem 8.26. *Let A be an affine domain. Then the normalization $\widetilde{A}$ of A is an affine domain, too.*

Proof. By Noether normalization (Theorem 8.19), we have a subalgebra $R \subseteq A$ which is isomorphic to a polynomial algebra, such that A is integral over R. In particular, $N := \mathrm{Quot}(A)$ is a finite field extension of $\mathrm{Quot}(R)$, and $\widetilde{A}$ is the integral closure of R in N. So the result follows from the following lemma. □

Lemma 8.27 (Integral closure in a finite field extension)**.** *Let R be a Noetherian domain and N a finite field extension of $L :=$ $\mathrm{Quot}(R)$. Assume that*

(a) R is normal and N is separable over L, or
(b) R is isomorphic to a polynomial ring over a field.

Then the integral closure S of R in N is finitely generated as an R-module (and therefore also as an R-algebra).

Proof. Choose generators of N as an extension of L, and let N' be the splitting field of the product of the minimal polynomials of the generators. Then N' is a finite normal field extension of L with $N \subseteq N'$, and if N is separable over L, so is N'. Since S is a submodule of the integral closure S' of R in N', it suffices to show that S' is a finitely generated R-module (use Theorem 2.10). So we may assume that N is normal over L. Let $G := \mathrm{Aut}_L(N)$ and consider the trace map

$$\mathrm{Tr}\colon N \to N^G,\ x \mapsto \sum_{\sigma \in G} \sigma(x).$$

It follows from the linear independence of homomorphisms into a field (see Lang [33, Chapter VIII, Theorem 4.1]) that Tr is nonzero. It is clearly L-linear. Let $b_1, \ldots, b_m \in N$ be an L-basis of N. By Lemma 8.15, there exists a power q of the characteristic of L (with $q = 1$ if N is separable over L) such that $\mathrm{Tr}(b_i)^q \in L$ for all i.

We first treat the (harder) case that $R \cong K[x_1, \ldots, x_n]$ with K a field. In fact, we may assume $R = K[x_1, \ldots, x_n]$. Let K' be a finite field extension of K containing qth roots of all coefficients appearing in $\mathrm{Tr}(b_i)^q$ as rational functions in the x_j. Then $\mathrm{Tr}(b_i) \in K'(x_1^{1/q}, \ldots, x_n^{1/q}) =: L'$. (For this containment to make sense without any homomorphism linking L' and N, it is useful to embed both fields in an algebraic closure of L.) So $R' := K'[x_1^{1/q}, \ldots, x_n^{1/q}]$ satisfies the following properties: (i) R' is finitely generated as an R-module, (ii) R' is normal (by Example 8.9(1)), and (iii) $\mathrm{Tr}(b_i) \subseteq \mathrm{Quot}(R')$ for all i.

In the case that R is normal and N is separable over L, these three properties are satisfied for $R' := R$.

Since $L \subseteq \mathrm{Quot}(R')$, property (iii) implies $\mathrm{Tr}(N) \subseteq \mathrm{Quot}(R')$. For $s \in S$, $\mathrm{Tr}(s)$ is integral over R, and therefore $\mathrm{Tr}(s) \in R'$ by (ii). Every $x \in N$ is algebraic over L, so there exists $0 \neq a \in R$ with $ax \in S$. Indeed, choosing a common denominator $a \in R$ of the coefficients of an integral equation of degree n for x over $L = \mathrm{Quot}(R)$ and multiplying the equation by a^n yields an integral equation for ax over R. Therefore we may assume that the basis elements b_i lie in S. So S is contained in the R-module

$$M := \{x \in N \mid \mathrm{Tr}(xb_i) \in R' \text{ for all } i = 1, \ldots, m\} \subseteq N.$$

There is an R-linear map

$$\varphi\colon M \to (R')^m,\ x \mapsto (\mathrm{Tr}(xb_1), \ldots, \mathrm{Tr}(xb_m)).$$

To show that φ is injective, let $x \in M$ with $\varphi(x) = 0$. By the L-linearity of the trace map, this implies $\mathrm{Tr}(xy) = 0$ for all $y \in N$, so $x = 0$ since $\mathrm{Tr} \neq 0$. So S is isomorphic to a submodule of $(R')^m$. But $(R')^m$ is finitely generated over R by the property (i) of R', and the result follows by Theorem 2.10. □

It is tempting to hope that for every Noetherian domain R, the normalization $\widetilde{R}$ is finitely generated as an R-module. However, Nagata [41, Appendix, Example E5] has an example in which $\widetilde{R}$ is not even Noetherian.

Corollary 8.28 (Normalization of an affine variety)**.** *Let X be an irreducible affine variety over an algebraically closed field K. Then there exists a normal affine variety $\widetilde{X}$ with a surjective morphism $f\colon \widetilde{X} \to X$ such that:*

(a) $\dim(\widetilde{X}) = \dim(X)$.
(b) *All fibers of f are finite, and if $x \in X$ is a point where the local ring $K[X]_x$ is normal, then the fiber of x consists of one point.*

Proof. By Theorem 8.26, the normalization $\widetilde{A}$ of the coordinate ring $A := K[X]$ is an affine domain, so by Theorem 1.25(b) there exists an affine variety $\widetilde{X}$ with $K[\widetilde{X}] \cong \widetilde{A}$. The inclusion $A \subseteq \widetilde{A}$ induces a morphism $f\colon \widetilde{X} \to X$. Clearly $\widetilde{X}$ is normal, and from Proposition 8.18 we obtain part (a), the surjectivity of f, and the statement on the fibers of points with A_x normal. The finiteness of the fibers follows from Theorem 8.12(c). □

The behavior of the morphism f from Corollary 8.28 can be observed very well in Example 8.9(4). Exercise 8.8 deals with a universal property of $\widetilde{X}$, as constructed in the above proof. Together with (a) and (b) of Corollary 8.28, this characterizes $\widetilde{X}$ up to isomorphism. The variety $\widetilde{X}$, or sometimes also $\widetilde{X}$ together with the morphism f, is called the **normalization** of X. In Section 14.1 we will see that if X is a curve, normalization is the same as desingularization.

Exercises for Chapter 8

8.1 (**Rings of invariants of finite groups).** In this exercise we prove that rings of invariants of finite groups are finitely generated under very general assumptions. The proof is due to Emmy Noether [43]. Let S be a ring with a subring $R \subseteq S$, and let $G \subseteq \mathrm{Aut}_R(S)$ be a finite group of automorphisms of S as an R-algebra (i.e., the elements of G fix R pointwise). Write

$$S^G := \{a \in S \mid \sigma(a) = a \text{ for all } \sigma \in G\} \subseteq S$$

for the **ring of invariants**. Observe that S^G is a sub-R-algebra of S.

(a) Show that S is integral over S^G. In particular, $\dim\left(S^G\right) = \dim(S)$.
(b) Assume that S is finitely generated as an R-algebra. Show that S^G has a finitely generated subalgebra $A \subseteq S^G$ such that S is integral over A.
(c) Assume in addition that R is Noetherian. Show that S^G is finitely generated as an R-algebra. In particular, S^G is Noetherian.

8.2 (**Rings of invariants are normal).** Let R be a normal ring, and let $G \subseteq \mathrm{Aut}(R)$ be a group of automorphisms of R. Show that R^G, the ring of invariants, is normal, too.

***8.3** (**The intersection of localizations).** Let R be a normal Noetherian domain. Show that

$$R = \bigcap_{\substack{P \in \operatorname{Spec}(R),\\ \operatorname{ht}(P)=1}} R_P.$$

(Notice that all R_P are contained in $\operatorname{Quot}(R)$, so the intersection makes sense.)
Hint: For $a/b \in \operatorname{Quot}(R) \setminus R$, consider an ideal P that is maximal among all colon ideals $(b) : (a')$ with $a' \in (a) \setminus (b)$.

8.4 (Quadratic extensions of polynomial rings). Let $f \in K[x_1, \ldots, x_n]$ be a polynomial with coefficients in a field of characteristic not equal to 2. Assume that f is not a square of a polynomial. Show that the ring $R := K[x_1, \ldots, x_n, y]/(y^2 - f)$ (with y a further indeterminate) is normal if and only if f is square-free.

8.5 (A normality criterion). Let R be a ring with an element $a \in R$ such that

(1) a is not a zero divisor.
(2) the ideal (a) is a radical ideal.
(3) the localization R_a is a normal domain.

Show that R is a normal domain.

Use this to show that for every field K and every positive integer q, the ring

$$K[x_1, x_2, y_1, y_2, z]/(z^q - (x_1 y_1)^{q-1} z - x_1^q y_2 - y_1^q x_2)$$

(with x_1, x_2, y_1, y_2, and z indeterminates) is a normal domain.

8.6 (Normalization). Assume that K contains a primitive third root of unity. Compute the normalization $\widetilde{R}$ of $R = K[x_1^3, x_1^2 x_2, x_2^3]$.
Hint: You may use Exercise 8.2. Alternatively, you may do the exercise without using the hypothesis on K.

***8.7 (Normalization of polynomial rings).** Let R be a Noetherian domain. Show that

$$\widetilde{R[x]} = \widetilde{R}[x]$$

(i.e., the normalization of the polynomial ring over R is equal to the polynomial ring over the normalization). Conclude that $R[x]$ is normal if and only if R is normal.
Hint: The hard part is to show that a polynomial $f = \sum_{i=0}^{n} a_i x_i \in \operatorname{Quot}(R)[x]$ that is integral over $R[x]$ lies in $\widetilde{R}[x]$. This can be done as follows: Show that there exists $0 \neq u \in R$ such that $u f^k \in R[x]$ for all $k \geq 0$. Conclude that $R[a_n]$ is finitely generated as an R-module. Then use induction on n.
Remark: The result is also true if R is not Noetherian. In fact, one can reduce to the Noetherian case by substituting R with a finitely generated subring in the above proof idea.
(Solution on page 222)

8.8 (The universal property of normalization). Show that the variety $\widetilde{X}$ constructed in the proof of Corollary 8.28 satisfies the following universal property. If Y is a normal affine K-variety with a dominant morphism $g: Y \to X$ (this means that the image $g(Y)$ is dense in X), then there exists a unique morphism $h: Y \to \widetilde{X}$ with $f \circ h = g$.

8.9 (Where going down fails). In this exercise we study an example of an integral extension of rings in which going down fails. Let K be a field of characteristic $\neq 2$, $S = K[x, y]$ the polynomial ring in two indeterminates, and

$$R := K[a, b, y] \subset S \quad \text{with} \quad a = x^2 - 1 \quad \text{and} \quad b = xa.$$

(a) Show that S is the normalization of R.
(b) Show that

$$P := \left(a - (y^2 - 1), b - y(y^2 - 1)\right)_R \subset R$$

is a prime ideal, and P is contained in the prime ideal

$$Q' := (x - 1, y + 1)_S \in \operatorname{Spec}(S).$$

(c) Show that the unique ideal $Q \in \operatorname{Spec}(S)$ with $R \cap Q = P$ is

$$Q := (x - y)_S$$

and conclude that going down fails for the inclusion $R \hookrightarrow S$.
(d) Compare this example to Example 8.9(4). Try to give a geometric interpretation to the failure of going down for $R \hookrightarrow S$.
Hint: The generators of R satisfy the equation $b^2 - a^2 \cdot (a + 1) = 0$.

8.10 (Noether normalization with linear combinations). Prove the statement in Remark 8.20.
Hint: Mimic the proof of Theorem 8.19, but set $y_i := x_i - \beta_i x_m$ with $\beta_i \in K$ $(i = 1, \dots, m - 1)$.

***8.11 (Noether normalization and systems of parameters).** Let $X \neq \emptyset$ be an equidimensional affine variety over a field K and let $c_1, \dots, c_n \in A := K[X]$ be as in Theorem 8.19. Let $x \in X$ be a point with corresponding maximal ideal $\mathfrak{m} := \{f \in A \mid f(x) = 0\}$. Show that

$$a_i := \frac{c_i - c_i(x)}{1} \in A_{\mathfrak{m}} \quad (i = 1, \dots, n)$$

provides a system of parameters of the local ring $K[X]_x = A_{\mathfrak{m}}$ at x. An interpretation of this result is that Noether normalization provides a global system of parameters or, from a reverse angle, that systems of parameters are a local version of Noether normalization.

Hint: With $I := (c_1 - c_1(x), \dots, c_n - c_n(x))_A$, first prove that A/I is Artinian. Then use Nakayama's lemma to show that $\mathfrak{m}_\mathfrak{m}^k \subseteq I_\mathfrak{m}$ for some k. *(Solution on page 223)*

8.12 (A Noetherian domain where Theorem 8.22 fails). Let $R = K[[x]]$ be a formal power series ring over a field, and $S = R[y]$ a polynomial ring. Exhibit two maximal ideals in $\operatorname{Spec}_{\max}(S)$ of different height. So S is a Noetherian domain for which Theorem 8.22 and Corollaries 8.23 and 8.24 fail.

8.13 (Hypotheses of Theorem 8.25). Use the following example to show that the hypothesis on equidimensionality cannot be dropped from Theorem 8.25:

$$A = K[x_1, x_2, x_3, x_4]/(x_1 - x_4, x_1^2 - x_2 x_4, x_1^2 - x_3 x_4)$$

and $a = \overline{x}_1 - 1$, the class of $x_1 - 1$ in A. Explain why this also shows that if $K[x_1, \dots, x_m]/(f_1, \dots, f_n)$ is a complete intersection, this need not imply that $K[x_1, \dots, x_m]/(f_1, \dots, f_{n-1})$ is a complete intersection, too.

8.14 (A dimension theorem). Let X and Y be two equidimensional affine varieties both of which lie in K^n. Show that every irreducible component Z of $X \cap Y$ satisfies

$$\dim(Z) \geq \dim(X) + \dim(Y) - n.$$

Hint: With $\Delta := \{(x, x) \mid x \in K^n\} \subset K^{2n}$ the diagonal, show that $X \cap Y \cong (X \times Y) \cap \Delta$ and conclude the result from that.

8.15 (Right or wrong?). Decide whether each of the following statements is true or false. Give reasons for your answers.

(a) Let K be a finite field and let X be a set. Then the ring $S = \{f\colon X \to K \mid f \text{ is a function}\}$ (with pointwise operations) is an integral extension of K (which is embedded into S as the ring of constant functions).
(b) If $R \subseteq S$ is an integral ring extension, then for every $P \in \operatorname{Spec}(R)$ the set $\{Q \in \operatorname{Spec}(S) \mid R \cap Q = P\}$ is finite.
(c) If A is an affine domain that can be generated by $\dim(A) + 1$ elements, then A is a complete intersection.
(d) If A is an affine algebra that can be generated by $\dim(A) + 1$ elements, then A is a complete intersection.
(e) If an affine domain is a complete intersection, it is normal.