Chapter 6
Localization

In commutative algebra, localization is a construction that is almost as important as the formation of quotient structures. In this chapter we define localization and give the basic properties. In particular, we will see what localization does to the spectrum of a ring. Localization naturally leads to the topics of local rings and the height of an ideal, which will be dealt with here.

The construction of $\mathbb{Q}$ from $\mathbb{Z}$ or, more generally, of $\operatorname{Quot}(R)$ from an integral domain R is a model for the following definition of localization. However, localization is more general in two ways: It allows one to make only a selection of ring elements invertible, which may include zero divisors, and we extend the definition to modules.

Definition 6.1. *Let R be a ring, M an R-module (where $M = R$ is an important special case), and $U \subseteq R$ a submonoid of the multiplicative monoid of R (i.e., $1 \in U$, and with $a, b \in U$ the product $a \cdot b$ also lies in U; we do* not *assume that $0 \notin U$). Such a set U is called a* **multiplicative subset** *of R. Define a relation $\sim$ on the Cartesian product $U \times M$ by*

$$(u_1, m_1) \sim (u_2, m_2) \quad :\Longleftrightarrow \quad \text{there exists } u \in U \text{ such that } uu_2m_1 = uu_1m_2.$$

(It is routine to check that this is indeed an equivalence relation.) We will write the equivalence class of $(u, m) \in U \times M$ as a fraction:

$$[(u, m)]_\sim =: \frac{m}{u}.$$

(This notation makes it clear that the intention of the equivalence relation is to allow the reduction of fractions, as well as the reverse process.) The **localization of M with respect to U**, *written as $U^{-1}M$, is the set of equivalence classes:*

$$U^{-1}M := (U \times M)/\sim \; = \left\{ \frac{m}{u} \mid m \in M,\ u \in U \right\}.$$

There is a **canonical map** *given by*

G. Kemper, *A Course in Commutative Algebra*, Graduate Texts in Mathematics 256, DOI 10.1007/978-3-642-03545-6_7,

$$\varepsilon\colon M \to U^{-1}M,\ m \mapsto \frac{m}{1}.$$

$U^{-1}M$ is made into an R-module by

$$\frac{m_1}{u_1} + \frac{m_2}{u_2} := \frac{u_2 m_1 + u_1 m_2}{u_1 u_2} \quad \textit{for} \quad m_i \in M \ \textit{and}\ u_i \in U$$

and

$$a \cdot \frac{m}{u} := \frac{am}{u} \quad \textit{for} \quad a \in R,\ m \in M,\ \textit{and}\ u \in U.$$

(Again, it is routine to check that these operations are well defined and the axioms of a module are satisfied.)

In the special case that $U = R \setminus P$ with $P \in \mathrm{Spec}(R)$, we write

$$U^{-1}M =: M_P$$

and call this the **localization of M at P**.

The generality and flexibility of localization are best demonstrated by examples.

Example 6.2. (1) Let R be an integral domain and $U = R \setminus \{0\}$. Then $U^{-1}R = \mathrm{Quot}(R)$, the field of fractions. $\mathrm{Quot}(R)$ is the localization of R at the prime ideal $\{0\}$.

(2) More generally, let R be any ring and

$$U := \{a \in R \mid a \text{ is not a zero divisor}\},$$

which is a multiplicative subset. $U^{-1}R$ is called the **total ring of fractions** of R. The canonical map $R \to U^{-1}R$ is injective, and U is maximal with this property. In fact, if $S \subseteq R$ is any multiplicative subset, then the canonical map $R \to S^{-1}R$ is injective if and only if $S \subseteq U$.

(3) Consider $R = \mathbb{Z}$ and $P = (2) \in \mathrm{Spec}(\mathbb{Z})$. Then R_P is (isomorphic to) the ring of all rational numbers with odd denominator. We have mentioned this ring on page 15 as an example of a non-Jacobson ring.

(4) Let $R = \mathbb{Z}/(6)$ and $P = (2) \in \mathrm{Spec}(R)$. In R_P, we have $\frac{2}{1} = \frac{0}{1}$, since $3 \notin P$ and

$$3 \cdot 1 \cdot 2 = 3 \cdot 1 \cdot 0.$$

With this, it is easy to see that $R_P \cong \mathbb{Z}/(2)$. So a localization can be "smaller" than the original ring.

Also notice that the total ring of fractions of $\mathbb{Z}/(6)$ is isomorphic to $\mathbb{Z}/(6)$.

(5) Let K be a field and $X \subseteq K^n$ an affine variety with coordinate ring $K[X]$. For $x \in X$, consider the maximal ideal $\mathfrak{m}_x \in \mathrm{Spec}_{\max}(K[X])$ of all regular functions vanishing at x. Then

$$K[X]_x := K[X]_{\mathfrak{m}_x}$$

consists of all fractions of regular functions on X whose denominator does not vanish at x. This example gives a first hint that localization has something to do with locality. A second hint is contained in Exercise 6.6. $K[X]_x$ is also called the **localization of $K[X]$ at x.**

(6) Let R be a ring and $a \in R$. Then $U = \{1, a, a^2, \ldots\} = \{a^k \mid k \in \mathbb{N}_0\} \subseteq R$ is a multiplicative subset. It is customary to write

$$M_a := U^{-1}M,$$

although this may sometimes lead to confusion. For example, with this notation $\mathbb{Z}_2$ is (isomorphic to) the ring of all rational numbers with a power of 2 as denominator.

(7) If $0 \in U$, then $U^{-1}M = \{0\}$ for every R-module M, including $M = R$. This follows from Definition 6.1.

(8) Let $(G, +)$ be a finite abelian group, which becomes a $\mathbb{Z}$-module by defining $a \cdot \sigma := \sum_{i=1}^{a} \sigma$ and $(-a) \cdot \sigma := -(a \cdot \sigma)$ for $\sigma \in G$ and $a \in \mathbb{Z}$ nonnegative. Let $U = \mathbb{Z} \setminus \{0\}$. Then $U^{-1}G$ is the zero module, since each $\sigma \in G$ has positive order $\operatorname{ord}(\sigma) \in U$, and $\operatorname{ord}(\sigma) \cdot \sigma = 0$. ◁

The following proposition is a collection of basic properties of localization. The proofs of all parts are straightforward but sometimes a little tedious. We leave them as an exercise for the reader, who should be prepared to spend a small pile of paper on them.

Proposition 6.3 (Properties of localization)**.** *Let R be a ring, $U \subseteq R$ a multiplicative subset, and M an R-module.*

(a) $U^{-1}R$ becomes a ring with the addition defined as in $U^{-1}M$, and multiplication defined as multiplying numerators and denominators.

(b) The canonical map $\varepsilon\colon R \to U^{-1}R$ is a homomorphism of rings. So $U^{-1}R$ becomes an R-algebra.

(c) $U^{-1}M$ becomes a $U^{-1}R$-module, with multiplication of an element of $U^{-1}R$ and an element of $U^{-1}M$ defined as multiplying numerators and denominators.

(d) All $\varepsilon(u)$ with $u \in U$ are invertible in $U^{-1}R$.

(e) Let $\varphi\colon R \to S$ be a ring homomorphism such that all $\varphi(u)$ with $u \in U$ are invertible in S. Then there exists a unique homomorphism $U^{-1}R \to S$ of R-algebras. This universal property tells us that $S^{-1}R$ is the "smallest" and "freest" R-algebra in which the elements from U become invertible.

(f) If R is an integral domain and $0 \notin U$, then $U^{-1}R$ is embedded in $\operatorname{Quot}(R)$ in the obvious way. Therefore we may (and often will) identify $U^{-1}R$ with a subalgebra of $\operatorname{Quot}(R)$.

(g) If $V \subseteq R$ is a multiplicative subset containing U, then

$$V^{-1}(U^{-1}M) \cong \varepsilon(V)^{-1}(U^{-1}M) \cong V^{-1}M$$

(isomorphisms of R-modules). So "step-by-step" localization is the same as "all-at-once" localization. For $M = R$, the second isomorphism is also

a ring isomorphism

$$\varepsilon(V)^{-1}(U^{-1}R) \cong V^{-1}R.$$

(h) *Let $N \subseteq M$ be a submodule. Then $U^{-1}N$ is isomorphic to a submodule of $U^{-1}M$. In fact, with $\varepsilon_M\colon M \to U^{-1}M$ the canonical map, the map*

$$U^{-1}N \to \Big(\varepsilon_M(N)\Big)_{U^{-1}R} = U^{-1}R \cdot \varepsilon_M(N),\ \frac{n}{u} \mapsto \frac{1}{u} \cdot \varepsilon_M(n),$$

is an isomorphism of $U^{-1}R$-modules. Therefore we may (and will) identify $U^{-1}N$ with $(\varepsilon_M(N))_{U^{-1}R} \subseteq U^{-1}M$. In particular, for an ideal $I \subseteq R$ we identify $U^{-1}I$ with the ideal $(\varepsilon(I))_{U^{-1}R} \subseteq U^{-1}R$.

(i) *Let $\mathfrak{N} \subseteq U^{-1}M$ be a $U^{-1}R$-submodule. With $\varepsilon_M\colon M \to U^{-1}M$ the canonical map, the preimage $N := \varepsilon_M^{-1}(\mathfrak{N}) \subseteq M$ is a submodule, and*

$$U^{-1}N = \mathfrak{N}.$$

In particular, if $\mathfrak{I} \subseteq U^{-1}R$ is an ideal, then

$$U^{-1}\varepsilon^{-1}(\mathfrak{I}) = \mathfrak{I}.$$

The above properties of localization will often be used without explicit reference to Proposition 6.3. As an immediate consequence of part (i), we get the following:

Corollary 6.4 (Localization preserves the Noether property)**.** *Let R be a ring, $U \subseteq R$ a multiplicative subset, and M an R-module. If M is Noetherian, then so is $U^{-1}M$ (as a $U^{-1}R$-module). In particular, if R is Noetherian, then so is $U^{-1}R$.*

The following result gives a description of the spectrum of a localized ring $U^{-1}R$. It is a counterpart of Lemma 1.22, which deals with quotient rings.

Theorem 6.5 (The spectrum of a localized ring)**.** *Let R be a ring and $U \subseteq R$ a multiplicative subset. Let $\varepsilon\colon R \to U^{-1}R$ be the canonical map and*

$$\mathcal{A} := \{Q \in \operatorname{Spec}(R) \mid U \cap Q = \emptyset\}.$$

Then the map

$$\operatorname{Spec}\left(U^{-1}R\right) \to \mathcal{A},\ \mathfrak{Q} \mapsto \varepsilon^{-1}(\mathfrak{Q})$$

is an inclusion-preserving bijection with inverse map

$$\mathcal{A} \to \operatorname{Spec}\left(U^{-1}R\right),\ Q \mapsto U^{-1}Q.$$

In particular, for a prime ideal $P \in \operatorname{Spec}(R)$, the prime ideals of R_P correspond to prime ideals $Q \in \operatorname{Spec}(R)$ with $Q \subseteq P$.

Proof. Since preimages of prime ideals under ring homomorphisms are always prime ideals, $\varepsilon^{-1}(\mathfrak{Q}) \in \mathrm{Spec}(R)$ for $\mathfrak{Q} \in \mathrm{Spec}\left(U^{-1}R\right)$. Moreover, $U \cap \varepsilon^{-1}(\mathfrak{Q}) = \emptyset$, since otherwise $\mathfrak{Q}$ would contain an invertible element from $U^{-1}R$. So $\varepsilon^{-1}(\mathfrak{Q}) \in \mathcal{A}$. By Proposition 6.3(i) we also have

$$U^{-1}\varepsilon^{-1}(\mathfrak{Q}) = \mathfrak{Q}.$$

Now let $Q \in \mathcal{A}$. We claim that

$$\varepsilon^{-1}\left(U^{-1}Q\right) = Q. \tag{6.1}$$

It is clear that $Q \subseteq \varepsilon^{-1}\left(U^{-1}Q\right)$. For the reverse inclusion, take $a \in \varepsilon^{-1}\left(U^{-1}Q\right)$. Then there exist $q \in Q$ and $u \in U$ with

$$\frac{a}{1} = \frac{q}{u},$$

so $u'ua = u'q$ with $u' \in U$. With the definition of $\mathcal{A}$, this implies $a \in Q$, proving (6.1).

We still need to show that $U^{-1}Q$ is a prime ideal. We see that $U^{-1}Q$ is an ideal by Proposition 6.3(h), and it follows from (6.1) that $U^{-1}Q \neq U^{-1}R$. Take $a_1, a_2 \in R$ and $u_1, u_2 \in U$ with

$$\frac{a_1}{u_1} \cdot \frac{a_2}{u_2} \in U^{-1}Q.$$

Then $\varepsilon(a_1a_2) \in U^{-1}Q$, so $a_1a_2 \in Q$ by (6.1). This implies that at least one of the a_i lies in Q, so $\frac{a_i}{u_i} \in U^{-1}Q$, and $U^{-1}Q$ is a prime ideal indeed.

It is immediately clear that our maps preserve inclusions. This completes the proof. □

In Exercise 6.5 it is shown that the bijections from Theorem 6.5 are actually homeomorphisms. Theorem 6.5 has two immediate consequences, Corollaries 6.6 and 6.8.

Corollary 6.6 (Dimension of a localized ring)**.** *Let R be a ring and $U \subseteq R$ a multiplicative set. Then*

$$\dim\left(U^{-1}R\right) \leq \dim(R).$$

Definition 6.7. *A ring R is called a* **local ring** *if it has precisely one maximal ideal.*

Corollary 6.8 (Localizing at a prime ideal gives a local ring)**.** *Let R be a ring and $P \subset R$ a prime ideal. Then the localization R_P is a local ring with P_P as unique maximal ideal.*

Example 6.9. The rings in Example 6.2(1), (3), and (5) are examples of local rings. We give a few more.

(1) Every field is a local ring.
(2) Let $K[x]$ be a polynomial ring over a field. Then $K[x]/(x^2)$ is a local ring with $(x)/(x^2)$ as unique maximal ideal.
(3) The formal power series ring $K[[x]]$ over a field is a local ring with (x) as unique maximal ideal (see Exercise 1.2).
(4) The zero ring $R = \{0\}$ is *not* a local ring. ◁

Definition 6.10. *Let R be a ring.*

(a) Let $P \subset R$ be a prime ideal. Then the **height** *of P is defined as*

$$\mathrm{ht}(P) := \dim(R_P) \in \mathbb{N}_0 \cup \{\infty\}.$$

So by Theorem 6.5, $\mathrm{ht}(P)$ is the maximal length n of a chain

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n = P$$

of prime ideals $P_i \in \mathrm{Spec}(R)$ ending with P.

(b) Let $I \subseteq R$ be an ideal. If $I \neq R$, the **height** *of I is defined as*

$$\mathrm{ht}(I) := \min \left\{ \mathrm{ht}(P) \mid P \in \mathcal{V}_{\mathrm{Spec}(R)}(I) \right\}.$$

If $I = R$, we set

$$\mathrm{ht}(I) := \dim(R) + 1.$$

Since the height, as defined in (a), gets smaller when we pass to a sub–prime ideal, the definitions in (a) and (b) are consistent.

Remark 6.11. (a) If $P \subset R$ is a prime ideal, then Lemma 1.22 tells us that $\dim(R/P)$ is the maximal length of a chain of prime ideals in $\mathrm{Spec}(R)$ *starting* with P. Therefore

$$\mathrm{ht}(P) + \dim(R/P) \leq \dim(R). \tag{6.2}$$

This is often an equality, for example in the case that $R = K[X]$ with X an equidimensional affine variety (see Corollary 8.23). For this reason, some authors use the term *codimension* for the height. Example 6.12(3) shows that the inequality (6.2) can also be strict.

(b) It is not hard to give a geometric interpretation of height. If X is an affine variety over an algebraically closed field, then the prime ideals in the coordinate ring $K[X]$ correspond to the closed, irreducible subsets of X (see Theorem 1.23 and Theorem 3.10(a)). So if $P \in \mathrm{Spec}(K[X])$ corresponds to $Y \subseteq X$, i.e., $Y = \mathcal{V}_X(P)$, then $\mathrm{ht}(P)$ is the maximal length k of a chain

$$Y = Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_k$$

of closed, irreducible subsets of X starting with Y. On the other hand, $\dim(K[X]/P)$ is the maximal length of a chain *ending* with Y. So $\mathrm{ht}(P) + \dim(K[X]/P)$ is the maximal length of a chain passing through Y. ◁

Example 6.12. (1) Every minimal prime ideal has height 0. If R is a Noetherian ring, then $\operatorname{ht}(\{0\}) = 0$. By Exercise 3.6, this is also true for R not Noetherian.

(2) Let K be a field, $(\xi_1, \ldots, \xi_n) \in K^n$, and $P = \mathcal{I}_{K[x_1,\ldots,x_n]}(\{(\xi_1, \ldots, \xi_n)\})$. Then $\operatorname{ht}(P) = n$, since we have a chain of prime ideals

$$\{0\} \subsetneqq (x_1 - \xi_1) \subsetneqq (x_1 - \xi_1, x_2 - \xi_2) \subsetneqq \cdots \subsetneqq (x_1 - \xi_1, \ldots, x_n - \xi_n) = P,$$

and on the other hand $\operatorname{ht}(P) \leq \dim(K[x_1, \ldots, x_n]) = n$ by (6.2).

(3) Let $X = Z_1 \cup Z_2$ be an affine variety over an algebraically closed field with irreducible components Z_1 and Z_2 such that $\dim(Z_1) < \dim(Z_2)$. Let x be a point of Z_1 not lying in Z_2. Then a chain of closed, irreducible subsets of X that starts with $\{x\}$ lies completely in Z_1, so for $P := \mathcal{I}_{K[X]}(x)$ we have

$$\operatorname{ht}(P) \leq \dim(Z_1).$$

(In fact, equality holds as a consequence of Corollary 8.24.) Since $\dim(K[X]/P) = 0$, we have

$$\operatorname{ht}(P) + \dim(K[X]/P) < \dim(K[X]),$$

so the inequality (6.2) is strict here. ◁

We conclude this chapter with a definition.

Definition 6.13. *Let R be a ring and M an R-module.*

(a) For an element $m \in M$, the **annihilator** *of m is*

$$\operatorname{Ann}(m) := \{a \in R \mid a \cdot m = 0\}.$$

This is an ideal in R.

(b) The **annihilator** *of M is*

$$\operatorname{Ann}(M) := \bigcap_{m \in M} \operatorname{Ann}(m) \subseteq R.$$

Clearly one can restrict this intersection to the elements m of a generating set of M.

(c) The **(Krull) dimension** *of M is*

$$\dim(M) := \dim(R/\operatorname{Ann}(M)),$$

where the dimension on the right-hand side denotes the Krull dimension of the ring. Readers should notice that for R a field and M a nonzero vector space, $\dim(M)$ is always 0, so this has nothing to do with dimension as a vector space.

(d) The **support** *of M is*

$$\operatorname{Supp}(M) := \{P \in \operatorname{Spec}(R) \mid M_P \neq \{0\}\} \subseteq \operatorname{Spec}(R).$$

So a $P \in \operatorname{Spec}(R)$ *lies in the support if and only if there exists* $m \in M$ *with* $\operatorname{Ann}(m) \subseteq P$.

Example 6.14. Let $I \subseteq R$ be an ideal in a ring, and consider the quotient ring $M := R/I$ as an R-module. Then it is easy to see that $\operatorname{Ann}(M) = I$ and $\operatorname{Supp}(M) = \mathcal{V}_{\operatorname{Spec}(R)}(I)$. A generalization can be found in Exercise 6.10. ◁

Exercises for Chapter 6

6.1 (**Properties of localization).** Check all assertions made in Definition 6.1 and Proposition 6.3.

6.2 (**Some examples of localization).** In each of the following examples, we give a ring R, a multiplicative subset $U \subseteq R$, and an R-module M. Give a description of the localization $U^{-1}M$. The letter K always stands for a field and x for an indeterminate.

(a) $M = R = K[x]$, $U = \{x^k \mid k \in \mathbb{N}_0\}$.
(b) $M = R = \mathbb{Z}$, $U = \{1\} \cup \{12n \mid n \in \mathbb{Z}, n > 0\}$.
(c) $R = \mathbb{Z}$, $U = \mathbb{Z} \setminus \{0\}$, $M = \mathbb{Z}[x]$.
(d) $R = K[x]$, $M = K[x]/(x^2)$, $U = \{x^k \mid k \in \mathbb{N}_0\}$.
(e) $R = K[x]$, $M = K[x]/(x^2)$, $U = K[x] \setminus (x)$.

6.3 (**Localization is an exact functor).** Let R be a ring and $U \subseteq R$ a multiplicative set. Let $\varphi\colon M \to N$ be a homomorphism of R-modules. Show that the map

$$U^{-1}\varphi\colon U^{-1}M \to U^{-1}N,\ \frac{m}{u} \mapsto \frac{\varphi(m)}{u},$$

is a homomorphism of $U^{-1}R$-modules. (Since passing from φ to $U^{-1}\varphi$ is compatible with composition of homomorphisms, this makes localization with respect to U into a functor from the category of R-modules to the category of $U^{-1}R$-modules.)

By an *exact sequence of R-modules*, we mean a sequence

$$\cdots \xrightarrow{\varphi_{-2}} M_{-1} \xrightarrow{\varphi_{-1}} M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} \cdots \tag{6.3}$$

with M_i modules over R and $\varphi_i\colon M_i \to M_{i+1}$ module homomorphisms such that $\operatorname{im}(\varphi_i) = \ker(\varphi_{i+1})$ for all $i \in \mathbb{Z}$. More formally, a sequence is a direct sum $M = \bigoplus_{i\in\mathbb{Z}} M_i$ of R-modules together with a homomorphism $\varphi\colon M \to M$ such that $\varphi(M_i) \subseteq M_{i+1}$ for all i, and exactness means that $\operatorname{im}(\varphi) = \ker(\varphi)$. Assume that the sequence (6.3) is exact and show that the localized sequence

$$\cdots \longrightarrow U^{-1}M_0 \xrightarrow{U^{-1}\varphi_0} U^{-1}M_1 \xrightarrow{U^{-1}\varphi_1} U^{-1}M_2 \longrightarrow \cdots \tag{6.4}$$

is also exact. We express this by saying that localization is an exact functor. *Remark:* Most exact sequences appearing in the real life of a mathematician are finite, meaning that only finitely many M_i are nonzero. The most frequent example is a "short exact sequence," in which all M_i except M_1, M_2, and M_3 are zero. In that case, exactness implies $M_3 \cong M_2/M_1$.

A consequence of this exercise is that injective (surjective) homomorphisms localize to injective (surjective) maps.

6.4 (Local–global principle). A *local–global principle* is a theorem that states that some property holds "globally" if and only if it holds everywhere "locally." Here are two examples.

(a) Let M be a module over a ring R with submodules $L, N \subseteq M$. Prove the equivalence

$$L \subseteq N \quad \Longleftrightarrow \quad L_{\mathfrak{m}} \subseteq N_{\mathfrak{m}} \quad \text{for all} \quad \mathfrak{m} \in \operatorname{Spec}_{\max}(R).$$

(b) Let $\varphi\colon M \to N$ be a homomorphism of modules over a ring R. For $\mathfrak{m} \in \operatorname{Spec}_{\max}(R)$ there is a homomorphism $\varphi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ as defined in Exercise 6.3. Show that φ is injective or surjective if and only if the same property holds for every $\varphi_{\mathfrak{m}}$ with $\mathfrak{m} \in \operatorname{Spec}_{\max}(R)$.

6.5 (Homeomorphisms). Show that the bijections from Theorem 6.5 are homeomorphisms. (Here $\mathcal{A}$ is equipped with the subset topology induced from the Zariski topology on $\operatorname{Spec}(R)$.)

6.6 (Localization hides components). Let $X = Y_1 \cup Y_2$ be an affine variety over a field K, decomposed as a union of two closed subsets. Let $x \in Y_1 \setminus Y_2$ be a point. Show that the restriction homomorphism $\varphi\colon K[X] \to K[Y_1]$ induces an isomorphism (of K-algebras)

$$\varphi_x\colon K[X]_x \to K[Y_1]_x, \ \frac{f}{u} \mapsto \frac{\varphi(f)}{\varphi(u)},$$

of the coordinate rings localized at x.
Remark: This result may be expressed thus: "Localization at x sees only those components in which x lies" – a further hint that localization has something to do with locality.

6.7 (A characterization of local rings). Let R be a ring. Prove the following.

(a) R is local if and only if the set of noninvertible elements of R is an ideal. Then this is the unique maximal ideal.
(b) Let $\mathfrak{m} \subsetneqq R$ be a proper ideal. Then R is local with $\mathfrak{m}$ as maximal ideal if and only if all elements from $R \setminus \mathfrak{m}$ are invertible.

6.8 (An alternative definition of Krull dimension). In this exercise we develop a prime-ideal-free definition of the Krull dimension of a ring. It is based on an article by Coquand and Lombardi [11], which was brought to my attention by Peter Heinig.

Let R be a ring. For $a \in R$, define the multiplicative set

$$U_a := \{a^m(1 + xa) \mid m \in \mathbb{N}_0,\ x \in R\}.$$

(a) For $a \in R$ and $P \in \operatorname{Spec}(R)$ a prime ideal, prove the following equivalence:

$$U_a \cap P = \emptyset \iff a \notin P \quad \text{and} \quad P + (a)_R \neq R.$$

(b) For $n \in \mathbb{N}_0$ a nonnegative integer, prove the following equivalence:

$$\dim(R) \leq n \iff \dim\left(U_a^{-1}R\right) \leq n - 1 \quad \text{for all} \quad a \in R.$$

(c) For $n \in \mathbb{N}_0$ a nonnegative integer, show that $\dim(R) \leq n$ holds if and only if for every $a_0, \ldots, a_n \in R$ there exist $m_0, \ldots, m_n \in \mathbb{N}_0$ such that

$$\prod_{i=0}^{n} a_i^{m_i} \in \left(a_j \cdot \prod_{i=0}^{j} a_i^{m_i} \,\middle|\, j = 0, \ldots, n\right)_R. \tag{6.5}$$

Remark: Part (c) provides the desired alternative definition of the Krull dimension. The condition (6.5) looks a bit messy at first glance, but it is easy to understand and to remember in terms of the lexicographic monomial ordering, which we will introduce in Example 9.2(1) on page 119. In fact, (c) says that $\dim(R) \leq n$ if and only if for every $a_0, \ldots, a_n \in R$ there exists a monomial in the a_i that can be written as an R-linear combination of lexicographically *larger* monomials in the a_i. As a nice application, it is easy to derive Theorem 5.5 and the first part of Corollary 5.7 from (c) using the lexicographic ordering (see Exercise 9.5). *(Solution on page 219)*

6.9 (Localizing an affine domain). Let A be an affine domain and $a \in A \setminus \{0\}$. Show that the localization A_a has the same dimension as A. Does this remain true for A an affine algebra or A an integral domain? Does it remain true if one localizes with respect to an arbitrary multiplicative subset $U \subseteq A \setminus \{0\}$?

6.10 (Support of modules). Let R be a ring and M an R-module.

(a) Assume that M is finitely generated and show that

$$\operatorname{Supp}(M) = \mathcal{V}_{\operatorname{Spec}(R)}(\operatorname{Ann}(M)).$$

In particular, $\operatorname{Supp}(M)$ is Zariski closed in $\operatorname{Spec}(R)$.

*(b) Give an example in which $\operatorname{Supp}(M)$ is not Zariski closed.

6.11 (Associated primes). Let R be a Noetherian ring and M an R-module. A prime ideal $P \in \operatorname{Spec}(R)$ is called an **associated prime** of M if there exists $m \in M$ with $P = \operatorname{Ann}(m)$. (But notice that not all annihilators of elements of M are prime ideals!) We write the set of all associated primes as $\operatorname{Ass}(M)$.

(a) Let I be an ideal that is maximal among all $\operatorname{Ann}(m)$ with $m \in M \setminus \{0\}$. Show that $I \in \operatorname{Ass}(M)$. So in particular $\operatorname{Ass}(M) \neq \emptyset$ if $M \neq \{0\}$.
(b) Let $U \subseteq R$ be a multiplicative subset and consider the R-module $U^{-1}M$. Show that

$$\operatorname{Ass}(U^{-1}M) = \{P \in \operatorname{Ass}(M) \mid U \cap P = \emptyset\}.$$

(c) Consider the special case $M = R/I$ with I a radical ideal. Show that $\operatorname{Ass}(M)$ is the set of all prime ideals that are minimal over I.
(d) Let $R = K[x_1, x_2]$ be a polynomial ring over a field and $M = R/I$ with $I := (x_1^2, x_1x_2)$. Determine $\operatorname{Ass}(M)$. Does the conclusion of part (c) hold?

Remark: Part (c) suggests that associated primes may be seen as a generalization of irreducible components. The theory of associated primes and primary decomposition is treated in most textbooks on commutative algebra, but not in this one.