Double Approximation and Complete Lattices

Taichi Haruna[1,2,⋆] and Yukio-Pegio Gunji[1]

[1] Department of Earth and Planetary Sciences, Graduate School of Science, Kobe University, 1-1, Rokkodaicho, Nada, Kobe, 657-8501, Japan
[2] PRESTO, JST, 4-1-8 Honcho Kawaguchi, Saitama, Japan
tharuna@penguin.kobe-u.ac.jp

Abstract. A representation theorem for complete lattices by double approximation systems proved in [Gunji, Y.-P., Haruna, T., submitted] is analyzed in terms of category theory. A double approximation system consists of two equivalence relations on a set. One equivalence relation defines the lower approximation and the other defines the upper approximation. It is proved that the representation theorem can be extended to an equivalence of categories.

Keywords: Rough sets, complete lattices, representation theorem, equivalence of categories.

1 Introduction

It is well-known that regular open sets in a topological space form a Boolean algebra. Recall that a regular open set U in a topological space X is an open set satisfying $\mathrm{Int}(\mathrm{Cl}(U)) = U$, where $\mathrm{Int}(Y)$ is the interior of Y and $\mathrm{Cl}(Y)$ is the closure of Y for a subset $Y \subseteq X$. An implicit assumption is that both the interior and the closure are taken under the same topology. What happens if the two operations are considered in different topologies? In this paper we consider this question not in topology but in rough set theory.

Given an equivalence relation R on a set X, rough set theory considers two approximations [5,6]. One is the R-lower approximation R_* which is an analog of the interior operation in topology. The other is the R-upper approximation R^* which is an analog of the closure operation in topology. A subset $Y \subseteq X$ satisfying $R_*R^*(Y) = Y$ is an analog of the notion of regular open set. However, $R_*R^*(Y) = Y$ if and only if $R^*(Y) = Y$ if and only if $R_*(Y) = Y$. Hence there is no analog of the distinction between open sets and closed sets. It is easy to see that $Y \subseteq X$ satisfies one of the above three conditions if and only if it can be written as a union of R-equivalence classes. Hence they form a field of sets, a Boolean algebra.

Now let us consider two equivalence relations R, S on a set X. We will call a triplet (X, R, S) *a double approximation system*. An analog of the notion of regular open set is a subset $Y \subseteq X$ satisfying $S_*R^*(Y) = Y$. We denote the set of all S_*R^* fixed subsets by $\mathrm{Fix}(S_*R^*)$. In previous work [2], it is shown

⋆ Corresponding Author.

P. Wen et al. (Eds.): RSKT 2009, LNCS 5589, pp. 52–59, 2009.

that one can make $\mathrm{Fix}(S_*R^*)$ a complete lattice by defining joins and meets in a suitable manner. Moreover, it is also proved that any complete lattice L can be represented as $\mathrm{Fix}(S_*R^*)$ for suitably chosen equivalence relations R, S on a set X obtained by information about L. The aim of this paper is to analyze this representation of complete lattices in terms of category theory. We extend the representation theorem to an equivalence of categories. In particular, we will see that the isomorphism that makes a complete lattice L isomorphic to $\mathrm{Fix}(S_*R^*)$ defines a natural isomorphism in the equivalence of categories.

This paper is organized as follows. Section 2 is preparations. We review the basic notions of rough set theory [7] and results of [2]. In section 3 we introduce the category of double approximation systems consisting of triplets (X, R, S), where X is a set, R and S are two equivalence relations on X. In section 4 we prove our main result mentioned above. Finally, concluding remarks are given in section 5.

2 Preparations

In this section we summarize the basic notions of rough set theory [7] and results of [2] relevant to this paper.

For an equivalence relation R on a set X, we write $\mathbb{F}(R)$ for a field of sets generated by R ordered by set inclusion $\subseteq$, that is, if $\mathbb{F}(R)$ is seen as a Boolean algebra by set theoretic intersections, unions and complements, its atoms are R-equivalence classes. However we will not consider the structure of Boolean algebra but treat them as just partially ordered sets. For a subset $Y \subseteq X$, its R-upper approximation is a set $R^*(Y) = \{x \in X | [x]_R \cap Y \neq \emptyset\}$ and its R-lower approximation is a set $R_*(Y) = \{x \in X | [x]_R \subseteq Y\}$, where $[x]_R$ is the R-equivalence class containing x. Both $R^*(Y)$ and $R_*(Y)$ are elements of $\mathbb{F}(R)$ for any $Y \subseteq X$.

Let R, S be two equivalence relations on a set X [2]. If the operation S_*R^* is seen as a map from $\mathbb{F}(S) \to \mathbb{F}(S)$ then one can show that it is a closure operator on $\mathbb{F}(S)$. For any $Y, Z \in \mathbb{F}(S)$, we have (i) $Y \subseteq S_*R^*(Y)$, (ii) if $Y \subseteq Z$ then $S_*R^*(Y) \subseteq S_*R^*(Z)$ and (iii) $S_*R^*(S_*R^*(Y)) = S_*R^*(Y)$. (iii) is a consequence of a more general equation $R^*S_*R^* = R^*$ held on $\mathbb{F}(S)$, which will be used repeatedly in the following sections (dually, we also have $S_*R^*S_* = S_*$ on $\mathbb{F}(R)$). However, one can show that (ii) and (iii) hold for any $Y, Z \subseteq X$ [2]. Let us denote the set of all fixed points of S_*R^* by $\mathrm{Fix}(S_*R^*)$. Since S_*R^* is a closure operator, one can make $\mathrm{Fix}(S_*R^*)$ a complete lattice by the usual way [1]. Dually, $\mathrm{Fix}(R^*S_*)$ is also a complete lattice, which is isomorphic to $\mathrm{Fix}(S_*R^*)$ [2]. The isomorphisms are given by restricting the maps $R^* : \mathbb{F}(S) \to \mathbb{F}(R)$ and $S_* : \mathbb{F}(R) \to \mathbb{F}(S)$ to $\mathrm{Fix}(S_*R^*)$ and $\mathrm{Fix}(R^*S_*)$, respectively.

Any triplet (X, R, S), where X is a set, R and S are equivalence relations on X, provides a complete lattice $\mathrm{Fix}(S_*R^*)$. How about the reverse direction? Can any complete lattice give a triplet (X, R, S)? What is the relationship between L and (X, R, S) obtained by L? The answer is given in [2]. Given any complete lattice L, one can construct a triplet (X, R, S) that satisfies $\mathrm{Fix}(S_*R^*) \cong L$.

Let $\leq$ be the associated order of a complete lattice L. Put $X = \not\leq \subseteq L \times L$. For $(x, y), (z, w) \in \not\leq$, define $(x, y)R(z, w) \Leftrightarrow y = w$ and $(x, y)S(z, w) \Leftrightarrow x = z$. It is clear that the two relations $R, S \subseteq \not\leq \times \not\leq$ are equivalence relations on $\not\leq$. For $x, y \in L$ put $\rho(y) = \{(z, y) | z \not\leq y, z \in L\}$ and $\sigma(x) = \{(x, z) | x \not\leq z, z \in L\}$. One can show that $\mathrm{Fix}(S_* R^*) = \{\sigma(\downarrow x) | x \in L\}$, where $\sigma(\downarrow x) = \bigcup_{z \leq x} \sigma(z)$. A map $\eta_L : L \to \mathrm{Fix}(S_* R^*)$ defined by $\eta_L(x) = \sigma(\downarrow x)$ for $x \in L$ is an isomorphism. We will show that η is a natural isomorphism with respect to categories and functors defined in the following sections. One can also show that $R^*(\sigma(\downarrow x)) = X - \rho(\uparrow x)$. Hence we have $\mathrm{Fix}(R^* S_*) = \{X - \rho(\uparrow x) | x \in L\}$, where $\rho(\uparrow x) = \bigcup_{x \leq z} \rho(z)$. One may expect that $S_*(X - \rho(\uparrow x)) = \sigma(\downarrow x)$ holds, which can be verified.

3 The Category of Double Approximation Systems

In this section we define the category of double approximation systems.

Definition 1. *The category of double approximation systems $\mathbb{D}$ consists of the following data:*

Objects: *Objects are triplets (X, R, S), where X is a set, R and S are equivalence relations on the set X. We call them double approximation systems.*

Morphisms: *Given two objects (X, R, S) and (X', R', S'), a morphism from (X, R, S) to (X', R', S') is an equivalence class of pairs of maps (f_R, f_S), where f_R is an inclusion-preserving map from $\mathbb{F}(R)$ to $\mathbb{F}(R')$ and f_S is an inclusion-preserving map from $\mathbb{F}(S)$ to $\mathbb{F}(S')$. The two maps f_R and f_S satisfy the following equations:*

$$R'^* S'_* f_R R^* = R'^* f_S S_* R^*,$$
$$S'_* R'^* f_S S_* = S'_* f_R R^* S_*.$$

Two pairs of maps (f_R, f_S) and (g_R, g_S) are equivalent if and only if the following two equations hold:

$$S'_* R'^* f_S S_* R^* = S'_* R'^* g_S S_* R^*,$$
$$R'^* S'_* f_R R^* S_* = R'^* S'_* g_R R^* S_*.$$

If (f_R, f_S) and (g_R, g_S) are equivalent then we write $(f_R, f_S) \sim (g_R, g_S)$. We denote the equivalence class containing (f_R, f_S) by $[(f_R, f_S)]$.

Note that the equivalence class containing (f_R, f_S) is determined by either f_R or f_S. Indeed, one can show that $(f_R, f_S) \sim (R'^* f_S S_*, f_S) \sim (f_R, S'_* f_R R^*)$. For example, $R'^* S'_* (R'^* f_S S_*) R^* S_* = R'^* f_S S_* R^* S_* = R'^* S'_* f_R R^* S_*$. The identity morphism associated with (X, R, S) in $\mathbb{D}$ is the equivalence class $[(\mathrm{id}_{\mathbb{F}(R)}, \mathrm{id}_{\mathbb{F}(S)})]$. The composition of two morphisms $[(f_R, f_S)] : (X, R, S) \to (X', R', S')$ and $[(g_R, g_S)] : (X', R', S') \to (X'', R'', S'')$ is defined by $[(g_R, g_S)] \circ [(f_R, f_S)] = [(g_R R'^* S'_* f_R, g_S S'_* R'^* f_S)] : (X, R, S) \to (X'', R'', S'')$. It is easy to check that the composition is well-defined.

We can associate a complete lattice with each double approximation system (X, R, S) in two ways: $\mathrm{Fix}(S_*R^*)$ and $\mathrm{Fix}(R^*S_*)$. Both ways give rise to functors from the category of double approximation systems to a category consisting of complete lattices defined in the next section. The above conditions on the relationship between f_R and f_S are introduced so that the two functors will be mutually conjugate. We will check this in the next section.

In order to see how the equivalence relation introduced on the set of pairs of maps (f_R, f_S) works, let us consider when two double approximation systems are isomorphic.

Proposition 1. *Two double approximation systems* (X, R, S) *and* (X', R', S') *are isomorphic if and only if* $\mathrm{Fix}(S_*R^*) \cong \mathrm{Fix}(S'_*R'^*)$.

Proof. Suppose $\mathrm{Fix}(S_*R^*) \cong \mathrm{Fix}(S'_*R'^*)$. Let us denote the isomorphism from $\mathrm{Fix}(S_*R^*)$ to $\mathrm{Fix}(S'_*R'^*)$ by f and its inverse by f^{-1}. Define a map $f_S : \mathbb{F}(S) \to \mathbb{F}(S')$ by $f_S = fS_*R^*$. We also define a map $f_S^{-1} : \mathbb{F}(S') \to \mathbb{F}(S)$ by $f_S^{-1} = f^{-1}S'_*R'^*$. f_R and f_R^{-1} can be defined similarly. We would like to show that $[(f_R^{-1}, f_S^{-1})] \circ [(f_R, f_S)] = [(\mathrm{id}_{\mathbb{F}(R)}, \mathrm{id}_{\mathbb{F}(S)})]$. However,

$$\begin{aligned} S_*R^*(f_S^{-1}S'_*R'^*f_S)S_*R^* &= S_*R^*f^{-1}S'_*R'^*S'_*R'^*fS_*R^*S_*R^* \\ &= S_*R^*f^{-1}S'_*R'^*fS_*R^* \\ &= S_*R^*f^{-1}fS_*R^* \\ &= S_*R^* = S_*R^*\mathrm{id}_{\mathbb{F}(S)}S_*R^*. \end{aligned}$$

$[(f_R, f_S)] \circ [(f_R^{-1}, f_S^{-1})] = [(\mathrm{id}_{\mathbb{F}(R')}, \mathrm{id}_{\mathbb{F}(S')})]$ can be shown similarly.

Conversely, given an isomorphism of double approximation systems $[(f_R, f_S)]$ from (X, R, S) to (X', R', S'), we define two maps $f : \mathrm{Fix}(S_*R^*) \to \mathrm{Fix}(S'_*R'^*)$ and $f^{-1} : \mathrm{Fix}(S'_*R'^*) \to \mathrm{Fix}(S_*R^*)$ by $f = S'_*R'^*f_S$ and $f^{-1} = S_*R^*f_S^{-1}$, where f_S^{-1} is chosen from the equivalence class which is the inverse of $[(f_R, f_S)]$. f or f^{-1} are not dependent on the choice of f_S or f_S^{-1}, respectively. Since $[(f_R^{-1}, f_S^{-1})] \circ [(f_R, f_S)] = [(\mathrm{id}_{\mathbb{F}(R)}, \mathrm{id}_{\mathbb{F}(S)})]$, we have $S_*R^*(f_S^{-1}S'_*R'^*f_S)S_*R^* = S_*R^*$. Therefore, for $Y \in \mathrm{Fix}(S_*R^*)$,

$$\begin{aligned} f^{-1}f(Y) &= S_*R^*f_S^{-1}S'_*R'^*f_S(Y) \\ &= S_*R^*f_S^{-1}S'_*R'^*f_SS_*R^*(Y) \\ &= S_*R^*(Y) = Y. \end{aligned}$$

Similarly, we have $ff^{-1} = \mathrm{id}_{\mathrm{Fix}(S'_*R'^*)}$. □

4 An Equivalence of Categories

In this section we show that the representation of complete lattices by double approximation systems can be extended to an equivalence of categories. First we define the category in which we work.

Definition 2. *The category of complete lattices with order-preserving morphisms $\mathbb{C}$ consists of the following data:*

Objects: *Complete lattices.*
Morphisms: *Order-preserving maps between complete lattices.*

Since an order-isomorphism between two partially ordered sets preserves all existing joins and meets [1], two complete lattices are isomorphic in $\mathbb{C}$ if and only if they are isomorphic as complete lattices.

Definition 3. *We define a functor G from $\mathbb{D}$ to $\mathbb{C}$ as follows. G sends each double approximation system (X, R, S) to a complete lattice $G(X, R, S) = \mathrm{Fix}(S_*R^*)$. A morphism $[(f_R, f_S)]$ from (X, R, S) to (X', R', S') in $\mathbb{D}$ is sent to an order-preserving map $G[(f_R, f_S)] = S'_*R'^*f_S$ from $\mathrm{Fix}(S_*R^*)$ to $\mathrm{Fix}(S'_*R'^*)$, where f_S in the right hand side is the restriction of f_S to $\mathrm{Fix}(S_*R^*)$.*

If $(f_R, f_S) \sim (g_R, g_S)$ then $S'_*R'^*f_S = S'_*R'^*g_S$ on $\mathrm{Fix}(S_*R^*)$, hence $G[(f_R, f_S)]$ is well-defined. Let us check G preserves compositions. Given two composable morphisms $[(f_R, f_S)] : (X, R, S) \to (X', R', S')$ and $[(g_R, g_S)] : (X', R', S') \to (X'', R'', S'')$ in $\mathbb{D}$, $G([(g_R, g_S)] \circ [(f_R, f_S)]) = G[(g_R R'^* S'_* f_R, g_S S'_* R'^* f_S)] = S''_* R''^* g_S S'_* R'^* f_S = G[(g_R, g_S)]G[(f_R, f_S)]$.

We can define a similar functor H from $\mathbb{D}$ to $\mathbb{C}$ by $H(X, R, S) = \mathrm{Fix}(R^*S_*)$ and $H[(f_R, f_S)] = R'^*S'_*f_R$. G and H are mutually conjugate in the following sense.

Proposition 2. *For a morphism $[(f_R, f_S)] : (X, R, S) \to (X', R', S')$ in $\mathbb{D}$, $H[(f_R, f_S)]R^* = R'^*G[(f_R, f_S)]$ and $G[(f_R, f_S)]S^* = S'^*H[(f_R, f_S)]$.*

Proof. For $Y \in \mathrm{Fix}(S_*R^*)$,

$$\begin{aligned} H[(f_R, f_S)]R^*(Y) &= R'^*S'_*f_R R^*(Y) \\ &= R'^* f_S S_* R^*(Y) \\ &= R'^* f_S(Y) \\ &= R'^* S'_* R'^* f_S(Y) = R'^* G[(f_R, f_S)](Y). \end{aligned}$$

$G[(f_R, f_S)]S^* = S'^*H[(f_R, f_S)]$ can be proved by the same way. □

Since R^* and S_* give rise to an isomorphism between $\mathrm{Fix}(R^*S_*)$ and $\mathrm{Fix}(S_*R^*)$, it follows that G and H are naturally isomorphic by proposition 2.

We would like to show that the two categories $\mathbb{C}$ and $\mathbb{D}$ are equivalent. Since we already know that each complete lattice is isomorphic to $G(X, R, S)$ for some double approximation system (X, R, S), it is sufficient to show that G is full and faithful [4].

Lemma 1. *The functor G is full and faithful.*

Proof. Given two morphisms $[(f_R, f_S)], [(g_R, g_S)] : (X, R, S) \to (X', R', S')$ in $\mathbb{D}$, suppose $G[(f_R, f_S)] = G[(g_R, g_S)]$. By the definition of G, this is equivalent to

$S'_*R'^*f_S = S'_*R'^*g_S$ on $\mathrm{Fix}(S_*R^*)$. Hence $S'_*R'^*f_SS_*R^* = S'_*R'^*g_SS_*R^*$, which implies that $(f_R, f_S) \sim (g_R, g_S)$. Thus G is faithful.

If $f : \mathrm{Fix}(S_*R^*) \to \mathrm{Fix}(S'_*R'^*)$ is an order-preserving map between two complete lattices then f can be extended to an inclusion-preserving map $f_S : \mathbb{F}(S) \to \mathbb{F}(S')$ by defining $f_S(Y) = fS_*R^*(Y)$ for $Y \in \mathbb{F}(S)$. Since $G[(R'^*f_SS_*, f_S)] = S'_*R'^*fS_*R^* = f$, G is full. □

Theorem 1. *The functor G gives rise to an equivalence of categories between $\mathbb{D}$ and $\mathbb{C}$.*

Proof. Just combine lemma 1 and the representation theorem in [2]. □

The counterpart of G is given by the following functor F.

Definition 4. *We define a functor F from $\mathbb{C}$ to $\mathbb{D}$ as follows. F sends each complete lattice L (whose associated order is denoted by $\leq$) to a double approximation system $FL = (X, R, S)$, where $X = \not\leq$, R and S are equivalence relations on X defined at the end of section 2. Each order-perserving map $f : L \to L'$ is sent to a morphsm in $\mathbb{D}$ $[(f_R, f_S)] : FL = (X, R, S) \to FL' = (X', R', S')$, where f_S is a map from $\mathbb{F}(S)$ to $\mathbb{F}(S')$ defined by $f_S(\bigcup_{i\in I} \sigma(x_i)) = \bigcup_{i\in I} \sigma(f(x_i))$.*

Note that any element in $\mathbb{F}(S)$ can be represented as $\bigcup_{i\in I} \sigma(x_i)$ for some subset $\{x_i\}_{i\in I} \subseteq L$. One can prove that F preserves the composition by using lemma 2 below repeatedly.

The following proposition shows that the isomorphism in the representation theorem for complete lattices by double approximation systems gives rise to the natural isomorphism from the identity functor on $\mathbb{D}$ to GF.

Proposition 3. *The pair (η_L, FL) is a universal morphism from a complete lattice L to the functor G, where η_L is an order-isomorphism from L to GFL defined by $\eta_L(x) = \sigma(\downarrow x)$ for $x \in L$.*

Proof. By the definition of universal morphism [4], we have to show that the following condition holds. Given any double approximation system (X', R', S') and any order-preserving map f from L to $G(X', R', S')$, there is a unique morphism $[(f_R, f_S)]$ from $FL = (X, R, S)$ to (X', R', S') such that $f = G[(f_R, f_S)]\eta_L$. That is, every morphism $f \in \mathbb{C}$ to G uniquely factors through the universal morphism η_L.

We define a map $f_S : \mathbb{F}(S) \to \mathbb{F}(S')$ by $f_S(\bigcup_{i\in I} \sigma(x_i)) = \bigcup_{i\in I} f(x_i)$. Thus we obtain a morphism $[(R'^*f_SS_*, f_S)]$ from FL to (X', R', S'). For $x \in L$ we have

$$\begin{aligned} G[(R'^*f_SS_*, f_S)]\eta_L(x) &= S'_*R'^*(f_S(\sigma(\downarrow x))) \\ &= S'_*R'^*(\bigcup_{z\leq x} f(z)) \\ &= S'_*R'^*(f(x)) \quad \text{(since } z \leq x \text{ then } f(z) \subseteq f(x)) \\ &= f(x) \quad \text{(since } f(x) \in \mathrm{Fix}(S'_*R'^*)). \end{aligned}$$

Now we prove the uniqueness. Suppose there is a morphism $[(g_R, g_S)] : (X, R, S) \to (X', R', S')$ such that $f = G[(g_R, g_S)]\eta_L$. We show that $S'_*R'^*f_S S_*R^*(Y) = S'_*R'^*g_S S_*R^*(Y)$ for any $Y \in \mathbb{F}(S)$. Then it follows that $(R'^*f_S S_*, f_S) \sim (g_R, g_S)$. Write $Y = \bigcup_{i\in I} \sigma(x_i)$ for some $\{x_i\}_{i\in I} \subseteq L$.

Lemma 2. $S_*R^*(Y) = \bigcup_{x \leq \bigvee_{i\in I} x_i} \sigma(x) = \sigma(\downarrow \bigvee_{i\in I} x_i)$.

Proof. For $(x, y) \in \not\leq$, we have

$$
\begin{aligned}
(x,y) \in S_*R^*(Y) &\Leftrightarrow \sigma(x) \subset R^*(Y) \\
&\Leftrightarrow \forall z \in L(x \not\leq z \Rightarrow \rho(z) \cap Y \neq \emptyset) \\
&\Leftrightarrow \forall z \in L(x \not\leq z \Rightarrow \exists i \in I \text{ such that } x_i \not\leq z) \\
&\Leftrightarrow \forall z \in L((\forall i \in I\ x_i \leq z) \Rightarrow x \leq z) \\
&\Leftrightarrow x \leq \bigvee_{i\in I} x_i \Leftrightarrow (x,y) \in \bigcup_{x \leq \bigvee_{i\in I} x_i} \sigma(x).
\end{aligned}
$$

□

From the lemma, we have

$$
\begin{aligned}
S'_*R'^*g_S S_*R^*(Y) &= S'_*R'^*(g_S(\sigma(\downarrow \bigvee_{i\in I} x_i))) \\
&= G[(g_R, g_S)]\eta_L(\bigvee_{i\in I} x_i) \\
&= f(\bigvee_{i\in I} x_i) \\
&= G[(R'^*f_S S_*, f_S)]\eta_L(\bigvee_{i\in I} x_i) = S'_*R'^*f_S S_*R^*(Y).
\end{aligned}
$$

□

5 Concluding Remarks

In this paper we analyzed a representation theorem for complete lattices by double approximation systems from the category theoretical point of view. Given any complete lattice L, there exists a double approximation system (X, R, S) such that $L \cong \mathrm{Fix}(S_*R^*)$. We extended this representation theorem to an equivalence of categories. We also proved that the isomorphism η_L from L to $\mathrm{Fix}(S_*R^*)$ gives rise to a natural isomorphism in the equivalence of categories.

One possible application of the representation theorem is a logical analysis of directed graphs. A directed graph is defined as a quadruplet $G = (A, O, \partial_0, \partial_1)$, where A is the set of edges, O is the set of nodes, and ∂_0 and ∂_1 are two maps from A to O. ∂_0 sends each edge to its source node. ∂_1 sends each edge to its target node. A double approximation system (A, R_0, R_1) arises here, where R_i is an equivalence relation on A defined by $fR_ig \Leftrightarrow \partial_i f = \partial_i g$ for $f, g \in A$. Thus we can associate a complete lattice with each directed graph. This direction is now under ongoing research by one of the authors.

Any double approximation system (X, R, S) can be transformed to a formal context (E_S, E_R, C_X), where E_S is the set of S-equivalence classes, E_R is the set of R-equivalence classes and $C_X \subseteq E_S \times E_R$ is defined by $(P, Q) \in C_X \Leftrightarrow P \cap Q \neq \emptyset$ for S-equivalence class P and R-equivalence class Q. Representation of complete lattices at this level of description seems to be possible by approximation operators used in [8]. The precise formulation for this direction is left as a future work.

The notion of point in our representation theorem would be interesting if it is compared to that in *locale* [3]. Locales are generalization of topological spaces by forgetting their points. However, there is an abstract notion of point. A point in a locale is a function from the set of open sets to $\{\text{true}, \text{false}\}$ (of course, everything should be considered under a suitable algebraic structure, called *frame*). The notion of point in locale is an abstraction of 'function' of a point in a topological space: classifying open sets by whether they include the point or not. On the other hand, the notion of point in our representation theorem is looser. Given a complete lattice L, the corresponding double approximation system is constructed as (X, R, S) with $X = \not\leq$. A point in X is a pair (x, y) with $x \not\leq y$, which can be interpreted as just a difference between two different sets. The exploration of this comparison is also left as a future work.

References

1. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, 2nd edn. Cambridge Univ. Press, Cambridge (2002)
2. Gunji, Y.-P., Haruna, T.: Non-Boolean Lattices Derived on the Basis of Double Indiscernibility (submitted)
3. Johnstone, P.T.: Stone spaces. Cambridge Univ. Press, Cambridge (1982)
4. MacLane, S.: Categories for the Working Mathematician, 2nd edn. Springer, New York (1998)
5. Pawlak, Z.: Information systems–theoretical foundations. Information Systems 6, 205–218 (1981)
6. Pawlak, Z.: Rough sets. Intern. J. Comp. Sci. 11, 341–356 (1982)
7. Polkowski, L.: Rough Sets, Mathematical Foundations. Physical-Verlag, Heidelberg (2002)
8. Yao, Y.Y.: Concept lattices in rough set theory. In: Fuzzy Information, 2004, Processing NAFIPS 2004, vol. 2, pp. 796–801 (2004)