Human Being and Mathematics Logical and Mathematical Thinking*

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Summary. *Logical thinking* as an expression of human reason grasps the actual reality by the basic forms of thinking: concept, judgment, and conclusion. *Mathematical thinking* abstracts from logical thinking to disclose a cosmos of forms of potential realities hypothetically. *Mathematics* as a form of mathematical thinking can therefore support *humans* within their logical thinking about realities which, in particular, promotes sensible actions. This train of thought has been convincingly differentiated by *Peirce's philosophical pragmatism* and concretized by a "*contextual logic*" invented by members of the mathematics department at the TU Darmstadt.

1 Logical Thinking

Already Pythagoras' pupil Alkmaion of Croton defined a human being as "zoon logon echon" (in latin: animal rationale), i.e. as "reasonable living being". This basic anthropological understanding of a human being has been lasted in western philosophy until Scheler's duality of "mind" and "body" and even further ([Fa73]; p.895). *"Reason"* is here understood as mental means of human beings to gain insights, to form judgments, and to act in accordance to those judgments ([Du95]; p.3694). Since those means are substantial for human beings, the formation of humans should achieve to learn thinking and acting in a reasonable manner. To what extent mathematics could play a role here, this shall be discussed in the following. In particular, the claim shall be examined that logical thinking can be supported by mathematics. How close are the meanings of "thinking logically" and "thingking reasonably", this may become clear by noticing that the meanings of both linguistic expressions are apprehended in English by one word, the verb "reason".

To understand what is meant by "logic thinking", one has to clarify what is meant by "logic". According to the "Duden: Das große Wörterbuch der deutschen Sprache", *logic* is the doctrine of the structure, the forms, and the laws of thinking ([Du95]; p.2145). Therefore, "logical thinking" means a thinking which activates logical (i.e. to logic belonging) structures, forms, and laws. In the philosophy since the 16th century, the basic forms of logical thinking are considered as the *concepts* (as basic units of thinking), the *judgments* (as connections between concepts), and the *conclusions* (as inferences gaining judgments from other

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judgments). How, founded on those basic forms, further logical structures, forms, and laws can be developed, this has already been shown 1662 by Antoine Arnauld and Pierre Nicole in their guiding book “La Logique ou l’Art de penser” [AN85]. Even when a new understanding has become dominant in the course of the stronger formalization of logic - after Logic was ascribed to the task of recognizing the laws of truthness (since the midle of the 19th century) - , the everyday uses of logical thinking has basically speaking not changed if one is orientated on general dictionaries.

This understanding of logical thinking, that it is based on concepts as the basic units of thinking, has been further deepened by *Jean Piaget* in his structure-genetic theory of cognition. For him, the logical thinking of a human being has its roots in the *coordinations of actions* which are already present before the development of the language; from those coordinations, mental operations and with them logical structures come into being in the psychic development ([Pi73], p.26ff). Piaget’s approach, by which he tries to clarify the question about the logic of conceptual thinking and the truthness of knowing, consequently run out according to Thomas Bernhard Seiler toward a theory which understands concepts as basic units of recognizing, thinking, and knowlege. Piaget identifies concepts with cognitive structures with which and through which the organism examines its environment in an acting manner, adapt to it, and in which the organism reconstructs the aspects of the environment relevant for its acting and thinking and which provide for it the basis for interpreting the meaning of signs (cf. [Se01]; p.164f.).

Most simple preforms of concepts are the *sensorimotor schemas* which arise already early out of coordinations of actions. The next step of evolution forms the *ideas* which abstract from the observed objects and correlated actions. If such structures of cognition can also be applied to new objects and other structures of cognition, then Piaget speaks of *preconcepts*. Structures of cognition have finally reached the step of *concepts* if they have been freed to a large extent from the intuitive view and have let coordinated to formal operations. Only the construction of complex concept systems and their systematic coordination allows a differentiated reconstruction of reality and leads to consistant concept orders, the availability of which is a necessary condition for the development of logical thinking. Piaget sees further conditions in the system properties of complex structures of action, the necessity of communicative negotiation and the compulsion to justify herself in the society ([Se01]; p.171).

Which priority meaning the logical thinking has for the recognition and action of human beings, this has been made distinct in particular by *Charles Sanders Peirce* in his philosophical pragmatism. For this the Cambridge Conferences Lectures are an impressive source which Peirce has given in 1898 about the theme *“Reasoning and the Logic of Things* [Pe92]. These lectures offer an introduction into Peirce’s late philosophy which tries to make it intelligible for all. Logic is understood in this lectures as normative science about forms and laws of thinking which, as a philosophical discipline, has as theme to make understandable the relationship between thinking and reality. Peirce sees the foundation for the understanding of forms of thinking in his *categories* of Firstness, Secondness, and Thirdness which he defines as follows ([Pe92]; p.146ff.): *Firstness* is the mode in which anything would be only for itself, irrespective of anything else; *Secondness* is the mode in which anything would be related to something else, irrespectively of anything third; *Thirdness* is the mode in which a First is joined with a Second by a Third. For instance, a concept as a third joins a concept word as a First with an object as a Second.

According to his categories, Peirce distinguishes between three kinds of logical conclusion: the abduction, the induction, and the deduction. The *abduction* creates out of the horizon of self-evidence a hypothesis as a First; the *induction* confirms a hypothesis by actually given facts as a Second; the *deduction* concludes a hypothesis out of valid premisses by logical laws as a Third. This means: "The deduction proves that something *must* be the case; the induction shows that something is *actually* efficient; the abduction only assumes that something *might be* the case ([Pe91]; p.400). In his Cambridge Lectures Peirce elucidates the three kinds of logical conclusions by the syllogistic figures of conclusion: the deduction by the figure Barbara, the induction by the figure Datisi, and the abduction (retroduction) by the figure Cesare ([Pe92]; p.141f.); with that he clarifies in particular that the three figures of conclusion distinghish essentually from each other, which Imanuel Kant challenced 1762 in his paper "Über die falsche Spitzfindigkeit der vier logistischen Figuren" ([Ka83a]; p.597ff.). With the reached understanding of the triadic nature of the logical conclusion, Peirce overcame the difficulties to express geometric and algebraic conclusions by syllogisms in the way that he extended the Boolean logic [Bo58] to the *logic of relations* ([Pe92]; p.150ff.), which was for him the formal foundation for all logical conclusions. The limitation of syllogisms, which was for Peirce essentially depend on their mechanistic nature, becomes surmounted in the logic of relations by an open diagrammatic conclusion which gives space for different types of conclusions.

Logical thinking was generally characterized by Peirce as follows: "Reasoning is the process by which we attain a belief which we regard as the result of previous knowledge" ([Pe98]; p.11). Peirce discusses in his first Cambridge Lecture about "Philosophy and the Conduct of Life" the logical thinking in everyday life, which succeeds for him as well without help by theoretical logic as with it. Primarily he sees the logical thinking detemined by the instinct and the sentiment of human beings and warns therefore for superficial logical conclusions that does not pay attention to instict and sentiment. As the logical thinking grows out of the human expierence, so instinct and sentiment develop in human beings from inner and outer experiences, and that takes place in a slow and deep process which brings out mental energy and vitality. Peirce considers this process as so important that he views *instinct and sentiment* as the real substance of the human mind ([Pe92]; p.110).

For this reason, the *"training in reasoning"* - so the theme of the fifth Cambridge Lecture - must, according to Peirce, concern the human mind as a whole; for this three mental operations are important for him: observation, experimentation, habituation.

Observation consists of two parts: the first as subconscious induction by which an associational potency arises on repeatedly reviewing an object of perception with a tendency to call up other ideas; the second as conscious formation of schematic ideas which are able to react on perceivable objects. The associational potency which arises out of the subconcious induction is according to Peirce the most important constituent of practical thinking, while the consciously formed schematic ideas are indispensable for theoretical thinking ([Pe92]; p.182). For logical thinking it is particularly important to train powers of discrimination; according to this, Peirce writes: "I never knew a man whose sagacity as a reasoner compelled my admiration without finding in him a considerably cultivated discrimination" ([Pe92]; p.183). For the observation the most important precondition is passivity, i.e. not to give way to the natural presure to immediately mix the observation with own ideas.

For the *experimentation* however, an active energy, a persistence, and a strong contribution of will is essential. For Peirce there is no doubt that, what ever strengthens the will also strengthens the power of logical thinking ([Pe92]; p.187). Experimentation needs furthermore a certain measure of resourcefullness, i.e. of movability of the creative imaginative faculty, of flair for significant questions and answers as well as of persistence to clarify advantages and disadvantages of different answers. For training logical thinking one should again and again be activated to experiment systematically; for this, systematic recordings are indispensable. In general, Peirce recommends to record on paper cards all what is noteworthy. For an eager student Peirce estimates approximately 20.000 paper cards per year by which he can built up a rich treasure of experience for his experimental thinking.

Habituation contains as mental operation the power of readily taking habits and of readily throwing them off; for Peirce there is no habit more useful than this habit taking up and easily throwing off mental habits ([**?**]). Important for logical thinking is to win new connections of thoughts; the necessary readiness to take up something new determines also the readiness to give up something old. For Peirce the learner of logical thinking has therefore to be like a child with all its uprightness and naivety of childlike imaginations and all of the plasticity of childlike states of mind. By reading a lot the aimed flexibility of thinking can be trained; for Peirce, reading 50 up to 100 books in a year would be desirable. The right way of reading consists in trying to understand the author and to assimilate his style of thinking. According to Peirce, the power of habituation can be improved in three directions: by exercises in distinguishing and classifying, by exercises in defining and logically analysing of ideas, and by excises in compressing theories and trains of thought ([Pe92]; p.192).

The distinct openess of logical thinking has worked out by Peirce mostly in his fourth Cambridge Lecture on "The first rule of logic". After this basic logical rule, logical thinking show a tendency to correct itself and that is not only by its conclusion, but also by its premisses ([Pe92]; p.165). The quality of self-correction, which already G. W. F. Hegel has considered as constitutive for the dialectic process of growing reason [He86], is important for the logical thinking of each kind of science. Peirce realizes that "research of every type, fully carry out, has the vital power of self-correction and of growth. This is a property so deeply saturating its inmost nature that it may truly be said that there is but one thing needfull for learning the truth, and that is a hearty and active desire to learn what is true" ([Pe92]; p.170).

The self-correction of logical thinking stands in the direct connection with another property of logical thinking, that is the principal *criticizability.* This property has, according to Peirce, to be understood first of all as sense-critics in the view of the *pragmatic maxim*, which in particular founds a connection between logic and ethics. Peirce writes in 1902/03: " ... which makes logic and ethics to peculiar normitive sciences is this: nothing can be logically true or morally good without a purpose in regard to that it can be named. Since a sentence and in particular the conclusion of an argument which would be only accidentally true, that is not logic" ([Ap75]; p.175). 1903 Peirce finished his Havard-Lectures about pragmatism with the maxim: "The elements of each concept enter into the logical thinking through the door of perception and go out again through the door of purposeful action; and all, what cannot be identified at the two doors, has to be detained as not authorized by the reason" ([Pe91]; p.420).

2 Mathematical Thinking

With the theme "Human Being and Mathematics" the relationship of logical thinking and mathematical thinking shall be examined in this contribution; therefore the *mathematical thinking* shall now be considered in more detail. To keep the connection with logical thinking in mind, it shall be first explained how Peirce makes mathematics and mathematical thinking in his Cambridge Lectures on *"Reasoning and the Logic of Things"* [Pe92] to his theme. For the authors of the extensive introduction for the first complete edition of these lectures, Keneth Laine Ketner and Hilary Putnam, the mathematics in the lectures play such a dominant role that they could even prefer the titel *"The Consequences of Mathematics"*. For this they stated several reasons: First Peirce had already planed and elaborated some provisional lectures as advanced contributions stimulated by the invitation to give a lecture series; these lectures were primarily planed mathematical. When he as well under the pressure of his promotor William James took back considerably the mathematical parts - because of the general understandability - , the basic character of mathematics however remained in the lectures. This links with a second reason that namely Peirce understood his philosophy, under which in the lectures also the logic is incorporated, as a consequence of mathematics. Thirdly Ketner and Putnam see in the expression "Consequences of Mathematics" an even deeper lying importence; they write: "Peirce argued that, epistemologically at any rate, mathematics was an observasional, experimental, hypothesis-confirming, inductive science that worked only with pure hypotheses without regard of their application in "real" life. Because it explored the consequences of pure hypotheses by experimenting upon representative diagrams, mathematics was the inspirational source for the pragmatic maxim, the jewel of the methodological part of semeotic, and the distinct feature of Peirce's thought" ([Pe92]; p.2).

At the end of his first Cambridge Lecture Peirce classifies the sciences ordered by the abstractness of its objects. He places *mathematics* as the most abstract of all sciences, because mathematics is for him the only science which is not concerned to explore what the actual facts are, but inquires hypotheses ([Pe92]; p.114). The objects of mathematics have consequently no actual existence, but are only modi of potential being. The goal that the pure mathematics approaches by making stepwise accessible an expending cosmos of forms of abstract thinking, that is - in the long run - the *potential world of reality.* As the formal science of potential reality, mathematics delivers formal-hypothetical foundations for all other sciences and humanities. In this sense logic is founding on mathematics. Thus Peirce judges: "All necessary logical reasoning is strictly speaking *mathematical reasoning*, that is to say, it is performed by *observing* something equivalent to a mathematical diagram" ([Pe92]; p.116). For the mathematical reasoning Peirce has developed as a kind of algebraic logic the mathematical logic of relations which he introductary explains in his third Cambridge Lectures.

To understand better how mathematical thinking is able to develop a mutual play between abstracting and concretizing, respectively, and to make it effective in the thinking and acting of human beings, the nature of mathematical thinking shall be made more understandable. For this, opinions and discoveries shall be used which Philip Kitcher explains in his book *"The nature of mathematical knowledge*" [Ki84]. For Kitcher there are three obvious insights: "First, we originally acquire much of our mathematical knowledge from teachers, on whose authority we accept not only basic principles but also conceptions of the nature of mathematical resoning. Second, some of this knowledge is acquired with the help of perceptions. Our early training is aided by the use of rods and beads; later, we appeal to diagrams. Third, mathematics has a long history. The origins of mathematical knowledge lie in the practical activities of Egyptians and Babylonians (or, perhaps, people historically are more remote)" ([Ki84]; p.91f.). Kitcher worked out these insights in his book to a convincing *Theory of the Mathematical Thoughts and Knowledge.* This process began in the earliest time with rudimentary perceptions and ideas which developed a first understanding of an arithmetic of small numbers and of a geometry of simple plane figures. Out of those roots, a mathematical thinking has been developed erected on existing knowledge, respectively, and renewed by changes for which Kitcher dicusses in detail the general activities of answering questions, generating questions, generalizing, rigorous changing and systematizing; in doing so, he examines the process of development of mathematical thinking in the sense of Kuhn's thesis that scientific change means a change of practice and not only of theory.

Kitcher explains the relationship of mathematical thinking to the real world in particular at *general actions of thinking* as collecting, segregating, combining, correlating etc. and their idealizations to mathematical operations of thinking. For example, he represents the set theory as an idealized theory of forming collections. How fruitful those mathematical idealizations of general actions of thinking are can be demonstrated by the action of thinking *"summerize to a whole"* which is based on Cantor's definition of sets. For instance, in the script of a lecture on "Linear Algebra I" mathematical elements have been summerize to sets as a whole which can be demonstrated as follows:

1. the real numbers to the whole $\mathbf{R}$ of all real numbers,
2. the triples of real numbers to the analytic representation $\mathbf{R^3}$ of the space of intuition,
3. the sections of the same length and direction to a vector,
4. scalars to a matrix,
5. the even numbers to the binary cipher 0 and the odd numbers to the binary cipher 1,
6. objects, attributes, and a joining relation to a formal context,
7. elements with the same properties to a set,
8. the subsets of a set S to the power set $\mathbf{P}(S)$,
9. a family of sets to their union, to their intersection, and to their direct product,
10. ordered pairs of sets to a relation,
11. equivalent elements to an equivalence class,
12. the equivalence of an equivalence relation to the appertaining quotient set,
13. relating arrows to a mapping,
14. the permutations of a set M and their concatenations $\circ$ to the symmetric group S_M,
15. the symmetries of a geometric figure F and their concatenations $\circ$ to to the symmetry group $Sym(F)$,
16. the cosets of a normal subgroup and the representational association to the appertaining quotient group,
17. the real numbers with addition and multiplication to the field $\mathbf{R}$ of the real numbers,
18. scalars to an n-tuple,
19. the n-tuples of elements of a field K and their componentwise additions and multiplication with a scalar to the vector space K^n,
20. the algebraic structures in which the vector space axioms are valid to the concept of the vector space,
21. elements of a vector space and the apppertaining scalars to a linear combination,
22. all linear combinations of elements $a_1, ..., a_k$ of a vector space to the subspace $< a_1, ..., a_k >$ generated by the given elements,
23. the elements of a vector space which a linear mapping ϕ maps on 0 to the subspace $Ker\phi$,
24. linear equations to a linear system of equations,
25. the solutions of a linear system of equations in n-variables to the affine subspace of the vector space K^n.

The mathematical examples make clear that the combination to a whole may end up quite different depending on what is formally mend by combining to the whole. What can hardly be differentiated in the common language may become transparent in the mathematical language: In 1., 2., 3., 5., 7., 8., 10., 11., 12., 13., 22., 23., 25., the forming of sets are of different nature; also 24. could be seen as a forming of sets, but the word "system" indicates that there is more what is expressed in equalities of variables. The formation of tuples and matrices in 18. and 4. are usually not considered as set formation, just as the structure formations in 9., 14., 15., 16., 17., and 19. In 6. and 21. one has sets and elements, respectively, which are formed by terms and in 20. by concepts. Further differentiations are obtained when the combined whole is mathematically characterized, which however shall not be elaborated. An extensive investigation of mathematical thinking in linear algebra has been presented by Katja Lengning and Susanne Prediger in [LP00].

On the basis of the rich treasure of mathematical forms, *the mathematical thinking* has the special ability to formally arrange and structure contents of thinking in great variety, by which more transparency and clearness can be usually gained. For Martin Heidegger this ability is even characteristic for modern thinking, and that is in the sense that not only the content is arranged by forms of thinking, but that also the content is understood at all by the corresponding forms of thinking. Heidegger sees this basic character of modern thinking and knowledge in the knowledge claim which he calls the *"mathematical"*. About this, Heidegger writes in his book "Die Frage nach dem Ding": "The mathematical is that basic position to the things in which we propose the things to what they are already given. The mathematical is therefore the basic assumption about the knowledge of the things" ([Hd62]; p.58). Mathematical thinking can hence not only be understood by the lexical meaning as the thinking belonging to mathematics, but more general as a thinking of forms able to the design which according to Heidegger is set "for which we actually consider the things, as what they are acknowledged in advance" ([Hd62]; p.71). Then the mathematical thinking is not explainable out of mathematics, but the *mathematics* is itself only a certain formation of mathematical thinking. Such an understanding of mathematics is closely related to the view which Reuben Hersh propagates in his book "What is Mathematics, Really?" [Hr97]. The historical, social-cultural forming of mathematics can be understood in such a way that out of figures and operations of mathematical form-thinking, which are again and again activated in communications, formal systems of thinking are formed in a process of a progressive conventionalizations and constituted out of this a culture of thinking which is called *"mathematics"* [Wi00a], [Wi01].

3 Human Being, Mathematics and Reality

The previous discusion about the relationship of human being and mathematics started from the understanding that it is intrinsic for a human being to think and to act reasonable, i.e. in particular to win insights, to form a judgment, and to follow after that in all actions. That mathematics supports the reasonable thinking and acting has its central reason in the *close connection of logical thinking and mathematical thinking.* Therefore the effort is worth to understand this connection between the logical and the mathematical and to make it effective. According to Peirce's pragmatic maxim, this means that logical thinking in his relationship to reality should be mathematized in a way that the connection of mathematizing with the manifold of potentially appertaining realities can be better understood and activated.

An attempt to that has been made in our "Darmstadt Research Group on Concept Analysis" with the elaboration of a *"contexual logic"* which is understood as a mathematization of the traditional philosophical logic with its doctrines of concept, judgment, and conclusion [Wi00b]. The basis of this philosophical logic underlies the view that the human recognition and thinking activates the basic logical structures concept, judgment, and conclusion by bringing realities under concepts, forming judgments from concepts, and concluding judgments out of other judgments. On this base, Gottlob Benjamin Jäsche makes clear in his introduction to the logic-lectures of Imanuel Kant (edited by Jäsche) with Kant's explicit explanation that "it is nothing else allowed to include in the actual treatise of logic and particularly in the elementary treatise as the theory of the three essential main functions of thinking - the concepts, the judgments, and the conclusions ([Ka83b]; p.424). Since the contextual logic is elaborated as a mathematical theory the basic structures of which are abstracted out of the traditional philosophical logic (cf. [Pr00]), the contextual logic is classified in a "contextual concept logic", a "contextual judgment logic", and a "contextual conclusion logic"; in its whole, the contextual logic is founded on the set-theoretic semantics of modern mathematics.

For the *contextual concept logic* it is first to answer the basic question: What is the properly abstracting linguistic set definition of the concept of concept? According to Piaget, concepts are cognitive structures which can only fulfill their task of the differentiating reconstruction of the reality, when they can be coordinated systematically and constructed by its complex concept systems; concepts are therefore formed in a relational structure which is constitutive for them. Therefore it counts first of all to introduce relational structures for creating abstract concepts as set structures in the greatest possible generality. That became successful - as rich experiences in the last thirty years have shown - with the conception of the *formal context* formed by objects, attributes, and a joining relation. A *"formal context"* is defined as a set structure (G, M, I) which consists of two sets G and M and a relation I between the sets G and M; the elements of G are called $(formal)$ *objects*, the elements of M are called $(formal)$ *attributes*, and the relational connection gIm is read: the object g has the attribute m. In the sense of Peirce's categories, an object is considered in a formal context as a First with an attribute as a Second which are linked by the context relation as a Third. ($Fig.$ 1)

Formal contexts can be understood as mathematization of real-world cross-tables. For instance, the *cross-table* presented in $Fig.$ 1, which is taken out of the publication "Kontrastive Untersuchung von Wortfeldern im Englischen und Deutschen" [Kr79], can be abstracted to a formal context (G_W, M_W, I_W) as follows: the object set G_W consists out of words of the investigated semantic

	natural	artificial	stagnant	running	inland	maritime	constant	temporary
tarn	×		×		×		×	
trickle	×			×	×		×	
rill	×			×	×		×	
beck	×			×	×		×	
rivulet	×			×	×		×	
runnel	×			×	×		×	
brook	×			×	×		×	
burn	×			×	×		×	
stream	×			×	×		×	
torrent	×			×	×		×	
river	×			×	×		×	
channel				×	×		×	
canal		×		×	×		×	
lagoon	×		×			×	×	
lake	×		×		×		×	
mere		×	×		×		×	
plash	×		×		×			×
pond		×	×		×		×	
pool	×		×		×		×	
puddle	×		×		×			×
reservoir		×	×		×		×	
sea	×		×			×	×	

Fig. 1. Formal context partly representing a lexical field "bodies of waters"

field "waters" in [Kr79] and the attribute set M_W consists of noems (smallest elements carring a meaning) by which the words are characterized according to their contents, and the relation I_W are grasped by the relationships which are indicated by the crosses; i.e. the mathematical expression "$puddle I_W temporary$" stands for the linguistic relationship "the word 'puddle' has the noem 'temporary' " indicated by a cross in the cross-table. In general, the cross-table has to be distinguished from the formal context which is abstracted from the cross-table; thus, a cross-table has a logical structure with which real relationships can be presented, but a formal context is a mathematical structure which first of all challenges the activation further mathematical structures and connections. In spite of their location, cross-table and formal context form a model for the close connection of logical and mathematical thinking.

For the mathematization of 'concept', the formal context as mathematization of the nessecary relational structure can now be assumed: A *formal concept* of a formal context (G, M, I) is defined as a pair (A, B) where A is a subset of G and B is a subset of M so that A consists of all those objects in G which have all attributes of B and B consists of all those attributes in M which apply to all objects in A; A is named the *extent* and B is named the *intent* of the formal concept (A, B). This mathematization proceeds from the philosophical understanding of concept; according to that, a concept is a unit of thought consisting of an extension and an intension, as it was already presented by the logic of Port Royal [AN85] in the 17th century (cf. also [Wa73], [Wi95]). A formal concept (A, B) of (G, M, I) is called a *subconcept* of a formal concept (C, D) in

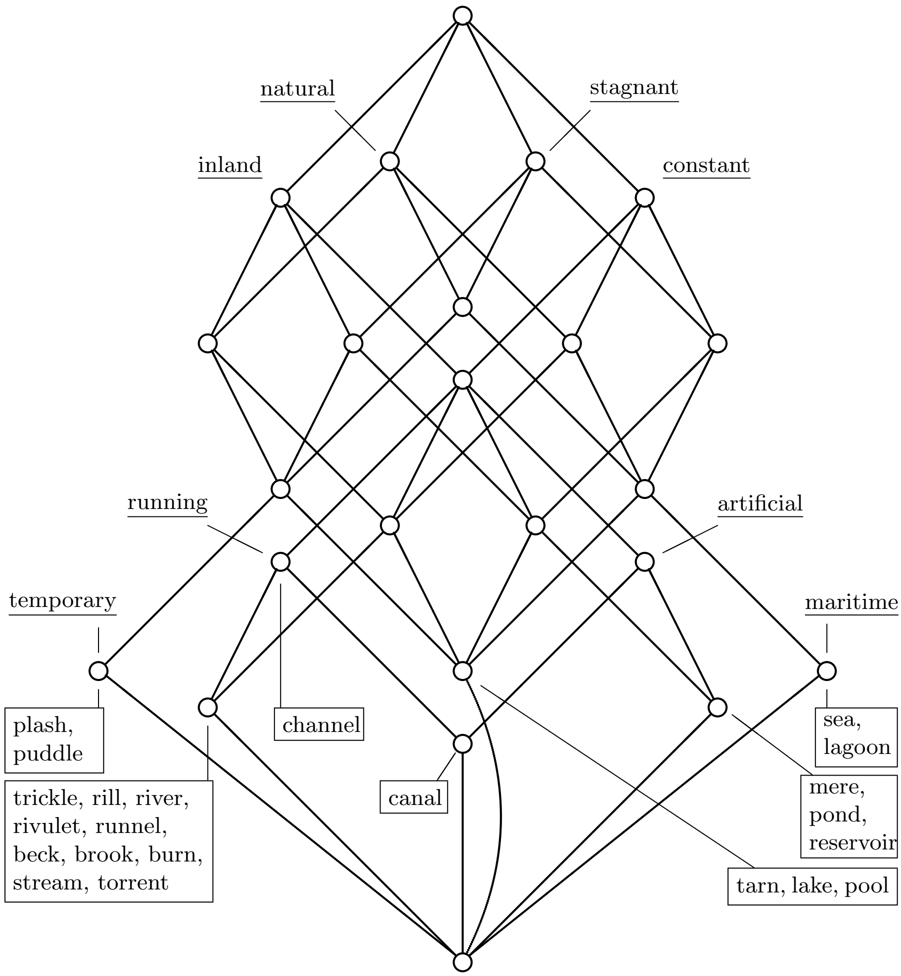


Fig. 2. Concept lattice of the formal context in Fig.1

(G, M, I) and (C, D) a *superconcept* of (A, B) if the extent A is contained in the extent C and, equivalently, if the intent B contains the intent D.

The *logical reciprocity* "the greater the concept extent the smaller the concept intent", which becomes visible by this equivalence, is winning conciseness and fruitfullness by the contextual mathematization of concept which lastingly moves the mathematical thinking. The reciprocity can be formulated by the definition of "derivation operators" of a formal context (G, M, I): For $X \subseteq G$ anf $Y \subseteq M$ the *derivation* is defined, respectively, by

$$X^I := \{m \in M | gIm \text{ for all } g \in X\} \text{ and } Y^I := \{g \in G | gIm \text{ for all } m \in Y\};$$

i.e. the derivation X^I is the set of all attributes out of M which all objects have, and the derivation Y^I is the set of all objects out of G which all attributes have. For $A \subseteq G$ and $B \subseteq M$, (A, B) is then obviously a formal concept of (G, M, I) if and only if $A = B^I$ and $B = A^I$. The logical reciprocity now finds its differentiated expression by the following mathematical discovery: For $U, V \subseteq G$ or $U, V \subseteq M$ we obtain:

$$(1)\text{: } U \subseteq V \text{ implies } U^I \supseteq V^I,\ (2)\text{: } U \subseteq U^{II},\ (3)\text{: } U^I = U^{III}.$$

For the task to determine the formal concepts of a formal context (G, M, I), the equation in (3) is basic because it follows from (3) that for $X \subseteq G$ and $Y \subseteq M$, respectively, the pairs (X^{II}, X^I) and (Y^I, Y^{II}) are formal concepts of (G, M, I); in particular, the special case of the *object concepts* $\gamma g := (\{g\}^{II}, \{g\}^I)$ and the *attribute concepts* $\mu m := (\{m\}^I, \{m\}^{II})$ are important. The mathematical potential of the derivation operators which become transparent by the relationships in (1), (2), and (3) cannot be estimated high enough; they represent mathematical connections which in general have been studied and activated multifariously as set-theoretic and logical dualities (also called *Galois conections*).

The set of all concepts of a formal context (G, M, I) forms with the subconcept-superconcept relation a mathematical structure of a complete lattice, which therefore is called the *concept lattice* of (G, M, I). The mathematical structure of a concept lattice can be made effectively accessible to logical thinking by (inscribed) *line diagrams*. The line diagram in Fig. 2 [KW87] represents the concept lattice of the formal concext which is presented by the cross-table in Fig. 1. The little circles of the line diagram represent the formal concepts of the appertaining formal context and the ascending line segment represent the subcontext-superconcept-relation. Hence the little circle in Fig. 2 to which the label "artificial" is assigned represents a subconcept of the concepts with the labels "inland" and "constant"; this indicates that, according to [Kr79], there is the logical relationship in English that each "artificial" water has the attributes "inland" and "constant". In general, the extent and intent of formal concepts can be read from the line diagram as follows: The concept extent consists of all objects the names of which are attached to a circle linked by an ascending sequence of line segments to the circle of the chosen concept. In Fig. 2, for instance, the little circle directly above the circle with the label "artificial" represents a concept the extent of which consists of the words "sea", "lagoon", "tarn", "lake", and "pool" and the intent of the noems "natural", "stagnant", and "constant". From this discussion it follows in particular that the underlying context can be reconstracted from the line diagram, i.e. no data are lost by the construction of the concept lattice and line diagram. Therefore the logical connections of the data represented in the cross-table can completely be reconstructed.

The logical connections which usually demand special interest are the *contextual implication between attributes*. From the line diagram of Fig. 2 one reads for instance that, according to [Kr79], each running water is always also constant and inland. Also of interest are the *classification of objects* by suitable combinations of attributes. The line diagram in Fig. 2 shows that the smallest of such classification consists of six concept extents: {*"plash"*, *"puddle"*}, {*"trickle"*, *"rill"*, *"river"*, *"rivulet"*, *"runnel"*, *"beck"*, *"brook"*, *"burn"*, *"stream"*, *"torrent"*}, {*"canal"*}, {*"tarn"*, *"lake"*, *"pool"*}, {*"meer"*, *"pond"*, *"reservoir"*}, and finally {"*sea*″, "*lagoon*″}. Respectively, a sequence of further forms of investigations and activations of logical connections in data contexts are treated in the papers [Wi87] and [Wi00c]. In the contrastive study in [Kr79] the *comparison of German and English semantic fields* with the same noems, respectively, are standing in the foreground. Remarkable is the finding that the first eight noems yield the same kind of concept lattices in German and English, which has the consequence that also the logical implications between the noems are equal. This is different at the classifications of objects, already because the English has considerably more words for waters as the German. This is also the reason for it, that further noems thoroughly result in different concept structures.

Line diagrams of concept lattices inspire again and again to *critics* and *self-correction* on the basis of background knowledge. A reseach project which provided multifarious examples for this was a common project of the Darmstadt research group on Formal Concept Analysis and of the ministry of building constructions and housing projects of the province "Nordrhein-Westfalen" [EKSW0]. The developed exploration system was supposed to support the administrative office with its supervision of building works to consider the legal regulations and technical determinations during the planing, examination, and execution of building projects in the necessary extent. For the exploration system an extensive data context was elaborated, the objects of which are the constructional relevant paragraphs or text-units of the pertinent laws and regulations and the attributes of which, understood as search words, are concerned with the structural components and their demands which are related to the text units. For the exploration system frequent concept lattices from the underlying data context were derived and represented by line diagrams to be able to use them as conceptual searching structures.

Already during the system development, line diagrams have multifariously fulfilled to make logical connections transparent. In this way the line diagrams have always again qualified the building experts to find mistakes in the extensive data contexts which has contributed to a conciderable improvement of the data quality. An instructtive case of *criticism and self-correction* has happened by means of the line diagram presented in Fig. 3, that makes available information to the theme "function rooms in a hospital": For testing the readability of such diagrams, a secretary was included into the meeting in the ministry. The secretary became much surprised that §51 of the "BauONW" ("Bauordnung Nordrhein-Westfalen"), which demands expansions necessary for handicapped people, was only attached to the circle with the label "toilet" (in the version of the diagram "function rooms in a hospital" at that time); she could not understand why the wash- and bathrooms do not have to meet requirements for handicapped people too. Even the experts became surprised when they checked again §51 and saw that only toilets are mentioned in connections with handicapped people. Only after a comprehensive discussion the experts came to the conclusion that, by superior aspects of law, §51 should apply also to wash- and bathrooms. Finally, by similar reasons, the consulting rooms and the residential rooms (bedrooms) were also included so that, in the underlying cross table, three

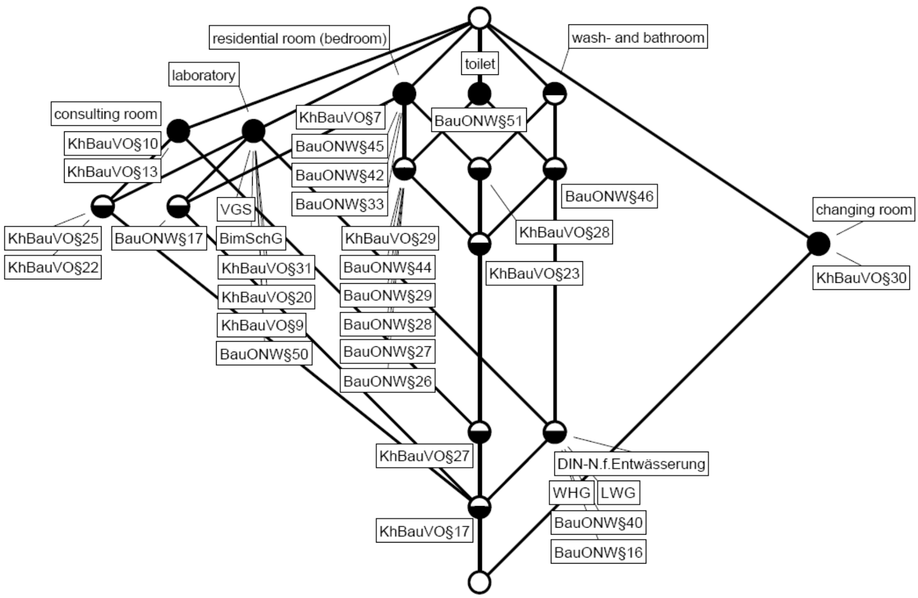


Fig. 3. Query structure "functional rooms in a hospital" of a TOSCANA information system about laws and regulations concerning building construction

more crosses were added in the row headed by "BauONW§51" so that, in the line diagram of Fig. 3, the label "BauONW§51" moved down to the circle with the label "KhBauVO§27".

The *Contextual Judgment Logic*, developped since 1996, builds up on the Contextual Concept Logic because judgments are formed by concepts. An elaborated informing Judgment Logic is already present since more than thirty years by Sowa's *Theory of Conceptual Graphs* [So84] which is founded on the logic of existential graphs of Charles Sanders Peirce and the logic of semantic networks of artificial intelligence. Conceptual graphs, as the simple example in Fig. 4 [So92], are understood as logical abstractions of linguistic expressions; they represend semantic judgments, i.e. valid statements. The conceptual graph in Fig. 4 represents the sentence "John is going to Boston by bus" as *judgment-logical structure*, where the sentence is logically further differentiated with assistance of background knowledge: John is identified as an instance of the concept "Person" and Boston as an instance of the concept "City"; furthermore, three valences of the concept "Go" are specified by the semantic relation "agent", "destination", and "instrument". The presented conceptual graph can be described in detail as follows: "There is some going which has as agent the person John, as destination the city Boston and as instrument some bus." The further *"logical differentiation*" discloses not only the background of a language, but supports further treatments as the translation to other languages, the preparation for document management etc. How a technical text can be judgment-logically processed has been, for instance, made clear in ([MSW99]; p.426) by

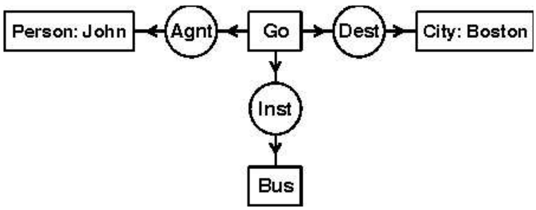


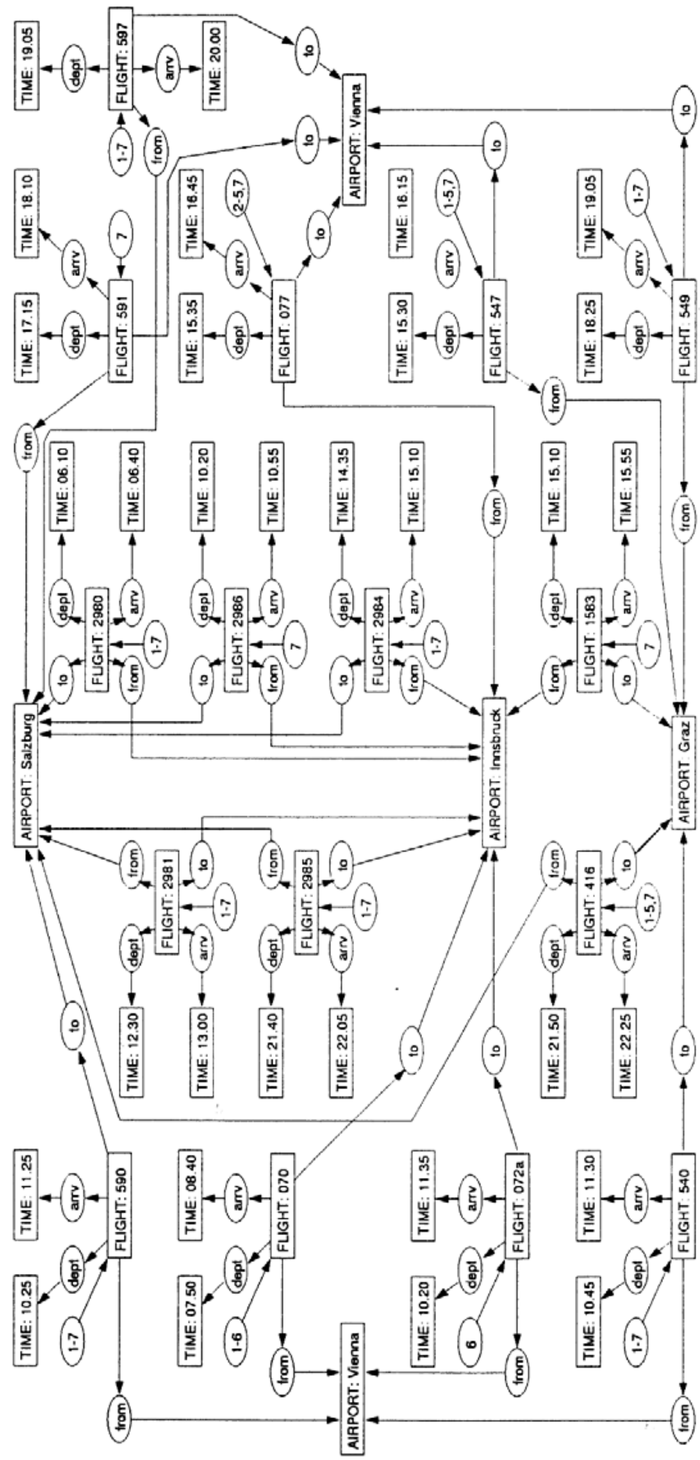
Fig. 4. A simple conceptual graph

the conceptual graph which represents the instruction for decalcifying a coffee machine.

The Contextual Judgment Logic adapts the Sowa graphs by taking the concepts and relations of the conceptual graphs as formal concepts of already given contexts; with that the conceptual graphs become mathematical structures for which the modified naming *"concept graphs"* has been chosen to distinguish between the mathematical and the logical (s. [Wi97], [Wi00b]). Within the Contextual Judgment Logic, judgments are represented by concept graphs which are therefore also named *formal judgments.* With the abstraction of judgments to mathematical structures, mathematical theories and methods can be activated for the judgment logic in a wide range. The promising method which, up to now, has been stimulated and made possible the mathematization of the judgment logic is the derivation of concept graphs out of relational data basis, which are mathematized contextual-logically in a suitable manner (s. [PW99]); i.e. expressed slogan-like: with this method, relational data bases can be "made speaking". Fig. 5 gives an insight into an informatoly application of this method: The upper diagram shows a concept graph derived out of a flight data base represented as a Sowa graph, which shows the possible flights of a weekend traveller from Vienna to Salzburg, Innsbruck, Graz, and back to Vienna; the lower diagram is a user-friendly representation of the same graph which uses more background knowledge of the traveller (s. [EGSW0], [Wi00c]).

The *Contextual Conclusion Logic* has already concept-logical and judgment-logical precursors by the Contextual Attribute Logic [GW99] and the Contextual Logic of Relations [Wi00d] which adapted the Peircean algebraic logic as reconstructed in [Bu91]. The *Contextual Conclusion Logic* however obtains its full foundation by the interplay of an elaborated syntax and semantics for concept graphs for which Susanne Prediger made available in [Pr98] convincing conceptions and results. Certainly, the interplay of mathematical structural thinking, the diagrammatic conclusions of Charles Sanders Peirce, and the logical thinking in general have to be understood even more deeper, in particular in relationship to the concrete intercourse with such a culture of thinking.

The close connection between the logical thinking and the mathematical thinking, which becomes visible in the frame of contextual logic, makes possible multifariously an effective support of logical thinking through mathematical thinking which can also be extended to *rich mathematical structures.* For example, the contextual-logical concept theory was already extended to an


FLIGHT: 597
TIME: 19.05
TIME: 20.00
FLIGHT: 591
TIME: 17.15
TIME: 18.10
FLIGHT: 077
TIME: 15.35
TIME: 16.45
2-5,7
FLIGHT: 547
TIME: 15.30
TIME: 16.15
1-5,7
FLIGHT: 549
TIME: 18.25
TIME: 19.05
AIRPORT: Vienna
FLIGHT: 2980
TIME: 06.10
TIME: 06.40
FLIGHT: 2986
TIME: 10.20
TIME: 10.55
FLIGHT: 2984
TIME: 14.35
TIME: 15.10
FLIGHT: 1583
TIME: 15.10
TIME: 15.55
AIRPORT: Salzburg
AIRPORT: Innsbruck
AIRPORT: Graz
FLIGHT: 2981
TIME: 12.30
TIME: 13.00
FLIGHT: 2985
TIME: 21.40
TIME: 22.05
FLIGHT: 416
TIME: 21.50
TIME: 22.25
FLIGHT: 590
TIME: 10.25
TIME: 11.25
FLIGHT: 070
TIME: 07.50
TIME: 08.40
1-6
FLIGHT: 072a
TIME: 10.20
TIME: 11.35
6
FLIGHT: 540
TIME: 10.45
TIME: 11.30
1-7
7
dept
arrv
from
to

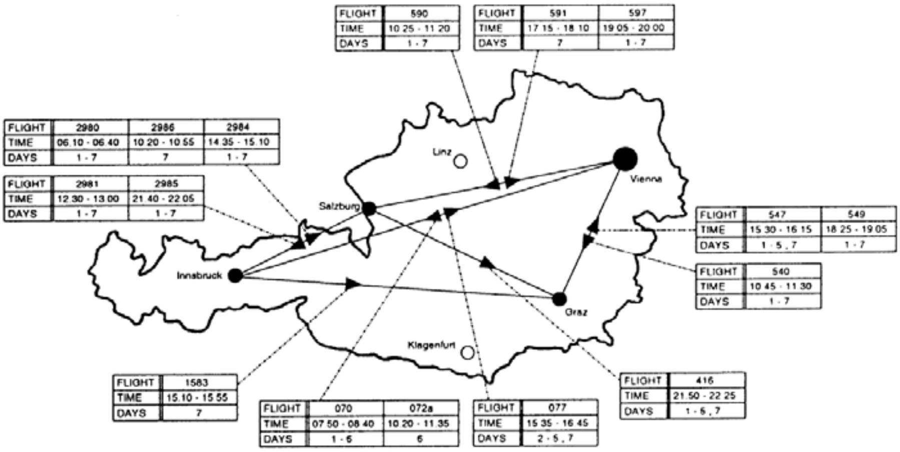


Fig. 5. Two representations of the same concept graph concerning flight connections

algebraic concept analysis [Vo94], to a contextual topology [Ha92], [Sa01], and to a relational concept analysis [Ps98]. These extensions correspond with the three structure types of the Bourbaki architecture of mathematics [Bo74] which Jean Piaget has recognized in the close connection to the structures discovered by him in the thinking of young children. In ([Pi73]; p.34f.), Piaget writes about the discussion with Jean Dieudonn, the founder of the Bourbaki-Group: "... to our great surprise we both found out that there exists a very direct connection between the three mathematical structures and the three structures of the operational thinking of children." Even if the activation of this relationship in the "New Math"-movement was exceeded one-sidedly, an appropriate presentation of that relationship would enrich the learning of mathematics and would contribute to an efficient connection from the mathematical to the logical thinking.

Naturally, the logical thinking with its reference to reality has also inversely a lasting effect on the development of mathematics by stimulating always further differentiations of mathematical thinking. Deputizing for the large manifold of such differentiations, it shall finally be mentioned a *new view on mathematics* which has been produced during the elaboration of the contextual logic under the influence of the triadic doctrine of categories of Charles Sanders Peirce: Different real world connections have shown that the elementary connection "an object has an attribute" should be specified in which way, under which conditions, by which arguments, on which purpose, in which situation such a connection is valid. This caused to extend the set structure of a formal context to a triadic structure [LW95], the appertaining concept structure of which was mathematically characterized by a so-called "trilattice" [Bi98]. The thereby possible mathematical *theory of triadic concepts* could already be applied within the Contextual Judgment Logic to receive mathematically the modal character of judgments [Wi98], [Pr98], [DW00]. To what extend the triadic view can generally be made productive for mathematics, this has to be explored by further research.

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