Supporting Fuzzy Rough Sets in Fuzzy Description Logics

Fernando Bobillo[1] and Umberto Straccia[2]

[1] Dpt. of Computer Science & Systems Engineering, University of Zaragoza, Spain
[2] Istituto di Scienza e Tecnologie dell'Informazione (ISTI - CNR), Pisa, Italy
fbobillo@unizar.es, straccia@isti.cnr.it

Abstract. Classical Description Logics (DLs) are not suitable to represent vague pieces of information. The attempts to achieve a solution have lead to the birth of fuzzy DLs and rough DLs. In this work, we provide a simple solution to join these two formalisms and define a fuzzy rough DL. We also show how to extend two reasoning algorithms for fuzzy DLs, which are implemented in the fuzzy DL reasoners FUZZYDL and DELOREAN.

1 Introduction

In the last years the interest in ontologies has significantly grown. An ontology is defined as an explicit and formal specification of a shared conceptualization [13], which means that ontologies represent the vocabulary of some domain. They have gained widespread popularity due to their success in several applications such as expert and multiagent systems or the Semantic Web. Description Logics (DLs) are a family of logics for representing structured knowledge [1]. They are the basis of most of the ontology languages, such as the current standard language OWL [16]. For instance, the logic behind the recent language OWL 2 is $\mathcal{SROIQ}(\mathbf{D})$ [8].

However, it is widely agreed that "classical" ontology languages are not appropriate to deal with *fuzzy/vague/imprecise knowledge*, which is inherent to several real world domains. With the aim of managing vagueness in ontologies, several extension of DLs have been proposed, being possible to group them in two categories. On the one hand, the combination with fuzzy logic [30] produced *fuzzy DLs*. Some notable works are [15,25,26,28], for a survey we refer to [21]. Under this approach, vagueness is quantified and expressed using a degree of membership to a vague concept. On the other hand, the combination with rough set theory [22] produced *rough DLs* [10,12,18,19,20,24]. These logics offer a qualitative approach to model vagueness. Instead of providing a degree of a membership, vague concepts are approximated by means of a couple of classical sets: an upper and a lower approximation. This approach is very useful when it is not possible to quantify the membership function of a vague concept.

Fuzzy logic and rough logic are complementary formalism to manage vagueness and hence it is natural to combine them by means of *fuzzy rough sets* [11,23].

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This is useful in several domains of application. For instance, in e-commerce, it is possible to combine rough concepts such as "potential buyer" (an individual which is possibly interested in some product) with fuzzy concepts such as "cheap price" (which can be modeled with a trapezoidal membership function). Another example is medicine, which combines rough concepts such as "possible patient" (an individual affected by some of the symptoms of some disease, and hence suspected of being patient) with fuzzy concepts such as "high blood pressure".

In this paper we follow this approach and extend a fuzzy DL with fuzzy rough sets. As we will see, the integration is seamless, as already pointed out by [24] for the classical semantics case, as the rough set component can mapped into the fuzzy DL component, with the non-negligible advantage that current fuzzy DLs reasoners can be used with minimal adaption.

Related works are [9], which presents a rough fuzzy ontology but without entering into the formal details of the subjacent logic, and [17], which considers a less expressive logic than ours and not dealing with implementation issues.

We proceed as follows. The next section provides some background on mathematical fuzzy logics and (fuzzy) rough set theory. Section 3 presents the definition of a extension of the DL $\mathcal{SROIQ}(\mathbf{D})$, the logic behind OWL 2, with fuzzy and fuzzy rough semantics. Section 4 describes two reasoning algorithms under two fragments of our logic. Finally, Section 5 sets out some conclusions.

2 Preliminaries

Mathematical Fuzzy Logic. In fuzzy logics, the convention prescribing that a statement is either true or false is changed. Changing the usual true/false convention leads to a new concept of statement, whose compatibility with a given state of facts is a matter of degree and can be measured on an ordered scale $\mathcal{S}$ that is no longer $\{0,1\}$, but, e.g., the unit interval $[0,1]$. This degree of fit is called *degree of truth* of the statement ϕ in the interpretation $\mathcal{I}$.

Fuzzy logics logics provide compositional calculi of degrees of truth, including degrees between "true" and "false". A statement is now not true or false only, but may have a truth degree taken from a *truth space* $\mathcal{S}$, usually $[0,1]$ (in that case we speak about *Mathematical Fuzzy Logic* [14]). In this paper, *fuzzy statements* will have the form $\phi \geqslant l$ or $\phi \leqslant u$, where $l, u \in [0,1]$ and ϕ is a statement, which encode that the degree of truth of ϕ is *at least* l resp. *at most* u.

Semantically, a *fuzzy interpretation* $\mathcal{I}$ maps each basic statement p_i into $[0,1]$ and is then extended inductively to all statements as follows:

$$\mathcal{I}(\phi \wedge \psi) = \mathcal{I}(\phi) \otimes \mathcal{I}(\psi), \mathcal{I}(\phi \vee \psi) = \mathcal{I}(\phi) \oplus \mathcal{I}(\psi),$$
$$\mathcal{I}(\phi \rightarrow \psi) = \mathcal{I}(\phi) \Rightarrow \mathcal{I}(\psi), \mathcal{I}(\neg \phi) = \ominus \mathcal{I}(\phi)\,,$$

where $\otimes$, $\oplus$, $\Rightarrow$, and $\ominus$ are so-called *combination functions*, namely, *triangular norms* (or *t-norms*), *triangular conorms* (or *t-conorms*), *implication functions*, and *negation functions*, respectively, which extend the classical Boolean conjunction, disjunction, implication, and negation, respectively, to the fuzzy case (see [14] for a formal definition of these functions and their properties). An important type of implication functions are *R-implications*, defined as $a \Rightarrow b = \sup \{c \mid a \otimes c \leq b\}$.

Table 1. Combination functions of various fuzzy logics

	Łukasiewicz Logic	Gödel Logic	Product Logic	Zadeh Logic
$a \otimes b$	$\max(a+b-1,0)$	$\min(a,b)$	$a \cdot b$	$\min(a,b)$
$a \oplus b$	$\min(a+b,1)$	$\max(a,b)$	$a+b-a\cdot b$	$\max(a,b)$
$a \Rightarrow b$	$\min(1-a+b,1)$	$\begin{cases}1 & \text{if } a \leq b\\ b & \text{otherwise}\end{cases}$	$\min(1,b/a)$	$\max(1-a,b)$
$\ominus a$	$1-a$	$\begin{cases}1 & \text{if } a = 0\\ 0 & \text{otherwise}\end{cases}$	$\begin{cases}1 & \text{if } a = 0\\ 0 & \text{otherwise}\end{cases}$	$1-a$

Several t-norms, t-conorms, implication functions, and negation functions have been proposed, giving raise to different fuzzy logics with different logical properties. In fuzzy logic, one usually distinguishes three different logics, namely Łukasiewicz, Gödel, and Product logic [14]. Zadeh logic (the fuzzy operators originally considered by Zadeh [30]) is a sublogic of Łukasiewicz logic. Łukasiewicz, Gödel, and Product logics have an R-implication, while Zadeh logic does not.

A *fuzzy set* R over a countable crisp set X is a function $R\colon X \to [0,1]$. A fuzzy set A is included in B (denoted $A \subseteq B$) iff $\forall x \in X, A(x) \leq B(x)$. The *degree of subsumption* between two fuzzy sets A and B is defined as $\inf_{x \in X} A(x) \Rightarrow B(x)$.

A (binary) *fuzzy relation* R over two countable crisp sets X and Y is a function $R\colon X \times Y \to [0,1]$. The *inverse* of R is the function $R^{-1}\colon Y \times X \to [0,1]$ with membership function $R^{-1}(y,x) = R(x,y)$, for every $x \in X$ and $y \in Y$. The *composition* of two fuzzy relations $R_1\colon X \times Y \to [0,1]$ and $R_2\colon Y \times Z \to [0,1]$ is defined as $(R_1 \circ R_2)(x,z) = \sup_{y \in Y} R_1(x,y) \otimes R_2(y,z)$. A fuzzy relation R is *reflexive* iff $\forall x \in X, R(x,x) = 1$. R is *symmetric* iff $\forall x \in X, y \in Y, R(x,y) = R(y,x)$. R is *transitive* iff $R(x,z) \geqslant (R \circ R)(x,z)$. A fuzzy *similarity relation* is a reflexive, symmetric and transitive relation.

A fuzzy interpretation $\mathcal{I}$ *satisfies* a fuzzy statement $\phi \geqslant l$ (resp., $\phi \leqslant u$) or $\mathcal{I}$ is a *model* of $\phi \geqslant l$ (resp., $\phi \leqslant u$), denoted $\mathcal{I} \models \phi \geqslant l$ (resp., $\mathcal{I} \models \phi \leqslant u$), iff $\mathcal{I}(\phi) \geqslant l$ (resp., $\mathcal{I}(\phi) \leqslant u$). The notions of satisfiability and logical consequence are defined in the standard way. $\phi \geqslant l$ is a *tight logical consequence* of a set of fuzzy statements $\mathcal{K}$ iff l is the infimum of $\mathcal{I}(\phi)$ subject to all models $\mathcal{I}$ of $\mathcal{K}$. Notice that the latter is equivalent to $l = \sup \{r \mid \mathcal{K} \models \phi \geqslant r\}$.

Rough Set and Fuzzy Rough Set Theories. The key idea in rough set theory [22] is the approximation of a vague concept by means of a pair a concepts: a sub-concept or *lower approximation* and a super-concept or *upper approximation*, describing the sets of elements which definitely and possibly belong to the vague set, respectively, as Figure 1 illustrates. The approximation is based on an indiscernibility equivalence relation (reflexive, symmetric and transitive) between elements of the domain. Given an indiscernibility relation R, the upper approximation of a set S is defined as: $\overline{S} = \{x \mid \exists y : (x,y) \in R \wedge y \in S\}$. Similarly, the lower approximation is defined as: $\underline{S} = \{x \mid \forall y : (x,y) \in R \rightarrow y \in S\}$.

A very natural extension is to consider a fuzzy similarity relation instead of an indiscernibility relation, which gives raise to *fuzzy rough sets* [11,23]. Given a fuzzy similarity relation R, a t-norm $\otimes$ and an implication function $\Rightarrow$, the upper

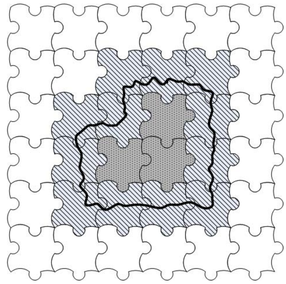

Fig. 1. Vague concept (bold line), upper approximation (striped line) and lower approximation (dotted line)

approximation of a fuzzy set S is given by the following membership function: $\forall x \in X, \overline{S}(x) = \sup_{y \in \Delta^{\mathcal{I}}}\{R(x,y) \otimes S(y)\}$. Similarly, the lower approximation is defined as: $\forall x \in X, \underline{S}(x) = \inf_{y \in \Delta^{\mathcal{I}}}\{R(x,y) \Rightarrow S(y)\}$.

3 The Fuzzy Rough DL $\mathcal{SROIQ}$(D)

In this section we describe a fuzzy rough extension of the fuzzy DL $\mathcal{SROIQ}(\mathbf{D})$, which is based on the fuzzy DLs presented in [5,7,28], and extended with upper and lower approximations of concepts. In the following, we assume $\bowtie \in \{\geq, >, \leq, <\}$, $\rhd \in \{\geq, >\}$, $\lhd \in \{\leq, <\}$, $\alpha \in (0,1]$, $\beta \in [0,1)$, $\gamma \in [0,1]$.

Syntax. A *fuzzy concrete domain* [27] $\mathbf{D}$ is a pair $\langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Delta_{\mathbf{D}}$ is a concrete interpretation domain, and $\Phi_{\mathbf{D}}$ is a set of fuzzy concrete predicates $\mathbf{d}$ with an arity n and an interpretation $\mathbf{d}_{\mathbf{D}} : \Delta_{\mathbf{D}}^n \rightarrow [0,1]$, which is an n-ary fuzzy relation over $\Delta_{\mathbf{D}}$. Usual functions for specifying fuzzy set membership degrees are the trapezoidal, the triangular, the L-function (left-shoulder function), and the R-function (right-shoulder function). For backwards compatibility, we also allow crisp intervals. These functions are defined over the set of non-negative rationals $\mathbb{Q}^+ \cup \{0\}$ For instance, we may define $\mathsf{Young}\colon \mathbb{N} \rightarrow [0,1]$ to be a fuzzy concrete predicate over the natural numbers denoting the degree of a person being young, as $\mathsf{Young}(x) = L(10, 30)$.

We further allow fuzzy modifiers, such as very, moreOrLess and slightly, which apply to fuzzy sets to change their membership function. Formally, a *modifier* is a function $f_m\colon [0,1] \rightarrow [0,1]$. We will allow modifiers defined in terms of linear hedges and triangular functions. For instance, $\mathsf{very}(x) = linear(0.8)$.

Similarly as for its crisp counterpart, fuzzy $\mathcal{SROIQ}(\mathbf{D})$ assumes three alphabets of symbols, for concepts, roles and individuals.

The *abstract roles* (denoted R) of the language can be built inductively as:

$$
\begin{aligned}
R \rightarrow\ & R_A \mid \text{(atomic role)} \\
& R^- \mid \text{(inverse role)} \\
& U \;\; \mid \text{(universal role)}
\end{aligned}
$$

Concrete roles are denoted T and cannot be complex.

Now, let n, m be natural numbers ($n \geqslant 0, m > 0$). The *concepts* (denoted C or D) of the language can be built inductively from atomic concepts (A), top concept $\top$, bottom concept $\bot$, named individuals (o_i), abstract roles (R), concrete roles (T), simple roles (S) [1] and fuzzy concrete predicates ($\mathbf{d}$) as:

$$
\begin{array}{rrl}
C, D \rightarrow & A \mid & \text{(atomic concept)} \\
& \top \mid & \text{(top concept)} \\
& \bot \mid & \text{(bottom concept)} \\
& C \sqcap D \mid & \text{(concept conjunction)} \\
& C \sqcup D \mid & \text{(concept disjunction)} \\
& \neg C \mid & \text{(concept negation)} \\
& \forall R.C \mid & \text{(universal quantification)} \\
& \exists R.C \mid & \text{(existential quantification)} \\
& \forall T.\mathbf{d} \mid & \text{(concrete universal quantification)} \\
& \exists T.\mathbf{d} \mid & \text{(concrete existential quantification)} \\
& \{o_1, \ldots, o_m\} \mid & \text{(nominals)} \\
& (\geq m\ S.C) \mid & \text{(at-least qualified number restriction)} \\
& (\leq n\ S.C) \mid & \text{(at-most qualified number restriction)} \\
& (\geq m\ T.\mathbf{d}) \mid & \text{(concrete at-least qualified number restriction)} \\
& (\leq n\ T.\mathbf{d}) \mid & \text{(concrete at-most qualified number restriction)} \\
& \exists S.\mathit{Self} & \text{(local reflexivity)}
\end{array}
$$

Assume m fuzzy similarity relations s_i ($i = 1, \ldots, m$). The above syntax is extended to include salient features of fuzzy DLs [3,7] as follows:

$$
\begin{array}{rrl}
C, D \rightarrow & \{\alpha_1/o_1, \ldots, \alpha_m/o_m\} \mid & \text{(fuzzy nominals)} \\
& C \rightarrow D \mid & \text{(fuzzy implication concept)} \\
& \alpha_1 C_1 + \cdots + \alpha_k C_k \mid & \text{(fuzzy weighted sum)} \\
& mod(C) \mid & \text{(modified concept)} \\
& [C \geq \alpha] \mid & \text{(cut concept)} \\
& [C \leq \beta] \mid & \text{(cut concept)} \\
& \overline{C}^i \mid & \text{(upper approximation)} \\
& \underline{C}_i & \text{(lower approximation)} \\
\\
\mathbf{d} \rightarrow & crisp(a, b) \mid & \text{(fuzzy crisp set)} \\
& L(a, b) \mid & \text{(fuzzy left-shoulder function)} \\
& R(a, b) \mid & \text{(fuzzy right-shoulder function)} \\
& triangular(a, b, c) \mid & \text{(fuzzy triangular function)} \\
& trapezoidal(a, b, c, d) & \text{(fuzzy trapezoidal function)} \\
\\
mod \rightarrow & linear(c) \mid & \text{(fuzzy linear modifier)} \\
& triangular(a, b, c) & \text{(fuzzy triangular modifier)} \\
\\
R \rightarrow & mod(R) \mid & \text{(modified role)} \\
& [R \geq \alpha] & \text{(cut role)}
\end{array}
$$

In the case of linear modifiers, we assume that $a = c/(c+1), b = 1/(c+1)$. Furthermore, for each of the connectives $\sqcap, \sqcup, \rightarrow$, we have indexed connectives

[1] Simple roles are needed to guarantee the decidability of the logic. Intuitively, simple roles cannot take part in cyclic role inclusion axioms (see [6] for a formal definition).

$\sqcap_X, \sqcup_X, \rightarrow_X$, where $X \in \{\text{Gödel}, \text{Łukasiewicz}, \text{Product}\}$, which are interpreted according to the semantics of the subscript.

Example 1. Concept $\mathsf{Human} \sqcap \exists \mathsf{hasAge}.L(10, 30)$ denotes the set of young humans, with an age given by $L(10, 30)$. If $linear(4)$ represents the modifier very, $Human \sqcap linear(4)(\exists \mathsf{hasAge}.L(10, 30))$ denotes the set of *very* young humans.

Furthermore, *abstract individuals* are denoted $a, b \in \Delta^{\mathcal{I}}$, while *concrete individuals* are denoted $v \in \Delta_{\mathbf{D}}$.

A *Fuzzy Knowledge Base* (KB) contains axioms organized in a fuzzy ABox $\mathcal{A}$, a fuzzy TBox $\mathcal{T}$ and a fuzzy RBox $\mathcal{R}$.

A *fuzzy ABox* consists of a finite set of *fuzzy assertions* of one of these types:

– a fuzzy concept assertion of the form $\langle a : C \bowtie \alpha \rangle$;
– a fuzzy role assertion, or constraint on the truth value of a role assertion, $\langle \Psi \bowtie \alpha \rangle$, where Ψ is of the form $(a, b) : R$, $(a, b) : \neg R$, $(a, v) : T$ or $(a, v) : \neg T$;
– an inequality assertion $\langle a \neq b \rangle$;
– an equality assertion $\langle a = b \rangle$.

A *fuzzy TBox* consists of a finite set of fuzzy General Concept Inclusions or *fuzzy GCIs*, which are expressions of the form $\langle C \sqsubseteq D \geq \alpha \rangle$ or $\langle C \sqsubseteq D > \beta \rangle$.

A *fuzzy RBox* consists of a finite set of *role axioms* of one these types:

– Fuzzy Role Inclusion Axioms or *fuzzy RIAs* $\langle w \sqsubseteq R \geq \alpha \rangle$, $\langle w \sqsubseteq R > \beta \rangle$, where $w = R_1 R_2 \dots R_m$ is a role chain, $\langle T_1 \sqsubseteq T_2 \geq \alpha \rangle$, or $\langle T_1 \sqsubseteq T_2 > \beta \rangle$;
– *transitive* role axioms $\mathtt{trans}(R)$;
– *disjoint* role axioms $\mathtt{dis}(S_1, S_2)$, $\mathtt{dis}(T_1, T_2)$;
– *reflexive* role axioms $\mathtt{ref}(R)$;
– *irreflexive* role axioms $\mathtt{irr}(S)$;
– *symmetric* role axiom $\mathtt{sym}(R)$;
– *asymmetric* role axioms $\mathtt{asy}(S)$.

Example 2. $\langle \mathsf{paul} : \mathsf{Tall} \geq 0.5 \rangle$ states that Paul is tall with at least degree 0.5. The fuzzy RIA $\langle \mathsf{isFriendOf}\ \mathsf{isFriendOf} \sqsubseteq \mathsf{isFriendOf} \geq 0.75 \rangle$ states that the friends of my friends can also be considered my friends with degree 0.75. □

A *fuzzy axiom* has a truth degree in [0,1]. A fuzzy axiom is *positive* (denoted $\langle \tau \rhd \alpha \rangle$) if it is of the form $\langle \tau \geq \alpha \rangle$ or $\langle \tau > \beta \rangle$. A fuzzy axiom is *negative* (denoted $\langle \tau \lhd \alpha \rangle$) if it is of the form $\langle \tau \leq \beta \rangle$ or $\langle \tau < \alpha \rangle$.

Semantics. A fuzzy interpretation $\mathcal{I}$ with respect to a fuzzy concrete domain $\mathbf{D}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) disjoint with $\Delta_{\mathbf{D}}$ and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:

– an *abstract individual* a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$;
– a *concrete individual* v onto an element $v_{\mathbf{D}}$ of $\Delta_{\mathbf{D}}$;
– a *concept* C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$;
– an *abstract role* R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$;

- a *concrete role* T onto a function $T^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}} \rightarrow [0,1]$;
- an n-ary *concrete fuzzy predicate* $\mathbf{d}$ onto the fuzzy relation $\mathbf{d}_{\mathbf{D}} : \Delta^{n}_{\mathbf{D}} \rightarrow [0,1]$;
- a *modifier mod* onto a function $f_{mod} : [0,1] \rightarrow [0,1]$.

Given arbitraries t-norm $\otimes$, t-conorm $\oplus$, negation function $\ominus$ and implication function $\Rightarrow$, the fuzzy interpretation function is extended to *complex concepts and roles* as shown in Table 2, and to *fuzzy axioms* as shown in Table 3.

Table 2. Semantics of the fuzzy concepts and roles in fuzzy $\mathcal{SROIQ}(\mathbf{D})$

Constructor	Semantics
$(\top)^{\mathcal{I}}(x)$	$= 1$
$(\bot)^{\mathcal{I}}(x)$	$= 0$
$(A)^{\mathcal{I}}(x)$	$= A^{\mathcal{I}}(x)$
$(C \sqcap D)^{\mathcal{I}}(x)$	$= C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$
$(C \sqcup D)^{\mathcal{I}}(x)$	$= C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$
$(\neg C)^{\mathcal{I}}(x)$	$= \ominus C^{\mathcal{I}}(x)$
$(\forall R.C)^{\mathcal{I}}(x)$	$= \inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\}$
$(\exists R.C)^{\mathcal{I}}(x)$	$= \sup_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$
$(\forall T.\mathbf{d})^{\mathcal{I}}(x)$	$= \inf_{v \in \Delta_{\mathbf{D}}} \{T^{\mathcal{I}}(x,v) \Rightarrow \mathbf{d}_{\mathbf{D}}(v)\}$
$(\exists T.\mathbf{d})^{\mathcal{I}}(x)$	$= \sup_{v \in \Delta_{\mathbf{D}}} \{T^{\mathcal{I}}(x,v) \otimes \mathbf{d}_{\mathbf{D}}(v)\}$
$(\{\alpha_1/o_1, \ldots, \alpha_m/o_m\})^{\mathcal{I}}(x)$	$= \sup_{i \mid x = o_i^{\mathcal{I}}} \alpha_i$
$(\geq m\ S.C)^{\mathcal{I}}(x)$	$= \sup_{y_1,\ldots,y_m \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{m} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \bigotimes (\otimes_{j<k} \{y_j \neq y_k\})]$
$(\leq n\ S.C)^{\mathcal{I}}(x)$	$= \inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k} \{y_j = y_k\})]$
$(\geq m\ T.\mathbf{d})^{\mathcal{I}}(x)$	$= \sup_{v_1,\ldots,v_m \in \Delta_{\mathbf{D}}} [(\min_{i=1}^{m} \{T^{\mathcal{I}}(x,v_i) \otimes \mathbf{d}_{\mathbf{D}}(v_i)\}) \bigotimes (\otimes_{j<k} \{v_j \neq v_k\})]$
$(\leq n\ T.\mathbf{d})^{\mathcal{I}}(x)$	$= \inf_{v_1,\ldots,v_{n+1} \in \Delta_{\mathbf{D}}} [(\min_{i=1}^{n+1} \{T^{\mathcal{I}}(x,v_i) \otimes \mathbf{d}_{\mathbf{D}}(v_i)\}) \Rightarrow (\oplus_{j<k} \{v_j = v_k\})]$
$(\exists S.Self)^{\mathcal{I}}(x)$	$= S^{\mathcal{I}}(x,x)$
$(mod(C))^{\mathcal{I}}(x)$	$= f_{mod}(C^{\mathcal{I}}(x))$
$([C \geq \alpha])^{\mathcal{I}}(x)$	$= 1$ if $C^{\mathcal{I}}(x) \geq \alpha$, 0 otherwise
$([C \leq \beta])^{\mathcal{I}}(x)$	$= 1$ if $C^{\mathcal{I}}(x) \leq \beta$, 0 otherwise
$(\alpha_1 C_1 + \cdots + \alpha_k C_k)^{\mathcal{I}}(x)$	$= \alpha_1 {C_1}^{\mathcal{I}}(x) + \cdots + \alpha_k {C_k}^{\mathcal{I}}(x)$
$(C \rightarrow D)^{\mathcal{I}}(x)$	$= C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$
$(\overline{C}^{i})^{\mathcal{I}}(x)$	$= \sup_{y \in \Delta^{\mathcal{I}}} {s_i}^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$
$(\underline{C}_i)^{\mathcal{I}}(x)$	$= \inf_{y \in \Delta^{\mathcal{I}}} {s_i}^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$
$(R_A)^{\mathcal{I}}(x,y)$	$= R_A^{\mathcal{I}}(x,y)$
$(R^-)^{\mathcal{I}}(x,y)$	$= R^{\mathcal{I}}(y,x)$
$(U)^{\mathcal{I}}(x,y)$	$= 1$
$(mod(R))^{\mathcal{I}}(x,y)$	$= f_{mod}(R^{\mathcal{I}}(x,y))$
$([R \geq \alpha])^{\mathcal{I}}(x,y)$	$= 1$ if $R^{\mathcal{I}}(x,y) \geq \alpha$, 0 otherwise
$(T)^{\mathcal{I}}(x,v)$	$= T^{\mathcal{I}}(x,v)$

$C^{\mathcal{I}}$ denotes the membership function of the fuzzy concept C with respect to the fuzzy interpretation $\mathcal{I}$. $C^{\mathcal{I}}(x)$ gives us the degree of being the individual x an element of the fuzzy concept C under $\mathcal{I}$.

Similarly, $R^{\mathcal{I}}$ denotes the membership function of the fuzzy role R with respect to $\mathcal{I}$. $R^{\mathcal{I}}(x,y)$ gives us the degree of being (x,y) an element of the fuzzy role R under $\mathcal{I}$.

A fuzzy interpretation $\mathcal{I}$ *satisfies* (is a *model* of):

- $\langle a : C \bowtie \gamma \rangle$ iff $(a : C)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a,b) : R \bowtie \gamma \rangle$ iff $((a,b) : R)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a,b) : \neg R \bowtie \gamma \rangle$ iff $((a,b) : \neg R)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a,v) : T \bowtie \gamma \rangle$ iff $((a,v) : T)^{\mathcal{I}} \bowtie \gamma$,

Table 3. Semantics of the fuzzy axioms in fuzzy $\mathcal{SROIQ}(\mathbf{D})$

Axiom	Semantics
$(a:C)^{\mathcal{I}}$	$= C^{\mathcal{I}}(a^{\mathcal{I}})$
$((a,b):R)^{\mathcal{I}}$	$= R^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}})$
$((a,b):\neg R)^{\mathcal{I}}$	$= \ominus R^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}})$
$((a,v):T)^{\mathcal{I}}$	$= T^{\mathcal{I}}(a^{\mathcal{I}},v_{\mathbf{D}})$
$((a,v):\neg T)^{\mathcal{I}}$	$= \ominus T^{\mathcal{I}}(a^{\mathcal{I}},v_{\mathbf{D}})$
$(C \sqsubseteq D)^{\mathcal{I}}$	$= \inf_{x\in\Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$
$(R_1 \dots R_m \sqsubseteq R)^{\mathcal{I}}$	$= \inf_{x_1,x_{n+1}\in\Delta^{\mathcal{I}}} \sup_{x_2\dots x_n\in\Delta^{\mathcal{I}}} (R_1^{\mathcal{I}}(x_1,x_2)\otimes\cdots\otimes R_n^{\mathcal{I}}(x_n,x_{n+1})) \Rightarrow R^{\mathcal{I}}(x_1,x_{n+1})$
$(T_1 \sqsubseteq T_2)^{\mathcal{I}}$	$= \inf_{x\in\Delta^{\mathcal{I}},v\in\Delta_{\mathbf{D}}} T_1^{\mathcal{I}}(x,v) \Rightarrow T_2^{\mathcal{I}}(x,v)$

- $\langle (a,v):\neg T \bowtie \gamma\rangle$ iff $((a,v):\neg T)^{\mathcal{I}} \bowtie \gamma$,
- $\langle a \neq b\rangle$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $\langle a = b\rangle$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- $\langle C \sqsubseteq D \rhd \gamma\rangle$ iff $(C \sqsubseteq D)^{\mathcal{I}} \rhd \gamma$,
- $\langle R_1 \dots R_m \sqsubseteq R \rhd \gamma\rangle$ iff $(R_1 \dots R_m \sqsubseteq R)^{\mathcal{I}} \rhd \gamma$,
- $\langle T_1 \sqsubseteq T_2 \rhd \gamma\rangle$ iff $(T_1 \sqsubseteq T_2)^{\mathcal{I}} \rhd \gamma$,
- $\mathtt{trans}(R)$ iff $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \geq \sup_{z\in\Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z)\otimes R^{\mathcal{I}}(z,y)$,
- $\mathtt{dis}(S_1,S_2)$ iff $\forall x,y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x,y) = 0$ or $S_2^{\mathcal{I}}(x,y) = 0$,
- $\mathtt{dis}(T_1,T_2)$ iff $\forall x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}}, T_1^{\mathcal{I}}(x,v) = 0$ or $T_2^{\mathcal{I}}(x,v) = 0$,
- $\mathtt{ref}(R)$ iff $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,x) = 1$,
- $\mathtt{irr}(S)$ iff $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$,
- $\mathtt{sym}(R)$ iff $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$,
- $\mathtt{asy}(S)$ iff $\forall x,y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$,
- a fuzzy KB iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

Reasoning. The notions of logical consequence and tight logical consequence are defined as in Sect. 2. Additionally, the *maximal satisfiability degree* [7] of a concept C w.r.t. a fuzzy KB $\mathcal{K}$ is defined as $glb(\mathcal{K},C) = \sup_{\mathcal{I}} \sup_{x\in\Delta^{\mathcal{I}}} C^{\mathcal{I}}(x)$.

Some logical properties. Due to the properties of fuzzy rough sets [23], in Zadeh, Gödel, Łukasiewicz and Product logics we have that:

- $\overline{\bot} \equiv \underline{\bot} \equiv \bot$, $\underline{\top} \equiv \overline{\top} \equiv \top$, $\underline{\underline{C}} \equiv \underline{C}$, $\overline{\overline{C}} \equiv \overline{C}$.
- $\neg\underline{C} \equiv \overline{\neg C}$, in Zadeh and Łukasiewicz logics.
- $\neg\overline{C} \equiv \underline{\neg C}$, in Zadeh and Łukasiewicz logics.
- $\overline{C \sqcap D} \subseteq \overline{C} \sqcap \overline{D}$, $\underline{C \sqcap D} \equiv \underline{C} \sqcap \underline{D}$,
- $\overline{C \sqcup D} \equiv \overline{C} \sqcup \overline{D}$, in Zadeh and Gödel logics.
- $\underline{C \sqcup D} \supseteq \underline{C} \sqcup \underline{D}$.

Note that fuzzy rough intersection and union are not truth-functional in general.

4 Reasoning and Implementation

In this section we will show how to extend two existing reasoning algorithms for fuzzy DLs so they can support fuzzy rough DLs, and how we have implemented them in the fuzzyDL system [7] and in the DeLorean system [4]. To this end, we recall that indeed we can map lower and upper approximation concepts into fuzzy DL concepts. This is not surprising as already pointed out by [24] for the crisp case. In fact, it is not difficult to see from the semantics of upper ($\overline{C}^i$) and lower ($\underline{C}_i$) approximation concepts, that these can be represented as fuzzy DL concepts $\exists s_i.C$ and $\forall s_i.C$, respectively. That is, we consider the transformation:

$$\overline{C}^i \mapsto \exists s_i.C \tag{1}$$

$$\underline{C}_i \mapsto \forall s_i.C \tag{2}$$

and, thus, we may replace upper and lover approximation concepts with ordinary fuzzy DL concepts. This is exactly the same transformation as provided in [24]. In the following, we show how two currently highly expressive fuzzy DL reasoners have been adapted to support our logic.

4.1 Tableau Rules and an Optimization Problem in fuzzyDL

fuzzyDL is a reasoner for fuzzy $\mathcal{SHIF}(\mathbf{D})$ extended with a lot of salient features of fuzzy DLs, under Zadeh, Łukasiewicz and Gödel logics [7]. It is available from `http://www.straccia.info`, and supports the logic defined in Sect. 3 without the additional elements of $\mathcal{SROIQ}$, i.e., fuzzy nominals, qualified cardinality restrictions, role assertions with a negated role, disjoint role axioms, complex fuzzy RIAs (with $w \neq R$), irreflexive role axioms and asymmetric role axioms.

Its reasoning algorithm combines a tableaux algorithm and a mixed integer linear optimization problem. The basic idea is to build a tableaux using a set of satisfiability preserving rules which generate new simpler fuzzy assertion axioms together with some inequations over $[0, 1]$-valued variables. Finally, an optimization problem through the set of inequations is solved. A detailed description of the reasoning algorithm cannot fit into this paper, but it can be found in [29].

To support upper and lower approximation concepts in fuzzyDL, essentially we need to support reflexive roles (symmetric and transitive roles are already supported). In particular, we firstly extend fuzzyDL with a couple of fuzzy role axioms. Reflexive and symmetric role axioms are of the form `(reflexive R)` and `(symmetric R)`, respectively, where R is a fuzzy role. Symmetric role axioms can already be simulated with fuzzyDL, and this axioms is just syntactic sugar. Indeed, axiom $R \sqsubseteq R^-$ implies that R is symmetric.

Then, we allow three additional concept constructors: upper approximations, lower approximations and local reflexivity concepts, which are of the form `(ua s_i C)`, `(la s_i C)` and `(self S)`, respectively, where $\mathsf{s_i}$ is a fuzzy similarity relation, S is a simple fuzzy role and C is a fuzzy concept. Local reflexivity concepts are not necessary for the rough extension, but adding them is easy (reasoning is similar to the case of reflexive roles).

Similarity relations must be previously defined using the following syntax: `(define-fuzzy-similarity s_i)`.

The reasoning algorithm is extended as follows:

- For every fuzzy similarity relation `(define-fuzzy-similarity` s_i`)` we assert s_i to be reflexive, symmetric and transitive by adding the following axioms: `(reflexive R)`, `(symmetric R)`, `(transitive R)`.
- Every symmetric role axiom `(symmetric R)` is replaced with an inverse role axiom `(inverse R invR)` and a role inclusion axiom `(implies-role R invR)`. Under an R-implication, it is well known that $\mathtt{sym}(\mathsf{R})$ is equivalent to $\mathsf{R} \sqsubseteq \mathsf{R}^-$.
- Every upper approximation concept `(ua` s_i `C)` is replaced with an existential restriction concept `(some` s_i `C)`.
- Every lower approximation concept `(la` s_i `C)` is replaced with a universal restriction concept `(all` s_i `C)`.
- The rule for a local reflexivity concept `(self S)` asserts that an individual is related to itself. Formally, in the calculus if $\langle \exists \mathsf{S}.\mathtt{Self}, l\rangle \in \mathcal{L}(v)$ (that is, if v is an instance of $\exists \mathsf{S}.\mathtt{Self}$ to degree not smaller than l) then append $\langle \mathsf{S}, l\rangle$ to $\mathcal{L}(\langle v, v\rangle)$ (that is, the pair $\langle v, v\rangle$ is an instance of S at least to degree l).
- The rule for reflexive roles `(reflexive R)` asserts that every individual is related to itself. Formally, if $\langle \mathtt{ref}(\mathsf{R})\rangle \in \mathcal{R}$ and v is a node to which this rule has not yet been applied then append $\langle \mathsf{R}, l\rangle$ to $\mathcal{L}(\langle v, v\rangle)$ (that is, the pair $\langle v, v\rangle$ is an instance of R to degree not smaller than l).

4.2 Reduction to Classical Description Logic in DeLorean

DeLorean is a reasoner for basic fuzzy $\mathcal{SROIQ}(\mathbf{D})$ [4] (not supporting the additional features of fuzzy DLs defined in Sect. 3) under Zadeh and Gödel (with an involutive negation) logics. The syntax of the supported language is in [2].

Its reasoning algorithm is based on a reduction to a classical DL, so current DL reasoners can be reused. A full description may be found in [3,5,6].

DeLorean already supported local reflexivity concepts, as well as reflexive and symmetric roles. Hence, it only remained to extend it with upper and lower approximations of the form `(upper` s_i `C)` and `(lower` s_i `C)`, where s_i is a fuzzy similarity relation and C is a fuzzy concept.

Now, the reasoning algorithm is extended as follows:

- Every concept `(upper` s_i `C)` is replaced with an existential restriction concept `(some` s_i `C)`. Furthermore, we add the following axioms if they do not exist in the fuzzy RBox: `(reflexive R)`, `(symmetric R)`, `(transitive R)`.
- Every concept `(lower` s_i `C)` is replaced with a universal restriction concept `(all` s_i `C)`. Once again, we add the following axioms in case they do not exist in the fuzzy RBox: `(reflexive R)`, `(symmetric R)`, `(transitive R)`.

5 Conclusions

In this paper we have studied a DL managing vagueness in two different but complementary ways, combining a fuzzy DL with fuzzy rough sets. In particular, we have presented a very expressive fuzzy rough extension of the DL $\mathcal{SROIQ}(\mathbf{D})$, the logic behind the language OWL 2. The rough extension is general (independent of the family of fuzzy operators) and uses m possible fuzzy similarity relations.

Reasoning under our general fuzzy rough DL is not currently possible, but we have extended and implemented two well-known reasoning algorithms for fuzzy DLs in order to deal with two important fragments of the logic. On the one hand, fuzzyDL implements a combination of a tableaux algorithm and a mixed integer linear optimization problem, and already supports fuzzy rough $\mathcal{SHIF}(\mathbf{D})$ (extended with salient features of fuzzy DLs) under Zadeh, Łukasiewicz and Gödel logics. On the other hand, DeLorean implements a translation to a crisp DL and supports fuzzy rough $\mathcal{SROIQ}(\mathbf{D})$ under Zadeh and Gödel (with an involutive negation) logics. Extending the of reasoning algorithms and the expressivity reasoners remains an open research problem.

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