Lattice Derived by Double Indiscernibility and Computational Complementarity

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Abstract. We here concentrate on equivalence relation, and show that the composition of upper approximation of one equivalence relation and the lower one of the other equivalence relation can form a lattice. We also show that this method can be used to define computational complementarity in automata.

Keywords: Indiscernibility, lattice theory, computational complementarity, automata.

1 Introduction

While rough set providing a method for data analysis based on the indiscernibility defined by equivalence relation [1-3], recent definition of rough set is generalized by binary relation. We here concentrate on the rough set based on equivalence relation, and define a pseudo-closure operator based on upper and lower approximations of different equivalence relations [4]. From this definition, it is shown that any lattice can be constructed by a collection of fixed point of pseudo-closure operator (i.e., existence of representation theorem).

On the other hand, lattice theory has been used to evaluate computational complementarity proposed to find quantum logic in computational process [5-7]. Although orthomodular lattice is found for a specific Moore automaton, the method constructing a lattice is not adequate since an observer can see through the hidden internal state of automata. Here we show that a rough set based on double indiscernibility is suitable to analyze automata in terms of lattice theory, and revise the computational complementarity.

2 Lattice Derived by Double Indiscernibility

It is easy to see that upper and lower closure forms a Galois connection [8], if a single equivalence relation is given. In fact, given a universal set U, and an equivalence relation $R \subseteq U \times U$, for a subset $X, Y \subseteq U$, Galois connection:

$$R^*(X) \subseteq Y \iff X \subseteq R_*(Y).$$

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It leads to the duality of fixed point expressed as

$$R^*(X)=X \Leftrightarrow R_*(X)=X.$$

From the duality it is shown that a partially ordered set such as $<P; \subseteq>$ with $P=\{X\subseteq U| R_*(X)=X\}$, is a set lattice, and that $<Q; \subseteq>$ with $Q=\{X\subseteq U| R^*(X)=X\}$ is also a set lattice. In fact, it is easy to verify that join and meet in $<P; \subseteq>$ and $<Q; \subseteq>$ can be defined by union and intersection, respectively.

The difference between upper and lower approximations is a central part of rough set theory. We are interested in how such a difference contributes to a lattice structure. To estimate the role of the difference, the composition of upper and lower approximation is introduced. Then, it can be verified that $<P; \subseteq>$ with $P=\{X\subseteq U| R_*(R^*(X))=X\}$ is a set lattice. Similarly, $<Q; \subseteq>$ with $Q=\{X\subseteq U| R^*(R_*(X))=X\}$ is also a set lattice. Thus, composition of lower and upper approximations is reduced to a single approximation. Even if objects are recognized depending on the approximation based on an equivalence relation, structure of a lattice is invariant, Boolean lattice. To obtain diversity of lattice structure, we have to break the Galois connection derived by a single equivalence relation.

We introduce two equivalence relations and the operator that is a composition of upper and lower approximations, where the upper approximation is based on the one relation and the lower one is based on the other relation.

Given a universal set U, R and $S\subseteq U\times U$ are defined as different equivalence relations. The operations T and S are defined by $T=S_*R^*$, $K= R^*S_*$. Then, for $X, Y\subseteq U$,

$$X \subseteq Y \Rightarrow T(X) \subseteq T(Y),\ K(X) \subseteq K(Y)$$

$$T(T(X)) = T(X), \qquad K(K(X)) = K(X).$$

It shows that T and K is similar with closure operator but is not in the strict sense. By using this operator, we can construct a lattice by the following. First we define a partially ordered set $<\boldsymbol{L}_T; \subseteq>$ with $\boldsymbol{L}_T =\{X\subseteq U| T(X)=X\}$. Similarly, $<\boldsymbol{L}_K; \subseteq>$ with $\boldsymbol{L}_K=\{X\subseteq U| K(X)=X\}$ is also constructed. Meet and join of these partially ordered set is defined by: for $X, Y\in \boldsymbol{L}_T$

$$X\wedge Y = T(X\cap Y), \quad X\vee Y = T(X\cup Y).$$

Similarly, for $X, Y\in \boldsymbol{L}_K$

$$X\wedge Y = K(X\cap Y), \quad X\vee Y = K(X\cup Y).$$

Under this condition it is easily proven that $<\boldsymbol{L}_T; \subseteq>$ and $<\boldsymbol{L}_K; \subseteq>$ are lattices. It is also proven that, for $X\subseteq U$,

$$T(X) = X \Rightarrow S_*(X) = X,$$

$$K(X) = X \Rightarrow R^*(X) = X.$$

Since two equivalence relations are independent of each other, Galois connection such that $R^*(X)\subseteq Y \Leftrightarrow X\subseteq S_*(Y)$ no longer holds. Despite double indiscernibility, Galois connection holds if subsets are chosen from a collection of fixed points such as $\boldsymbol{L}_T=\{X\subseteq U| T(X)=X\}$. It reveals that a collection of fixed points with respect to T or K can constitute a structure stably observed.

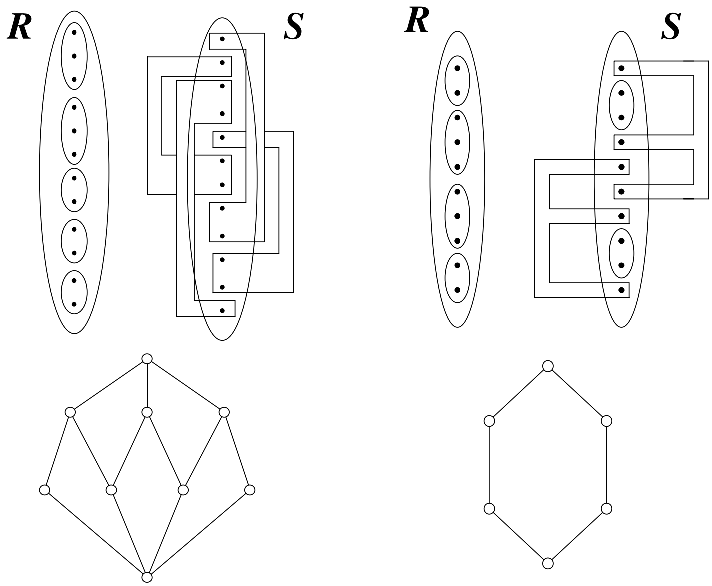

Fig. 1. Hasse diagram of a lattice (below) defined by a collection of $T(X) = X$. Equivalence classes of R are represented by loops, and those of S are represented by polygons (above).

Fig. 1 shows two examples of a lattice defined by $\langle \mathbf{L}_T; \subseteq\rangle$ with $\mathbf{L}_T = \{X\subseteq U| T(X)=X\}$. Given two equivalence relations, a collection of fixed points with respect to T constitutes a lattice. It is easy to see that there exists a lattice isomorphism between $\mathbf{L}_T$ and $\mathbf{L}_K$ In fact, we can show that a map $\varphi: \mathbf{L}_T \rightarrow \mathbf{L}_K$ is a lattice homomorphism, where for $X\in \mathbf{L}_T$, $\varphi(X) = R^*(X)$, and $\mathbf{L}_T = \{X\subseteq U| T(X)=X\}$ and $\mathbf{L}_K = \{X\subseteq U| K(X)=X\}$.

We can verify any lattice can be represented by a collection of fixed points with respect to operator T or K. Let $\langle L; \leq\rangle$ be a lattice. A universal set $U_L \subseteq L\times L$ derived from L is defined by $U_L = \{\langle x, y\rangle \in L\times L | \langle x, y\rangle \notin \leq\}$. Two equivalence classes derived from L, denoted by R and $S\subseteq U_L\times U_L$, are defined by $\langle x, y\rangle R\langle z, y\rangle$ and $\langle x, y\rangle S\langle x, w\rangle$. Let $\langle L; \leq\rangle$ be a lattice. Given x in L, we obtain

$$R^*(X_{x,}{}^l) = U - X_{x,}{}^u,$$
$$S_*(U - X_{x,}{}^u) = X_x{}^l,$$

where $X_x{}^l = \{\langle y, z\rangle \in U_L | y\leq x\}$ and $X_x{}^u = \{\langle y, z\rangle \in U_L | x\leq z\}$. Then the map $\eta: \langle L; \leq\rangle \rightarrow \langle \mathbf{L}_T; \subseteq\rangle$ defined by $\eta(x) = X_x{}^l$ for $x\in L$, is an isomorphism of L onto $\mathbf{L}_T$. That is a representation theorem.

3 Computational Complementarity

Computational complementarity was first investigated by Moore and was also found in attempting to construct logics from experimentally obtained propositions about automata [6]. Svozil proposes the method to construct a lattice for a given automaton, and shows that Moore automaton revealing computational complementarity is expressed as an orthocomplented lattice [5].

An automaton is defined by a transition of internal state, $\delta: Q\times\Sigma\rightarrow Q$ and an output function, f: $Q\rightarrow O$, where Σ is an input alphabet, Q is a finite set of states, O is a finite set of output symbols. Computational complementarity found in some specific automata is defined as follows: There exists an automaton such that any pair of its states is distinguishable, but there is no experiment which can determine in what state the automaton was at the beginning of the experiment.

Moore automaton is known as an example revealing computational complementarity, where Q={1, 2, 3, 4}, Σ={0, 1}, O={0, 1}, and the transition is defined by $\delta_0(1)= \delta_0(3)=4$, $\delta_0(2)=1$, $\delta_0(4)=2$, $\delta_1(1)= \delta_1(2)=3$, $\delta_1(3)=4$, $\delta_1(4)=2$, and the output function is defined by $f(1)=f(2)=f(3)=0$ and $f(4)=1$. The transition δ_0 and δ_1 represent the transition due to input 0 and 1, respectively.

Intrinsic propositional calculus is expressed as a partition of states due to the transition. A partition under the input k, is expressed as $\{\{a, b\}, \{c, d\}\}$ if and only if $\delta_k(a) = \delta_k(b)$ and $\delta_k(c) = \delta_k(d)$. In the Moore automaton mentioned above, we obtain two partitions,

$$\{\{1, 3\}, \{2\}, \{4\}\}, \{\{1, 2\}, \{3\}, \{4\}\},$$

dependent on input symbol. The first partition is derived by δ_0, and the second one is derived by δ_1. Each partition can reveal a set lattice whose atoms are represented by elements of partitions. Svozil's method [5] to construct a lattice is based on pasting Boolean lattices. A lattice for the intrinsic propositional calculus of an automaton is constructed by collecting all elements of set lattices derived by partitions, and order is defined by inclusion. In a Boolean lattice derived by {{1, 3}, {2}, {4}}, co-atoms are described as {{1, 2, 3}, {1, 3, 4}, {2, 4}}. In the other one derive by {{1, 2}, {3}, {4}}, co-atoms are described as {{1, 2, 3}, {1, 2, 4}, {3, 4}}. Thus, in the pasting lattice, we find not the order {2}≤{1, 2, 4} but {4}≤{1, 2, 4} and {1, 2}≤{1, 2, 4}, because {1, 2, 4} is constructed by union of {1, 2} and {4} in the second lattice. Since an element {4} exists in both partitions, some information of sub-lattices is lost in pasting two Boolean lattices, and that entails an orthocomplemented lattice.

Although Svozil calls his method "intrinsic" propositional calculus, an observer who constructs a lattice knows any state of automaton. Thus, he can deal each partition dependent of each other, and can paste them. An intrinsic observer is, however, destined to know states of automaton only through partition depending on input. Therefore, we cannot assume that an observer can prepare states of automaton, and that two experiments dependent on input cannot be conducted in a parallel fashion. It shows that two partitions cannot be distributed and has to be ordered in a sequence.

Under this idea, two partitions can be regarded as two kinds of equivalence relations (i.e. elements of a partition is an equivalence class). As for the Moore automaton, we think that the equivalence relation S and R are defined by {{1, 3}, {2}, {4}} and {{1, 2}, {3}, {4}}, respectively. Since two experiments (partitions) has to be ordered in a sequence, the lattice derived by the Moore automaton can be obtained by a collection of fixed points such as $\boldsymbol{L}_T =\{X\subseteq U|\ T(X)=X\}$. The lattice is just a distributive lattice (Heyting algebra) not showing an orthocomplemented lattice.

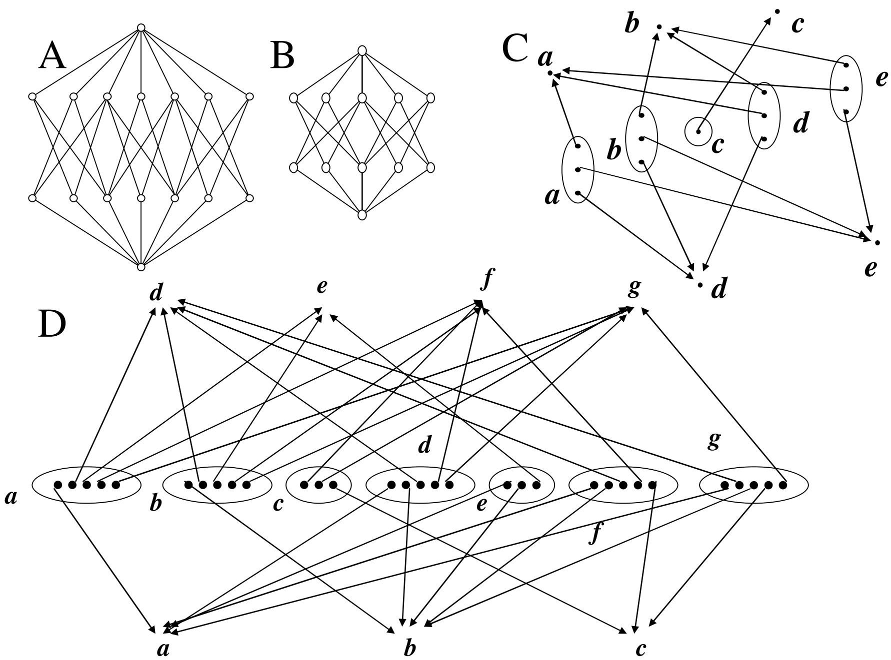

Fig. 2. Hasse diagram of a lattice (A and B) defined by a collection of $T(X)=X$ for an automaton defined by the transition (D and C, respectively). In the diagram C and D, the one partition represented by loops is the one equivalence relation R and the other partition represented by arrows is the other equivalence relation S.

Conversely, if experiments of automaton have to be ordered in a sequence, and if the lattice is obtained only by the composition of two partitions, an orthocomplemented lattice reveals more complicated automaton. Fig, 2 shows two examples of automata revealing an orthocomplemented lattice. Fig. 2D shows two kind of partitions. The one represented by loops shows the transition δ_0, and the other partition represented by arrows shows the transition δ_1. These partitions have the following remarkable features: (i) There exists x in Q such that $\delta_0(x)=\delta_1(x)$, (ii) if $\delta_0(x)=a$ and $\delta_1(x)=b$ then there exists y in Q such that $\delta_0(x)=b$ and $\delta_1(x)=a$. We can show that the automaton satisfying these features reveals complemented lattice, where two equivalence relations R and S are defined by δ_0 and δ_1 such that $R=\{<x, y>\in Q\times Q \mid \delta_0(x)=\delta_0(y)\}$ and $S=\{<x, y>\in Q\times Q \mid \delta_1(x)=\delta_1(y)\}$.

Previously computational complementarity is defined by an automaton having specific complementary structure. Our finding shows much more universal complementarity defined by the property (ii). It may explore new kind of computational complementarity in complex systems.

4 Conclusion

We here concentrate on an equivalence relation, since indiscernibility is a central notion in a rough set. Due to indiscernibility, approximation operators form a Galois connection that shows a strong bondage between two perspectives. As a result, a collection of a fixed point with respect to approximation operator forms a trivial set lattice. There is no diversity in terms of lattice structure.

Diversity of lattice structure results from discrepancy between two equivalence relations. Collecting fixed points results in loss of information, and then join and meet in a lattice cannot be defined by union and intersection, respectively. It can provide a wide variety of lattices. We here shows that any lattice can be analyzed with respect to two kind of transitions dependent on environments.

In addition we show that the idea of computational complementarity can be explored by the lattice derived by double indiscernibility. Since the order of measurements is essential for quantum physics, the composition of two equivalence relations can be congruent to complementarity. It can be adapted in studies of cellular automata and complex systems.

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