Introduction

How To Use This Book

The main intention of this book is to provide material for an introductory graduate course of one or two semesters. The duration of the course clearly depends on such parameters as speed and teaching hours per week and on how much material is covered. In the book, I have indicated three options for skipping material. For example, one possibility is to omit Chapter 10 and most of Section 7.2. Another is to skip Chapters 9–11 almost entirely. But apart from these options, interdependencies in the text are close enough to make it hard to skip material without tearing holes into proofs that come later. So the instructor can best limit the amount of material by choosing where to stop. A relatively short course would stop after Chapter 8, while other natural stopping points are after Chapters 11 or 13.

The book contains a total of 143 exercises. Some of them deal with examples that illustrate definitions (such as an example of an Artinian module that is not Noetherian) or shed some light on the necessity of hypotheses of theorems (such as an example in which the principal ideal theorem fails for a non-Noetherian ring). Others give extensions to the theory (such as a series of exercises that deal with formal power series rings), and yet others invite readers to do computations on examples. These examples often come from geometry and also serve to illustrate the theory (such as examples of desingularization of curves). Some exercises depend on others, as is usually indicated in the hints for the exercise. However, no theorem, proposition, lemma, or corollary in the text depends on results from the exercises. I put a star by some exercises to indicate that I consider them more difficult. Solutions to all exercises are collected in a solutions manual, which is available for instructors.

Although the ideal way of using the book is to read it from beginning to end (every author desires such readers!), an extensive subject index should facilitate a less linear navigation.

G. Kemper, *A Course in Commutative Algebra*, Graduate Texts in Mathematics 256, DOI 10.1007/978-3-642-03545-6_1,

Prerequisites

Readers should have taken undergraduate courses in linear algebra and abstract algebra. Everything that is assumed is contained in Lang's book [33], but certainly not everything in that book is assumed. Specifically, readers should have a grasp of the following subjects:

- definition of a (commutative) ring,
- ideals, prime ideals, and maximal ideals,
- zero divisors,
- quotient rings (also known as factor rings),
- subrings and homomorphisms of rings,
- principal ideal domains,
- factorial rings (also known as unique factorization domains),
- polynomial rings in several indeterminates,
- finite field extensions, and
- algebraically closed fields.

In accordance with the geometric viewpoint of this book, it sometimes uses language from topology. Specifically, readers should know the definitions of the following terms:

- topological space,
- closure of a set,
- subspace topology, and
- continuous map.

All these can be found in any textbook on topology, for example Bourbaki [6].

Contents

The first four chapters of the book have a common theme: building the "Algebra–Geometry Lexicon," a machine that translates geometric notions into algebraic ones and vice versa. The opening chapter deals with Hilbert's Nullstellensatz, which translates between ideals of a polynomial ring and affine varieties. The second chapter is about the basic theory of Noetherian rings and modules. One result is Hilbert's basis theorem, which says that every ideal in a polynomial ring over a field is finitely generated. The results from Chapter 2 are used in Chapter 3 to prove that affine varieties are made up of finitely many irreducible components. That chapter also introduces the Zariski topology, another important element of our lexicon, and the notion of the spectrum of a ring, which allows us to interpret prime ideals as generalized points in a more abstract variant of geometry. Chapter 4 provides a summary of the lexicon.

In any mathematical theory connected with geometry, dimension is a central, but often subtle, notion. The four chapters making up the second part of the book relate to this notion. In commutative algebra, dimension is defined by the Krull dimension, which is introduced in Chapter 5. The main result of the chapter is that the dimension of an affine algebra coincides with its transcendence degree. Chapter 6 is an interlude introducing an important construction that is used throughout the book: localization. Along the way, the notions of local rings and height are introduced. Chapter 6 sets up the conceptual framework for proving Krull's principal ideal theorem in Chapter 7. That chapter also contains an investigation of fibers of morphisms, which leads to the nice result that forming a polynomial ring over a Noetherian ring increases the dimension by 1. Chapter 8 discusses the notions of integral ring extensions and normal rings. One of the main results is the Noether normalization theorem, which is then used to prove that all maximal chains of prime ideals in an affine domain have the same length.

The third part of the book is devoted to computational methods. Theoretical and algorithmic aspects go hand in hand in this part. The main computational tool is Buchberger's algorithm for calculating Gröbner bases, which is developed in Chapter 9. As a first application, Gröbner bases are applied to compute elimination ideals, which have important geometric interpretations. Chapter 10, the second chapter of this part, continues the investigation of fibers of morphisms started in Chapter 7. This chapter contains a constructive version of Grothendieck's generic freeness lemma. This is one of the main ingredients of an algorithm for computing the image of a morphism of affine varieties, probably a novelty. The chapter also contains Chevalley's result that the image of a morphism is a constructible set. The results of Chapter 10 are not used elsewhere in the book, so there is an option to skip that chapter and the parts of Chapter 7 that deal with fibers of morphisms. Finally, Chapter 11 deals with the Hilbert function and Hilbert series of an ideal in a polynomial ring. The main result, whose proof makes use of Noether normalization, states that the Hilbert function is eventually represented by a polynomial whose degree is the dimension of the affine algebra given by the ideal. This result leads to an algorithm for computing the dimension of an affine algebra, and it also plays an important role in Chapter 12 (which belongs to the fourth part of the book). Nevertheless, it is possible to skip the third part of the book almost entirely by modifying some parts of the text, as indicated in an exercise.

The fourth and last part of the book deals with local rings. Geometrically, local rings relate to local properties of varieties. Chapter 12 introduces the associated graded ring and presents a new characterization of the dimension of a local ring. Chapter 13 studies regular local rings, which correspond to nonsingular points of a variety. An important result is the Jacobian criterion for calculating the singular locus of an affine variety. A consequence is that an affine variety is nonsingular almost everywhere. The final chapter deals with topics related to rings of dimension one. The starting point is the observation that a Noetherian local ring of dimension one is regular if and only if it is normal. From this it follows that affine curves can be desingularized. After an excursion to multiplicative ideal theory for more general rings, the attention is focused to Dedekind domains, which are characterized as "rings with a perfect multiplicative ideal theory." The chapter closes with an application that explains how the group law on an elliptic curve can be defined by means of multiplicative ideal theory.

Further Reading

The contents of a book may also be described by what is missing. Since this book is relatively short and concentrates on the central issues, it pays a price in comprehensiveness. Homological concepts and methods should probably appear at the top of the list of what is missing. In particular, the book does not treat syzygies, resolutions, and Tor and Ext functors. As a consequence, depth and the Cohen–Macaulay property cannot be dealt with sensibly (and would require much more space in any case), so only one exercise touches on Cohen–Macaulay rings. Flat modules are another topic that relates to homological methods and is not treated. The subject of completion is also just touched on. I have decided not to include associated primes and primary decomposition in the book, although these topics are often regarded as rather basic and central, because they are not needed elsewhere in the book.

All the topics mentioned above are covered in the books by Matsumura [37] and Eisenbud [17], which I warmly recommend for further reading. Of these books, [37] presents the material in a more condensed way, while [17] shares the approach of this book in its focus on the geometric context and in its inclusion of Gröbner basis methods. Eisenbud's book, more than twice as large as this one, is remarkable because it works as a textbook but also contains a lot of material that appeals to experts.

Apart from deepening their knowledge in commutative algebra, readers of this book may continue their studies in different directions. One is algebraic geometry. Hartshorne's textbook [26] still seems to be the authoritative source on the subject, but Harris [25] and Smith et al. [47] (to name just two) provide more recent alternatives. Another possible direction to go in is computational commutative algebra. A list of textbooks on this appears at the beginning of Chapter 9 of this book. I especially recommend the book by Cox et al. [12], which does a remarkable job of blending aspects of geometry, algebra, and computation.