Chapter 6

Stability, Sensitivity and Robustness

Stability is desirable,
Sensitivity is a sign of responsiveness,
Robustness provides a feeling of safety
but great stability, low sensitivity and extreme robustness
will make for a boring life!

6.1 Introduction and Motivation

We are interested in the dynamic behavior of systems, more particularly we are interested in changing the behavior of a system, but before we attempt to synthesize desired behavior it pays to analyze system properties.

One way that changes can be effected is by tuning some parameters, like changing a gain. Another more interesting mechanism through which we may change the overall behavior is by interconnecting the system of interest with another system, which we will often refer to as a control system; the interconnected system becomes a controlled system. Our analysis will need to cater for both mechanisms.

This chapter is devoted to the notions of *stability*, *sensitivity* and *robustness*. Stability plays an important role in the study of system dynamics. It captures the idea that *small causes have small consequences forever*. This is obviously a desirable property in particular when we want to predict future behavior of a system with a degree of certainty and accuracy. All *controlled* systems should be stable.

Of course, our notion of small has to be made precise, and how we determine the size of signals and systems plays an important role in defining what we mean by stability. Also we may expect many different notions of stability to distinguish between the different types of cause and effect we are interested in. As usual, we will only consider a few of the options.

Most important to the notion of stability is the fact that it relates to how the system evolves over time into the indefinite future. Simply requiring that small causes have small effects is called *continuity*, and more often than not we will take this property for granted.

Stability is first a qualitative property, in design however it is important to be able to quantify how stable a system is. A measure of stability, like the range of parameters we can accept in a system without losing stability, is critically important in any design problem. To this end, in systems' analysis the notions of sensitivity and robustness are used.

Sensitivity measures how important signals, like the input and output of a system, depend on other external signals affecting the system, like measurement noise or actuator disturbances. If a small noise signal can change the input to the plant significantly, we say that the input is very sensitive to this noise, a highly undesirable situation.

Robustness indicates how changes in the parameters or the environment of a system may affect the system behavior. Typically we like the response of a system not to vary too much despite the fact that the system may undergo some changes.

Most natural and engineered systems behave in a stable manner and do not rely on control for the purpose of stability. There are important exceptions though and some

P. Albertos et al., *Feedback and Control for Everyone*

examples are presented in the next section. In any case, control will be called for if the measure of stability, as also reflected in sensitivity and robustness properties, does not meet expectations.

This chapter is organized as follows. First we look at some simple examples, to motivate and to set the scene somewhat. Then we look at the notion of stability, from its mechanical origins' point of view, for systems without inputs, so-called *autonomous systems*. From a control point of view systems without inputs are rather boring, but they form a good place to start. Next we provide some more detail for linear systems. Then we consider stability from a general system dynamics perspective, particularly dealing with inputs. The ideas of robustness and sensitivity come natural within this context.

6.2 Some Examples

By way of setting the scene, consider a direct current electrical motor, as shown in Fig. 6.1.

Whenever we apply a voltage to the armature, E_a, after some *transient* period, the motor will reach a stable shaft speed, ω. For each applied voltage, a particular motor torque results, P_m. Depending on the mechanical load, L, a particular final speed is attained. From this point of view the motor behaves in a stable manner.

One notion of sensitivity captures how big a voltage change is required to effect a particular change in rotation speed. Sensitivity is high when small changes in the input voltage produce corresponding and large variations in the output speed. In the same context a notion of robustness captures how big a change in speed is observed for a particular change in mechanical load on the motor axle. More robustness would mean that a large change in mechanical load only results in a minor variation of the speed, a very desirable property in many applications. This would be the case for a highly geared motor, where the motor speed is much higher than the output shaft speed (see for example Sect. 3.4).

Any description of stability, sensitivity and robustness in this context would only apply within what are called the operational limits for the electrical motor. For example due to *saturation* effects in the magnetic material used in the motor construction there exists a voltage level beyond which a further increase no longer yields an

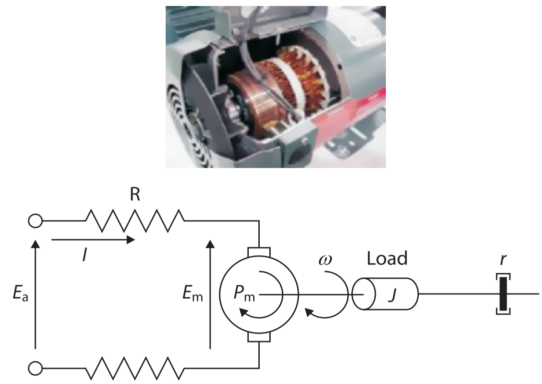

Fig. 6.1. A DC motor

increase in motor speed. Even worse, there is a voltage beyond which the motor will be irreparably damaged, and start to smell unpleasantly. Normally system models for control purposes will only describe the system of interest, here the electrical motor, for signals well within these operational limits. In other words, the model's validity (and consequently the usefulness of any notions such as stability, sensitivity and robustness) will break down well before the system does.

Not all systems are stable. There are inherently unstable systems. A familiar example of a mechanically unstable system that needs careful control to function properly is our body. We need control to keep our balance and our posture whilst walking, standing even sitting. In order to walk, we all must learn how to keep our body, in particular our head, in an upright position above our feet. Mechanically (in the presence of gravity) our upright position is somewhat unnatural, it is an *unstable* position. On all fours we are rather more stable, but far less mobile. Mobility and stability always require a trade-off.

A similar situation, not so easily managed, is to balance a slim long rod upright in the palm of your hand, as the general picture of an inverted pendulum (Fig. 6.2)[1] shows. Exactly the same problem occurs when launching a rocket for say a space shuttle. The long rocket body must be balanced against gravity using a thrust force at the bottom of the rocket. This requires careful control of the orientation of this thrust force.

In some chemical processes an unstable equilibrium (that is without control it is unstable) is desired because the yield is much higher at this equilibrium as compared to the possible naturally stable equilibria. Similarly, in a mechanical system, an instability (in the uncontrolled system of course) may lead to high maneuverability and hence be desirable.

The consequences of letting an unstable process evolve unchecked can be disastrous, as is well-known to us from the Chernobyl nuclear power plant's catastrophic accident. Graphite moderated nuclear fission reactions are unstable at low power levels, and stable at high power output, where they should be operated. The lack of proper control and sufficient fail safe mechanisms in the low power regime in the Chernobyl nuclear reactor caused a massive explosion in April 1986.

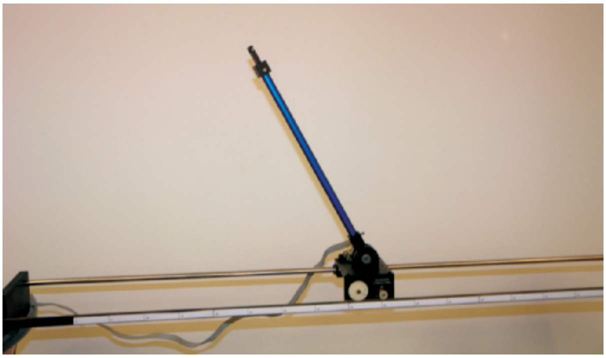

Fig. 6.2. An inverted pendulum

[1] Taken from Quanser®.

6.3 Stability of Autonomous Systems

The concept of stability was first coined in mechanics, which is the study of motion of objects in three-dimensional space (like robotics or the motion of satellites).

Stability is a local property attached to a particular response (motion) in a system. What happens to the system response when at a particular instance in time a small deviation (in position or velocity) is introduced? Is the subsequent evolution close to the unperturbed response? If it is, we say the response (motion) is stable, if not the response (motion) is unstable.

More often than not we discuss this property only for very specific system responses. Sometimes, when there is no room for confusion, we may say that the system is stable, meaning that any response in the system has this property.

Consider the example of the motion of a ball in some undulating terrain, subject to gravity and rolling friction. The situation is somewhat abstractly represented in Fig. 6.3. The *system* here is the ball rolling over the surface subject to gravity. If we assume that the ball is always in contact with the surface, it is fully characterized by its longitudinal position and speed.

Intuitively it is clear that the lowest position in the valley, **a** in Fig. 6.3 is a special point. It is called an *equilibrium point*. If the ball is there initially at rest (that is without velocity), and no external force, other than gravity and friction are applied (which implies that the ball's acceleration in the horizontal direction is zero), it will remain there forever as predicted by Newton's laws. A small perturbation away from this point (either in position, or by giving a small push, or by providing an initial horizontal velocity) is not going to cause much concern, the ball will remain in the neighborhood of the equilibrium **a**, or its motion will be close to the equilibrium at all times in the future. This equilibrium is called *stable*.

The position **c**, on top of a hill, is also an equilibrium point but it is rather different from the equilibrium **a** in the valley. A small perturbation (either by a non-zero horizontal velocity or a small horizontal force) away from the position **c** will result in the ball running down, away from **c**. Moreover the ball will not return back to **c**. We capture this, by saying that the equilibrium **c** is *unstable*.

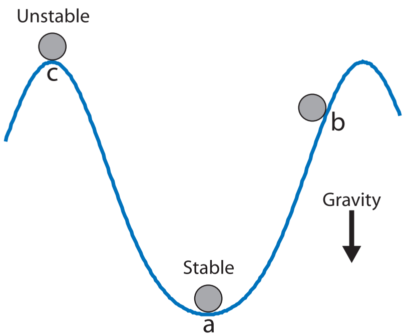

Fig. 6.3. A ball, subject to gravity (assumed directed from top to bottom of the page), on an undulating surface

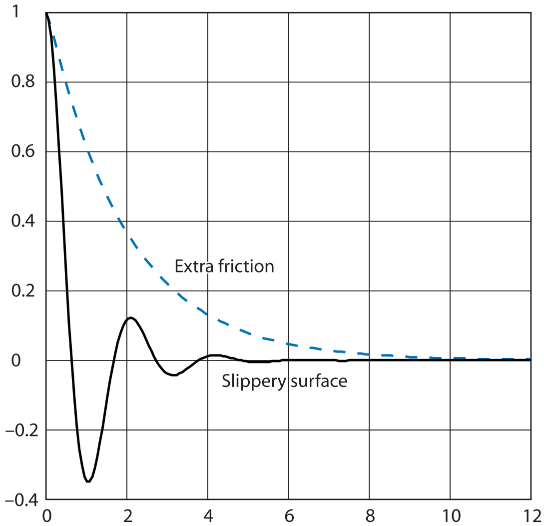

Fig. 6.4. Trajectories starting from the same point, with different friction

When the ball is released from an initial position **b**, with zero initial velocity, the ball will roll down under the influence of gravity and start oscillating back and forth around the position **a**. Because of friction the ball will eventually settle at the point **a**. We summarize this as the point **a** is a *stable and attracting* equilibrium point. It is also clear that the equilibrium **a** is only locally stable. Indeed if we give the ball a sufficiently large initial velocity it will be able to pass the position **c**, never to return.

The trajectory of the ball, projected on the x-axis, that is, on the horizontal, will strongly depend on the friction coefficient between the ball and the surface it rolls over. The trajectory may be oscillatory, if there is little friction to damp the oscillations, or without oscillations if the ball is loosing much energy as it moves, as pointed out in Fig. 6.4.

Stability (instability) is a local property, a qualitative descriptor for the behavior of the motion of the ball near a particular motion (here we used equilibria). It considers the motion of the ball in the future, so from the time point of view it is a global property, that is what makes it so special. That stability is a local in the motion space is quite clear from the example, as the ball has multiple equilibria with different stability properties.

The stability property that the equilibrium **a** possesses is very strong. First, we may perturb the starting point, or provide a small initial velocity, or a small push to any trajectory in a neighborhood of **a**, and any such generated trajectory or motion will remain close to **a** and eventually will converge to **a**. It is said that the equilibrium is *asymptotically stable*. Asymptotic stability requires that the motion is not only stable, but that is also attractive, nearby motions converge to the motion under consideration.

Stability of an Autonomous System

Consider a dynamic system as described by the collection of responses from all possible initial conditions.

An equilibrium is a constant response.

We say that the equilibrium is *stable* if for any initial condition close to the equilibrium, the corresponding system trajectory remains close to it for all future time. Otherwise, we say that the equilibrium is unstable.

We say that the equilibrium is *attractive* if for any initial condition close to it the corresponding system trajectory converges to it as time progresses.

We say that the equilibrium is *asymptotically stable* if it is both stable and attractive.

We say that the properties hold globally if there is no restriction on the initial condition.

A *trajectory* (system response) is *stable* if small initial deviations away lead to small deviations in the indefinite future. It is asymptotically stable if it is stable with the added property that the future deviations become vanishingly small.

6.4 Linear Autonomous Systems

In the case the dynamics are linear, the question of stability can be settled using linear algebra. Most stability questions, including synthesis questions, are computationally tractable, even if the state dimension is large. Substantial computer-aided design software tools are available.

Though few systems are linear, most systems allow for a linear system that approximates it well enough over the operational range of interest to test for stability even in the nonlinear situation. After all stability is a local property. This is in a nutshell the content of the famous Hartman-Grobman theorem in dynamical systems. Moreover, in a control context, the whole aim is typically to model for control, and most likely control requires that the controlled system exhibits a desirable behavior. It is clear that good performance would mean that the controlled system should be close to this desired behavior. The deviations away from this desired behavior can often be well approximated by a linear system.

Consider the simplest autonomous dynamic system model

$$x(k+1) = ax(k), \quad k = 0, 1, 2, \ldots \tag{6.1}$$

where $x(k)$ and a are scalars. a is called the **pole** or also *eigenvalue* of the System 6.1. It is easy to compute the collection of all future solutions, by simply iterating (6.1):

$$\{x(k) = x_0 a^k \quad k = 0, 1, 2, \ldots\}$$

where x_0 is the initial condition $x(0) = x_0$.

It is also clear that $x_0 = 0$ is an equilibrium.

Clearly this equilibrium is globally asymptotically stable provided $|a| < 1$, the pole must be less than one in magnitude. Indeed, under this condition all solution stay close to the equilibrium and converge to it as time progresses.

With $a = 1$ or $a = -1$ the equilibrium is stable, but not attractive. For $a = 1$ all solutions remain constant, so what starts close remains close, but clearly no solution gets closer to another. For $a = -1$ all solutions, apart from the equilibrium are two-periodic: $x(0) = x_0$, $x(1) = -x_0$ and $x(2) = x_0$ and so on; clearly the equilibrium is stable but solutions are not attracted towards the origin.

If $a > 1$ or $a < -1$, the equilibrium is unstable, and all solutions (apart from the solution that stays at the equilibrium) diverge.

Using the Z-transform, the System 6.1 can be rewritten as

$$zX(z) - x_0 = X(z), \quad \text{or} \quad X(z) = \frac{1}{z-a}x_0 \tag{6.2}$$

The system operator[2] $1/(z-a)$ captures the stability of the system.

6.4.1 General Linear Autonomous Systems, Discrete Time[3]

General linear systems in discrete time can be decomposed in a set of first order equations, like Eq. 6.1, with variables x_i for $i = 1, 2, \ldots, n$. Obviously, in this case the system will have n poles or eigenvalues.

Another model for a general linear discrete time system, without input, as introduced in Eq. 5.16 is

$$x(k+1) = Ax(k), \quad k = 1, 2, \ldots \quad x(0) = x_0 \tag{6.3}$$

where $x(k)$ is the state vector at time k, and A is called the system matrix, or state transition matrix.

The stability of the equilibrium $x = 0$ can be settled by considering the n eigenvalues of the matrix A.[4] It can be established that stability is guaranteed when all eigenvalues of the matrix A have a magnitude less than 1.

This is illustrated in Fig. 6.5, which shows a number of eigenvalue locations, with the associated time functions, clearly delineating stable and unstable behavior.

Another representation of a linear system is through the linear difference equation:

$$y(k+n) + \alpha_1 y(k+n-1) + \ldots + \alpha_n y(k) = 0 \tag{6.4}$$

or, by means of the Z-transform

$$(z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n)y(z) = 0 \tag{6.5}$$

the stability is determined by the roots, a_i, also called *poles*, of the *characteristic equation*, namely

$$z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n = \prod_{i=1}^{n}(z - a_i) = 0 \tag{6.6}$$

[2] The Z-transform of the sequence $x(k)$ is by definition given by $\Sigma_{k=1}^{\infty} x(k)z^{k-1}$. Using the explicit expression for the solution $x(k)$ we obtain $\Sigma_{k=1}^{\infty} a^{k-1}x_0 z^{k-1} = 1/(z-a)x_0$.

[3] This section may be skipped in a first reading.

[4] A scalar λ is an eigenvalue of the matrix A if there exists non zero ξ such that $\lambda\xi = A\xi$. Eigenvalues can be complex numbers. Selecting ξ as an initial condition, we obtain the solution $x(k) = \lambda^{k-1}\xi$. Clearly the equilibrium cannot be stable as soon as there is an eigenvalue such that $|\lambda| > 1$.

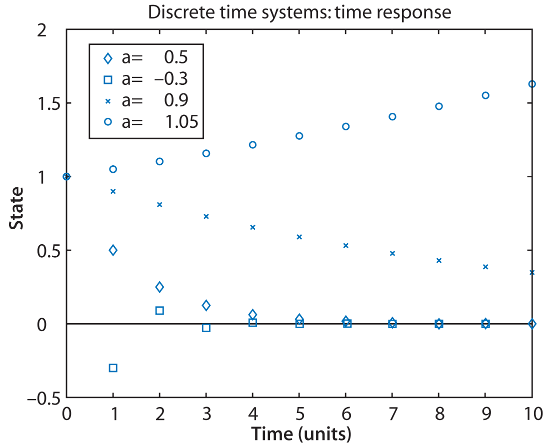

Fig. 6.5. Stability of linear systems in discrete time can be verified by the location of the eigenvalues of system matrix A

The system is stable if $|a_i| < 1, \forall i$. The system is stable if the absolute value of each pole of the system is less than one. These poles are complex numbers and, stability follows if all the poles are located inside the unit circle in the complex plane.

6.4.2 Continuous Time, Linear System

If a continuous time first order autonomous (without input) system is considered, its model, similar to Eq. 6.1, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \dot{x}(t) = ax(t), \quad x(0) = x_0 \tag{6.7}$$

This model was already introduced in Eq. 4.3, when dealing with exponential signals, and we know that the solution of this equation, the trajectories of the system from a given initial condition x_0, is

$$x(t) = \mathrm{e}^{at}x_0$$

It is clear that $x_0 = 0$ is an equilibrium point and that in order for this equilibrium to be (globally, asymptotically) stable the pole, a, must be negative. Otherwise, the solutions will exponentially diverge.

Using the Laplace transform[5], the System 6.7 can be rewritten as

$$sX(s) - x_0 = X(s), \quad \text{or} \quad X(s) = \frac{1}{s-a}x_0 \tag{6.8}$$

Again, the system operator $1/(s-a)$ captures the stability of the system.

[5] See Laplace transform property (Eq. 4.12).

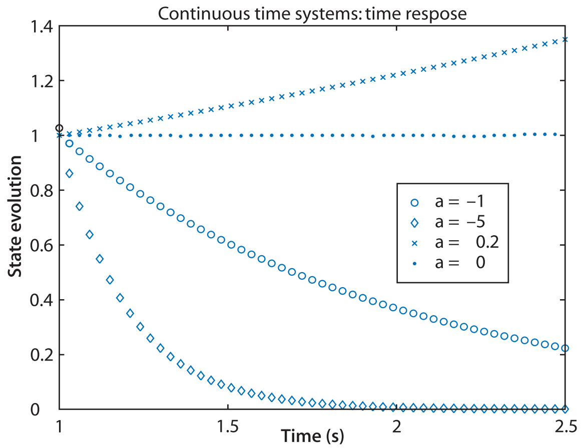

Fig. 6.6. Stability of linear systems in continuous time can be verified by the location of the eigenvalues of system matrix A

For general linear systems in continuous time, the model can be also decomposed in a set of first order equations, like Eq. 6.7, and the set of coefficients a_i, called poles or eigenvalues of the system, will determine the stability condition of the system. The s-operator in this case is a composition of terms $1/(s - a_i)$, globally expressed as a rational function which denominator is $(s - a_1)(s - a_2) \ldots (s - a_n)$, is denoted as the characteristic polynomial of the system.

The system operator captures the stability of the system as well and hence the system is stable provided the poles of the transfer operator are strictly in the left half of the complex plane (the real part is negative). This is illustrated in Fig. 6.6, which shows a number of eigenvalue locations, with the associated time functions, clearly delineating stable and unstable behavior.

6.4.3 Exploring Stability

In designing control systems the minimal property to establish is stability. It is therefore important to understand how the stability property depends on system parameters. Is stability a sensitive property? Can small variations in parameters cause system stability to change? The example of the ball in the valley indicated that this may not be the case. On the other hand, if a system is unstable, the main goal of control is to be able to change instability into stability (that is we must be able to create a valley where there used to be a crest). There must be a transition from stable to unstable somewhere as parameters vary. How this happens, and what actually happens in the behavior of a particular system as parameters in the system vary belongs to the realm of *bifurcation theory*. In great generality it can be shown that system behavior varies typically smoothly with parameter changes, apart from some singular combinations of parameters. Around such *bifurcation* points, qualitative properties such as stability may change abruptly, because equilibria are created or disappear or interact in some manner.

An Example

We go back to the example of the tank that we considered in Chap. 2, see in particular also the Sect. 2.4.4.

Consider a water tank, described as follows

$$x(k+1) = x(k) - c(k) + u(k); \quad u(k) = \alpha(x(k) - r) \tag{6.9}$$

where

- $x(k)$ is the water volume in the tank at the k^{th} instant of observation,
- $c(k) \geq 0$ is the total volume of water removed from the tank over the period between the k^{th} and $k+1^{\text{st}}$ observation. It is assumed to be independent of the tank volume,
- $u(k)$ is a volume of water supplied or extracted over the same period,
- $r > 0$ is the desired water volume in the tank,
- α is a *control* parameter adjusting how much water is taken from or added to the tank[6].

Assume for the moment that the volume of water taken from the tank over a sample period is constant $c(k) = c > 0$ and that the desired water volume r is also constant. We are going to analyze the behavior of the tank based on the control parameter α. How does it influence the dynamics of the system?

It will be useful to rewrite the system description in the following manner:

$$x(k+1) = (1+\alpha)x(k) - c - \alpha r \tag{6.10}$$

Equilibrium Point

First there is an equilibrium point x_e, which follows from:

$$x_e = x_e - c + \alpha(x_e - r), \quad \text{or} \quad x_e = \frac{c}{\alpha} + r \tag{6.11}$$

The physical interpretation is that, at equilibrium, the water added to or taken from the tank through the control term $u(k) = \alpha(x_e - r)$ is constant, and exactly compensates for the water extracted c. Under this condition, the tank's water volume stays constant. Notice that this immediately implies that the equilibrium water level cannot be equal to the desired level (Eq. 6.11). The control term $u(k)$, based on feedback (observation of the tank level, relative to the desired level), needs an error $x_e - r \neq 0$ in order to be able to take a corrective action.

The System 6.10 can be rewritten in yet another equivalent format:

$$(x(k+1) - x_e) = (1+\alpha)(x(k) - x_e) \tag{6.12}$$

showing the evolution of the difference between the tank volume and the equilibrium point.

[6] This kind of control is denoted as proportional control, as described later in Chap. 10, the volume of water added is proportional to how much the tank volume deviates from the desired volume.

Design Question

The design question is "When is the equilibrium stable?". How do we need to select the parameter α to ensure stability, as well as an equilibrium that is close to the desired level[7]?

The equilibrium x_e is to be close to the target level $r > 0$. This requires that a large value for α (either positive or negative) is the way to go. But, to be stable, the pole of the linear system, (Eq. 6.12), $1 + \alpha$ should be less than one in magnitude, which severely limits the range of acceptable α

$$0 > \alpha > -2$$

(see the previous Sect. 6.4). It follows that stability requirement implies negative feedback $\alpha < 0$. If the water fill volume is too high, i.e. $x(k) - r > 0$, water is taken from the tank by the control action $u(k)$ as $\alpha < 0$.

It is interesting to visualize the evolution of the water volume by plotting it in the plane whose horizontal axis represents the current state, and the vertical axis the next state. Equation 6.12 is handy for this purpose.

A very special case, described as *dead-beat* control, happens when $\alpha = -1$. The equilibrium is reached after the first control action.

In Fig. 6.7, the trajectory of the water level is traced for $\alpha = -1.5$. Similar graphs can be plotted for all α, changing α is reflected in the picture through a changed slope for the off-diagonal line (do that as an exercise).

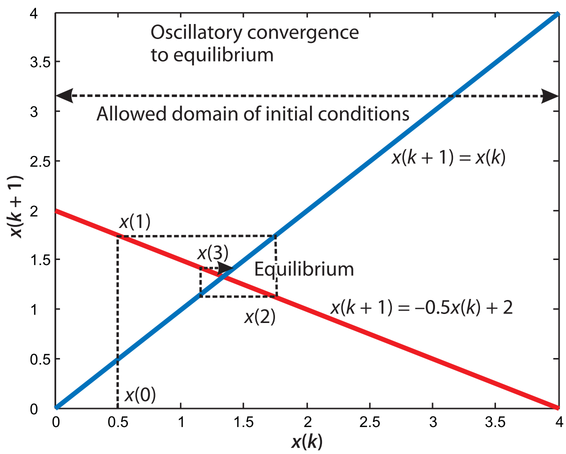

Fig. 6.7. Oscillatory behavior of the water level for $-1 > \alpha > -2$

[7] The fact we are asking two questions, and have only one degree of freedom to play with, immediately implies that there will have to be a trade-off between these two objectives.

Bifurcations

Most importantly the example illustrates how we change stability into instability through feedback and how this transition occurs. All of these phenomena may be summarized in a single diagram, in the parameter plane Fig. 6.8. This figure indicates for each of the possible choices of water reference level r and control parameter α a specific qualitative system behavior. Such a diagram is known as a *bifurcation* diagram. It identifies the loci in parameter space where the qualitative properties of the system (or more precisely its trajectories) change abruptly (or bifurcate).

We conclude this example with a few generalizations.

- Stability properties change at discrete parameter values (here $\alpha = 0, -1, -2$). Away from these special conditions a small change in a parameter only affects the trajectories, their stability and performance attributes in a small way. This is true in great generality. It is one of the foundational observations in so-called *bifurcation* theory, which is all about understanding these special points and organizing them so that we get a clear overview of what happens (or may happen) to the trajectories of a dynamical system.
- The dynamics in the example are linear. However, as discussed in the next section, and because of the Hartman-Grobman theorem, the results carry over to equilibria of nonlinear systems as well. In a small neighborhood of equilibria, a linear system approximation can represent most of the local behavior as well as the nonlinear system.
- Performance in steady state (here the difference between the equilibrium and the desired level) and performance in transient (here how fast the equilibrium is reached) are coupled, but they do not go hand in hand. There is a trade-off. Good steady state per-

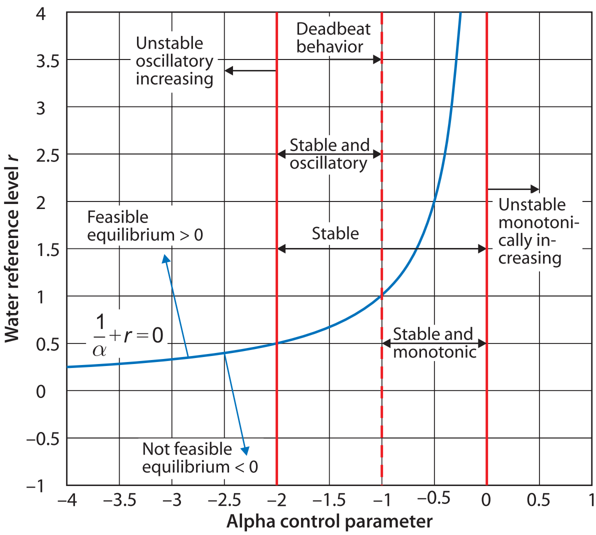

Fig. 6.8. The changes in behavior for the water level in a tank subject to a constant water flow drain $c = 1$, for variable water reference r and control parameter α

formance would require large α, but the largest α which also guarantees stability is -2. Moreover, the closer α is to -2, the slower the transients die out, or the longer it takes to get to the equilibrium. The system structure is too simple to allow us to design for transient as well as steady state performance. More elaborate *control* over the supply rate must be considered to avoid or improve on the encountered trade-off. It is precisely the existence of these trade-offs that make (control) design interesting.

6.5 Nonlinear Systems: Lyapunov Stability

The development of the notion of stability is intimately associated with the name of Lyapunov[8], a Russian mathematician interested in the stability of motion as studied in mechanics. For his master research program he was given the topic to investigate the shape a viscous liquid would assume when rotating around an axis of symmetry. This was and still is an important question in understanding the shape of planets. The problem proved rather difficult (it is still an interesting object of research) and after he had defended his Master's thesis Lyapunov abstracted, and simplified the problem to the study of the motion of rigid bodies (as opposed to deformable ones) with an arbitrary but finite number of degrees of freedom and subject to given force fields. He developed a theory, now called Lyapunov theory, that allowed one to infer stability without the need to know the precise motions of interest. The latter was extremely important as in general it is not possible to express the motions of interest in an analytically tractable form (apart from linear systems). Even today, with all the computing power we have available, Lyapunov's ideas are still relevant. Indeed stability is not easily captured through numerical simulation studies as we need to be able to verify an infinity of possibilities (for example all the possible initial positions of the ball in Fig. 6.3). His results and ideas have influenced the study of systems theory ever since, and even today they play an important role in the design and verification of nonlinear control systems.

6.5.1 Lyapunov's First Method

The first great contribution that Lyapunov made was to clearly understand that the stability of an equilibrium is a local property and that it can be studied using a linear approximation to the dynamics of interest. This is the so-called first method of Lyapunov for stability. To analyze the stability of point **a**, in the ball example, we only need to investigate the ball behavior in a neighborhood of this point. Despite the fact that stability property requires us to consider the time evolution of all trajectories in a neighborhood of the equilibrium into the indefinite future, Lyapunov showed that under very mild conditions to be imposed on the linearization, the stability properties enjoyed by the linear approximation are the same as the stability properties of the system itself. Moreover for linear dynamics Lyapunov provided a purely algebraic character-

[8] Aleksandr Lyapunov 1857–1918, studied mathematics in St. Petersburg University. He was influenced by L. Chebyshev to study the stability of ellipsoidal forms of rotating fluids, and obtained his Masters degree in 1884 on this topic. In 1892 he delivered his now famous Stability of Motion dissertation, on the basis of which he received his Doctorate. Another mathematician with a lunar feature named after him.

ization of stability, through the solution of a set of linear equations that now carry his name. This is a great advance, because we now have a very simple computationally efficient manner to verify the stability of equilibria.

6.5.2 Energy and Stability: Lyapunov's Second Method

The study of stability is very much linked to the notion of energy. Returning back to the example of the ball subject to gravity, we notice that the stable equilibria correspond to local minima in the potential energy for the ball, i.e. the ball at rest in a valley. Also, the local maxima in the potential energy for the ball, that is the ball at rest on a crest, correspond to unstable equilibria for the ball.

If we follow the trajectory of the ball released from point **b** with zero velocity, it is clear that the ball gains in kinetic energy (speeds up) as it loses potential energy. The potential energy that the ball possessed in point **b** is converted into kinetic energy, and some heat, a consequence of friction with the ground. When the ball reaches point **a** its potential energy achieves a local minimum, and its kinetic energy reaches a local maximum. As the ball continues to roll, it converts its kinetic energy back into potential energy, reaches a maximum position (local maximum in potential energy, local minimum in kinetic energy) and rolls backwards; and so on. Eventually the ball comes at rest at point **a**. From the first law of thermodynamics, we know that energy is conserved. Hence the mechanical energy in the ball, the sum of its potential energy and kinetic energy is decreasing over time, as friction dissipates energy in the form of heat which must come at the expense of the available mechanical energy in the ball. Moreover, as the total mechanical energy is bounded from below, because clearly the mechanical energy has a global minimum if the ball is at rest in the deepest valley, it must follow that the mechanical energy of the ball reaches a minimum along the trajectory of the ball. At that point the ball is precisely in a point like **a**, at the bottom of a valley, and possesses no kinetic energy. Natural systems evolve towards stable equilibria. (This makes a nice link with the maximum entropy principle in thermodynamics; Gyftopolous and Beretta 1991.)

For most physical systems the total energy in a system and in particular its evolution over time is always an excellent starting point to understand the system's stability. Even then, for complex systems, or system dynamics not rooted in physics, like social or economic systems, it may not be possible to think in energy terms. To overcome such problems, these energy ideas have been beautifully generalized by Lyapunov to allow one to study arbitrary abstract systems. This was Lyapunov's second important contribution to the study of the stability of motion, called *Lyapunov's second method*.

Lyapunov's Second Method for Stability

Consider an equilibrium point x_e of a system of interest.

Suppose there is an energy-like function of the system state, $V(x)$, such that $V(x) \geq 0$ in a neighborhood of the equilibrium point x_e, and is such that it only vanishes at this point. This captures the notion that the bottom of the valley is an equilibrium.

The system will be stable if, for all initial conditions x_0 in some neighborhood of x_e along a trajectory starting at x_0 the function value $V(x(t))$ (where $x(t)$ is the trajectory starting at x_0) is not increasing with time.

In Lyapunov's second method, identifying stability is now shifted to finding an energy like function V, called Lyapunov function with all the desired properties: bounded below (with a minimum at the equilibrium), and decreasing along solutions. If such a function is found then stability can be inferred.

Do such Lyapunov functions always exist when the equilibrium in question is stable? It would be important to know that a search for such a function is not in-vain. Lyapunov demonstrated that *if* an equilibrium is (locally) stable and attractive, then indeed such a Lyapunov function exists. Nevertheless, such existence results provide little solace in the quest for finding appropriate Lyapunov functions as the Lyapunov functions constructed in these existence results typically involve the collection of all trajectories of the system, which is not a computable object.

The weakness of the second or direct method of Lyapunov is exactly in finding the Lyapunov function. This is hard, and failure to find an appropriate function just means that we are unable to find it. It does not prove anything.

The strength of the Lyapunov idea lies in the fact that a conclusion about stability can be reached without precise knowledge or even being able to compute the trajectories of the system. Indeed all we need to establish is that the scalar valued Lyapunov function V) is decreasing along the evolutions of the system. It captures the behaviour of *all* trajectories (at least in a neighbourhood of the equilibrium of interest) at once, without the need of computing all trajectories. This can be established without knowing the solutions. Suppose that the system is described by an ordinary differential equation $\dot{x} = f(x)$. We may not be able to solve it, but we can check if a $V(x)$ is decreasing along the solutions, as all we have to do is to compute the derivative of V along the solutions, and this is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x) = \frac{\partial V(x)}{\partial x} f(x)$$

An example illustrates the point.

Nonlinear System Example

Consider the system defined by

$$\dot{x} = -x^3$$

A Lyapunov function candidate is

$$V(x) = x^2$$

because it is always positive, and zero at the equilibrium, moreover as $||x||$ grows larger so does $V(x)$. The time derivative of V along the solutions satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x,t) = 2x\dot{x} = -2x^4 \leq 0$$

which is always negative, except at the equilibrium. The equilibrium is stable, even asymptotically stable, and because there are no restrictions on our calculations we can even decide that the stability property holds true globally.

6.6 Non-Autonomous Systems

So far we have explored how the stability of an equilibrium, a particular solution of a system, can be determined using Lyapunov ideas, and how these stability properties change as we modify some parameters that appear in the description of the system. In this discussion all the parameters are assumed to be constants, they do not vary over time, and we observe how these constants shape the behavior of the system. In many other situations however, these *parameters* are not fixed but may vary from time instant to time instant; for example it is natural to expect that the water consumption may depend on time, as the amount of water drained from the tank will depend on the opening of a valve and the latter can change over time. Time variations complicate matters substantially but also create interesting possibilities.

Another important question in the study of stability is: *How does the interconnection of systems affect the stability of the overall system?* To discover this it does not suffice to study the stability of equilibria of autonomous systems. Interconnecting systems means sharing of signals between systems, hence it is imperative to study system stability from a signal or input perspective.

6.6.1 Linear Systems

Linearity simplifies matters considerably. Indeed, in general because in non-autonomous systems, the solutions are a consequence of both the initial conditions as well as the external inputs. We expect that in order to deal with stability in the presence of external signals we will also need to keep track of how initial conditions and external signals interact. However, in the linear system case, the solution can always be written as a linear combination of two solutions, one without initial conditions, only due to the external signal and one without external input, only due to the initial condition. That means input/output stability can be analyzed assuming null initial conditions. Moreover, using transfer operator notation, it is easy to see that the system stability does not depend on the input (Eq. 6.6).

Consider again the example of the tank. If the input variables, such as the reference r or the water consumption c, are time varying, the only result is that the equilibrium point will change, but still the system dynamics will be characterized by the parameter α.

In a more general setting we can consider the input/output behavior determined by the transfer function operator, Eq. 5.8 in the continuous time case and Eq. 5.15 in the discrete time setting[9].

[9] Almost any discrete time transfer function can be expressed as

$$G(z) = \sum_{i=1}^{n} \frac{b_i}{z - a_i}$$

The parameters a_i and b_i may be complex.

Linear Systems Stability

A linear system $x(k+1) = Ax(k) + Bu(k)$ or $\dot{x}(t) = Ax(t) + Bu(t)$ is stable if the poles (eigenvalues of A) of the System 6.6 are:

- inside the unit circle (smaller than one) if the system evolves in discrete time, see Fig. 6.5,
- strictly to the left of the imaginary axis in the complex plane (real part negative) if the system evolves in continuous time, see Fig. 6.6.

Stability does not depend on the input. It is an intrinsic property of the system.

There are some computationally efficient mechanisms (using Lyapunov equations for example) to conclude whether or not the eigenvalues of a matrix lead to a stable linear system, without necessarily computing the matrix eigenvalues explicitly.

6.6.2 Nonlinear Systems

In trying to come to an understanding of how an input into a system, affects its overall behavior the main difficulty is that there is so much that can change over time. So in order to make some progress, we are going to limit the possibilities of what we want to consider. One reasonable question would be to ask how the size of an input affects the size of a signal of interest in a system. In stability literature this is captured by the notions of *input-to-state stability*, or **ISS**), and the concept of input-to-state gain[10] (function).

In general even this simple question is difficult. For example in the tank example, if the input is neither the reference nor the outflow but the control parameter α, the system becomes nonlinear, as there is the product of the input and the state. We may guess that, in this case, the system will become unstable as soon as the input (α) goes out of the interval $\{0, -2\}$.

To discuss the general case, we start with the discrete state space description of the System 5.17.

The initial value for the state is given at time t_0 as x_0. Recall that the nature of the state is such that given the information about the initial state and the model equation (which includes knowledge of the time variations and the input function into the indefinite future) we are in a position to compute the future behavior of the state, that is for all time after the initial time $t > t_0$.

A typical input-to-state stability property takes the form:

$$\|x(t)\| \leq g_x\left(\|x_0\|, t\right) + g_u\left(\|u\|\right), \quad \forall t > t_0 \tag{6.13}$$

Here $\|.\|$ is a measure for the size. g_x is a function both of the size of the initial condition and time. It reflects how initial condition effects disappear in time, that is we expect $g_x(., t) \rightarrow 0$ as time progresses. Also, $g_x(0, t) = 0$.

[10]The concept of gain is much more general than the one described in Sect. 5.5. It expresses the ratio between some output size and some input size. These signal sizes can be for example the largest absolute value reached by a signal, or its total energy. There are many different measurements of size for signals.

g_u is also a gain function, that expresses how the input affects the state. We want that $g_u(0) = 0$ and because it is a gain function, we expect that its value increases with the argument, that is $g_u(a) < g_u(b)$ whenever $0 < a < b$ and similarly $g_x(a, t) < g_x(b, t)$ whenever $a < b$. Implicitly, by stating Eq. 6.13 it is clear that there is an equilibrium at zero. So, a statement like this is a very strong statement about stability.

Clearly, ISS approaches stability both from an initial condition *and* input aspect. Indeed an appropriate ISS concept must capture an initial condition stability property, because it must capture the no input condition as a special case.

An ISS property is geared towards characterizing the stability of an equilibrium. This is the most frequently encountered stability question, but by no means the most general.

By demanding that all trajectories in some neighborhood of the equilibrium remain close and also eventually converge to this equilibrium under zero input conditions, ISS recovers the classical notion of an asymptotically stable equilibrium. In fact one can show that the previous definition and the above expression are equivalent.

In addition, ISS deals with the input and expresses that the input cannot drive the state of the system too far from the equilibrium. How large the state can become under the influence of the input is captured by an operator or *gain function* g_u, which bounds how big the state can become under the influence of a signal of a given size. This plays the role of the transfer function operator in linear systems.

For instance, referring back to the motion of the ball in the basin, suppose the ball is in the equilibrium point **a**, and it is softly kicked. Intuitively, the ball will move away from the equilibrium but because it is a soft kick return and settle again at **a**. However if it had received a strong kick, sufficient to pass over the point **c**, the ball will never return. ISS can capture all of this by limiting the domain of where the Expression 6.13 is valid.

We may expect that an ISS concept based on the size of a signal will necessarily be conservative. The absolute size is not the only crucial information about a signal. On the other hand, because size is such a practical measure, ISS is very practical, in particular when considering the interconnection of systems. Roughly speaking, from this point of view, each system is replaced by a gain function, and the topology of the interconnection dictates how signal sizes are transformed. This allows us to readily verify if a system constructed from subsystems that are input-to-state stable keeps this property, and moreover it allows us to quantify how large the overall system's state can become a function of all the external inputs influencing the system. We explore this in the following sections.

Input-to-State Stability

A system, described by its state space model, either (5.18) or (5.17) is said to be ISS provided that for any initial condition x_0 and any bounded input function u (with bounded size $||u||$) the size of the state x can be bounded as

$$\|x(t)\| \leq g_x\left(\|x_0\|, t\right) + g_u\left(\|u\|\right), \quad \forall t > t_0 \tag{6.14}$$

The function *gx* captures the transient effect of the initial conditions and it increases with the initial state and decrease to zero with time.

The function *gu* captures the effect of the input size on the size of the state, it is called the input-to-state gain function. It must be non-decreasing.

ISS expresses a very strong, hence also very practical, and very desirable notion of stability. In particular, observe that for zero initial condition and zero input, both terms on the right side of the Inequality 6.14 are zero, which implies that the state is identically zero. In other words, $x(t) = 0$ is an equilibrium. Moreover, it is the only equilibrium.

There are ways of estimating the functions g_x and g_u for classes of non-linear systems based on ideas of Lyapunov stability. In fact ISS and Lyapunov stability are closely related.

For linear in the state and the input(s) systems[11], the notions of input-to-state stability and equilibrium stability coincide. Hence, as already mentioned, for linear system dynamics it is indeed quite legitimate to talk about a stable or unstable system. For nonlinear systems this is in general not at all the case, as we realized with the tank example as well as the ball in the basin.

Input-to-state stability is useful in that it allows us to infer stability properties for interconnected systems from knowledge of the ISS properties of the systems that are being interconnected. It is particularly suitable when the cause-and-effect direction is obvious, as the underlying implicit assumption in ISS is precisely that the information flow is from the input to the state, the former affects the future of the latter. When the information flow is clear, there are essentially but two main mechanisms for building large systems from subsystems, either by cascading two subsystems or by forming feedback loops of two subsystems.

Bounded-Input-Bounded-Output Stability

A particular case of ISS, precisely when the output coincides with the state, is the notion of Bounded-Input-Bounded-Output stability, BIBO for short. This concept expresses the idea that bounded inputs lead to bounded outputs.

6.6.3 Input-to-State Stability and Cascades

Consider a cascade of two systems. The idea is illustrated in Fig. 1.6. In a cascade of two systems the state, or an output of the first system ($y_1 = C_1 x_1$) serves as input to the second system.

If the first system S_1 (see bottom of Fig. 1.6) is input-to-state stable from input u to state (output) y_1 with input-to-state gain function G_1 and the second system (S_2) is also input-to-state stable from input (u_2) to state (output) y_2 with input-to-state gain function G_2, then the overall cascade with $u_2 = y_1$ is also input-to-state stable from input (u_1) to the state of the cascade which is (y_1, y_2). Moreover it is possible to estimate an input-to-state gain function. The gain function of the cascade is closely related to the composition of the gain functions of the subsystems, i.e., dealing with operators

$$G_{\text{cascade}} = G_1 G_2$$

Clearly, if both tanks separately are ISS, then the cascade will also be ISS.

[11] This means $f(x, u, t) = A(t)x + B(t)u$ in Eq. 5.18 or 5.17.

Interesting examples of cascades in system models have been encountered in Sect. 3.2 and 3.3, where we discussed the production of ceramic tiles and gravity fed irrigation systems respectively. In manufacturing processing, typically the output of each stage feeds into the next stage, creating naturally cascaded subsystems in the description of the overall system. Very similarly, in the irrigation channels, the immediate upstream water level will affect the water level in the downstream pool again producing a natural cascade of subsystems (pools) that describe the entire system (a channel). If each subsystem in isolation is input-to-state stable then we can conclude that the whole system is also input-to-state stable. Moreover we can quantify what effect a disturbance occurring in a subsystem will have on all downstream subsystems, and the system as a whole. In this sense input-to-state stability is a powerful concept. For example it is sufficient to understand input-to-state stability for a single subsystem in a cascade of identical or almost identical subsystems in order to deduce relevant information about the entire cascade.

Stability of Cascade Systems

To determine the stability of a cascade of systems we only need to analyze the stability of each subsystem: the cascade is stable if all the subsystems are. If any of the subsystems is unstable, the cascade is also unstable.

This is quite clear in the case of linear systems. The transfer function of the cascade is just the product of the transfer function of the subsystems, as can be easily derived by using the block-diagrams algebra (See 5.6.2). Thus, say for two systems in series, represented by

$$G_1(z) = \frac{N_1(z)}{D_1(z)} \quad \text{and} \quad G_2(z) = \frac{N_2(z)}{D_2(z)}$$

$G(z) = G_1(z)G_2(z)$ represents the cascade. Clearly the denominator of the cascade is the product $D_1(z)D_2(z)$, and the collection of its roots is the union of the set of roots from the subsystem transfer functions. The cascade is stable only if both systems are stable.

6.7 Beyond Equilibria

So far our discussion has concentrated on the stability of equilibria. In many engineering applications this suffices, but there are also many systems where the real object of interest is not an equilibrium but rather a time varying object. For example, in the antennae tracking problem considered in Sect. 3.4, the issue is to drive the antennae such as to track a star in the sky, whose position can be represented as a quadratic function of time. In other instances a system has to behave in a periodic fashion (like a clock, a wheel, a hard disk drive), or it naturally behaves in a periodic or almost periodic manner (like our solar system, the tide, our heart beat). In fact periodic and almost periodic phenomena are prevalent in nature and engineering systems alike.

The main tool that is used to analyze a time varying object is actually to attempt to map the object of interest to an equilibrium, perhaps for a different system. This equilibrium may be analyzed using the above techniques, in particular the input-to-state stability ideas, especially when the transformation into an equilibrium involve approximations. Finally our understanding is transformed back onto the original description for interpretation.

6.7.1 Limit Cycles and Chaos

Some systems may present different equilibrium points or even more complicated stationary behavior. Let us analyze a model for a population of rabbits (without predators). At a given moment, the number of rabbits is expressed by $x(k)$. We assume that the rate of birth is related to the number of animals and the rate of death, due to a lack of food (and old age), is proportional to the square of the number of rabbits. That is

$$x(k+1) = r.x(k) - d.x^2(k)$$

If we defined a new variable z, such that $z(k) = (d/r)\,x(k)$, the new equation[12] only depends on the parameter r:

$$z(k+1) = r.z(k)\big[1 - z(k)\big] \tag{6.15}$$

There are two possible equilibrium points, at $z_{e1} = 0$ and $z_{e2} = 1 - 1/r$ (which is only a realistic equilbrium for $r > 1$). Varying the parameter r, in this normalized model, leads from an overdamped stable equilibrium to oscillatory and even chaotic behavior.

For $0 \le r \le 1$ the evolution is overdamped and rabbit population goes to extinction, i.e. a final value of $z_e = 0$ is reached. For $1 \le r \le 3$ the evolution is underdamped and the rabbits do not become extinct (unless there were none to start with). The population reaches the final value of $z_e = 1 - 1/r$. At $r = 1$ there is a change in the local stability properties of the zero population. It is a bifurcation parameter.

For $r = 3$ there is another bifurcation, the so-called flip bifurcation. The system does not reach any equilibrium but instead it remains oscillating with period 2. For $r = 3.0$ and any initial normalized population $z(0) < 1$, the final population jumps between 0.6179 and 0.7131, as shown in Fig. 6.9a. Almost all populations will reach a *limit cycle* (apart from starting at the equilibria z_{e1} and z_{e2} which are now both unstable).

If we further increase the value of r, new oscillations appear and for $r \ge 3.57$ the behavior becomes *chaotic*. In the chaotic regime, the population evolution is very sensitive to the initial conditions, as can be seen in the graph depicted in Fig. 6.9b, where the coefficient $r = 3.8$ and the initial conditions are $z_1(0) = 0.5$ for the evolution marked with $\circ$, and $z_2(0) = 0.5001$ marked with with $\cdot$. We realize that after 25 periods the evolution is completely different and the difference (marked by crosses) starts to be more and more significant.

[12]This equation is called the *logistic equation* and it shows very interesting properties related to bifurcations and sensitivity.

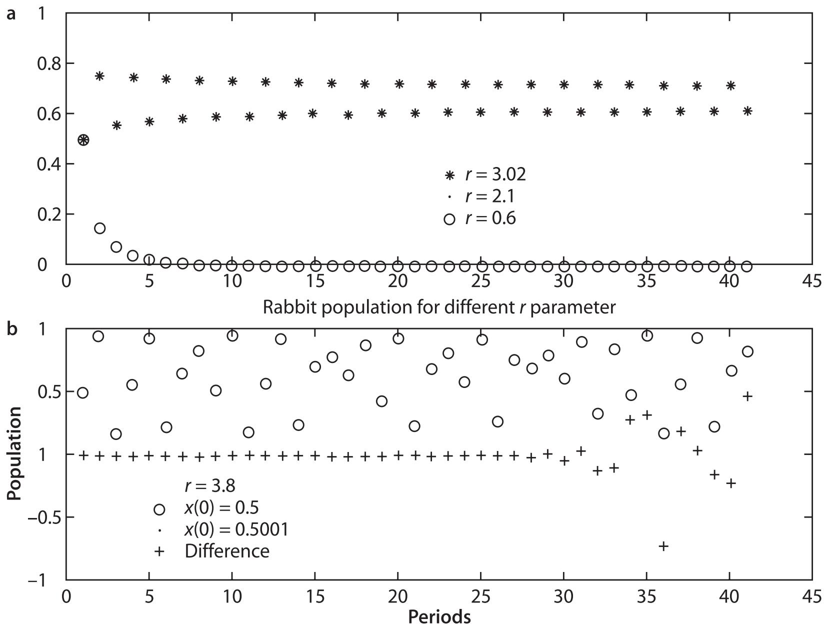

Fig. 6.9. The chaotic evolution of a rabbit population

This chaotic behavior could be interpreted as stochastic, but it is not. A chaotic behavior is totally deterministic, as the future state value can be exactly computed if the model and the current state are known. The main characteristic of a chaotic behavior is that the system evolution is extremely sensitive to small perturbations in the state value. From this point of view, a stochastic interpretation of this deterministic system may well be useful (there is sense in this madness).

Few systems exhibit such extreme *sensitivity*. In general, if the system is stable, small changes in either the system parameters, the input signals or new disturbances produce small changes in the system behavior. Qualitative changes, as marked by bifurcations, only appear in rare circumstances.

Because bifurcations are rare, understanding what can be and cannot be is important. Normal experimentation with a system may never show any trace of looming bifurcation points. Careful experiments must be planned to exhibit the phenomena of interest.

6.8 Sensitivity

In the example of the rabbit colony, the sensitivity of the behavior was with respect to an internal parameter (related to birth and death rates). To introduce the setting for the notions of sensitivity and robustness, we start with some additional simple examples.

Static Measurement/Sensor Inaccuracies

Let us consider a simple tachometer, as presented in Fig. 7.9. In principle the device is characterized by a static model such as $V = f(\omega)$, where ω is the rotational speed of the motor and V is the output voltage. If the tachometer axis slips relative to the axis whose rotational speed we are interested in, an error results:

$$V + \Delta V = f(\omega + \Delta\omega)$$

We call $S_\omega = (\Delta V)/(\Delta\omega)$ the input sensitivity function of the tachometer. In general, it is a function of the input. If the tachometer function is linear, the sensitivity function is constant and equal to the tachometer gain. This idea is represented in Fig. 6.10a.

Possibly, the actual measurement function is different from the one we have considered to be the model. That is, the measurement is $V = F(\omega) \neq f(\omega)$. This would be the result of an incomplete calibration of the instrument, for example. In this case we do not know the true function F, but we may know something about the difference, because we have information from the calibration of the instrument. Let the difference be

$$W(\omega) = F(\omega) - f(\omega)$$

Denote $\hat{V}$ as the expected value of the voltage, we can use a block diagram as shown in Fig. 6.10b, to represent the tachometer. The function $W(\omega)$ represents the uncertainty in the knowledge of the model, here represented as an *additive uncertainty*.

The notion of how close two functions are is not a simple one. One (conservative) measure for the difference between two functions could be identify it with the maximum possible deviation over all possible measurements.

Another model for measurement uncertainty is

$$V = F\big(f(\omega)\big); \quad V = f\big(F(\omega)\big) \tag{6.16}$$

Here F represents a function, presumably close to unity that captures the uncertainty. On the left the uncertainty acts on the output of the *ideal* device, on the right it acts on the object to be measured directly, i.e. the input of the instrument. The latter is called an input uncertainty, the former an output uncertainty. How much F deviates from unity indicates how little we know about the actual measurement instrument f. In a linear system context, this uncertainty is described as a multiplicative uncertainty. In either the linear or nonlinear case, it is represented as a cascade of systems, only in the nonlinear case the order of the sequence matters.

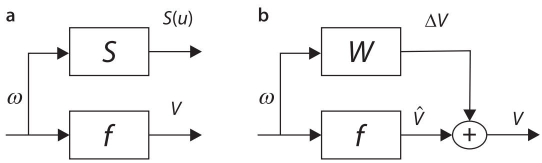

Fig. 6.10. Sensitivity of a static measurement device

In any measurement device there are many different sources that lead to errors. There may be systematic errors, or random errors due, for instance, to quantization. In all control applications, it is important to know about all the sources of uncertainty. More preferable is to have some information as to the possible size of these sources of uncertainty so that acceptable performance expectations can be quantified.

Unmodeled Dynamics

Let us consider a compact disk (CD) player. The laser beam is attached to an arm moved by a very fast motor (a voice coil motor or VCM). A simple model of the motor may be a double integrator, from force to position. In doing so we assume that the arm is rigid, that the disk is rotating at a fixed and known speed and that there are no external disturbances. The real system however has mechanical resonances as it is not perfectly rigid (fast movement, means light, means not rigid). Moreover the arm is impacted by external vibrations and the relative position of beam and track on the rotating CD is subject to the eccentricity in the disk, which will vary in magnitude from disk to disk. All these are disturbances away from the ideal model.

The way these disturbances affect the behavior of the CD player are quite different, mainly because their range of frequencies, spectral content, is very different. The eccentricity of the CD rotating has a perfectly well-defined frequency and can be tracked and compensated for. The mechanical resonances are typically in the very high frequency range and hence are only relevant in fast transient when moving the pointer from one track to another. Typically they are avoided by ensuring that the transients are not exciting the resonances. The external mechanical vibrations are much lower in frequency, related to the sound being produced out of the speakers for example, and the controlled system has to be insensitive to these disturbances.

This example is very much like the tracking antennae in Sect. 3.4.

If we go back to the tachometer, another kind of disturbances may be considered. The brush sweeping effect in the rotating machine will produce a high frequency noise. This noise, if not filtered, will be fed back to the process and will disturb the operation of the system to which the tachometer is attached. This issue will be considered in Chap. 7.

6.8.1 Robustness

Typically the term sensitivity is reserved to describe the effect of small variations in parameters in the model on a particular signal in the system, or the effect on such signals caused by small variations in external signals and so on. Robustness refers to a similar cause-effect relationship, but considers large variations belonging to some predetermined set of allowable perturbations. Nevertheless, both terms are often used interchangeably. In a colloquial way, robustness refers to a lack of sensitivity. Thus, a system signal, or system property may be robust with respect to variations in the input signal, meaning that the allowed perturbations away from the nominal input is not going to cause a significant departure in the response or property of the system. Or, a system is robust with respect to a system variation, i.e. the actual system is different from the model, but this difference has little impact in that the behavior of the system is pretty much like the behavior of the model. In each case robustness refers to the insensitivity of a particular property with respect to a perturbation that belongs to a predetermined set of perturbations to be considered.

Robustness is not a universally desirable property. A system is designed for a purpose, and responsiveness may be exactly what is required. Robustness must only hold in the sense that the desired behavior is insensitive to changes in the rest of the system, or signals impacting on the system.

Robustness of stability reflects to what extent system parameters are allowed to vary without affecting the desired stability. This is so-called *robust stability*.

Stronger than robust stability, because stability is really a must have property in a controlled system, is *robust performance*. In this case, the requirement is to keep the controlled system performing within acceptable limits, despite a range of variations in the system or external signals impacting on the system. The aim is to have only a minor change in performance, despite a large perturbation somewhere in the system.

By way of example, a hard disk drive has a guaranteed seek time, the maximum time it takes to go from one track to another. This performance specification has to be met by each hard disk drive mechanism. The control algorithm has to ensure this, despite the fact that the mechanical resonant frequencies may vary significantly between different units (even produced on the same manufacturing line). The seek time is robust with respect to resonant frequency.

6.8.2 Sensitivity Computation

Typically the following uncertainties or perturbations must be considered in system design:

- *External disturbances*, external signals entering the system, some measured (like the variable composition of the feed-material in the ceramic tile factory), some unmeasured (like the noise in the music recording, or the wind load on the antennae).
- *System variations in parameters*, often referred to as *structural uncertainty* or *parametric uncertainty* (like the resonance frequency in the hard disk drive).
- *System model errors*, often referred to as *unstructured uncertainty*. Due to the fact that the model does not capture all of the system dynamics. For example in the antennae servo design, the resonances are not part of the control model.

With respect to a single loop control system, as depicted in Fig. 6.11, the design engineer will verify the sensitivity or robustness of all the signals in the loop with respect to the above uncertainties. In Fig. 6.11 d_e represents an external disturbance, becoming d in the system, and r_f is a filtered reference.

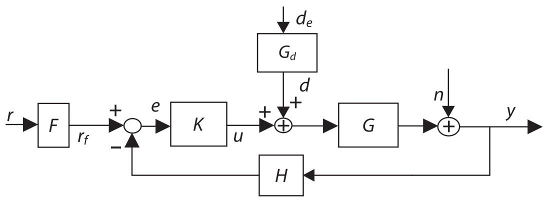

Fig. 6.11. A typical closed-loop control system

6.8.3 General Approach

When designing for robustness, the operational boundaries of a system are first defined. Typically no bifurcations are allowed within the normal operating envelope of the system. Bifurcations will be considered as part of the analysis of the behavior under fault conditions. Within the operational envelope of the system it is safe to assume that all cause-effect considerations, robustness and sensitivity can be captured through operators whose properties vary smoothly with parameters, signals and so on.

From the block diagram in Fig. 6.11, assuming for simplicity $F = 1$ and $G_\mathrm{d} = 0$, and an ideal, unitary gain sensor, $H = 1$, the operators of main interest are:

$$\begin{aligned} T = S_{yr} &= \frac{y}{r} = \frac{GK}{1+GK}; \qquad S_{yd} = \frac{y}{d} = \frac{G}{1+GK} \\ S_{ur} &= \frac{u}{r} = \frac{K}{1+GK}; \qquad S = S_{yn} = \frac{y}{n} = \frac{1}{1+GK} \end{aligned} \tag{6.17}$$

S_{yr} represents the response of the output y to the reference signal r, and so on. As can be observed, all these operators have the same denominator. We could think that just making this denominator somehow large enough all the sensitivities would be small, regardless of which signals we consider.

There are constraints though. In particular for the so-called *system sensitivity*, $S = S_{yn}$, indeed it is trivial to observe that

$$S_{yn} + S_{yr} = 1 \tag{6.18}$$

This equation tells us that system sensitivity $S = S_{yn}$ and the so-called *complementary sensitivity* $T = S_{yr}$, cannot both be small at once, as their sum adds up to the unity operator. We have to accept some sensitivity in a control loop. This is a fundamental constraint[13].

All sensitivity functions play an important role in control design.

6.8.4 Sensitivity with Respect to System Dynamics Variations

The Operators 6.17 describe the response of the system variables $y, u, \ldots$ to the external inputs $r, n, \ldots$. They define the performances of the system. In design, it is precisely these operators that are to be designed. It is therefore important to analyze what happens if the operator G (describing the system under control) changes, in this way we explore what happens to our design when the real system differs from the model used in design.

In particular, let us see the influence of changing G on the reference signal to output operator, the complementary sensitivity, T. Assume $H = 1$, for simplicity, and take the *derivative* of T with respect to G in Expression 6.17[14], it is easy to see that

[13] It should be noticed that the fundamental limit imposed by the equality $S + T = 1$ allows both S and T to be larger than one. Think of both as a complex number (that varies with frequency), or one positive and large and the other negative and large.

$$\frac{\mathrm{d}T}{T} = S\frac{\mathrm{d}G}{G} \tag{6.19}$$

That is, the ratio of the relative change in the closed-loop operator $\mathrm{d}T/T$ due to a relative change in the open-loop operator $\mathrm{d}G/G$ is precisely the sensitivity function S, which is also the response from the noise to the output. This underscores the importance of the sensitivity function.

6.8.5 Sensitivity Measurements

We expressed sensitivity using operators, allowing us to see how the sensitivity depends on particular system parameters. It is also apparent that there is an inherent trade-off due to the fundamental constraints, and sensitivities cannot be arbitrarily specified.

Information such as the maximum gain of these sensitivity operators is important information about the system behavior:

$$M_s = \max_\omega |S(j\omega)|; \quad M_T = \max_\omega |T(j\omega)|; \quad M_{yd} = \max_\omega |S_{yd}(j\omega)| \tag{6.20}$$

and are useful to characterize system sensitivity.

There are positive and negative forms of sensitivity. For instance, high sensitivity is important in measurement, to detect special signals, like the frequency selection in a radio. On the other hand high sensitivity may be dangerous, as in a hyper allergic reaction. Similarly high sensitivity to resonance or noise could be disastrous to the radio telescope or hard disk drive mechanism.

Summary of Stability and Robustness

The general concept of stability can be expressed as:

A system is stable if its behavior (which includes all initial conditions) has the property that for all bounded inputs, all conceivable signals are bounded.

We discussed different forms of stability:

- *Local stability* of an equilibrium (orbit) in an autonomous system requires all system responses starting in initial conditions close to the equilibrium (orbit) to remain close to the equilibrium (orbit);
- *Asymptotic stability* of an equilibrium, if the responses are both stable and asymptotically approach the equilibrium;
- *Global stability* of an equilibrium in an autonomous system, if the above holds true for any initial condition;
- *Robust stability*, if the stability property is maintained under changes in the system parameters;
- A system response is *sensitive* to some parameters, input signals or operators if the system response varies (significantly) with changes in these parameters, input signals, operators.

For linear systems, stability is an intrinsic property of the system.

[14] $\frac{\mathrm{d}T}{\mathrm{d}G} = \frac{K(1+GK) - KGK}{(1+GK)^2} = \frac{K}{(1+GK)^2} = \frac{T}{G}S$.

6.9 Comments and Further Reading

Stability is a well-studied topic, originally motivated by mechanics and such lofty questions as "Is our solar system stable?". The early work by Lyapunov laid the foundations for a more general study of stability. His treatise is still an excellent introduction to the topic (Lyapunov 1992). Stability is dealt with in detail in nonlinear systems texts such as Khalil (2002) and Willems (1970), which also provide a modern introduction to Luyapunov's first and second method. The celebrated Hartman-Grobman theorem, extending Lyapunov's first method is described in Guckenheimer and Holmes (1986).

The notion of input-to-state stability, and its connections to Lyapunov stability is very important in systems engineering as it allows us to deal with systems with inputs, and hence interconnection of systems. The work by Sontag is key in this area, see for example Sontag (1998).

Bifurcation theory is a branch of dynamical systems theory. It seeks to bring order in the domain of closed system behavior. Typical books include Wiggins (2003), Guckenheimer and Holmes (1986) and with a greater emphasis on computational ideas Kuznetsov (2004). One-dimensional dynamics, like the logistic equation, are well understood. The generic behavior of such dynamics as well as its robustness are comprehensively treated in de Melo and van Strien (1991). A dynamical systems approach to control systems is not for the faint hearted. Results in this direction are summarized in Colonius and Kliemann (2000).

Robustness and sensitivity are important notions in all engineered and general systems. These notions require extensive computational resources to be fully explored. There are fundamental constraints imposed by the system interconnection structure, and hence the many iterations in designing new systems. In the context of linear systems there is a well-established theoretical framework to deal with robustness and sensitivity issues (Green and Limebeer 1995; Boyd and Barratt 1991), even for large scale systems.