Chapter 4
Collective Choice for Simple Preferences

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4.1 Introduction

Individual preferences often take simple structures in some restricted environments. The so-called universal domain assumption in the three impossibility results by Arrow (1951), Sen (1970a,b), and Gibbard (1973) and Satterthwaite (1975) have been scrutinized and (partially) abandoned in numerous later studies, which do not intend to identify well-behaved social welfare functions that "would be universal in the sense that it would be applicable to any community" (Arrow 1951, p. 24). Important breakthroughs have been made in this line of research: Gaertner (2002) provides a comprehensive survey of the literature on domain restrictions.

Of our central interest in this survey are simple preferences with few indifference classes such as the so-called dichotomous or trichotomous preferences as studied by Inada (1964, 1969, 1970)[1]. Later investigations on collective choice with dichotomous preferences have been closely connected to studies of the normative and strategic advantages of majority and approval voting systems and of their axiomatic foundation: see Brams and Fishburn (2002) for an extensive survey of this literature as well as Brams and Fishburn (1978) and Fishburn (1978a,b, 1979) among others. This survey connects old and recent theoretical developments in this literature with a single but comprehensive perspective.

The survey starts with a brief overview of the classical impossibility results. Section 4.2 discusses some possibility results on several domains of dichotomous preferences. Section 4.3 discusses axiomatic foundations for majority and approval voting systems. We investigate the logical relationship among the existing axiomatic characterizations. In the process, we discover ways of strengthening existing results and we offer new characterization results. Readers are referred to Xu (2010) in this

[1]Dichotomous preferences are also considered by Bogomolnaia and Moulin (2004) and Bogomolnaia et al. (2005) in their investigation of well-behaved randomization mechanisms.

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volume for a compact overview of the literature on axiomatic characterizations of majority voting.[2] Section 4.4 discusses strategic voting and the robustness of voting systems. Some results associated with the Condorcet principle and realizability of Condorcet winners in strategic voting environments are included. Section 4.5 discusses some recent developments in unconstrained multi-issue problems with separable preferences. The section deals with strategy-proof voting schemes and shows the conflict between Pareto efficiency and strategy-proofness on the entire domain of separable preferences and on restricted domains of "dichotomous" or "trichotomous" preferences. Section 4.6 discusses dichotomous opinion aggregation problems that have drawn some attention recently among scholars interested in group identification.

4.1.1 Preliminaries

Let $\mathbb{X}$ be the set of all alternatives. There are infinitely many "potential" agents, identified with the natural numbers in $\mathbb{N}$. Let $\mathcal{X}$ and $\mathcal{N}$ be the set of finite subsets of $\mathbb{X}$ and of $\mathbb{N}$ respectively. Each agent $i \in \mathbb{N}$ has a preference ordering R_i that is a complete, reflexive, and transitive binary relation over $\mathbb{X}$. Let $\mathcal{R}$ be the set of all preference orderings over $\mathbb{X}$. We sometimes consider binary relations that are not necessarily transitive. Let $\underline{\mathcal{R}}$ be the set of all complete and reflexive binary relations. For each $N \in \mathcal{N}$ and each $X \in \mathcal{X}$, let $\mathcal{R}^N$ be the set of profiles of preference orderings of agents in N and let $\mathcal{U}_{N,X} \equiv \mathcal{R}^N \times \{X\}$. Let $\mathcal{U}_N \equiv \bigcup_{X \in \mathcal{X}} \mathcal{U}_{N,X}$, $\mathcal{U}_X \equiv \bigcup_{N \in \mathcal{N}} \mathcal{U}_{N,X}$, and $\mathcal{U} \equiv \bigcup_{N \in \mathcal{N}, X \in \mathcal{X}} \mathcal{U}_{N,X}$. Subsets of $\mathcal{U}_{N,X}$, $\mathcal{U}_N$, $\mathcal{U}_X$, and $\mathcal{U}$ are denoted respectively by $\mathcal{D}_{N,X}$, $\mathcal{D}_N$, $\mathcal{D}_X$, and $\mathcal{D}$. Elements of $\mathcal{R}^N$ are denoted by R_N, R'_N, R''_N, etc., and also by R, R', R'', etc., when N is clear from the context. Elements of $\mathcal{R}$ are denoted by R_0, R'_0, R''_0, etc., and also by R_i, R'_i, R''_i, etc., when they belong to agent i.

A *social decision function* on $\mathcal{D}_{N,X}$ is a function $f : \mathcal{D}_{N,X} \to \underline{\mathcal{R}}$ associating with each profile $(R, X) \in \mathcal{D}_{N,X}$ a social preference relation $f(R, X) \in \underline{\mathcal{R}}$. A *social welfare function* on $\mathcal{D}_{N,X}$ is a function $f : \mathcal{D}_{N,X} \to \mathcal{R}$ associating with each profile $(R, X) \in \mathcal{D}_{N,X}$ a social preference *ordering* $f(R, X) \in \mathcal{R}$. We often denote a social preference relation by $\succeq$, its strict counterpart by $\succ$, and its indifference by $\sim$, in order to distinguish them from individual preference relations.

Let $P(X)$ be the set all subsets of X and $\bar{P}(X)$ the set of all non-empty subsets of X. A social preference relation $\succeq$ generates a choice rule $C(\cdot; \succeq) : P(X) \to P(X)$ as follows: for all $Y \subseteq X$,

$$C(Y; \succeq) \equiv \{x \in Y : \text{ for all } y \in Y, \ x \succeq y\}. \quad (4.1)$$

By finiteness of X, if $\succeq$ is transitive, the choice rule is non-empty valued at each non-empty $Y \subseteq X$. For non-empty valuedness, each of the following weaker conditions

[2] Thomson (2001) offers an extensive survey and discussion on the axiomatic method in Social Choice Theory and Game Theory.

is also sufficient. Preference relation $\succeq$ is *quasi-transitive* if its strict counterpart $\succ$ is transitive, that is, for all $x, y, z \in X$, $x \succ y$ and $y \succ z$ imply $x \succ z$. It is *acyclic* if there is no sequence of finite alternatives, $x_1, \ldots, x_T \in X$ such that $x_1 \succ x_2$, $x_2 \succ x_3, \ldots, x_{T-1} \succ x_T, x_T \succ x_1$. Clearly, transitivity implies quasi-transitivity, which implies acyclicity**.** Quasi-transitivity is sufficient but not necessary for the non-emptiness of the choice rule in (4.1) (so is transitivity). Acyclicity is necessary and sufficient for the non-emptiness of the choice rule (Sen 1970a,b, Lemma 1*1).

A *collective choice quasi-rule* on $\mathcal{D}_{N,X}$ is a function $c\colon \mathcal{D}_{N,X} \times \bar{P}(X) \to P(X)$ associating with each profile $(R, X, Y) \subseteq \mathcal{D}_{N,X} \times \bar{P}(X)$ a subset of Y, that is, $c(R, X, Y) \subseteq Y$. A *collective choice rule* on $\mathcal{D}_{N,X}$ is a non-empty valued collective choice quasi-rule, namely a function $c\colon \mathcal{D}_{N,X} \times \bar{P}(X) \to \bar{P}(X)$ associating with each profile $(R, X, Y) \subseteq \mathcal{D}_{N,X} \times \bar{P}(X)$ a nonempty subset of Y, that is, $\emptyset \neq c(R, X, Y) \subseteq Y$. We sometimes use notation $c_{R,X}(Y) \equiv c(R, X, Y)$.

Each choice rule $C : \bar{P}(X) \to \bar{P}(X)$ generates a binary relation $R(C)$ as follows:

$$xR(C)y \text{ if and only if } x \in C(\{x, y\}).$$

Call $R(C)$ the *base relation* of $C(\cdot)$ (as in Herzberger 1973). Choice rule C is *normal* if $C(\cdot) = C(\cdot; R(C))$. Unless specified otherwise, we consider collective choice rules generating normal choice rules. Necessary and sufficient conditions for a choice rule to be normal are summarized in Sen (1977, pp. 64–65, Propositions 8 and 9).[3]

4.1.2 Classical Impossibility Results

Consider social decision functions or collective choice rules over $\mathcal{D}_{N,X}$. Here are some basic axioms for social decision functions considered in Arrow (1951), Sen (1970a,b), Gibbard (1973) and Satterthwaite (1975). In defining the axioms, we will only state the properties needed for a social decision function f. That is, instead of stating "f is said to satisfy Axiom A if it satisfies property A," we simply state property A.

Unrestricted Domain: $\mathcal{D}_{N,X} = \mathcal{U}_{N,X}$.

Transitive Social Preferences, (briefly, **Transitivity):** For all $R \in \mathcal{D}_{N,X}$, the social preference relation at R, $f(R, X)$, is transitive.

Replacing transitivity with quasi-transitivity or acyclicity, we define the axioms of *quasi-transitive social preferences* (briefly, *quasi-transitivity*) and *acyclic social preferences* (briefly, *acyclicity*), respectively.

Weak Pareto: For all $x, y \in X$, if everyone strictly prefers x to y at R, then x is strictly preferred to y under the social preference relation $f(R, X)$.

Non-dictatorship: There is no person $i \in N$ - such a person would be a dictator - such that for all $R \in \mathcal{D}_{N,X}$ and all $x, y \in X$, if i strictly prefers x to y, then x is strictly preferred to y under the social preference relation $f(R, X)$.

[3] One necessary and sufficient condition in citetSen77 (properties $\alpha 2$ and $\gamma 2$) is the following: for all $Y \in \bar{P}(X)$, $x \in C(Y)$ if and only if for all $y \in Y$, $x \in C(\{x, y\})$.

Note that if a collective choice rule $c(\cdot)$ generates a dictatorial base relation $R(c)$, then by normality, the dictator's preferred choices constitute $c(R, X, Y)$ for all $Y \in \bar{P}(X)$.

Independence of Irrelevant Alternatives: For all $R, R' \in \mathcal{D}_{N,X}$ and all $Y \subseteq X$, if individual preferences of both profiles R and R' are identical over Y, then the two social preference relations at the two profiles generate the same choice over Y, that is, $C(Y; f(R, X)) = C(Y; f(R', X))$.[4]

Replacing $f(R, X)$ in the above axioms with the base relation $R(c)$ generated from a collective choice rule $c(\cdot)$, we define the corresponding axioms for collective choice rules. The same names are used for these axioms.

4.1.2.1 Arrow's Theorem

Arrow (1951) investigates the existence of social decision functions satisfying the five basic axioms in the previous section. When there are at least three alternatives, such a function does not exists.

Theorem 4.1.1 (Arrow's Impossibility Theorem). *If there are at least three alternatives, then no social decision function* (*or collective choice rule*) *satisfies unrestricted domain, transitive social preferences, weak Pareto, non-dictatorship, and independence of irrelevant alternatives.*[5]

When the axiom of transitive social preferences is weakened to quasi-transitivity, there does exist a social decision function satisfying the other four axioms. For example, "Pareto dominance" gives a quasi-transitive, but not necessarily transitive, social preference relation. Later works in this direction (Gibbard 1969; Guha 1972; Mas-Colell and Sonnenschein 1972) deliver a characterization of "oligarchic" social decision functions where a group, namely oligarchy, is decisive and each member of the group has veto power. Replacing quasi-transitivity with acyclicity leads to a larger family of social decision functions that may not be oligarchic but close to oligarchy, in the sense that all decisive groups share some core members as shown by Brown (1975) and Banks (1995).

Further progress has been made in the line of research that focuses on restricted preferences in some specialized environments. Gaertner (2002) provides a comprehensive survey of the literature on restricted domains. Sections 4.2–4.6 give an overview of results pertaining to dichotomous preferences. Section 4.2 provides some possibility results on dichotomous domains. We list several definitions of dichotomous domains that admit some social decision functions satisfying Arrow's axioms except for the axiom of unrestricted domain. Moreover, as we will see in Sect. 4.3, majority decision stands out among other decision functions as the unique one satisfying Arrow's axioms and other standard axioms in the literature.

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[4] See (4.1) for the definition of the choice rule generated by a social preference relation.

[5] Note that normality assumption for collective choice rules allows us to state this result for both social decision function and collective choice rule at once.

4.1.2.2 Gibbard–Satterthwaite Theorem

Here we consider a domain $\mathcal{D}_{N,X} \subseteq \mathcal{U}_{N,X}$ such that for some $\bar{\mathcal{R}} \subseteq \mathcal{R}$, $\mathcal{D}_{N,X} = \bar{\mathcal{R}}^N \times \{X\}$. Preferences are primitive variables for collective choice or social decision but they are often unobservable. Agents or voters seek their own private interests and may vote untruthfully whenever advantageous. Collective choice procedures may not work properly unless they have a certain embedded property in themselves preventing untruthful voting. An important line of research has been devoted to the search for truthful collective choice procedures. The seminal work of Gibbard (1973) and Satterthwaite (1975) show that when there are at least three alternatives, there is no truthful procedure that also satisfies unrestricted domain, non-dictatorship, and the full-range condition.

A collective choice rule $c : \mathcal{D}_{N,X} \times \bar{P}(X) \to \bar{P}(X)$ is *resolute* if it always picks a single alternative, that is, for all $(R, X) \in \mathcal{D}_{N,X}$ and all $Y \in \bar{P}(X)$, $c(R, X, Y)$ is a singleton. For truthful procedures, Gibbard (1973) and Satterthwaite (1975) require that for all possible reported preferences of others, each agent $i \in N$ always prefers the outcome that results from the truthful announcement of his preferences to any outcome that he could obtain by lying.

Strategy-Proofness: For all $R \in \mathcal{D}_{N,X}$, all $Y \in \bar{P}(X)$, all $i \in N$, and all $R'_i \in \bar{\mathcal{R}}$, $c((R_i, R_{-i}), X, Y)\ R_i\ c((R'_i, R_{-i}), X, Y)$.

An extension of strategy-proofness for set-valued rules is discussed in Sect. 4.4.

Theorem 4.1.2 (Gibbard–Satterthwaite Theorem). *If there are at least three alternatives, no resolute collective choice rule satisfies unrestricted domain, non-dictatorship, strategy-proofness and the full-range condition.*

Important positive results are derived in later works pertaining to specialized environments that accommodate some natural restrictions on preferences. We will survey the results pertaining to dichotomous domains in Sects. 4.2 and 4.4 and the domain of separable preferences in Sect. 4.5. Moulin (1980) characterizes a large family of strategy-proof rules on the domain of single-peaked preferences over public alternatives that are ordered on a line. Any such rule chooses a "generalized Condorcet winner." In the case of private good rationing model with single peaked preferences, the family of strategy-proof rules is much more restricted as shown by Sprumont (1991).

4.1.2.3 Sen's Paretian Liberal Paradox

Sen (1970a,b) investigates the existence of a social decision function that satisfies weak Pareto and a minimal form of liberalism, as well as the condition of acyclic social preferences and unrestricted domain. Again the result is negative.

His minimal notion of liberalism requires that there should be at least two agents who are decisive when making social comparison of a pair of alternatives. Formally, we say that agent i is *decisive for x and y with $x \neq y$*, if for all $R \in \mathcal{D}_{N,X}$, $xP_i y$ implies $x \succ_{f(R,X)} y$.

Minimal Liberalism: There are at least two agents who are decisive for a pair of alternatives.

His main result, known as the Paretian liberal paradox, is the following.

Theorem 4.1.3 (Sen's Paradox). *No social decision function* (*or collective choice rule*) *satisfies unrestricted domain, acyclic social preferences, weak Pareto, and minimal liberalism.*

Gibbard (1974) pushes this negative result to the most extreme form by showing that Sen's liberalism, properly extended in the model of collective decision with personal components, cannot be well-defined. He provides a simple preference profile for which any choice of an alternative necessarily violates at least one liberal right: this is known as Gibbard's paradox.

We will ask whether Sen's paradox holds on the dichotomous preferences domain for the problems of unconstrained choice of multiple issues in Sects. 4.5 and 4.6.

4.2 Possibility Results on Some Dichotomous Domains

Consider $\mathcal{D}_{N,X} \subseteq \mathcal{U}_{N,X}$ such that for some $\bar{\mathcal{R}} \subseteq \mathcal{R}$, $\mathcal{D}_{N,X} = \bar{\mathcal{R}}^N \times \{X\}$. Throughout this section, we consider several examples of "dichotomous" domains. On these domains, there do exist some social decision functions satisfying Arrow's axioms (in Theorem 4.1.1) except for unrestricted domain. This is shown by some existing results that we overview here. We also offer some characterizations imposing Arrow's axiom of independence of irrelevant alternatives together with other standard axioms.

For all $R \in \mathcal{D}_{N,X}$ and all $x, y \in X$, let $N_{x,y}(R) \equiv \{i \in N : x\ P_i\ y\}$ be the set of agents who prefer x to y (or vote for x against y) and $n_{x,y}(R) \equiv |\{i \in N : x\ P_i\ y\}|$ the number of agents who prefer x to y (or the number of votes x wins against y). Independence of irrelevant alternatives can be restated as follows:

Independence of Irrelevant Alternatives: For all $R, R' \in \mathcal{D}_{N,X}$ and all $x, y \in X$, if $N_{x,y}(R) = N_{x,y}(R')$ and $N_{y,x}(R') = N_{y,x}(R)$, then $x \succeq_{f(R)} y$ implies $x \succeq_{f(R')} y$.

The next axiom is stronger and is crucial for strategy-proofness.

Monotonicity: For all $R, R' \in \mathcal{D}_{N,X}$ and all $x, y \in X$, if $N_{x,y}(R) \subseteq N_{x,y}(R')$ and $N_{y,x}(R') \subseteq N_{y,x}(R)$, then $x \succeq_{f(R)} y$ implies $x \succeq_{f(R')} y$.

Applying monotonicity when $N_{x,y}(R) = N_{x,y}(R')$ and $N_{y,x}(R') = N_{y,x}(R)$ yields independence of irrelevant alternatives.

The next axiom, considered by MaY (1952), plays a key role in his and other axiomatic characterizations of majority decision.

Positive Response: For all $R, R' \in \mathcal{D}_{N,X}$ and all $x, y \in X$, if $N_{x,y}(R) \subseteq N_{x,y}(R')$, $N_{y,x}(R') \subseteq N_{y,x}(R)$, and at least one of the two inclusions is strict, then $x \succeq_{f(R)} y$ implies $x \succ_{f(R')} y$.

Note that when there are only two alternatives, say x and y, positive response implies monotonicity because then, $N_{x,y}(R) = N_{x,y}(R')$ and $N_{y,x}(R') = N_{y,x}(R)$, which is the case relevant not to positive response but to monotonicity, imply $R = R'$ and so application of monotonicity in this case is trivial. However, with more than two alternatives, this implication no longer holds, and there is no logical relation between the two axioms.

The next two axioms require symmetric treatment of individuals and of alternatives, respectively.

Anonymity: For all permutations on N, $\pi : N \to N$, and all $R \in \mathcal{D}_{N,X}$, $f(R) = f(R_\pi)$, where $R_\pi \in \mathcal{D}_{N,X}$ is such that for all $i \in N$, $R_{\pi i} = R_{\pi(i)}$.

Neutrality: For all $R \in \mathcal{D}_{N,X}$ and all $x, y, x', y' \in X$, if $R' \in \mathcal{D}_{N,X}$ is the preferences profile obtained after relabeling x and y in profile R with x' and y' respectively, then $x \succeq_{f(R)} y$ if and only if $x' \succeq_{f(R')} y'$.

The best known decision function satisfying the above axioms is *majority decision function* $f_{MAJ}(\cdot)$, which maps each $R \in \mathcal{D}_{N,X}$ into a social preference relation $\succeq_{f_{MAJ}(R)}$ defined as follows: for all $x, y \in X$,

$$x \succeq_{f_{MAJ}(R)} y \text{ if and only if } n_{x,y}(R) \geq n_{y,x}(, R).$$

In fact, there is a large family of monotonic decision functions, of which the special example is majority decision. In order to define this family, we need the following notation and concepts. Let $\mathfrak{d}^* \equiv \{(L_1, L_2) : L_1, L_2 \in P(N), L_1 \cap L_2 = \emptyset\}$ be the set of all pairs of disjoint subsets of N. A *decisive structure for a pair* $x, y \in X$, $\mathfrak{d}_{x,y}$ is a non-empty subset of $\mathfrak{d}^*$such that for all $(L_1, L_2), (L_1', L_2') \in \mathfrak{d}^*$,

$$\text{if } (L_1, L_2) \in \mathfrak{d}_{x,y},\ L_1 \subseteq L_1',\ \text{and } L_2' \subseteq L_2,\ \text{then } (L_1', L_2') \in \mathfrak{d}_{x,y}. \tag{4.2}$$

Call this property $\mathfrak{d}$-*monotonicity*. A *decisive structure* $\mathfrak{d} \equiv (\mathfrak{d}_{x,y})_{x,y \in X}$ is a profile of decisive structures for pairs of alternatives such that for all $x, y \in X$ and all $(L_1, L_2) \in \mathfrak{d}^*$,

$$\text{if } (L_1, L_2) \notin \mathfrak{d}_{x,y},\ \text{then } (L_2, L_1) \in \mathfrak{d}_{y,x}. \tag{4.3}$$

Call this property $\mathfrak{d}$-*completeness*. A decisive structure $\mathfrak{d}$ represents the social decision function $f^{\mathfrak{d}}$ defined as follows: for all $R \in \mathcal{D}_{N,X}$ and all $x, y \in X$, $x \succeq_{f^{\mathfrak{d}}(R)} y$ if and only if $(N_{x,y}(R), N_{y,x}(R)) \in \mathfrak{d}_{x,y}$.

Note that by (4.2), $f^{\mathfrak{d}}$ is monotonic and that by (4.3), the social preference relations chosen by $f^{\mathfrak{d}}$ are complete. It is easy to show that neutrality of $f^{\mathfrak{d}}$ requires $\mathfrak{d}_{x,y} = \mathfrak{d}_{x',y'}$ for all $x, y, x', y' \in X$. Conversely, any monotonic social decision function f generates a decisive structure $\mathfrak{d}^f$ and is represented by it. To show this, define $\mathfrak{d}^f_{x,y}$ as follows: for all $(L_1, L_2) \in \mathfrak{d}^*$, $(L_1, L_2) \in \mathfrak{d}^f_{x,y}$ if and only if for some $R \in \mathcal{D}_{N,X}$, $x \succeq_{f(R)} y$, $N_{x,y}(R) \subseteq L_1$, and $L_2 \subseteq N_{y,x}(R)$. To prove (4.3), suppose $(L_1, L_2) \in \mathfrak{d}^* \backslash \mathfrak{d}^f_{x,y}$. Consider $R \in \mathcal{D}_{N,X}$ such that $N_{x,y}(R) = L_1$ and

$N_{y,x}(R) = L_2$.[6] Then not $x \succsim_{f(R)} y$ and by completeness of $f(R)$, $y \succ_{f(R)} x$. This shows $(L_2, L_1) \in \mathfrak{d}^f_{y,x}$. Monotonicity of f directly implies (4.2). Therefore we obtain:

Proposition 4.2.1. *A social decision function satisfies monotonicity if and only if it is represented by a decisive structure.*

For anonymous and neutral decision functions, decisive structures representing them take a simple form. Let $\mathfrak{n}^* \equiv \{(n_1, n_2) : n_1, n_2 \in \{0, 1, \ldots, n\}$, and $n_1 + n_2 \leq n\}$. A *decisive index structure for a pair* x, y, $\mathfrak{n}_{x,y}$ is a non-empty subset of $\mathfrak{n}^*$ such that for all $(n_1, n_2), (n'_1, n'_2) \in \mathfrak{n}^*$,

$$\text{if } (n_1, n_2) \in \mathfrak{n}_{x,y},\ n_1 \leq n'_1, \text{ and } n'_2 \leq n_2, \text{ then } (n'_1, n'_2) \in \mathfrak{n}_{x,y}. \tag{4.4}$$

Call this $\mathfrak{n}$-*monotonicity*. A *decisive index structure* $\mathfrak{n} \equiv (\mathfrak{n}_{x,y})_{x,y \in X}$ is a profile of decisive index structures for pairs of alternatives such that for all $x, y \in X$ and all $(n_1, n_2) \in \mathfrak{n}^*$,

$$\text{if } (n_1, n_2) \notin \mathfrak{n}_{x,y}, \text{ then } (n_2, n_1) \in \mathfrak{n}_{y,x}. \tag{4.5}$$

Call this $\mathfrak{n}$-*completeness* Note that for neutral social decision functions represented by a decisive index structure $\mathfrak{n}$, neutrality and $\mathfrak{n}$-completeness imply the following: for all $k \in \{0, 1, \ldots, [n/2]\}$ and all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$,

$$\mathfrak{n}_{x,y} = \mathfrak{n}_{x',y'} \text{ and } (k, k) \in \mathfrak{n}_{x,y}, \tag{4.6}$$

where $[n/2]$ is the greatest integer that is less than or equal to $n/2$. Call this $\mathfrak{n}$-*neutrality*. This property and $\mathfrak{n}$-monotonicity together imply that for all $k \in \{0, 1, \ldots, [(n-1)/2]\}$ and all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, $(k+1, k) \in \mathfrak{n}_{x,y} = \mathfrak{n}_{x',y'}$. Combining this with (4.6), we get: for all $(n_1, n_2) \in \mathfrak{n}^*$ and all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$,

$$\mathfrak{n}_{x,y} = \mathfrak{n}_{x',y'}, \text{ and if } n_1 \geq n_2, \text{ then } (n_1, n_2) \in \mathfrak{n}_{x,y}. \tag{4.7}$$

Therefore we obtain:

Proposition 4.2.2. *A social decision function satisfies monotonicity and anonymity if and only if it is represented by a decisive index structure. Adding neutrality, we characterize the subfamily of social decision functions represented by an* $\mathfrak{n}$*-neutral decisive index structure. Moreover, these* $\mathfrak{n}$*-neutral index structures satisfy* (4.7).

When preferences are linear (no indifference), Propositions 4.2.1 and 4.2.2 give characterizations of what are known as "monotonic simple games." Since we will mostly focus on dichotomous domains where indifference is prevalent, decisive structures are more relevant to our later discussion.

[6] Existence of such R is the basic richness assumption for $\mathcal{D}_{N,X}$ that we need in order to obtain Proposition 4.2.1.

4.2.1 Two Alternatives

The simplest example of dichotomous domains is of course when there are only two alternatives, say a, b (that is, $X = \{a, b\}$). Then any social decision function satisfies *transitive social preferences* trivially. There are numerous social decision functions satisfying all other axioms in Arrow's theorem. For example, the social decision functions represented by a monotonic and non-dictatorial decisive structure satisfy all of Arrow's axioms. There are also numerous strategy-proof and non-dictatorial collective choice functions. An important property for strategy-proofness in this binary choice framework is monotonicity. Since there are only two alternatives, Sen's minimal liberalism is hardly satisfied unless the set of admissible preferences is extremely restricted.

Majority decision function stands out among other well-behaved social decision functions, as shown by MaY (1952). The key axiom in his axiomatic characterization of majority decision is positive response.

Now, to find out the implication of positive response, consider a function f represented by decisive structure $\mathfrak{d}$. Let R, R' be the two profiles in the premise of the axiom of positive response. Assume $x \succeq_{f(R)} y$, that is, $(N_{x,y}(R), N_{y,x}(R)) \in \mathfrak{d}_{x,y}$. Positive response, then requires $x \succ_{f(R')} y$, which implies $(N_{y,x}(R'), N_{x,y}(R')) \notin \mathfrak{d}_{y,x}$. Thus positive response implies the following extra condition on decisive structures: for all $x, y \in X$ and all $(L_1, L_2), (L_1', L_2') \in \mathfrak{d}^*$ with $x \neq y$ and $(L_1, L_2) \neq (L_1', L_2')$,

$$\text{if } (L_1, L_2) \in \mathfrak{d}_{x,y},\ L_1 \subseteq L_1',\ \text{and } L_2' \subseteq L_2,\ \text{then } (L_2', L_1') \notin \mathfrak{d}_{y,x}. \tag{4.8}$$

For decisive index structures, this condition can be written as: for all $x, y \in X$ and all $(n_1, n_2), (n_1', n_2') \in \mathfrak{n}^*$ with $x \neq y$ and $(n_1, n_2) \neq (n_1', n_2')$,

$$\text{if } (n_1, n_2) \in \mathfrak{n}_{x,y},\ n_1 \leq n_1',\ \text{and } n_2' \leq n_2,\ \text{then } (n_2', n_1') \notin \mathfrak{n}_{y,x}. \tag{4.9}$$

For a neutral social decision function represented by a decisive index structure $\mathfrak{n}$, if there is $(n_1, n_2) \in \mathfrak{n}^*$ such that $n_1 < n_2$ and $(n_1, n_2) \in \mathfrak{n}_{x,y}$, then by (4.9), $([\frac{n_1+n_2}{2}] + 1, [\frac{n_1+n_2}{2}]) \notin \mathfrak{n}_{y,x}$, which contradicts to (4.7). Therefore, neutrality and positive response together imply the following: for all $x, y \in X$ and all $(n_1, n_2) \in \mathfrak{n}^*$,

$$\text{if } n_1 < n_2,\ \text{then } (n_1, n_2) \notin \mathfrak{n}_{x,y}. \tag{4.10}$$

Combining (4.7) and (4.10), we obtain:

Theorem 4.2.1 (MaY 1952). *When there are two alternatives, a social decision function on $\mathcal{U}_{N,X}$ satisfies anonymity, neutrality, and positive response if and only if it is majority decision function.*

An extended version of this result with more than two alternatives is provided in Theorem 4.3.1. Aşan and Sanver (2002) replaces positive response with the combination of "path independence" and Pareto (if no voter prefers b to a and some voter prefers a to b, then a should be socially preferred to b). In the same framework, Sanver (2009) imposes weak Pareto, anonymity, neutrality, and monotonicity, together with some additional axioms, and characterizes variants of majority decision function.

4.2.2 Two Fixed Indifference Classes

In this section, we assume that there are two types of alternatives and all alternatives of a type are indifferent. This assumption is formulated by the following domain property.

Definition 4.2.1. A domain $\mathcal{D}^{2fic}_{N,X} \subseteq \mathcal{U}_{N,X}$ has the property of *two-fixed-indifference-class* if alternatives are partitioned into two fixed classes and for all $R \in \mathcal{D}^{2fic}_{N,X}$ and all $i \in N$, the two classes constitute the two indifference sets of R_i. Let $a, b \in X$ be two representative alternatives and the two fixed classes are denoted by X_a and X_b.

On such a domain, majority decision function do satisfy transitivity (as is implied by Theorem 4.2.2). Hence there does exist a social decision function satisfying all of Arrow's axioms. A characterization of a family of transitive social decision functions is provided in the next proposition.

Let $f : \mathcal{D}^{2fic}_{N,X} \to \underline{\mathcal{R}}$ be a social decision function satisfying monotonicity and transitivity. Let $\mathfrak{d}$ be a decisive structure representing f. Suppose that for some $x, y \in X_a$ and some $R \in \mathcal{D}^{2fic}_{N,X}$, $x \succ_{f(R)} y$. Then by monotonicity, the strict social ranking holds at all other preference profiles, that is, for all $R' \in \mathcal{D}^{2fic}_{N,X}$, $x \succ_{f(R')} y$. This is because $N_{x,y}(R) = N_{x,y}(R') = N_{y,x}(R) = N_{y,x}(R') = \emptyset$. Then the ranking between x and y can be decided by $\mathfrak{d}_{x,y}$ and $\mathfrak{d}_{y,x}$ such that $(\emptyset, \emptyset) \in \mathfrak{d}_{x,y}$ and $(\emptyset, \emptyset) \notin \mathfrak{d}_{y,x}$. Similarly, if for some $R \in \mathcal{D}^{2fic}_{N,X}$, $x \sim_{f(R)} y$, then this social indifference holds at all other preference profiles and $(\emptyset, \emptyset) \in \mathfrak{d}_{x,y}$ and $(\emptyset, \emptyset) \in \mathfrak{d}_{y,x}$. Therefore, there is a fixed social preference relation over alternatives in X_a and over alternatives in X_b, which holds at all $R \in \mathcal{D}^{2fic}_{N,X}$. Since social decision function f satisfies transitivity, we may order elements in the two sets X_a and X_b in the same order of their fixed social rankings; that is, elements of X_a are $a_1 \succeq a_2 \succeq \cdots \succeq a_q$ and elements of X_b are $b_1 \succeq b_2 \succeq \cdots \succeq b_r$. Strict ranking among $a_1, \ldots, a_q$ or among $b_1, \ldots, b_r$ is excluded when we require the following mild axiom:

Indifference Unanimity: For all R and all $x, y \in X$, if for all $i \in N$, $x \, I_i \, y$, then $x \sim_{f(R)} y$.

The next result characterizes a family of functions satisfying transitivity, monotonicity, and indifference unanimity.

Proposition 4.2.3. *Consider a domain with the property of two-fixed-indifference-class. Denote two representative alternatives in the two fixed classes by a and b and* *the two fixed classes by X_a and X_b. A social decision function satisfies transitivity, monotonicity, and indifference unanimity if and only if it is represented by a decisive structure $\mathfrak{d} \equiv (\mathfrak{d}_{x,y})_{x,y \in X}$ such that for all $x, x' \in X_a$ and all $y, y' \in X_b$, $(\emptyset, \emptyset) \in \mathfrak{d}_{x,x'} = \mathfrak{d}_{y,y'}$ and for all $x \in X_a$ and all $y \in X_b$, $\mathfrak{d}_{x,y} = \mathfrak{d}_{a,b}$ and $\mathfrak{d}_{y,x} = \mathfrak{d}_{b,a}$.*

Proof. By indifference unanimity, for all $R \in \mathcal{D}^{2fic}_{N,X}$, alternatives in X_a are all socially indifferent and similarly for X_b. Define $\mathfrak{d}_{x,y}$ as follows: for all $(L_1, L_2) \in \mathfrak{d}^*$, $(L_1, L_2) \in \mathfrak{d}_{x,y}$ if and only if for some $R \in \mathcal{D}^{2fic}_{N,X}$, $x \succeq_{f(R)} y$, $N_{x,y}(R) \subseteq L_1$, and $L_2 \subseteq N_{y,x}(R)$. By indifference unanimity, for all $x, x' \in X_a$ and all $y, y' \in X_b$, $(\emptyset, \emptyset) \in \mathfrak{d}_{x,x'} = \mathfrak{d}_{y,y'}$. Now let $x \in X_a$ and $y \in X_b$. For all $R \in \mathcal{D}^{2fic}_{N,X}$, since $x \sim_{f(R)} a$ and $y \sim_{f(R)} b$, then by transitivity, $x \succeq_{f(R)} y$it if and only if $a \succeq_{f(R)} b$. This and the construction of $\mathfrak{d}$ imply $\mathfrak{d}_{x,y} = \mathfrak{d}_{a,b}$. Similarly, $\mathfrak{d}_{y,x} = \mathfrak{d}_{b,a}$. □

4.2.3 Two Indifference Classes

We now consider domains where individual preferences can have at most two indifference classes.

Definition 4.2.2. A domain $\mathcal{D}^{2ic}_{N,X} \subseteq \mathcal{U}_{N,X}$ has the property of *two-indifference-class* if for all $R \in \mathcal{D}^{2ic}_{N,X}$, all triples $x, y, z \in X$ and all $i \in N$, R_i partitions $\{x, y, z\}$ into at most two indifference classes.

Clearly any domain with the property of two-fixed-indifference-class has this property, but not vice versa. On such domains, majority decision function always generates a transitive social preference relation.

Theorem 4.2.2 (Inada 1964). *On any domain with the property of two-indifference-class, majority decision function satisfies transitivity.*

Proof. Let $x, y, z \in X$ be three distinct alternatives. If $R \in \mathcal{D}^{2ic}_{N,X}$, then for all $i \in N$, R_i is one of the following seven "dichotomous" preference orderings: (1) xI_iyI_iz, (2) xI_iyP_iz, (3) xP_iyI_iz, (4) xI_izP_iy, (5) yP_ixI_iz, (6) yI_izP_ix, (7) zP_ixI_iy. Let $n_1, \ldots, n_7$ be the numbers of agents of each type. Note that $n_{x,y}(R) = n_3 + n_4$, $n_{y,x}(R) = n_5 + n_6$, $n_{y,z}(R) = n_2 + n_5$, $n_{z,y}(R) = n_4 + n_7$, $n_{x,z}(R) = n_2 + n_3$, and $n_{z,x} = n_6 + n_7$. To show transitivity of social preference relation, suppose $x \succeq_{f_{MAJ}(R)} y$ and $y \succeq_{f_{MAJ}(R)} z$. Then

$$n_3 + n_4 \geq n_5 + n_6 \text{ and } n_2 + n_5 \geq n_4 + n_7. \tag{4.11}$$

Combining the two inequalities, we obtain $n_2 + n_3 + n_4 + n_5 \geq n_4 + n_5 + n_6 + n_7$, that is,

$$n_2 + n_3 \geq n_6 + n_7. \tag{4.12}$$

This implies $n_{x,z}(R) \geq n_{z,x}(R)$. Therefore, $x \succeq_{f_{MAJ}(R)} z$. □

In fact, majority decision is the only transitive social decision function satisfying monotonicity, anonymity, and neutrality, except for *degenerate indifference function* that is the constant social decision function taking the complete indifference as its value (alternatives are all indifferent).

Theorem 4.2.3 (Ju 2009b). *Consider a domain with the property of two-indifference-class.*[7] *A social decision function satisfies monotonicity, anonymity, neutrality and transitivity if and only if it is either majority decision function or degenerate indifference function.*

Proof. By Theorem 4.2.2, majority decision function is transitive. It also satisfies the other axioms by Proposition 4.2.2. In order to prove the converse, let f be a social decision function on $\mathcal{D}^{2ic}_{N,X}$ satisfying monotonicity, anonymity, neutrality and transitivity. By Proposition 4.2.2, f is represented by an $\mathfrak{n}$-neutral index structure $\mathfrak{n} \equiv (\mathfrak{n}_{x,y})_{x,y\in X}$ and the index structure satisfies (4.7). Let $\mathfrak{n}_0 \equiv \mathfrak{n}_{x,y}$ for all $x, y \in X$ with $x \neq y$. Throughout the proof, we follow the same classification of dichotomous preferences over $\{x, y, z\}$ as in the proof of Theorem 4.2.2. For all $k = 1, \ldots, 7$, let n_k the number of persons with the dichotomous preferences of type k. Recall $n_{x,y}(R) = n_3 + n_4$, $n_{y,x}(R) = n_5 + n_6$, $n_{y,z}(R) = n_2 + n_5$, $n_{z,y}(R) = n_4 + n_7$, $n_{x,z}(R) = n_2 + n_3$, and $n_{z,x}(R) = n_6 + n_7$.

Step 1: If $(p, q) \in \mathfrak{n}_0$ *and* $p, q \geq 1$*, then* $(p-1, q-1) \in \mathfrak{n}_0$.

Let $(p, q) \in \mathfrak{n}_0$ be such that $p, q \geq 1$. Consider a profile R consisting of $p-1$ agents of type 3, $q-1$ agents of type 6, 1 agent of type 4 and type 5, and $n-(p+q)$ agents of type 1 (thus there is no type 1 agent if $p + q = n$). That is, at R, $n_1 = n - (p+q)$, $n_2 = 0$, $n_3 = p-1$, $n_4 = 1$, $n_5 = 1$, $n_6 = q-1$, and $n_7 = 0$. Then $n_{x,y}(R) = p$, $n_{y,x}(R) = q$, $n_{y,z}(R) = 1$, $n_{z,y}(R) = 1$, $n_{x,z}(R) = p-1$, and $n_{z,x}(R) = q-1$. Since $(p, q) \in \mathfrak{n}_0$, $x \succeq_{f(R)} y$. By (4.7), $(1, 1) \in \mathfrak{n}_0$ and so $y \succeq_{f(R)} z$. Then by transitivity, $x \succeq_{f(R)} z$, which means $(p-1, q-1) \in \mathfrak{n}_0$.

Step 2: $\max\{q - p : (p, q) \in \mathfrak{n}_0\} = n$ *or* 0.

Let $(p^*, q^*) \in \mathfrak{n}_0$ be such that

$$q^* - p^* = \max\{q - p : (p, q) \in \mathfrak{n}_0\}. \tag{4.13}$$

Suppose $q^* - p^* \neq 0$. Then applying Step 1 repeatedly p^*-times, we show $(0, q^* - p^*) \in \mathfrak{n}_0$. Then since $q^* - p^* \geq 1$, by $\mathfrak{n}$-monotonicity, $(0, 1) \in \mathfrak{n}_0$.

Suppose by contradiction $q^* - p^* \neq n$. Then evidently $q^* \leq n - 1$. Thus there is a profile R consisting of $q^* - p^*$ agents of type 6, 1 agent of type 7, and the rest of $n - (q^* - p^* + 1)$ agents of type 1 (note that $q^* - p^* + 1 \leq q^* + 1 \leq n$ and so the number of agents of type 1 is a non-negative integer and the total number of agents is n). Then at R, $n_1 = n - (q^* - p^* + 1)$, $n_2 = n_3 = n_4 = n_5 = 0$, $n_6 = q^* - p^*$, and $n_7 = 1$. Thus $n_{x,y}(R) = 0$, $n_{y,x}(R) = q^* - p^*$, $n_{y,z}(R) = 0$,

[7] A stronger property, adding a domain richness to the property of two-indifference-class, is needed to prove this result. See Ju (2009b) for details.

$n_{z,y}(R) = 1, n_{x,z}(R) = 0$, and $n_{z,x}(R) = q^* - p^* + 1$. Since $(0, q^* - p^*), (0, 1) \in \mathfrak{n}_0$, then $x \succsim_{f(R)} y$ and $y \succsim_{f(R)} z$. By transitivity, $x \succsim_{f(R)} z$, which implies $(0, q^* - p^* + 1) \in \mathfrak{n}_0$, contradicting (4.13).

Step 3: f is either majority decision function or degenerate indifference function.

When $q^* - p^* = 0$, this and (4.7) imply that f is majority decision function. When $q^* - p^* = n$, $(p^*, q^*) = (0, n)$. Thus by $\mathfrak{n}$-monotonicity, $\mathfrak{n}_0 = \mathfrak{n}^*$. Hence for all $R \in \mathcal{D}^{2ic}_{N,X}$ and all $x, y \in X$, $(n_{x,y}(R), n_{y,x}(R)), (n_{y,x}(R), n_{x,y}(R)) \in \mathfrak{n}_0$; so $x \sim_{f(R)} y$. Therefore, f is degenerate indifference function. □

Remark 4.2.1. Maskin (1995) proved that on the domain of *linear* preference profiles with an *odd* number of voters, majority decision function is "most transitive" among social decision functions satisfying monotonicity, anonymity, and neutrality (in fact, he considers independence of irrelevant alternatives and weak Pareto instead of monotonicity). A similar result without the odd-number-assumption is obtained by Campbell and Kelly (2000). These results rely on some domain richness properties that our dichotomous domain does not have; e.g., Campbell and Kelly's characterization relies on the availability of single-peaked preferences in the domain. In addition, dichotomous preferences do not have linearity assumed in the above two papers. Moreover, our result is with transitivity on the "entire domain under consideration" and for both odd or even numbers of voters.

Other social decision functions satisfying monotonicity, anonymity, and neutrality violate transitivity. However, all these functions satisfy acyclicity.

Theorem 4.2.4 (Ju 2009b). *On any domain with the property of two-indifference-class, all social decision functions with monotonicity, anonymity, and neutrality satisfy acyclicity.*

Proof. Let f be a social decision function on $\mathcal{D}^{2ic}_{N,X}$ satisfying the three axioms. By Proposition 4.2.2, f is represented by an $\mathfrak{n}$-neutral index structure $\mathfrak{n} \equiv (\mathfrak{n}_{x,y})_{x,y \in X}$ which satisfies (4.7). For all $x, y \in X$ with $x \neq y$, let $\mathfrak{n}_0 \equiv \mathfrak{n}_{x,y}$.

Step 1: For all $x, y \in X$ and all $R \in \mathcal{D}^{2ic}_{N,X}$, if $x \succ_{f(R)} y$, then $n_{x,y}(R) > n_{y,x}(R)$.

This follows directly from (4.7).

Step 2: For all $x, y, z \in X$ and all $R \in \mathcal{D}^{2ic}_{N,X}$, if $n_{x,y}(R) > n_{y,x}(R)$ and $n_{y,z}(R) > n_{z,y}(R)$, then $n_{x,z}(R) > n_{z,x}(R)$.

The proof of this step uses a similar argument as in the proof of Theorem 4.2.2. Let $n_{x,y}(R) > n_{y,x}(R)$ and $n_{y,z}(R) > n_{z,y}(R)$. Then the two inequalities in (4.11) hold with strict inequality and from them, (4.12) is obtained as a strict inequality, which means $n_{x,z}(R) > n_{z,x}(R)$.

Step 3: If $R \in \mathcal{D}^{2ic}_{N,X}$ and a sequence of finite alternatives, $x_1, \ldots, x_T \in X$ are such that $x_1 \succ_{f(R)} x_2$, $x_2 \succ_{f(R)} x_3, \ldots, x_{T-1} \succ_{f(R)} x_T$, then $x_1 \succsim_{f(R)} x_T$; thus $x_T \succ_{f(R)} x_1$ does not hold.

If $x_1 \succ_{f(R)} x_2$ and $x_2 \succ_{f(R)} x_3$, then by Step 1, $n_{x_1,x_2}(R) > n_{x_2,x_1}(R)$ and $n_{x_2,x_3}(R) > n_{x_3,x_2}(R)$, which imply by Step 2, $n_{x_1,x_3}(R) > n_{x_3,x_1}(R)$. Applying this argument iteratively, we obtain, $n_{x_1,x_T}(R) > n_{x_T,x_1}(R)$, which implies, by (4.7), $x_1 \succsim_{f(R)} x_T$. □

4.2.4 Two Fixed Classes Separated by Strict Preferences

Now we consider a domain where alternatives are separated into two fixed subsets and any alternative in one is always preferred to any alternative in the other. Formally:

Definition 4.2.3. A domain $\mathcal{D}_{N,X}^{2fcs} \subseteq \mathcal{U}_{N,X}$ has the property of *two-fixed-class-separation* if for all distinct triples $x, y, z \in X$, there is a nonempty proper subset $A \subsetneq \{x, y, z\}$ such that for all $R \in \mathcal{D}_{N,X}^{2fcs}$ and all $i \in N$, either [for all $a \in A$ and all $b \in \{x, y, z\} \backslash A$, $aP_i b$] or [for all $a \in A$ and all $b \in \{x, y, z\} \backslash A$, $bP_i a$].[8]

Theorem 4.2.5 (Inada 1964). *On any domain with the property of two-fixed-class-separation, if the number of agents is odd, majority decision function satisfies transitivity.*

Proof. Let $x, y, z \in X$ be three distinct alternatives. Without loss of generality, assume that the two fixed classes are $A \equiv \{x\}$ and $B \equiv \{y, z\}$. Let $R \in \mathcal{D}_{N,X}^{2fcs}$. To prove transitivity, we need to consider the following six cases: (1) $x \succeq_{f_{MAJ}(R)} y$ and $y \succeq_{f_{MAJ}(R)} z$, (2) $x \succeq_{f_{MAJ}(R)} z$ and $z \succeq_{f_{MAJ}(R)} y$, (3) $y \succeq_{f_{MAJ}(R)} z$ and $z \succeq_{f_{MAJ}(R)} x$, (4) $z \succeq_{f_{MAJ}(R)} y$ and $y \succeq_{f_{MAJ}(R)} x$, (5) $y \succeq_{f_{MAJ}(R)} x$ and $x \succeq_{f_{MAJ}(R)} z$, (6) $z \succeq_{f_{MAJ}(R)} x$ and $x \succeq_{f_{MAJ}(R)} y$. Arguments for (1) and (2) are similar and also the arguments for (3) and (4) and for (5) and (6) are similar. Thus we only consider (1), (3), and (5) below.

Note that by the property of two-fixed-class-separation, $N_{x,y}(R) = N_{x,z}(R)$ and $N_{y,x}(R) = N_{z,x}(R)$. Thus by independence of irrelevant alternatives and neutrality of f_{MAJ},

$$x \succeq_{f_{MAJ}(R)} y \iff x \succeq_{f_{MAJ}(R)} z. \tag{4.14}$$

Case 1: $x \succeq_{f_{MAJ}(R)} y$ and $y \succeq_{f_{MAJ}(R)} z$.

By (4.14), $x \succeq_{f_{MAJ}(R)} y$ implies $x \succeq_{f_{MAJ}(R)} z$.

Case 2: $y \succeq_{f_{MAJ}(R)} z$ and $z \succeq_{f_{MAJ}(R)} x$.

By (4.14), $z \succeq_{f_{MAJ}(R)} x$ implies $y \succeq_{f_{MAJ}(R)} x$.

Case 3: $y \succeq_{f_{MAJ}(R)} x$ and $x \succeq_{f_{MAJ}(R)} z$.

By (4.14), $y \succeq_{f_{MAJ}(R)} x$ implies $z \succeq_{f_{MAJ}(R)} x$. Hence $z \sim_{f_{MAJ}(R)} x$, which implies $n_{z,x}(R) = n_{x,z}(R)$. By the property of two-fixed-class-separation, $n_{z,x}(R) + n_{x,z}(R) = n$. Therefore n is an even number, contradicting the initial assumption. Therefore, Case 3 does not occur on the domain. □

With a stronger condition on the domain, we can show that except for degenerate indifference function, majority decision function is the only transitive social decision function satisfying the three standard axioms.

[8] Sakai and Shimoji (2006) study "dichotomous domains" that are close to domains with two-fixed-class-separation. Assuming that the domain of individual preferences can be either dichotomous or universal, they find some domain conditions for the existence of Arrovian social welfare function.

Theorem 4.2.6 (Ju 2009b). *Consider a domain with the property of two-fixed-class-separation.*[9] *Assume that all preferences in this domain are linear and that there is an odd number of agents. Then a social decision function satisfies monotonicity, anonymity, neutrality, and transitivity if and only if it is either majority decision function or degenerate indifference function.*

Proof. Let f be a social decision function on $\mathcal{D}_{N,X}$ satisfying the four axioms. By Proposition 4.2.2, f is represented by a decisive structure $\mathfrak{n} \equiv (\mathfrak{n}_{x,y})_{x,y \in X}$ satisfying (4.7). Let $\mathfrak{n}_0 \equiv \mathfrak{n}_{x,y}$ for all distinct $x, y \in X$. By (4.7) and the assumption that all preference orderings on domain $\mathcal{D}_{N,X}$ are linear, in order to show that f is majority decision function, we only have to show that for all $(n_1, n_2) \in \mathfrak{n}^*$ with $n_1 + n_2 = n$, if $n_1 < n_2$, then $(n_1, n_2) \notin \mathfrak{n}_0$. Suppose that f is not majority function and so for some $(n_1, n_2) \in \mathfrak{n}^*$, $n_1 + n_2 = n$, $n_1 < n_2$, and $(n_1, n_2) \in \mathfrak{n}_0$. Let $x, y, z \in X$ be three distinct alternatives. Without loss of generality, assume that the two fixed classes are $A \equiv \{x\}$ and $B \equiv \{y, z\}$. Let $R \in \mathcal{D}_{N,X}$ be such that $n_{x,y}(R) = n_2$, $n_{y,x}(R) = n_1$, and for all $i \in N$, $yP_i z$. Then by the property of two-fixed-class-separation, $n_{x,z}(R) = n_2$ and $n_{z,x}(R) = n_1$. Since $(n_1, n_2) \in \mathfrak{n}_0$, $y \succeq_{f(R)} x$ and $z \succeq_{f(R)} x$. By (4.7), the reverse relations also hold and therefore $y \sim_{f(R)} x$ and $z \sim_{f(R)} x$. Finally by transitivity, $y \sim_{f(R)} z$. Since every agent prefers y to z at R by construction, this implies that $(0, n) \in \mathfrak{n}_0$, which means that f is degenerate indifference function. □

When there are even number of agents, the result does not hold, as shown by the following example due to Inada (1964). There are four agents with $xP_i yP_i z$ and four agents with $yI_i zP_i x$. Then majority decision gives $x \sim_{f_{MAJ}(R)} y$, $y \succ_{f_{MAJ}(R)} z$, and $x \sim_{f_{MAJ}(R)} z$, violating transitivity. However, note that this social preference relation is quasi-transitive. In fact, for quasi-transitivity, we do not need the odd number assumption. Moreover, any social decision function satisfying monotonicity and neutrality is quasi-transitive.

Theorem 4.2.7 (Ju 2009b). *On any domain with the property of two-fixed-class-separation, all social decision functions with monotonicity and neutrality satisfy quasi-transitivity.*

Proof. The proof is similar to the proof of the above theorem with the replacement of weak majority preference relation with the strict one. Note that the arguments used for Cases 1–2 in the above proof do not depend on the fact that the social decision function is majority decision function. The same arguments go through for any social decision function as long as it is represented by a decisive structure and is neutral. Case 3 will not occur now because $y \succ_{f(R)} x$ implies $z \succ_{f(R)} x$, which contradicts $x \succ_{f(R)} z$. □

Note that monotonicity in Theorem 4.2.7 can be weakened to independence of irrelevant alternatives.

[9] A stronger property, adding a domain richness to two-fixed-class-separation, is needed to prove this result. See Ju (2009b) for details.

4.3 Axiomatic Foundations for Majority Decision and Approval Voting

Throughout this section, assume that the set of alternatives X is fixed. Assume further that as in Sect. 4.2.3, preferences can have at most two indifference classes. These preferences are called *dichotomous preferences.* Each dichotomous preference is characterized by the set of best, or preferred alternatives. Thus we use $B_0 \in \bar{P}(X)$ to denote the dichotomous preference of which the set of preferred alternatives is B_0 and use $\mathfrak{D} \equiv \bar{P}(X)$ to denote the set of dichotomous preferences. In what follows, we fix the feasibility set to be equal to X and focus on characteristics of collective choice rules on the set of admissible preference profiles. Thus given a domain of dichotomous preferences $\mathcal{D} \subseteq \mathfrak{D}$, a *collective choice rule in this section* is a non-empty valued correspondence $c\colon \bigcup_{N\in\mathcal{N}} \mathcal{D}^N \to \bar{P}(X)$. Similarly a *collective choice quasi-rule* is a correspondence $c\colon \bigcup_{N\in\mathcal{N}} \mathcal{D}^N \to P(X)$ that may take the empty set as its value.

Rule $c(\cdot)$ is *anonymous* if the identities of persons are inessential, that is, for all $N, N' \in \mathcal{N}$ with $|N| = |N'|$ and all one-to-one functions $\lambda\colon N \to N'$, $c((B_i)_{i\in N}) = c((B_{\lambda(i)})_{i\in N})$. A profile of dichotomous preferences may be reduced to a function $\pi: \mathcal{D} \to \{0, 1, 2, \ldots\}$ mapping each dichotomous preference in the domain to the number of agents who have this preference. Let $\Pi(\mathcal{D})$ be the set of all such functions. With a slight abuse, we refer to elements in $\Pi(\mathcal{D})$ preference profiles. We often denote an anonymous rule (or quasi-rule) $c\colon \Pi(\mathcal{D}) \to P(X)$ as a function on $\Pi(\mathcal{D})$ instead of its original domain $\bigcup_{N\in\mathcal{N}} \mathcal{D}^N$.

A *voting system* is a pair of a set of valid ballots $\mathfrak{B} \subseteq P(X)$ and a non-empty valued correspondence $\phi\colon \bigcup_{N\in\mathcal{N}} \mathfrak{B}^N \to \bar{P}(X)$ on the set of all possible ballot profiles. We call $\phi(\cdot)$ a *ballot aggregator.* Voting system $(\mathfrak{B}, \phi)$ is *anonymous* if for all $N, N' \in \mathcal{N}$ with $|N| = |N'|$ and all one-to-one functions $\lambda\colon N \to N'$, $\phi((B_i)_{i\in N}) = \phi((B_{\lambda(i)})_{i\in N})$. For an anonymous voting system, the identities of voters are inessential. Reducing this information, a *ballot response profile* $\pi: \mathfrak{B} \to \{0, 1, 2, \ldots\}$ maps each valid ballot into the number of voters casting this ballot. Let $\Pi(\mathfrak{B})$ be the set of all ballot response profiles. For an anonymous voting system $(\mathfrak{B}, \phi)$, for all pairs $N, N' \in \mathcal{N}$, if $(B_i)_{i\in N}$ and $(B'_i)_{i\in N'}$ generate the same ballot response profile, then $\phi((B_i)_{i\in N}) = \phi((B'_i)_{i\in N'})$. Therefore we may define a ballot aggregator ϕ as a function φ on the set of ballot response profiles $\Pi(\mathfrak{B})$. Conversely, any such function $\varphi\colon \Pi(\mathfrak{B}) \to \bar{P}(X)$ defines an anonymous ballot aggregator. We call $\varphi\colon \Pi(\mathfrak{B}) \to \bar{P}(X)$ a *voting rule*. When voters have dichotomous preferences and reveal their true preferences using ballot response profiles in $\Pi(\mathfrak{B})$, a voting system $(\mathfrak{B}, \varphi)$ gives the collective choice rule identical to the voting rule φ.

Throughout Sects. 4.3 and 4.4, we assume that ballot space $\mathfrak{B}$ satisfies the *basic richness*, consisting of the following two properties: for all distinct pairs $x, y \in X$ and all permutations $\lambda\colon X \to X$,

$$\text{There is } B_0 \in \mathfrak{B} \text{ such that } x \in B_0 \text{ and } y \notin B_0. \tag{4.15}$$

$$\text{For all } B_0 \in \mathfrak{B},\ \lambda(B_0) \in \mathfrak{B}. \tag{4.16}$$

Given a profile $\pi \in \Pi(\mathcal{D})$, for all $x \in X$, let $n(x, \pi) \equiv \sum_{B_0 \in \mathcal{D}: x \in B_0} \pi(B_0)$ be the number of votes x wins at π. *Majority rule* on $\mathcal{D}$, $c_{MAJ}\colon \Pi(\mathcal{D}) \to \bar{P}(X)$, maps each profile $\pi \in \Pi(\mathcal{D})$ into $c_{MAJ}(\pi) \equiv \{x \in X : \text{for all } y \in X, n(x, \pi) \geq n(y, \pi)\}$. In the case of voting systems, majority rule is denoted by ϕ_{MAJ} or φ_{MAJ}. Note that for dichotomous preferences, there always exists a Condorcet winner since majority decision function is transitive (Theorem 4.2.2). Thus the Condorcet rule $CW(\cdot)$ mapping each preference profile into the set of Condorcet winners is well-defined, and it coincides with majority rule. In general, any transitive social decision function $f : \mathcal{D}^N \to \mathcal{R}$ on the restricted domain of dichotomous preferences $\mathcal{D} \subseteq P(X)$ generates a collective choice rule as in (4.1). Since we fix the set of alternatives X in our definition of collective choice rules, not all social decision functions can be generated by collective choice rules. A collective choice rule can be considered as generating a social decision function of which the social preferences are dichotomous.

In the following two subsections, we overview some important axiomatic characterizations for majority rule and approval voting. A more focused overview of the literature considering the ballot space $\mathfrak{B} = \bar{P}(X)$ and approval voting is provided in Xu (2010) in this volume. Most of the characterizations we overview are accompanied by some conditions on ballot space $\mathfrak{B}$ that are sufficient for the characterization. Thus, we will clarify to what ballot spaces (or voting procedures) each characterization of majority rule applies, which was not all clear in the literature. We will find that some of the results apply to a very wide variety of ballot spaces (voting procedures) and others apply only to the ballot space for approval voting.

Throughout this section, our discussion is focused on voting systems. However, most results on voting systems also apply to collective choice rules after the straightforward extension of axioms and conditions we state for voting systems. When there is no need of distinguishing ballot space $\mathfrak{B}$ and the same domain of dichotomous preferences, we use $\mathfrak{B}$ to denote both the ballot space and the preference domain.

4.3.1 Characterizations of Majority Voting Systems

4.3.1.1 Basic Axioms in the Fixed Population Model

In this section, we define basic axioms for voting systems in a fixed population framework. Let $N \equiv \{1, 2, \ldots, n\}$ be the set of voters.

The first axiom says that alternatives should be treated equally. In other words, changing their labels should not make any essential change in the voting outcome.

Neutrality: For all $B \in \mathfrak{B}^N$ and all permutations $\lambda\colon X \to X$, $\lambda(\phi(B)) = \phi(\lambda(B))$.

The next axiom introduced by Baigent and Xu (1991) has the flavor of anonymity. It embodies the condition that each vote for an alternative by a voter has the same weight independently of what other alternatives are in his ballot.

Independence of Vote Exchange: For all $B \in \mathfrak{B}^N$ and all $i, j \in N$, if $x \in B_i \backslash B_j$ and $y \in B_j \backslash B_i$, then letting $B_i' \equiv [B_i \backslash \{x\}] \cup \{y\}$ and $B_j' \equiv [B_j \backslash \{y\}] \cup \{x\}$,[10] $\phi(B_i', B_j', B_{-\{i,j\}}) = \phi(B)$.[11]

When only singleton ballots are available, this axiom coincides with anonymity. The next axiom pertains to even more drastic vote reallocations than vote exchange.

Independence of Vote Reallocation: For all $B, B' \in \mathfrak{B}^N$, if for all $x \in X$, $n(x, B) = n(x, B')$, then $\phi(B) = \phi(B')$.

Clearly, independence of vote reallocation implies independence of vote exchange.

The next axiom says that when two alternatives win the same number of votes, they should be treated equally.

Equal Treatment of Equal Votes: For all $B \in \mathfrak{B}^N$ and all $x, y \in X$, if $n(x, B) = n(y, B)$, then $x \in \phi(B)$ if and only if $y \in \phi(B)$.[12]

This axiom is an implication of neutrality and independence of vote exchange as shown by the next lemma. Baigent and Xu (1991) obtain this implication in a richer setting with choice aggregation procedures.

Lemma 4.3.1. *Neutrality and independence of vote exchange together imply equal treatment of equal votes.*

Proof. Let $B \in \mathfrak{B}^N$ and $x, y \in X$ be such that $n(x, B) = n(y, B)$. Since $n(x, B) = n(y, B)$, then $N(x, B) \backslash N(y, B)$ and $N(y, B) \backslash N(x, B)$ have the same cardinality. Thus it is possible to exchange one x-vote and one y-vote between agents in the former set and agents in the latter set one by one. Let $B' \in \mathfrak{B}^N$ be the profile obtained after these vote exchanges. Applying the reverse iterative vote exchanges at B', we return to B.

It is clear that B' can also be obtained after the transposition of x and y at B, that is, letting $\tau\colon X \to X$ be such that $\tau(x) = y$, $\tau(y) = x$, and $\tau(z) = z$ for all $z \in X \backslash \{x, y\}$, we have $B' = \tau B \equiv (\tau(B_i))_{i \in N}$. Clearly, $\tau B' = B$.

By neutrality, $x \in \phi(B)$ if and only if $\tau(x) = y \in \phi(\tau B) = \phi(B')$. By independence of vote exchange, $y \in \phi(B')$ if and only if $y \in \phi(B)$. Therefore, $x \in \phi(B)$ if and only if $y \in \phi(B)$. □

Baigent and Xu (1991) reformulate May's (1952) positive response for social decision function in the current framework as follows.

Positive Response to Vote Addition: For all $B \in \mathfrak{B}^N$ and all $i \in N$, if $x \notin B_i$ and $B_i' \equiv B_i \cup \{x\} \in \mathfrak{B}$, then $x \in \phi(B)$ implies $\phi(B_i', B_{-i}) = \{x\}$.

Note that this axiom has bite when the ballot space $\mathfrak{B}$ is closed under the addition of an alternative (vote) to any ballot. For example, if $\mathfrak{B} \equiv \{\{x\} : x \in X\}$, any ballot

[10] The two ballots B_i', B_j' are admissible in $\mathfrak{B}$ because of assumption (4.16).

[11] Xu (2010) in this volume and Baigent and Xu (1991) call this axiom "independence of symmetric substitution."

[12] The same axiom is called as "equal treatment" in Xu (2010) in this volume.

aggregator satisfies this axiom trivially. The next axiom is an alternative formulation that has a wider applicability.

Positive Response* to Vote Addition: For all $B \in \mathfrak{B}^N$, all $i \in N$, and all $x, y \in X$, if $x \notin B_i$, $x \in B_i' \in \mathfrak{B}$, and $B_i \cap \{y\} = B_i' \cap \{y\}$, then $x \in \phi(B)$ implies $\phi(B_i', B_{-i}) \cap \{x, y\} = \{x\}$.

The next axiom says that any additional vote for another alternative does not do any good for alternative x.

Negative Response to Competing Vote Addition: For all $B \in \mathfrak{B}^N$, all $i \in N$, and all $x, y \in X$, if $y \notin B_i$ and $B_i \cup \{y\} \in \mathfrak{B}$, then $[x \notin \phi(B)$ or $y \in \phi(B)]$ implies $x \notin \phi(B_i \cup \{y\}, B_{-i})$ (i.e., $x \in \phi(B_i \cup \{y\}, B_{-i})$ implies $x \in \phi(B)$ and $y \notin \phi(B)$).

Equivalently, for all $B \in \mathfrak{B}^N$, all $i \in N$, and all $x, y \in X$, if $y \in B_i$ and $B_i \backslash \{y\} \in \mathfrak{B}$, then $x \notin \phi(B_i \backslash \{y\}, B_{-i})$ or $y \in \phi(B_i \backslash \{y\}, B_{-i})$ implies $x \notin \phi(B)$ (i.e., $x \in \phi(B)$ implies $x \in \phi(B_i \backslash \{y\}, B_{-i})$ and $y \notin \phi(B_i \backslash \{y\}, B_{-i})$). Like positive response, this axiom has bite when the ballot space is closed under the addition of an alternative. Here is an alternative formulation with wider applicability.

Negative Response* to Competing Vote Addition: For all $B \in \mathfrak{B}^N$, all $i \in N$, all $B_i' \in \mathfrak{B}$, and all $x, y \in X$, if $y \notin B_i$, $y \in B_i'$, and $B_i \cap \{x\} = B_i' \cap \{x\}$, then $[x \notin \phi(B)$ or $y \in \phi(B)]$ implies $x \notin \phi(B_i', B_{-i})$ (equivalently, $x \in \phi(B_i', B_{-i})$ implies $x \in \phi(B)$ and $y \notin \phi(B)$).

4.3.1.2 Characterization Results: Voting Systems

We first show that May's Theorem (Theorem 4.2.1) for the binary choice framework can be extended in the current framework in a fairly straightforward manner. This result is based on Propositions 4.2.1 and 4.2.2. Since there can be more than two alternatives, we need independence of irrelevant alternatives in addition to May's three axioms.

Theorem 4.3.1. *A social decision function on $\mathfrak{B}^N$ satisfies independence of irrelevant alternatives, anonymity, neutrality, and positive response if and only if it is majority decision function on $\mathfrak{B}^N$. Moreover, majority decision function on $\mathfrak{B}^N$ satisfies transitivity and generates majority voting system $(\mathfrak{B}, \phi_{MAJ})$ as its choice rule.*

Proof. By Theorem 4.2.2, majority decision function satisfies transitivity on dichotomous domain $\mathfrak{B}$ as well as the other three axioms. To prove the converse, let f be a social decision function on $\mathfrak{B}^N$ satisfying the four stated axioms. Independence of irrelevant alternatives and positive response together imply monotonicity. Due to the richness of ballot space $\mathfrak{B}$ stated in (4.15) and (4.16), Proposition 4.2.2

holds, and f can be represented by a decisive structure. By anonymity, neutrality, and Proposition 4.2.2, f can be represented by a decisive index structure satisfying (4.7). Following the same argument as is given before Theorem 4.2.1, we show (4.10). □

As a corollary, we obtain:

Corollary 4.3.1. *A voting system* $(\mathfrak{B}, \phi)$ *is generated by a social decision function on* $\mathfrak{B}^N$ *satisfying independence of irrelevant alternatives, anonymity, neutrality, and positive response if and only if it is a majority voting system, that is,* $\phi = \phi_{MAJ}$. *Thus when* $\mathfrak{B} = \bar{P}(X)$, *it is approval voting system.*

In the framework of collective aggregation procedures, Baigent and Xu (1991) obtain a similar axiomatic characterization of *approval* voting imposing positive response to vote addition. In the current framework, their result can be stated as follows:

Theorem 4.3.2 (Baigent and Xu 1991). *Assume that ballot space* $\mathfrak{B}$ *is closed under the addition of a single vote, that is, for all* $B_0 \in \mathfrak{B}$ *and all* $x \in X$, $B_0 \cup \{x\} \in \mathfrak{B}$.[13] *Then the following are equivalent*:

(i) *Voting system* $(\mathfrak{B}, \phi)$ *satisfies neutrality, independence of vote exchange, and positive response to vote addition.*[14]
(ii) *Voting system* $(\mathfrak{B}, \phi)$ *satisfies equal treatment of equal votes and positive response to vote addition.*
(iii) *Voting system* $(\mathfrak{B}, \phi)$ *is a majority voting system,* $\phi = \phi_{MAJ}$.

Proof. Lemma 4.3.1 shows that (i) implies (ii). It is easy to show (iii) implies (i). We only prove (ii) implies (iii) below. Let $\mathfrak{B}$ be the ballot space with the stated property.

Let ϕ be the ballot aggregator in part (ii). Let $B \in \mathfrak{B}^N$. We need to show that $x \in \phi(B)$ if and only if for all $y \in X$, $n(x, B) \geq n(y, B)$. By equal treatment of equal votes, we only have to show the "only if" part. Suppose to the contrary that $x \in \phi(B)$ and for some $y \in X, n(y, B) > n(x, B)$. Then $N(y, B) \backslash N(x, B) \neq \emptyset$ and there are at least $[n(y, B) - n(x, B)]$ agents in this set. Change ballots of these agents from B_i to $B_i' \equiv B_i \cup \{x\}$. For all other i's, let $B_i' \equiv B_i$. Thus by construction, $n(x, B') = n(y, B')$. By positive response to vote addition, $\phi(B') = \{x\}$. On the other hand, since $n(x, B') = n(y, B')$, then by equal treatment of equal votes, $y \in \phi(B') = \{x\}$, which is a contradiction. □

Unlike Theorem 4.3.1, this result uses the assumption that the ballot space is closed under vote addition.

[13] Thus we need to allow $X \in \mathfrak{B}$. The assumption is needed to prove that (ii) implies (iii). It is not needed for other implications.

[14] Universal domain axiom is added in Baigent and Xu (1991).

Remark 4.3.1. Instead of the assumption on $\mathfrak{B}$ in the above theorem, require that for all distinct pairs $x, y \in X$, there is $B_0 \in \mathfrak{B}$ such that $x, y \in B_0$. Then the same result holds if positive response in parts (i) and (ii) is replaced with positive response*. To prove this, we only replace B_i' in the proof with $B_i' \in \mathfrak{B}$ such that $x, y \in B_i'$ and replace $\phi(B')$ in the proof with $\phi(B') \cap \{x, y\}$. The rest of the proof is the same.

The equivalence between (i) and (iii) is also stated in Theorem 5 of Xu (2010) in this volume, focusing on $\mathfrak{B} = P(X)\backslash\{\emptyset\}$. In fact, as stated in Theorem 4.3.2, the equivalence holds for a broader set of ballot spaces. The equivalence between (ii) and (iii) is somewhat close to Theorem 4 of Xu (2010) in this volume, focusing on $\mathfrak{B} = P(X)\backslash\{\emptyset\}$.

An alternative characterization with negative response to competing vote addition is obtained with a different assumption on $\mathfrak{B}$.

Theorem 4.3.3. *Assume that ballot space $\mathfrak{B}$ is closed under the deletion of a single vote, that is, for all $B_0 \in \mathfrak{B}$ and all $x \in B_0$, $B_0\backslash\{x\} \in \mathfrak{B}$.*[15] *Then the following are equivalent:*

(i) *Voting system $(\mathfrak{B}, \phi)$ satisfies neutrality, independence of vote exchange, and negative response to competing vote addition.*
(ii) *Voting system $(\mathfrak{B}, \phi)$ satisfies equal treatment of equal votes and negative response to competing vote addition.*
(iii) *Voting system $(\mathfrak{B}, \phi)$ is a majority voting system, $\phi = \phi_{MAJ}$.*

Proof. By Lemma 4.3.1, (i) implies (ii). We only prove that (ii) implies (iii). Let $\mathfrak{B}$ be given as stated above.

Let ϕ be the ballot aggregator in part (ii). Let $B \in \mathfrak{B}^N$. We need to show that $x \in \phi(B)$ if and only if for all $y \in X$, $n(x, B) \geq n(y, B)$. By equal treatment of equal votes, we only have to show the "only if part." Suppose to the contrary that $x \in \phi(B)$ and for some $y \in X$, $n(y, B) > n(x, B)$. Then $N(y, B)\backslash N(x, B) \neq \emptyset$ and there are at least $[n(y, B) - n(x, B)]$ agents in this set. Change ballots of these agents from B_i to $B_i' \equiv B_i\backslash\{y\}$ (this is possible by the assumption on $\mathfrak{B}$). For all other i's, let $B_i' \equiv B_i$. Thus by construction, $n(x, B') = n(y, B')$. Applying negative response to competing vote addition repeatedly, we show $x \in \phi(B')$ and $y \notin \phi(B')$, contradicting equal treatment of equal votes for $n(x, B') = n(y, B')$. □

Remark 4.3.2. Assume instead that for all distinct pairs $x, y \in X$, there is $B_0 \in \mathfrak{B}$ such that $B_0 \cap \{x, y\} = \emptyset$. Then the same result holds if negative response in parts (i) and (ii) is replaced with negative response*. To prove this, we only replace B_i' in the proof with $B_i' \in \mathfrak{B}$ such that $B_i' \cap \{x, y\} = \emptyset$. The rest of the proof is the same.

4.3.1.3 Extension in the Variable Population Framework

We now consider voting systems on a variable population domain. All the axioms defined in the fixed population framework can be extended to that framework by

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[15] Thus we need to allow $\emptyset \in \mathfrak{B}$.

simply adding the quantifier "for all $N \in \mathcal{N}$." All results in the previous section can be extended in the variable population framework. In particular, Theorem 4.3.2 can be so extended. Moreover, the result applies to more variety of ballot spaces by adding the extra, but mild condition that the empty ballot is allowed ($\emptyset \in \mathfrak{B}$), and the following natural axiom pertaining to the effect of the empty ballot. It says that adding an empty (abstention) vote does not affect the voting outcome.

Null Consistency: For all $N \in \mathcal{N}$ and all $B \in \mathfrak{B}^N$, if $i \notin N$ and $B_i = \emptyset$, then $\phi(B) = \phi(B, B_i)$.

The next two results extend Theorem 4.3.2 in the variable population framework.

Theorem 4.3.4. *Assume that $\emptyset \in \mathfrak{B}$ and for all $x \in X$, $\{x\} \in \mathfrak{B}$. Then on the domain $\bigcup_{N \in \mathcal{N}} \mathfrak{B}^N$, the following are equivalent*:

(i) *Voting system $(\mathfrak{B}, \phi)$ satisfies null consistency, neutrality, independence of vote exchange, and positive response to vote addition.*
(ii) *Voting system $(\mathfrak{B}, \phi)$ satisfies null consistency, equal treatment of equal votes, and positive response to vote addition.*
(iii) *Voting system $(\mathfrak{B}, \phi)$ is a majority voting system, $\phi = \phi_{MAJ}$.*

Proof. The proof is similar to the proof of Theorem 4.3.2. To prove that (ii) implies (iii), suppose to the contrary that $x \in \phi(B)$ and for some $y \in X$, $n(y, B) > n(x, B)$. Let N' be a set of $[n(y, B) - n(x, B)]$ agents such that $N' \cap N = \emptyset$. Let $B^0 \equiv (\emptyset, \ldots, \emptyset) \in \mathfrak{B}^{N'}$ and $B' \in \mathfrak{B}^{N'}$ be such that for each $i \in N'$, $B'_i = \{x\}$. By construction, $n(x, (B, B')) = n(y, (B, B'))$ and by null consistency, $\phi(B, B^0) = \phi(B)$ and so $x \in \phi(B, B^0)$. Applying positive response to vote addition repeatedly at (B, B^0), we get $\phi(B, B') = \{x\}$. On the other hand, by equal treatment of equal votes, $y \in \phi(B, B') = \{x\}$, which is a contradiction. □

Replacing positive response with positive response* to vote addition, we obtain a similar result. Unlike in Theorem 4.3.4, we do not need any assumption on the ballot space except for the availability of the empty ballot.

Theorem 4.3.5. *Assume that $\emptyset \in \mathfrak{B}$. On the domain $\bigcup_{N \in \mathcal{N}} \mathfrak{B}^N$, the following are equivalent*:

(i) *Voting system $(\mathfrak{B}, \phi)$ satisfies null consistency, neutrality, independence of vote exchange, and positive response* to vote addition.*
(ii) *Voting system $(\mathfrak{B}, \phi)$ satisfies null consistency, equal treatment of equal votes, and positive response* to vote addition.*
(iii) *Voting system $(\mathfrak{B}, \phi)$ is a majority voting system, $\phi = \phi_{MAJ}$.*

Proof. To prove this, we only have to replace $B'_i = \{x\}$ in the proof of Theorem 4.3.4 with $B'_i \in \mathfrak{B}$ such that $x \in B'_i$ and $y \notin B'_i$ (such B'_i exists by (4.15)) and replace $\phi(B, B')$ with $\phi(B, B') \cap \{x, y\}$. The rest of the proof is the same. □

4.3.2 Characterizations of Majority Voting in the Variable Population Framework

In this section, we consider anonymous voting systems in the variable population framework. Recall that such a voting system can be described by a pair of a ballot space $\mathfrak{B}$ and a voting rule (an anonymous ballot aggregator) $\varphi\colon \Pi(\mathfrak{B}) \to \bar{P}(X)$. We will use the following concept and notation.

A *null response profile* is a profile π where no alternative is supported by anyone, that is, for all $A \in \bar{P}(X)$, $\pi(A) = 0$. The *empty response profile* $\pi^{\emptyset}$ is the null response profile with no vote, that is, for all $A \in P(X)$, $\pi^{\emptyset}(A) = 0$. For all $A \in P(X)$, let π_A be such that $\pi_A(A) = 1$ and for all other ballots $B \in P(X)\backslash\{A\}$, $\pi_A(B) = 0$. For all $\pi \in \Pi(\mathfrak{B})$ and all $x \in X$, let $n(x,\pi) \equiv \sum_{A:x\in A} \pi(A)$ and $n(\pi) \equiv \sum_{x\in X} n(x,\pi)$.

4.3.2.1 Basic Axioms of Voting Rules

Let $m \equiv |X|$ be the number of alternatives. The following axioms have been considered by numerous authors in the literature on approval voting.

First, if there is only one voter, that voter's ballot should be fully respected.

Faithfulness: For all $A \in \mathfrak{B}\backslash\{\emptyset\}$, $\varphi(\pi_A) = A$.

Neutrality can be defined in the same way in the current framework as in earlier sections. A much weaker axiom requires that decisions at a null response profile should be neutral.

Null-Neutrality: For all null response profiles $\pi \in \Pi(\mathfrak{B})$, $\varphi(\pi) = X$.

The next axiom plays a key role in some characterizations of approval voting to be presented later. It pertains to a merger of two groups of voters. If a rule has a common recommendation for the two groups before the merger, the common recommendation should be the recommendation after the merger.

Consistency: For all $\pi, \pi' \in \Pi(\mathfrak{B})$, if $\varphi(\pi) \cap \varphi(\pi') \neq \emptyset$, then $\varphi(\pi + \pi') = \varphi(\pi) \cap \varphi(\pi')$.[16]

The next one is a weaker version of consistency considered by Sertel (1988).

Weak Consistency: For all $\pi \in \Pi(\mathfrak{B})$ and all $A \in \mathfrak{B}$, if $\varphi(\pi) \cap \varphi(\pi_A) \neq \emptyset$, then $\varphi(\pi + \pi_A) = \varphi(\pi) \cap \varphi(\pi_A)$.

The next axiom says that when there are two voters casting disjoint ballots, a voting rule should recommend the union of the two ballots.

Disjoint Equality: For all $A, B \in \mathfrak{B}\backslash\{\emptyset\}$, if $A \cap B = \emptyset$, then $\varphi(\pi_A + \pi_B) = A \cup B$.

The next axiom proposed by Sertel (1988) captures a similar idea but in a much stronger form.

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[16] This axiom and other axioms of consistency were studied also by Ching (1996) and Yeh (2006) for characterizations of plurality voting rule on the standard domain of preferences.

Sertel Disjoint Equality: For all $\pi \in \Pi(\mathfrak{B})$ and all $A \in \mathfrak{B}$ if $\varphi(\pi) \cap \varphi(\pi_A) = \emptyset$, then $x \in \varphi(\pi + \pi_A)$ if and only if $x \in \varphi(\pi)$ or $[x \in \varphi(\pi_A)$ and $\max_{y \in \varphi(\pi)} n(y, \pi) = 0]$ or $[x \in \varphi(\pi_A)$ and $n(x, \pi) = \max_{y \in \varphi(\pi)} n(y, \pi) - 1 \geq 0]$.

The next axiom pertains to a special case of ballot responses where all alternatives receive the same number of votes. It requires in this case that a voting rule should treat all alternatives equally by recommending all of them.

Cancellation: For all $\pi \in \Pi(\mathfrak{B})$, if all alternatives receive the same number of votes at π, that is, for all $x, y \in X$, $n(x, \pi) = n(y, \pi)$, then $\varphi(\pi) = X$.

Cancellation implies that the choice at any null response profile should be X as in approval voting.

The next axiom requires that a voting rule should make the same decision when two voters merge their ballots and cast the merged ballot as a single voter.

Independence of Pairwise Vote Merge: For all $\pi \in \Pi(\mathfrak{B})$ and all $A, B \in \mathfrak{B}$, if $A \cap B = \emptyset$ and $A \cup B \in \mathfrak{B}$, then $\varphi(\pi + \pi_A + \pi_B) = \varphi(\pi + \pi_{A \cup B})$.

Vote merge is a type of vote reallocation. The next independence axiom pertains to more drastic vote reallocations.

Independence of Vote Reallocation: For all $\pi, \pi' \in \Pi(\mathfrak{B})$, if for all $x \in X$, $n(x, \pi) = n(x, \pi')$, then $\varphi(\pi) = \varphi(\pi')$.

Note that independence of pairwise vote merge together with faithfulness and consistency imply cancellation.[17]

4.3.2.2 Scoring Rules

Majority rule is an example in the large family of voting rules based on scoring methods. Characterization of this family is quite useful for our later discussion of majority or approval voting.

A score function $s \colon \{1, \ldots, m\} \to \mathbb{R}$ maps each natural number of a ballot size into a real number (the score of the ballot). For all $\pi \in \Pi(\mathfrak{B})$ and all $x \in X$, let

$$p(x, \pi; s) \equiv \sum_{B \in \mathfrak{B} : x \in B} s(|B|)\pi(B) = \sum_{k=1}^{m} \sum_{\substack{B \in \mathfrak{B} : x \in B, \\ |B| = k}} s(k)\pi(B)$$

be the total points x wins at π under score function s. A voting rule φ is a *scoring rule* if there is a score function $s \colon \{1, \ldots, m\} \to \mathbb{R}$ such that for all $\pi \in \Pi(\mathfrak{B})$,

[17] To show this let $\pi \in \Pi(\mathfrak{B})$ be such that for all $x, y \in X$, $n(x, \pi) = n(y, \pi)$. Note that by independence of pairwise vote merge (when $B = \emptyset$), we may assume that $\pi(\emptyset) = 0$. Let $n \equiv n(x, \pi)$ for all $x \in X$. Applying this axiom again repeatedly, we obtain $\varphi(\pi) = \varphi(\sum_{x \in X} n\pi_{\{x\}}) = \varphi(n \sum_{x \in X} \pi_{\{x\}}) = \varphi(n\pi_X)$. By faithfulness and consistency, $\varphi(n\pi_X) = X$.

$$\varphi(\pi) \equiv \{x \in X : p(x,\pi;s) \geq p(y,\pi;s) \text{ for all } y \in X\}.$$

Note that when φ is represented by score function s, φ is also represented by $a \times s$ for all $a > 0$. Majority rule is the scoring rule represented by a positive and constant score function s such as $s(k) = 1$ for all $k = 1,\ldots,m$. When $\mathfrak{B} = P(X)$ or $P(X)\backslash\{\emptyset\}$ or $P(X)\backslash\{\emptyset, X\}$, majority rule on $\Pi(\mathfrak{B})$ is called as approval voting rule.[18]

Given a finite sequence of score functions $s^1,\ldots,s^T$, for all $x,y \in X$ and all $\pi \in \Pi$, $(p(y,\pi;s^t))_{t=1}^T$ *lexicographically dominates* $(p(x,\pi;s^t))_{t=1}^T$ if there is $t_0 \in \{1,\ldots,T\}$ such that $p(y,\pi;s^{t_0}) > p(x,\pi;s^{t_0})$ and for all $t = 1,\ldots,t_0-1$, $p(y,\pi;s^t) \geq p(x,\pi;s^t)$. A voting rule φ is a *lexicographic scoring rule* if there are $T \geq 1$ score functions $s^1,\ldots,s^T$ such that for all $\pi \in \Pi(\mathfrak{B})$, $x \in \varphi(\pi)$ if and only if there is no $y \in X$ such that $(p(y,\pi;s^t))_{t=1}^T$ lexicographically dominates $(p(x,\pi;s^t))_{t=1}^T$.

Young (1975) characterizes (lexicographic) scoring rules in the framework of ranked voting procedures, where voters can express their preferences in their ballots. The key axioms in his result are neutrality and consistency. In the current "non-ranked" voting procedures, the next two results are counterparts of Young's characterization.

Theorem 4.3.6 (Fishburn 1979). *Given a ballot space $\mathfrak{B} \subseteq P(X)\backslash\{\emptyset, X\}$, a voting rule satisfies neutrality and consistency if and only if it is a lexicographic scoring rule. Moreover, the number of score functions representing the rule is at most the number of possible sizes of ballots, namely, $|\{|B| : B \in \mathfrak{B}\}|$.*

Due to the nature of lexicographic comparison, "overwhelming majority" may not be enough to influence the voting outcome under lexicographic scoring rules. In order to avoid this unnatural feature, we impose the next axiom.[19]

Continuity: For all $\pi,\pi' \in \Pi(\mathfrak{B})$ and all $x \in X$, if $x \notin \varphi(\pi)$, then there is an integer $K > 0$ such that for all $k \geq K$, $x \notin \varphi(k\pi + \pi')$.

It is clear that scoring rules satisfy continuity since increasing k, the difference between the score of x and the score of another winning alternative at π gets arbitrarily larger. No other lexicographic scoring rules can satisfy continuity and we obtain:

Theorem 4.3.7 (Fishburn 1979). *Given a ballot space $\mathfrak{B} \subseteq P(X)\backslash\{\emptyset, X\}$, a voting rule satisfies neutrality, consistency, and continuity if and only if it is a scoring rule.*

Suppose that a lexicographic scoring rule is represented by score functions $s^1,\ldots,s^T$, and for some $k \in \{|B| : B \in \mathfrak{B}\}$ and $t \in \{1,\ldots,T\}$, $s^1(k) = \cdots = s^{t-1}(k) = 0$

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[18] Admissibility of $\emptyset$ or X in the ballot space does not make any essential difference in the choices made by majority rule.

[19] Myerson (1995) calls it "overwhelming majority."

and $s^t(k) < 0$. Then for any $B \in \mathfrak{B}$ with $|B| = k$ and any $b \in B$, $p(b, \pi_B; s^1) = \cdots = p(b, \pi_B; s^{t-1}) = 0$ and $p(b, \pi_B; s^t) = s^t(k) < 0$. Then b is not chosen by this lexicographic scoring rule, violating faithfulness. Thus faithfulness implies that the first non-zero component in $(s^1(k), \ldots, s^T(k))$ is positive. Consequently, a scoring rule satisfies faithfulness if and only if it is represented by a positive score function.

Corollary 4.3.2. *Assume that* $\mathfrak{B} \subseteq P(X)\backslash\{\emptyset, X\}$. *Then*:

(i) *A voting rule satisfies neutrality, consistency, and faithfulness if and only if it is a lexicographic scoring rule represented by a finite sequence of score functions* $s^1, \ldots, s^T$ *such that for all* $k \in \{1, \ldots, m\}$, *there is* $t \in \{1, \ldots, T\}$ *such that* $s^1(k) = \cdots = s^{t-1}(k) = 0 < s^t(k)$.
(ii) *A voting rule satisfies neutrality, consistency, continuity, and faithfulness if and only if it is a scoring rule represented by a positive score function.*

4.3.2.3 Characterizations of Majority Voting

If a positive score function s gives different score points for different ballot sizes, then there is $\pi \in \Pi$ such that all alternatives win the same number of votes but the alternatives winning a ballot with a higher score point have the greatest total score point. These alternatives are chosen and other alternatives are not chosen. For example, when $s(1) > s(2)$, let π be such that $\pi(\{a, b\}) = 1$, for all $x \in X\backslash\{a, b\}$, $\pi(\{x\}) = 1$, and for all other ballots Y, $\pi(Y) = 0$. Then the scoring rule will choose $X\backslash\{a, b\}$, which is a violation of cancellation. Thus, in order to satisfy cancellation, score function s must be constant. Therefore, the scoring rule represented by s is majority rule. The next result is similar to Young's characterization of the Borda rule for linear preferences.

Theorem 4.3.8 (Fishburn 1979). *Given a ballot space* $\mathfrak{B} \subseteq P(X)\backslash\{\emptyset, X\}$, *a voting rule satisfies neutrality, consistency, faithfulness, and cancellation if and only if it is majority rule.*

Note that this result holds for any arbitrary ballot space satisfying the richness conditions (4.15) and (4.16). For example, the ballot space consisting of only singleton ballots is rich. When the ballot space has no restriction on ballot sizes, the theorem yields a characterization of approval voting. The proof of Theorem 4.3.8 is relatively long. A much simpler proof is provided by Alos-Ferrer (2006) for unrestricted ballot space $\mathfrak{B} = \bar{P}(X)$. Moreover, he shows that neutrality in Fishburn's result can be dropped. The next theorem is based on the main results in Alos-Ferrer (2006).

Theorem 4.3.9. *Assume* $\mathfrak{B} \equiv \bar{P}(X)$. *Consider a voting rule* φ *on* $\Pi(\mathfrak{B})\backslash\{\pi^{\emptyset}\}$. *The following are equivalent*:

(i) *Voting rule* φ *satisfies faithfulness, consistency, and cancellation.*
(ii) *Voting rule* φ *satisfies faithfulness, consistency, and independence of pairwise vote merge.*

(iii) *Voting rule φ satisfies faithfulness, consistency, and independence of vote reallocation.*
(iv) *Voting rule φ is a majority rule, $\varphi = \varphi_{MAJ}$.*

Proof. It is easy to show (iv) $\Rightarrow$(i). In what follows, show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
Step 1: (i) $\Rightarrow$ (ii)

We only have to show that the three axioms in (i) imply independence of pairwise vote merge. Let $A, B \in P(X)$ be such that $A \cap B = \emptyset$. By cancellation,

$$\varphi(\pi_{A\cup B} + \pi_{X\setminus(A\cup B)}) = X = \varphi(\pi_A + \pi_B + \pi_{X\setminus(A\cup B)}). \tag{4.17}$$

Hence,

$$\begin{aligned}\varphi(\pi + \pi_A + \pi_B) &= \varphi(\pi + \pi_A + \pi_B) \cap \varphi(\pi_{A\cup B} + \pi_{X\setminus(A\cup B)}); \\ \varphi(\pi + \pi_{A\cup B}) &= \varphi(\pi + \pi_{A\cup B}) \cap \varphi(\pi_A + \pi_B + \pi_{X\setminus(A\cup B)}).\end{aligned} \tag{4.18}$$

Then by consistency,

$$\begin{aligned}&\varphi(\pi + \pi_A + \pi_B) \cap \varphi(\pi_{A\cup B} + \pi_{X\setminus(A\cup B)}) \\ &\quad = \varphi(\pi + \pi_A + \pi_B + \pi_{A\cup B} + \pi_{X\setminus(A\cup B)}); \\ &\varphi(\pi + \pi_{A\cup B}) \cap \varphi(\pi_A + \pi_B + \pi_{X\setminus(A\cup B)}) \\ &\quad = \varphi(\pi + \pi_{A\cup B} + \pi_A + \pi_B + \pi_{X\setminus(A\cup B)}).\end{aligned} \tag{4.19}$$

Finally, since $\pi+\pi_A+\pi_B+\pi_{A\cup B}+\pi_{X\setminus(A\cup B)} = \pi+\pi_{A\cup B}+\pi_A+\pi_B+\pi_{X\setminus(A\cup B)}$, then (4.18) and (4.19) give $\varphi(\pi + \pi_A + \pi_B) = \varphi(\pi + \pi_{A\cup B})$.

Step 2: (ii) $\Rightarrow$ (iii)

We only have to show that the three axioms in (ii) imply independence of vote reallocation. Let $\pi \in \Pi$. By independence of pairwise vote merge, we may assume that $\pi(\emptyset) = 0$. Iterative application of independence of pairwise vote merge gives $\varphi(\pi) = \varphi(\sum_{A\in \bar{P}(X)} \pi(A) \sum_{x\in A} \pi_{\{x\}})$. Since $\sum_{A\in \bar{P}(X)} \pi(A) \sum_{x\in A} \pi_{\{x\}} = \sum_{x\in X} n(x, \pi)\pi_{\{x\}}$,

$$\varphi(\pi) = \varphi(\sum_{x\in X} n(x, \pi)\pi_{\{x\}}).$$

Thus $\varphi(\pi)$ depends only on $n(x, \pi)$. Therefore, when π and π' satisfy $n(x, \pi) = n(x, \pi')$ for all $x \in X$, $\varphi(\pi) = \varphi(\pi')$.

Step 3: (iii) $\Rightarrow$ (iv)

Let $\pi \in \Pi(\mathfrak{B})$ and $K \equiv max_{x\in X} n(x, \pi)$. Since $\emptyset \notin \mathfrak{B}$ and $\pi^{\emptyset}$ is assumed to be out of the domain, $K > 0$.[20] For each $k \in \{1, \ldots, K\}$, let $X_k \equiv \{x \in X :$

[20] If $\pi^{\emptyset}$ is in the domain, neither independence of pairwise vote merge nor independence of vote reallocation implies $\varphi(\pi^{\emptyset}) = \varphi_{MAJ}(\pi^{\emptyset}) = X$, while cancellation does. Thus the equivalence

$n(x,\pi) = k\}$. Then $X_0, X_1, \ldots, X_K$ partition X. Now construct a non-decreasing sequence of subsets as follows:

$$\begin{aligned}
Y_K &\equiv X_K \\
Y_{K-1} &\equiv X_K \cup X_{K-1} \\
Y_{K-2} &\equiv X_K \cup X_{K-1} \cup X_{K-2} \\
\vdots \quad & \quad \vdots \qquad \vdots \\
Y_1 &\equiv X_K \cup X_{K-1} \cup X_{K-2} \cup \cdots \cup X_1
\end{aligned}$$

Note that $Y_K \cap Y_{K-1} = X_K, Y_K \cap Y_{K-1} \cap Y_{K-2} = X_K, \cdots, Y_K \cap Y_{K-1} \cap Y_{K-2} \cap \cdots \cap Y_1 = X_K$. By faithfulness,

$$\varphi(\pi_{Y_K}) = Y_K, \varphi(\pi_{Y_{K-1}}) = Y_{K-1}, \ldots, \varphi(\pi_{Y_1}) = Y_1.$$

Applying consistency,

$$\begin{aligned}
\varphi(\pi_{Y_K} + \pi_{Y_{K-1}} + \cdots + \pi_{Y_1}) &= \varphi(\pi_{Y_K}) \cap \varphi(\pi_{Y_{K-1}}) \cap \cdots \cap \varphi(\pi_{Y_1}) \\
&= Y_K \cap Y_{K-1} \cap \cdots \cap Y_1 \\
&= X_K.
\end{aligned}$$

Finally, since π and $\pi_{Y_K} + \pi_{Y_{K-1}} + \cdots + \pi_{Y_1}$ give the same number of votes for each alternative, by independence of vote reallocation, $\varphi(\pi) = X_K = \varphi_{MAJ}(\pi)$. □

The proof relies heavily on the richness of the ballot space $\mathfrak{B} \equiv \bar{P}(X)$. In particular, the ballot space is closed under union.[21] Therefore the result cannot be applied to restricted ballot spaces such as the space of singleton ballots. The equivalence between (i) and (iv) is also stated in Theorem 1 in Xu (2010) of this volume.

The next characterization of approval voting rule uses disjoint equality. Unlike Theorem 4.3.8, the result applies only to the ballot space $P(X)\backslash\{\emptyset, X\}$.

Theorem 4.3.10 (Fishburn 1978a, 1979). *Assume that* $\mathfrak{B} = P(X)\backslash\{\emptyset, X\}$ and consider voting rules over $\Pi(\mathfrak{B})\backslash\{\pi^{\emptyset}\}$.

(i) *Assume* $|X| = 2$. *Then a voting rule satisfies neutrality, consistency, and faithfulness if and only if it is majority rule.*
(ii) *Assume* $|X| \geq 3$. *Then a voting rule satisfies neutrality, consistency, and disjoint equality if and only if it is majority rule.*

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cannot be established. If $\pi^{\emptyset}$ is in the domain, the result may be changed by replacing cancellation with a slightly weaker version by requiring $\pi \neq \pi^{\emptyset}$ in the definition of the axiom and weakening (iv) by allowing for any arbitrary choice at $\pi^{\emptyset}$.

[21] Alos-Ferrer (2006) assumes $X \notin \mathfrak{B}$. But then Y_1 in the above proof may not be an admissible ballot (the ballot space is not closed under union) and the proof does not go through. This is why we assume $X \in \mathfrak{B}$.

Sertel (1988) replaces disjoint equality with a stronger axiom, Sertel disjoint equality, and characterizes the rule that coincides with majority rule except when the empty set is the only ballot response. In this case, the rule selects the empty set (taking the empty value is allowed in his definition of voting rules). Although Sertel disjoint equality is less primitive and harder to motivate than disjoint equality, his proof is remarkably simpler than the proof of Theorem 4.3.10. Here we present his result in a different way in order to give a more clear comparison. Unlike Theorem 4.3.10, Sertel's characterization holds with an arbitrary ballot space $\mathfrak{B}$ and with null-neutrality, which is much weaker than neutrality.

Theorem 4.3.11. *Given any ballot space $\mathfrak{B}$, a voting rule φ satisfies null-neutrality, faithfulness, weak consistency, and Sertel disjoint equality if and only if $\varphi = \varphi_{MAJ}$.*

Remark 4.3.3. Sertel's faithfulness says that when there is only one ballot response A that is possibly the empty set, voting rule must choose A (recall that in our definition, faithfulness pertains to non-empty A). Clearly φ_{MAJ} does not satisfy this axiom. Sertel (1988) shows that his approval voting rule (identical to the standard approval voting rule except at null response profiles) is the only quasi-rule satisfying his faithfulness together with neutrality, weak consistency, and Sertel disjoint equality. In fact, dropping the requirement of non-empty valuedness in the definition of voting rule (thus among quasi-rules) and replacing null-neutrality in Theorem 4.3.11 either with neutrality or with "$\varphi(\pi) = \emptyset$ or X at all null response profiles π," we obtain a joint characterization of the two rules, Sertel's approval voting rule and the standard approval voting rule. The proof is essentially the same.

Proof. Let $\mathfrak{B}$ be a ballot space and φ a rule on $\Pi(\mathfrak{B})$ satisfying the four axioms. In what follows, for all $k \in \mathbb{N}$, we prove the claim that for all $\pi \in \Pi(\mathfrak{B})$ with $n(\pi) \leq k$, $\varphi(\pi) = \varphi_{MAJ}(\pi)$. The proof is by induction on k. The claim with $k = 1$ follows directly from null-neutrality and faithfulness. Let $k \geq 2$. Suppose by induction that for all $\pi \in \Pi(\mathfrak{B})$ with $n(\pi) \leq k$, $\varphi(\pi) = \varphi_{MAJ}(\pi)$. Let $\pi \in \Pi(\mathfrak{B})$ be such that $n(\pi) = k + 1$. We prove that $\varphi(\pi) = \varphi_{MAJ}(\pi)$. Note that there are $\pi' \in \Pi(\mathfrak{B})$ and $A \in \mathfrak{B}$ such that $n(\pi') = k$ and $\pi = \pi' + \pi_A$. Then by the induction hypothesis, $\varphi(\pi') = \varphi_{MAJ}(\pi')$ and $\varphi(\pi_A) = \varphi_{MAJ}(\pi_A)$.

Case 1: $\varphi(\pi') \cap \varphi(\pi_A) \neq \emptyset$. Then $\varphi_{MAJ}(\pi') \cap \varphi_{MAJ}(\pi_A) \neq \emptyset$. Since both φ and φ_{MAJ} satisfy weak consistency, $\varphi(\pi' + \pi_A) = \varphi(\pi') \cap \varphi(\pi_A)$ and $\varphi_{MAJ}(\pi' + \pi_A) = \varphi_{MAJ}(\pi') \cap \varphi_{MAJ}(\pi_A)$. Since $\varphi(\pi') = \varphi_{MAJ}(\pi')$, $\varphi(\pi_A) = \varphi_{MAJ}(\pi_A)$, and $\pi = \pi' + \pi_A$, then $\varphi(\pi) = \varphi_{MAJ}(\pi)$.

Case 2: $\varphi(\pi') \cap \varphi(\pi_A) = \emptyset$. Then $\varphi_{MAJ}(\pi') \cap \varphi_{MAJ}(\pi_A) = \emptyset$. By Sertel disjoint equality of φ, $x \in \varphi(\pi' + \pi_A)$ if and only if (i) $x \in \varphi(\pi')$ or (ii) $x \in \varphi(\pi_A)$ and $max_{y \in \varphi(\pi')} n(y, \pi') = 0$ or (iii) $x \in \varphi(\pi_A)$ and $n(x, \pi') = max_{y \in \varphi(\pi')} n(y, \pi') - 1 \geq 0$. Note that since $\varphi(\pi') = \varphi_{MAJ}(\pi')$ and $\varphi(\pi_A) = \varphi_{MAJ}(\pi_A)$, then (i), (ii), and (iii) are equivalent respectively to (i′) $x \in \varphi_{MAJ}(\pi')$, (ii′) $x \in \varphi_{MAJ}(\pi_A)$ and $max_{y \in \varphi_{MAJ}(\pi')} n(y, \pi') = 0$, and (iii′) $x \in \varphi_{MAJ}(\pi_A)$ and $n(x, \pi') = max_{y \in \varphi_{MAJ}(\pi')} n(y, \pi') - 1 \geq 0$. Therefore since both φ and φ_{MAJ} satisfy Sertel disjoint equality, $x \in \varphi(\pi' + \pi_A)$ if and only if $x \in \varphi_{MAJ}(\pi' + \pi_A)$. Since $\pi = \pi' + \pi_A$, we obtain $\varphi(\pi) = \varphi_{MAJ}(\pi)$. □

4.4 Strategic Voting and Condorcet Principle

In this section, we overview important supports for approval voting from the point of view of robustness to strategic voting as well as satisfying the Condorcet principle.

Consider a ballot space $\mathfrak{B}$. For all profiles of dichotomous preferences $\pi \in \Pi(\mathfrak{B})$, the *Condorcet set* $CW(\pi) \equiv \{x : n(x,\pi) \geq n(y,\pi),$ for all $y \in X\}$ is the set of Condorcet winners at π. By Theorem 4.2.2, the Condorcet set is non-empty and it coincides with the choice made by majority voting rule, $\varphi_{MAJ}(\pi) = CW(\pi)$. The Condorcet principle requires that a voting rule should select the Condorcet set.

Condorcet: For all $\pi \in \Pi(\mathfrak{B})$, $\varphi(\pi) = CW(\pi)$.

Evidently, a voting rule on $\mathfrak{B}$ satisfies Condorcet if and only if it is the majority rule on $\mathfrak{B}$. A weaker requirement is that a voting rule should select some Condorcet winners.

Weak Condorcet: For all $\pi \in \Pi(\mathfrak{B})$, $\varphi(\pi) \cap CW(\pi) \neq \emptyset$.

A voting rule φ on $\mathfrak{B}$ is *minimally selective* if for some $\pi \in \Pi(\mathfrak{B})$, $\varphi(\pi) \neq X$. Clearly, any non-constant voting rule is minimally selective. Fishburn (1979) obtains the following characterization of majority voting rule.

Theorem 4.4.1 (Fishburn 1979). *A voting rule on $\mathfrak{B}$ satisfies neutrality, consistency, continuity, minimal selectiveness, and weak Condorcet if and only if it is majority voting rule.*

In the strategic voting environment, Condorcet, not to speak of weak Condorcet, does not guarantee a Condorcet winner to be a final voting outcome. To investigate strategic voting behavior under a voting rule that sometimes produces tied outcomes, understanding how voters evaluate subsets of alternatives is needed. For a dichotomous preference relation $B \in \mathfrak{D}$ of voter i there are five natural assumptions about its extension over subsets of alternatives. Denote the extended preference relation of B by R_i^B. The five assumptions are as follows: for all $x, y \in X$,

P1. $\{x\}P_i^B\{y\}$ if and only if $x \in B$ and $y \notin B$;
P2. $\{x\}P_i^B\{x,y\}$ and $\{x,y\}P_i^B\{y\}$ if $x \in B$ and $y \notin B$;
P3. AR_i^BA' if $A \subseteq B$ or $A' \subseteq X \backslash B$ or $[A\backslash A' \subseteq B$ and $A'\backslash A \subseteq X\backslash B]$,
P4. $A \cup \{a\} I_i^B A \cup \{a'\}$ if $a, a' \notin A \cup B$ or $a, a' \in B \backslash A$,
P5. $AP_i^B X\backslash B$ if $A \cap B \neq \emptyset$; $BP_i^B A$ if $A \cap [X\backslash B] \neq \emptyset$, where P_i^B is the strict counterpart of R_i^B.

The first three assumptions, P1–P3, are considered by Fishburn (1979). Two additional assumptions, P4–P5, are needed to extend his result on the unrestricted ballot space to general ballot spaces.

4.4.1 Strategic Voting Under Anonymous Voting Systems

Given a ballot space $\mathfrak{B}$ and an agent with dichotomous preference $B \in \mathfrak{D}$, a ballot response $A \in \mathfrak{B}$ is *dominated* by ballot response $A' \in \mathfrak{B}$ if for all

$\pi \in \Pi(\mathfrak{B}) \cup \{\pi^{\emptyset}\}$, $\varphi(\pi + \pi_{A'}) R^B \varphi(\pi + \pi_A)$ with strict relation for at least one $\pi \in \Pi(\mathfrak{B}) \cup \{\pi^{\emptyset}\}$. Let $\mathfrak{B}^{ud}(B, \varphi)$ be the set of all undominated ballots in $\mathfrak{B}$. Since $\mathfrak{B}$ is a finite set, there exists at least one undominated ballot and $\mathfrak{B}^{ud}(B, \varphi) \neq \emptyset$. For each dichotomous preference profile $\pi \in \Pi(\mathfrak{D})$, let $\Pi^{ud}(\mathfrak{B})(\pi, \varphi)$ be the set of all ballot response profiles consisting of undominated ballots of all agents. A voting system $(\mathfrak{B}, \varphi)$ and the correspondence of undominated ballot response profiles $\Pi^{ud}(\mathfrak{B})(\cdot, \varphi)$ generate a collective choice rule for dichotomous preferences in $\mathcal{D} \subseteq \mathfrak{D}$, $c\colon \Pi(\mathcal{D}) \to P(X)\backslash\{\emptyset\}$ defined as follows: for all $\pi \in \Pi(\mathcal{D})$, $c(\pi) \equiv \bigcup\{\varphi(\hat{\pi}) : \hat{\pi} \in \Pi^{ud}(\mathfrak{B})(\pi, \varphi)\}$. It is natural to assume that each voter will not cast a dominated ballot and that the outcomes from strategic voting will be within the set of outcomes from undominated ballot profiles, that is, $c(\pi)$. Strategic voting is not an issue for agents who have complete indifference over all outcomes, namely agents with *unconcerned* dichotomous preference X because any two ballots will be indifferent independently of others' ballots. In what follows, we will focus on *concerned* agents who have dichotomous preferences with a preferred set $B \neq X$.

The set of undominated outcomes $c(\pi)$ may be quite different from the set of outcomes from truthful voting, $\varphi(\pi)$ and so the voting system $(\mathfrak{B}, \varphi)$ may lead to too different an outcome from the truthful outcome. Particular attention has been paid to voting systems that do not have this problem. A voting rule φ (or an anonymous collective choice rule c) on domain $\Pi(\mathcal{D})$ is *realizable in undominated strategies* by voting system $(\mathfrak{B}, \varphi)$ if for all profiles of concerned preferences $\pi \in \Pi(\mathcal{D}\backslash\{X\})$, $\hat{\varphi}(\hat{\pi}) \subseteq \varphi(\pi)$ for all undominated ballot response profiles $\hat{\pi} \in \Pi(\mathfrak{B})$ *at* π. Voting rule $\varphi(\cdot)$ on $\Pi(\mathcal{D})$ is *strategy-proof* if for all profiles of concerned preferences $\pi \in \Pi(\mathcal{D}\backslash\{X\})$, it is realizable in undominated strategies by voting system $(\mathfrak{B}, \varphi)$ and there is a unique undominated ballot response profile at π. We say that voting system $(\mathfrak{B}, \varphi)$ is *strategy-proof on* $\mathcal{D}$ if it always has a unique undominated profile at all $\pi \in \Pi(\mathcal{D}\backslash\{X\})$. Formally:

Strategy-Proofness on $\mathcal{D}$: For all $\pi \in \Pi(\mathcal{D}\backslash\{X\})$, $\Pi^{ud}(\mathfrak{B})(\pi, \varphi) = \{\pi'\}$ and $\varphi(\pi') \subseteq \varphi(\pi)$.

The next lemma shows that if a neutral and faithful voting system is strategy-proof on $\mathcal{D}$, then there should be no constraint on expressing one's concerned preferences in $\mathcal{D}$. That is,

No Ballot Constraint on $\mathcal{D}$: $\mathcal{D}\backslash\{X\} \subseteq \mathfrak{B}$.

Now we are ready to state the lemma.

Lemma 4.4.1. *If a voting system $(\mathfrak{B}, \varphi)$ satisfies neutrality, faithfulness, and strategy-proofness on $\mathcal{D}$, then it has no ballot constraint and for all $B \in \mathcal{D}\backslash\{X\}$, B is the unique undominated strategy for dichotomous preference B.*

Proof. Let B' be the undominated strategy for a concerned preference B and $B' \notin \{B, X\backslash B, \emptyset, X\}$. Then there exist $c, d \in X$ such that (i) $c \in B' \cap [X\backslash B]$ and $d \in [X\backslash B]\backslash B'$ or (ii) $c \in B' \cap B$ and $d \in B\backslash B'$. Consider the first case (i) (similar argument applies to case (ii)). Let $\lambda : X \to X$ be such that $\lambda(c) = d$, $\lambda(d) = c$, and for all other $x \in X\backslash\{c, d\}$, $\lambda(x) = x$. Then since B' is the only undominated strategy for B, for some π, $\varphi(\pi_{B'}+\pi)P^B\varphi(\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi)$. Note that $\pi_{B'}\lambda = \pi_{[B'\backslash\{c\}]\cup\{d\}}$ and $\pi_{B'} = \pi_{[B'\backslash\{c\}]\cup\{d\}}\lambda$. Hence $\varphi((\pi_{B'}+\pi)\lambda) = \varphi(\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi\lambda)$ and $\varphi((\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi)\lambda) = \varphi(\pi_{B'}+\pi\lambda)$. By neutrality, $\varphi(\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi\lambda) = \lambda(\varphi(\pi_{B'}+\pi))$ and $\varphi(\pi_{B'}+\pi\lambda) = \lambda(\varphi(\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi))$. By P4, $\lambda(\varphi(\pi_{B'}+\pi))I^B\varphi(\pi_{B'}+\pi)$ and $\lambda(\varphi(\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi))I^B\varphi(\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi)$. Hence $\varphi(\pi_{[B'\backslash\{c\}]\cup\{d\}}+\pi\lambda)P^B\varphi(\pi_{B'}+\pi\lambda)$, which shows that $[B'\backslash\{c\}]\cup\{d\}$ is not dominated by B', contradicting that B' is the unique undominated strategy for B.

If $X\backslash B$ is the unique undominated strategy for B, $\varphi(\pi_{X\backslash B}+\pi^{\emptyset})R^B\varphi(\pi_A+\pi^{\emptyset})$ for any $A\in\mathfrak{B}$ with $A\cap B\neq\emptyset$. If φ is faithful, then $X\backslash BR^BA$, contradicting P5.

Therefore, if a neutral and faithful voting system $(\mathfrak{B},\varphi)$ is strategy-proof on $\mathcal{D}$, then for all concerned preference $B\in\mathcal{D}\backslash\{X\}$, B should be the unique undominated strategy; so $B\in\mathfrak{B}$. Hence there should be no constraint in expressing one's concerned preferences in $\mathcal{D}$. □

Brams and Fishburn (1978, Theorems 2 and 6) offer a necessary and sufficient conditions for undominated ballots under majority voting systems. For dichotomous preferences, their condition roughly says that undominated ballots for each dichotomous preference B are the ballots that best approximate B either from above or from below in the ballot space $\mathfrak{B}$. Formally:

Lemma 4.4.2 (Brams and Fishburn 1978). *Given a majority voting system* $(\mathfrak{B},\varphi_{MAJ})$*, for each dichotomous preference* $B\in P(X)\backslash\{\emptyset,X\}$*, a ballot* $\hat{B}\in\mathfrak{B}$ *is undominated if and only if* (i) $\hat{B}\subseteq B$ *and there is no* $A\in\mathfrak{B}\backslash\{\hat{B}\}$ *such that* $\hat{B}\subsetneq A\subseteq B$ *or* (ii) $B\subseteq\hat{B}$ *and there is no* $A\in\mathfrak{B}\backslash\{\hat{B}\}$ *such that* $B\subseteq A\subsetneq\hat{B}$.

Thus if dichotomous preference B is in ballot space $\mathfrak{B}$, then B is the only undominated strategy, $\mathfrak{B}^{ud}(B,\varphi_{MAJ}) = \{B\}$. Similarly, if $\pi\in\Pi(\mathfrak{B})$, $\Pi^{ud}(\mathfrak{B})(\pi,\varphi_{MAJ}) = \{\pi\}$. Thus if $\mathcal{D}\subseteq\mathfrak{B}$, majority voting system $(\mathfrak{B},\ \varphi_{MAJ})$ is strategy-proof on $\mathcal{D}$. Conversely, if $\mathcal{D}\nsubseteq\mathfrak{B}$, then by Lemma 4.4.2, majority voting system $(\mathfrak{B},\varphi_{MAJ})$ has more than one undominated ballot response profiles at a profile π consisting of some B in $\mathcal{D}\backslash\mathfrak{B}$. Therefore, we obtain:

Theorem 4.4.2. *Majority voting system* $(\mathfrak{B},\varphi_{MAJ})$ *is strategy-proof on a subdomain of dichotomous preferences* $\mathcal{D}\subseteq\mathfrak{D}$ *if and only if there is no ballot constraint, i.e.,* $\mathcal{D}\subseteq\mathfrak{B}$*. Thus approval voting is the only strategy-proof majority voting system on the entire domain of dichotomous preferences,* $\mathfrak{D}$.

Note that Condorcet winners at π coincide with the alternatives selected by majority voting rule at π. Thus when π is in the space of ballot response profiles, by Lemma 4.4.2, π is the only undominated ballot response profile and thus any undominated ballot response profile at π gives the set of Condorcet winners. However, if a voter has a dichotomous preference that is not in the ballot space, then this equivalence between the set of Condorcet winners and the set of alternatives obtained by an undominated strategy profile in the majority voting system fails. Moreover, the failure can be so drastic that some undominated ballot response profile does not give any Condorcet winner. A stronger version of this claim is established for voting systems satisfying neutrality and the following basic axiom.

Strong Pareto: For all $\pi \in \Pi(\mathfrak{B})$ and all $x, y \in X$, if for all $A \in \mathfrak{B}$ with $\pi(A) > 0$, $x \in A$ or $x, y \in X \backslash A$, and there is $A \in \mathfrak{B}$ with $\pi(A) > 0$ such that $x \in A$ and $y \in X \backslash A$, then $y \notin \varphi(\pi)$.[22]

Lemma 4.4.3 (Fishburn 1979, Theorem 9). *Consider a voting system* $(\mathfrak{B}, \varphi)$ *satisfying neutrality and strong Pareto. Assume that for a dichotomous preference,* $B \in P(X) \backslash \{\emptyset, X\}$, *there is an undominated ballot* A *different from* B *(i.e.,* $A \in \mathfrak{B}^{ud}(\varphi, B)$ *and* $A \neq B$*), then there is a profile of dichotomous preferences* $\pi \in \Pi$ *such that for some undominated ballot response profile* $\hat{\pi} \in \Pi^{ud}(\mathfrak{B})(\pi, \varphi)$, $\varphi(\hat{\pi}) \cap CW(\pi) = \emptyset$.

Proof. Consider a voting system $(\mathfrak{B}, \varphi)$ satisfying neutrality and strong Pareto, and a dichotomous preference $B \in P(X) \backslash \{\emptyset, X\}$. Suppose $A \in \mathfrak{B}^{ud}(\varphi, B)$ and $A \neq B$.

Case 1: *There is* $b \in B \backslash A$.

Let π be such that for all $B' \in P(X)$ with $b \in B'$ and $|B'| = |B|$, $\pi(B') = 1$ and for all other $C \in P(X)$, $\pi(C) = 0$. Since $A \in \mathfrak{B}^{ud}(\varphi, B)$, then for all $B' \in P(X) \backslash \{\emptyset, X\}$ with $b \in B'$ and $|B'| = |B|$, by neutrality, there is $A(B') \in \mathfrak{B}$ such that $A(B') \in \mathfrak{B}^{ud}(f, B')$ and $b \in B' \backslash A(B')$. Let $\hat{\pi}$ be such that for all B' with $b \in B'$ and $|B'| = |B|$, $\hat{\pi}(A(B')) = 1$ and for all other $C \in \mathfrak{B}$, $\hat{\pi}(C) = 0$. Then by construction of π, $CW(\pi) = \{b\}$. Also by construction, $\hat{\pi} \in \Pi^{ud}(\mathfrak{B})(\varphi, \pi)$ and by strong Pareto, $b \notin \varphi(\hat{\pi})$. Therefore, $\varphi(\hat{\pi}) \cap CW(\pi) = \emptyset$.

Case 2: *There is* $a \in A \backslash B$.

Let π be such that for all $A' \in \mathfrak{B}$ with $a \in A'$ and $|A'| = |A|$, $\pi(A') = 1$ and for all other ballots $B' \in \mathfrak{B}$, $\pi(B') = 0$. Since $A \in \mathfrak{B}^{ud}(\varphi, B)$, then by neutrality, for all $A' \in \mathfrak{B}$ with $a \in A'$ and $|A'| = |A|$, there is $B(A') \in P(X) \backslash \{\emptyset, X\}$ such that $A' \in \mathfrak{B}^{ud}(\varphi, B(A'))$ and $a \in A' \backslash B(A')$. Let $\hat{\pi}$ be such that for all A' with $\pi(A') > 0$, $\hat{\pi}(B(A')) = 1$ and for all other ballots $C \in P(X)$, $\hat{\pi}(C) = 0$. Then by strong Pareto, $\varphi(\pi) = \{a\}$. Also by construction of $\hat{\pi}$, $\pi \in \Pi^{ud}(\mathfrak{B})(\varphi, \hat{\pi})$ and $a \notin CW(\hat{\pi})$. Therefore, $\varphi(\pi) \cap CW(\hat{\pi}) = \emptyset$. □

We now return to the Condorcet principle in the strategic voting environment.

Condorcet realizability on $\mathcal{D}$: For all $\pi \in \Pi(\mathcal{D})$ and all $\hat{\pi} \in \Pi^{ud}(\mathfrak{B})(\pi, \varphi)$, $\varphi(\hat{\pi}) = CW(\pi)$.

The next axiom is weaker and corresponds to weak Condorcet.

Weak Condorcet realizability on $\mathcal{D}$: For all $\pi \in \Pi(\mathcal{D})$ and all $\hat{\pi} \in \Pi^{ud}(\mathfrak{B})(\pi, \varphi)$, $\varphi(\hat{\pi}) \cap CW(\pi) \neq \emptyset$.

The next lemma is important for establishing the next characterization of majority voting.

[22] If for all $A \in \mathfrak{B}$ with $\pi(A) > 0$, $y \notin A$, then using any $x \in A$ for some $A \in \mathfrak{B}$ with $\pi(A) > 0$, we can show that the premise of strong Pareto is met. Thus in this case $y \notin \varphi(\pi)$.

Lemma 4.4.4. *If voting system* $(\mathfrak{B}, \varphi)$ *satisfies neutrality, strong Pareto, and weak Condorcet realizability on* $\mathcal{D}$*, then the system has no ballot constraint on* $\mathcal{D}$ *(that is,* $\mathcal{D} \subseteq \mathfrak{B}$*), strategy-proofness on* $\mathcal{D}$ *and weak Condorcet. Replacing weak Condorcet realizability with Condorcet realizability, we obtain* $\varphi = \varphi_{MAJ}$*. Thus when* $\mathcal{D} = P(X)\backslash\{\emptyset, X\}$*, it is approval voting.*

Proof. For any voting system $(\mathfrak{B}, \varphi)$ satisfying neutrality and strong Pareto, if there is an undominated strategy A that differs from the voter's dichotomous preference $B \in \mathcal{D}$, by Lemma 4.4.3, voting system $(\mathfrak{B}, \varphi)$ violates weak Condorcet realizability. Note that there can be such an undominated strategy $A \neq B$ if $B \in \mathcal{D}\backslash\mathfrak{B}$ or the voting system is not strategy-proof. Hence, neutrality, strong Pareto, and weak Condorcet realizability together imply both no ballot constraint, $\mathcal{D} \subseteq \mathfrak{B}$, and strategy-proofness. Moreover, the unique undominated strategy for $B \in \mathcal{D}$ is B itself. Therefore, at all dichotomous preference profiles $\pi \in \Pi(\mathcal{D})$, $\Pi^{ud}(\mathfrak{B})(\pi, \varphi) = \{\pi\}$ and weak Condorcet realizability implies $\varphi(\pi) \cap CW(\pi) \neq \emptyset$, that is, the voting system satisfies weak Condorcet. If $(\mathfrak{B}, \varphi)$ satisfies Condorcet realizability, then the last conclusion is strengthened to $\varphi(\pi) = CW(\pi)$, that is the voting system satisfies Condorcet; it is majority voting. □

We now obtain the following characterization of majority voting based on Condorcet realizability and strategy-proofness.

Theorem 4.4.3. *Consider a subdomain of dichotomous preferences* $\mathcal{D} \subseteq \mathfrak{D}$ *and a ballot space* $\mathfrak{B} \subseteq \mathcal{D}$*. The following are equivalent.*

(i) *Voting system* $(\mathfrak{B}, \varphi)$ *satisfies neutrality, strong Pareto, and Condorcet realizability on* $\mathcal{D}$*.*
(ii) *Voting system* $(\mathfrak{B}, \varphi)$ *satisfies neutrality, consistency, continuity, minimal selectiveness, and weak Condorcet realizability on* $\mathcal{D}$*.*
(iii) *Voting system* $(\mathfrak{B}, \varphi)$ *satisfies neutrality, consistency, faithfulness, and strategy-proofness on* $\mathcal{D}$*.*
(iv) *Voting system* $(\mathfrak{B}, \varphi)$ *is majority voting without ballot constraint on* $\mathcal{D}$*.*

Proof. The proof of the equivalence between (i) and (iv) is established using Theorem 4.4.2 and Lemma 4.4.4. The equivalence between (ii) and (iv) is obtained from Lemma 4.4.4 and Theorem 4.4.1. Finally, the next lemma states that (iii) implies (iv), and the converse follows from Theorem 4.4.2. □

The next lemma is an extension of a result in Fishburn (1979, Theorem 10, pp. 216–217), which is for the ballot space $\bar{P}(X)\backslash\{X\}$. Our result is for any arbitrary ballot space satisfying the richness conditions, (4.15) and (4.16).

Lemma 4.4.5. *Consider a subdomain of dichotomous preferences* $\mathcal{D} \subseteq \mathfrak{D}$ *and a ballot space* $\mathfrak{B} \subseteq \mathcal{D}$*. If voting system* $(\mathfrak{B}, \varphi)$ *satisfies neutrality, consistency, faithfulness, and strategy-proofness on* $\mathcal{D}$*, then it is majority voting without ballot constraint on* $\mathcal{D}$*.*

Proof. By Theorem 4.3.6, there are scoring functions $s^1, \ldots, s^T$ that represent φ as the lexicographic scoring rule. The case where all scoring functions are zero functions can be treated easily.[23] Excluding this case, without loss of generality, assume that no s^t is uniformly zero.

By Lemma 4.4.1, the voting system has no ballot constraint ($\mathcal{D}\backslash\{X\} = \mathfrak{B}\backslash\{X\}$) and for all $B \in \mathcal{D}\backslash\{X\}$, B is the unique undominated strategy for dichotomous preference B.

Suppose that for some $B \in \mathcal{D}\backslash\{\emptyset, X\}$, $s^1(|B|) < 0$. Then $\varphi(\pi_B) = X\backslash B$ (since $B \neq \emptyset, X$ and so $X\backslash B \neq \emptyset, X$), contradicting faithfulness.

Suppose that for some $A, B \in \mathcal{D}\backslash\{\emptyset, X\}$, $0 \leq s^1(|A|) < s^1(|B|)$. Without loss of generality, assume there exist $a \in A\backslash B$ and $b \in B\backslash A$. Consider dichotomous preference A. Construct a ballot response profile π such that for all $C \in \mathfrak{B}$, if $\pi(C) > 0$, then $|C| = |B|$, and $n(b,\pi) = n(a,\pi) + 1$ and $n(x,\pi) \leq n(a,\pi) - 1$ for all $x \in X\backslash\{a,b\}$.[24] Existence of such a profile is guaranteed by (4.15) and (4.16) because $\mathcal{D} \subseteq \mathfrak{B}$. Since $s^1(|A|) < s^1(|B|)$, $\varphi(\pi + \pi_A) = \{b\}$. On the other hand, $\varphi(\pi + \pi_{[B\backslash\{b\}]\cup\{a\}}) = \{a,b\}$, which is preferred to $\{b\}$ by the agent with preferences A. Therefore, A does not dominate $[B\backslash\{b\}] \cup \{a\}$, which implies that A is not the only undominated ballot, contradicting Lemma 4.4.1.

The above argument shows that $s^1(\cdot)$ is a constant and positive valued function over $\{|B| : B \in \mathcal{D}\backslash\{\emptyset, X\}\}$. The same argument can be used to show that the remaining score functions $s^2(\cdot), \ldots, s^T(\cdot)$ are constant functions over $\{|B| : B \in \mathcal{D}\backslash\{\emptyset, X\}\}$, which is sufficient to conclude that $\varphi = \varphi_{MAJ}$. □

It follows from Theorem 4.4.3 and Lemma 4.4.5 that:

Corollary 4.4.1. *Consider the domain of all dichotomous preferences, $\mathfrak{D}$. Then the following are equivalent.*

(i) *Voting system $(\mathfrak{B}, \varphi)$ satisfies neutrality, strong Pareto, and Condorcet realizability on $\mathfrak{D}$.*
(ii) *Voting system $(\mathfrak{B}, \varphi)$ satisfies neutrality, consistency, continuity, minimal selectiveness, and weak Condorcet realizability on $\mathfrak{D}$.*
(iii) *Voting system $(\mathfrak{B}, \varphi)$ satisfies neutrality, consistency, faithfulness, and strategy-proofness on $\mathfrak{D}$.*
(iv) *Voting system $(\mathfrak{B}, \varphi)$ is approval voting.*

4.5 Unconstrained Multi-issue Problems and Voting by Committees

In this section, we consider a collective decision model where there are multiple issues and for each issue, a binary decision needs to be made. This model is studied

[23] If all scoring functions are zero functions, then φ will always choose X, in which case no ballot is dominated and all ballots are undominated.

[24] Consider a ballot response profile π such that $\pi(B) = 1$, $\pi([B\backslash\{x\}] \cup \{a\}) = 2$ for each $x \in B\backslash\{b\}$, and $\pi(B') = 0$ for all other ballots B'. Then $n(b,\pi) = 2(|B|-1)+1 = n(a,\pi)+1$, and for each $x \in B\backslash\{b\}$, $n(x,\pi) = 2(|B|-2) + 1 = n(a,\pi) - 1$.

by Barberà et al. (1991, 1997), etc., and an extended model by Barberà et al. (1993), Le Breton and Sen (1999), etc.

Let $M \equiv \{1, \ldots, m\}$ be the set of issues. A collective decision is a vector of 1 or -1, that is,

$$x \equiv (x_1, \ldots, x_m) \in \{-1, 1\}^M,$$

where 1 in the kth component means accepting the kth issue and -1 means rejecting it. Thus $S_x \equiv \{k \in M : x_k = 1\}$ is the set of accepted issues at x. Let $X \equiv \{-1, 1\}^M$ be the set of all possible decisions. There is no constraint on the set of accepted issues in this model and any number or none of the issues can be accepted.[25]

On the unrestricted domain of preferences, the three impossibility results, Theorems 4.1.1–4.1.3, apply. Moreover, an even more disturbing paradox, known as Gibbard's paradox, holds (Gibbard 1974): the mere assignment of Sen's liberal rights to each person cannot be made coherently under any collective choice function. Sen (1983, p. 14) points out that Gibbard's paradox does not hold on the restricted domain of preferences for which each issue affects a person's welfare separately from other issues, the so-called separable preferences. Moreover, on the restricted domain of separable and linear preferences, Gibbard–Satterthwaite theorem does not hold and there do exist non-dictatorial and well-behaved strategy-proof rules (Barberà et al. 1991).

Formally, a preference R_0 is *separable* if for all $x, x' \in X$ and all $k \in M$, $(x_k, x_{-k}) R_0 (-x_k, x_{-k})$ if and only if $(x_k, x'_{-k}) R_0 (-x_k, x'_{-k})$. An issue $k \in M$ is a *good* (resp. a *bad* or a *null*) if for all $x \in X$ with $x_k = 1$, $(1, x_{-k}) P_0 (-1, x_{-k})$ (resp. $(-1, x_{-k}) P_0 (1, x_{-k})$ or $(1, x_{-k}) I_0 (-1, x_{-k})$). Let $G(R_0)$ be the set of goods for R_0 and $B(R_0)$ the set of bads. Let $\mathcal{S}$ be the set of separable preferences and $\mathcal{S}_L$ the set of *linear* separable preferences. Given a domain of separable preference profiles, $\mathcal{D} \subseteq \mathcal{S}^N$, a *collective choice function* $c : \mathcal{D} \to X$ associates with each preference profile a single collective decision.

Collective choice functions that can be practiced through a simple non-ranked voting procedure have been of central interest in the literature. A *voting scheme* is a collective choice function that only uses information about which issues are good or bad and so can be applied through a voting procedure under which voters express, in their ballots, which issues are goods and which are bads. That is, a voting scheme is a collective choice function satisfying:

Votes-Only: For all $R, R' \in \mathcal{S}^N$, if for all $i \in N$, $G(R_i) = G(R'_i)$ and $B(R_i) = B(R'_i)$, then $c(R) = c(R')$.

On the domain of linear separable preferences $\mathcal{S}_L^N$, this property is known as the "tops-only" property because $G(R_i)$ is the top alternative for $R_i \in \mathcal{S}_L$. When there are nulls, adding some or all nulls to $G(R_i)$ makes no difference from $G(R_i)$

[25] Barberà et al. (2005) consider a similar model with some constraints on the number of issues to be accepted.

and makes another top alternative for R_i. Thus there are multiple top alternatives consisting of all goods, some or all nulls, and no bads.

The following two additional axioms pin down an important family of voting schemes. The first axiom says that if for each agent, there are more goods and less bads, then more issues should be accepted.

Issues Monotonicity: For all $R, R' \in \mathcal{S}^N$, if for all $i \in N$, $G(R_i) \subseteq G(R'_i)$ and $B(R_i) \supseteq B(R'_i)$, then $c(R) \leq c(R')$.

The next axiom says that decisions on each issue should be made independently from the other issues, relying on who is in favor of the issue and who is against it. For all $R \in \mathcal{S}^N$ and all $k \in M$, let $N_k^G(R)$ be the set of agents for whom issue k is a good, and $N_k^B(R)$ the set of agents for whom issue k is a bad.

Issues Independence: For all $R, R' \in \mathcal{S}^N$ and all $k \in M$, if $N_k^G(R) = N_k^G(R')$ and $N_k^B(R) = N_k^B(R')$, then $c_k(R) = c_k(R')$.

Note that each of the above two axioms implies votes-only. The family of collective choice functions satisfying the two axioms can be represented by an issue-wise decisive structure, similar to the decisive structures in Sect. 4.2, defined as follows. For all $k \in M$, a *decisive structure for issue* k is a nonempty subset of $\mathfrak{d}^*$, $\mathfrak{d}_k \subseteq \mathfrak{d}^*$ satisfying: for all $(L_1, L_2), (L'_1, L'_2) \in \mathfrak{d}^*$,

$$\text{if } (L_1, L_2) \in \mathfrak{d}_k,\ L_1 \subseteq L'_1 \text{ and } L_2 \supseteq L'_2,\ \text{then } (L'_1, L'_2) \in \mathfrak{d}_k.$$

Call this property $\mathfrak{d}$*-monotonicity*, as in Sect. 4.2. An *issue-wise decisive structure* is a list of decisive structures for all issues, $\mathfrak{d} = (\mathfrak{d}_k)_{k \in M}$. An issue-wise decisive structure $\mathfrak{d} = (\mathfrak{d}_k)_{k \in M}$ represents the function $c(\cdot)$ defined as follows: for all $R \in \mathcal{S}^N$ and all $k \in M$, $c_k(R) = 1$ if and only if $(N_k^G(R), N_k^B(R)) \in \mathfrak{d}_k$.

Proposition 4.5.1 (Ju 2003). *A collective choice function on $\mathcal{S}^N$ (or on $\mathcal{S}_L^N$) satisfies issues monotonicity and issues independence if and only if it is represented by an issue-wise decisive structure.*

The proof is similar to the proof of Proposition 4.2.1, and it holds on numerous subdomains of $\mathcal{S}^N$ (Ju 2003). On the domain of linear separable preferences $\mathcal{S}_L^N$, voting schemes represented by an issue-wise decisive structure consisting of *proper* subsets $\mathfrak{d}_k$ of $\mathfrak{d}^*$ for all $k \in M$ are called *schemes of voting-by-committees* (Barberà et al. 1991). Note that because for all $k \in M$, $\mathfrak{d}_k$ is a proper subset of $\mathfrak{d}^*$, schemes of voting-by-committees have full-range. *Issue-wise majority voting scheme* is the voting scheme represented by $\mathfrak{d} \equiv (\mathfrak{d}_k)_{k \in M}$ such that for all $k \in M$, $(L_1, L_2) \in \mathfrak{d}_k$ if and only if $|L_1| > |L_2|$. An axiomatization of issue-wise majority voting scheme can be established with the combination of issues monotonicity, issues independence, anonymity, neutrality, and a duality-type axiom, on the domain of separable linear preferences with an odd number of agents.

4.5.1 Strategy-Proofness and Separable Preferences

Barberà et al. (1991) show that on the domain of linear separable preferences $\mathcal{S}_L^N$, all strategy-proof collective choice functions with full-range satisfy the votes-only property, and so they are voting schemes. Based on this result, they also show that strategy-proofness and the full-range condition together imply issues independence as well as issues monotonicity. This leads to the following characterization of voting-by-committees.

Theorem 4.5.1 (Barberà et al. 1991). *A collective choice function on the domain of linear separable preferences satisfies the full-range condition and strategy-proofness if and only if it is a scheme of voting-by-committees.*

Barberà et al. (1993) and Le Breton and Sen (1999) generalize this result in the extended model of multi-issue problems where more than two alternatives are available on each issue. In particular, Le Breton and Sen (1999) identify a general domain condition under which their characterization holds. The key argument is to prove an extended version of issues independence, called "decomposability," of strategy-proof collective choice functions. All these works assume linearity of preferences, which plays a crucial role.

When preferences are not linear, as shown by Le Breton and Sen (1995), issues independence of a strategy-proof collective choice function is not guaranteed, which makes it hard to obtain a result like Theorem 4.5.1 on the domain of separable "weak" orderings $\mathcal{S}^N$. In fact, we need an additional axiom to characterize voting schemes represented by an issue-wise decisive structure.

Null-Independence: For all $i \in N$, all $k \in M$, all $R_i, R_i' \in \mathcal{S}$, and all $R_{-i} \in \mathcal{S}^{N\setminus\{i\}}$, if k is a null issue for both R_i and R_i', then $c_k(R_i, R_{-i}) = 1$ if and only if $c_k(R_i', R_{-i}) = 1$.

Among voting schemes, the combination of strategy-proofness and null-independence is equivalent to the combination of issues monotonicity and issues independence (Ju 2003, Proposition 4, p. 485). Thus it follows from Proposition 4.5.1 that:

Theorem 4.5.2 (Ju 2003). *A voting scheme on the domain of separable preferences satisfies strategy-proofness and null-independence if and only if it is represented by an issue-wise decisive structure.*

After identifying a domain $\mathcal{D}$ where well-behaved strategy-proof functions exist, it is important to understand whether this existence result may be extended to a larger domain. In fact, as shown by Barberà et al. (1991, Theorem 3), the domain of separable linear preferences is the unique maximal "rich" domain (of linear preferences) where well-behaved strategy-proof functions exist. Dropping the linearity assumption, yet focusing on voting schemes, Ju (2003, Theorem 3) shows that the domain of separable preferences (weak orderings) is the unique maximal "rich" domain where well-behaved strategy-proof *voting schemes* exist. Maximal domain results are also established in the extended model of multi-issue problems by Serizawa (1995) and Le Breton and Sen (1999).

4.5.2 Strategy-Proofness Versus Efficiency and Domain Restrictions

Although strategy-proof collective choice functions on the domain of separable preferences are numerous, only dictatorial ones are efficient.

Theorem 4.5.3 (Barberà et al. 1991). *When there are at least three issues, a collective choice function on the domain of linear separable preferences is strategy-proof and efficient if and only if it is dictatorial.*

Le Breton and Sen (1999) obtain this result in their extended model of multi-issue problems. Shimomura (1996) weakens efficiency by requiring it on the subdomain where agents' preferences bear some degrees of resemblance and pins down a small family of schemes of voting by committees, which includes non-dictatorial voting schemes. When the set of alternatives is variable, Ju (2005b) requires efficiency only for problems with at most two alternatives. He shows that only those voting schemes that are quite close to the issue-wise majority function satisfy this restricted notion of efficiency as well as strategy-proofness, anonymity, and two additional axioms pertaining to agenda variation.

Ju (2005a) further restricts the domain of separable preferences to domains of "dichotomous" preferences. An *additive* preference is a separable preference represented by a utility vector $u \equiv (u_1, \ldots, u_m) \in \mathbb{R}^m$ as follows: for all $x, x' \in \{-1, 1\}^M$, xR_0x' if and only if $u{\cdot}x \geq u{\cdot}x'$. An additive preference is *trichotomous* if all goods are indifferent and all bads are indifferent. A trichotomous additive preference is *dichotomous* if all issues are either goods or bads. Although we use the same term as in the earlier sections, dichotomous preferences here are not dichotomous in the sense we use in Sect. 4.3. Dichotomous preferences in this section have at most two indifference sets of issues but may have more than two indifference sets in the alternative space $\{-1, 1\}^M$. Considering some examples of restricted domains consisting of dichotomous or trichotomous additive preferences, Ju (2005a) proves that only those voting schemes that are very close to issue-wise majority voting scheme satisfy efficiency as well as issues independence, anonymity, and neutrality (neutrality is needed only in the case of dichotomous preferences). Whether this result or a similar result holds for other simple domains such as the domain of dichotomous separable preferences that are not necessarily additive is open for future research.

4.6 Simple Opinion Aggregation and Decision by Powers and Consent

In this section, we consider the problem of aggregating dichotomous or trichotomous opinions, introduced by Wilson (1975) and further studied by Rubinstein and Fishburn (1986), Aleskerov et al. (2007), and Ju (2005a, 2008, 2010). The model is similar to the unconstrained multi-issue problems except that here, we deal with opinions rather than preferences. For separable linear preferences, issues are either goods or bads. Thus opinions can be interpreted as restricted preference revelations.

A special example of opinion aggregation is the problem of group identification. A finite number of potential members have to decide who among themselves belong to a certain collective through aggregating their dichotomous or trichotomous opinions. This problem is introduced by Kasher and Rubinstein (1997) and further studied by Samet and Schmeidler (2003), Sung and Dimitrov (2003), Dimitrov et al. (2007), Çengelci and Sanver (2010), Miller (2008), and Ju (2008, 2009a, 2010).

We continue using the same notation as in Sect. 4.5. Each person $i \in N$ has his *opinion* on issues in M, represented by an $1 \times m$ row vector V_i consisting of 1, 0, or -1. A *problem* is an $n \times m$ opinion matrix V consisting of n row vectors $V_1, \ldots, V_n$. Let $\mathcal{V}_{\text{Tri}}$ be the set of all opinion matrices, called, the *trichotomous* (opinion) *domain*. An *alternative* is a vector of 1 and -1, $x \equiv (x_1, \ldots, x_m) \in \{-1, 1\}^M$, where 1 (resp. -1) in the kth component means accepting the kth issue (resp. rejecting the kth issue). For all $V \in \mathcal{V}_{\text{Tri}}$ and all $k \in M$, V^k denotes the kth column vector of V. Let

$$||V^k_+|| \equiv \sum_{i \in N : V_{ik} = 1} V_{ik}, \quad ||V^k_-|| \equiv \sum_{i \in N : V_{ik} = -1} -V_{ik}, \quad \text{and } ||V^k_{+,-}|| \equiv ||V^k_+|| + ||V^k_-||.$$

For example, in the *group identification* model, $M = N$ and an alternative describes who belongs to the collective and who does not.

Let $\mathcal{V}_{\text{Di}}$ be the subset of $\mathcal{V}_{\text{Tri}}$, consisting of the opinion matrices whose entries are either 1 or -1, called the *dichotomous* (opinion) *domain*. Let $\mathcal{D}$ be either one of the two domains. Samet and Schmeidler (2003) consider the dichotomous domain of the group identification model.[26] A *collective choice function* on $\mathcal{D}$, $c : \mathcal{D} \to \{-1, 1\}^M$, associates with each problem in the domain a single alternative. A collective choice function satisfies *non-degeneracy* if for each $i \in N$, there exist $V, V' \in \mathcal{D}$ such that $c_i(V) = 1$ and $c_i(V') = -1$.

Section 4.5 provides the definition of a collective choice function represented by an issue-wise decisive structure. The same definition applies here, treating all issues k with $V_{ik} = 1$ as goods for person i and all issues l with $V_{il} = -1$ as bads, and all other issues as nulls. In the same way, we can extend the definitions of issues monotonicity and issues independence, which together characterize the family of collective choice functions represented by an issue-wise decisive structure (Proposition 4.5.1).

A subfamily of these collective choice functions plays an important role in the literature on opinion aggregation. In particular, an *issue-wise dictatorial function* $c(\cdot)$ is represented by an issue-wise decisive structure conferring on a person the full decision power over an issue: that is, for each $k \in M$, there is $i \in N$ such that

[26] Dichotomous opinions in Samet and Schmeidler (2003) are described by vectors of 1 and 0, where 0 has the same meaning as -1 in our model.

for all $V \in \mathcal{D}$ with $V_{ik} \in \{-1, 1\}$, $c_k(V) = V_{ik}$. In group identification problems with dichotomous opinions, the issue-wise dictatorial function where person i has the decision power on his own qualification is called as the *liberal* function. A milder notion of decision powers is discussed in the next subsection.

4.6.1 Social Decision by Powers and Consent for Dichotomous or Trichotomous Opinion Aggregation

Here we define a milder notion of decision powers. We first focus on dichotomous opinions. After this, we give the general definition.

Given a collective choice function c defined on the dichotomous domain $\mathcal{V}_{\text{Di}}$, person $i \in N$ has the "power to influence the social decision on the kth issue", briefly, the *power on the k-th issue* if the decision on the kth issue is made following person i's opinion whenever person i's opinion obtains sufficient consent from society: formally, there exist $q_+, q_- \in \{1, \ldots, n+1\}$ such that for all $V \in \mathcal{V}_{\text{Di}}$,

$$\begin{array}{l}\text{(i) when } V_{ik} = 1,\ c_k(V) = 1 \Leftrightarrow ||V_+^k|| \geq q_+ ;\\ \text{(ii) when } V_{ik} = -1,\ c_k(V) = -1 \Leftrightarrow ||V_-^k|| \geq q_- .\end{array} \tag{4.20}$$

The two numbers q_+ and q_- are called *consent-quotas*. The greater q_+ or q_- is, the higher degree of social consent is required for the exercise of the power. There are three extreme cases. When $q_+ = q_- = 1$, i's opinion determines social decision independently of social consent. Thus the power is *decisive*. When $q_+ = n + 1$ and $q_- = n + 1$, the power is *anti-decisive* because i's opinion is reflected reversely in the social decision. When $q_+ + q_- = n + 1$, the two parts in (4.20) coincide and all persons can have the same powers as person i (changing i with any j in (4.20) makes no difference). In this case, all persons have the equal power on the same issue; so such a power is said to be non-exclusive (formal definition will be provided later).[27]

The total number of positive or negative votes, denoted by ν, always equals n on the dichotomous domain. However, on the trichotomous domain, it is variable. We allow consent-quotas to vary relative to the total number of votes. Given a collective choice function c defined on $\mathcal{V}_{\text{Tri}}$, a person $i \in N$ has the *power on the k-th issue* if there exist three functions $q_+ : N \cup \{0\} \to N \cup \{0, n+1\}$, $q_0 : N \cup \{0\} \to N \cup \{0, n+1\}$, and $q_- : N \cup \{0\} \to N \cup \{0, n+1\}$ such that for all $\nu \in N \cup \{0\}$, and all $V \in \mathcal{V}_{\text{Tri}}$ with $||V_{+,-}^k|| = \nu$,

$$\begin{array}{l}\text{(i) when } V_{ik} = 1,\ c_k(V) = 1 \Leftrightarrow ||V_+^k|| \geq q_+(\nu);\\ \text{(ii) when } V_{ik} = 0,\ c_k(V) = 1 \Leftrightarrow ||V_+^k|| \geq q_0(\nu);\\ \text{(iii) when } V_{ik} = -1,\ c_k(V) = -1 \Leftrightarrow ||V_-^k|| \geq q_-(\nu).\end{array} \tag{4.21}$$

[27] Consent quotas are closely related with the power index by Shapley and Shubik (1954) as discussed in Ju (2010).

We call the list of the three functions $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ the *consent-quotas function*. The power is *decisive* if for all $\nu \in N$, both $q_+(\nu)$ and $q_-(\nu)$ take the value of 1. The power is *anti-decisive* if for all ν, both $q_+(\nu)$ and $q_-(\nu)$ take the value of $\nu + 1$. To avoid unnecessary complication, we assume that for all $\nu \in N$,

$$q_+(\nu), q_-(\nu) \in \{1, \ldots, \nu + 1\},\ q_0(\nu) \in \{0, 1, \ldots, \nu + 1\},\ \text{and}\ q_0(0) \in \{0, 1\},$$

and

$$q_+(0) = q_+(1),\ q_-(0) = q_-(1),\ \text{and}\ q_0(n) = q_0(n-1).$$

Let $\mathcal{Q}$ be the family of consent-quota functions satisfying these assumptions.

Definition 4.6.1 (System of Powers). A *system of powers* representing a collective choice function c on $\mathcal{V}_{\text{Tri}}$ is a function $W\colon M \to N \times \mathcal{Q}$ mapping each issue $k \in M$ into a pair of the person, $W_1(k)$, who has the power on the kth issue, and the consent-quotas function, $W_2(k) = (q_+(\cdot), q_0(\cdot), q_-(\cdot))$, associated with the power. That is, when $W_1(k) = i$, for all $\nu \in \{0, 1, \ldots, n\}$ and all $V \in \mathcal{V}_{\text{Tri}}$ with $||V^k_{+,-}|| = \nu$, the social decision on the kth issue is made as described in (4.21).

The power on the kth issue is (fully) *exclusive* if there is a person i who has the power on the kth issue and no one else does. It is (fully) *non-exclusive* if all persons have the "equal" power on the kth issue associated with a single consent-quotas function (or, on the dichotomous domain, a list of consent-quotas). The power on an issue is either exclusive or non-exclusive (Remark 1 of Ju (2010)). Thus either only one person has the power or all persons have equal power. Two systems of powers W and W' are *equivalent*, denoted by $W \sim W'$, if for all k with $W_1(k) \neq W'_1(k)$, the power on the kth issue is *non-exclusive* (so, $W_2(k) = W'_2(k)$); otherwise, $W_2(k) = W'_2(k)$. If a collective choice function is represented by a system of powers, the system of powers is unique up to this equivalence relation (Ju 2010, Proposition 2). The following two extreme systems are notable. Under a *non-exclusive system of powers*, everyone has non-exclusive power on every issue. Under a *monocentric system of powers*, one person has exclusive power on every issue.

A necessary and sufficient condition for *issues monotonicity* is composed of the following two properties of consent-quotas functions (Ju 2010, Proposition 3). A consent-quotas function $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ has the *component ladder property* if for all $\nu \in \{1, \ldots, n\}$, the following three inequalities hold:

$$\begin{aligned} &\text{(i) } q_+(\nu-1) \leq q_+(\nu) \leq q_+(\nu-1) + 1;\\ &\text{(ii) } q_0(\nu-1) \leq q_0(\nu) \leq q_0(\nu-1) + 1;\\ &\text{(iii) } q_-(\nu-1) \leq q_-(\nu) \leq q_-(\nu-1) + 1. \end{aligned} \tag{4.22}$$

When this property fails, the decision may not respond monotonically after other persons' opinions become more favorable. The function has the *intercomponent ladder property* if for all $\nu \in \{1, \ldots, n\}$,

$$q_+(\nu) \leq q_0(\nu-1) + 1 \leq \nu - q_-(\nu) + 2. \tag{4.23}$$

When this property fails, the issue, initially accepted, may be rejected after the person who has the power on the issue becomes more favorable. For example, any anti-decisive power which has $q_{+}(\nu) = q_{-}(\nu) = \nu + 1$ for all ν violates intercomponent ladder property. On the dichotomous domain, component ladder property has no bite and intercomponent ladder property reduces to $q_{+} + q_{-} \leq n + 2$. The *ladder property* refers to the conjugation of the two ladder properties.

In the Arrovian framework, Sen (1970a,b, 1976, 1983) and many of his critics formulate individual rights based on (1) the existence of the so-called recognized personal spheres and (2) 'how the recognition of the personal spheres of different individuals should be reflected in the choices made by the society' (Gaertner et al. 1992, p. 162). Here, to formulate such recognized personal spheres, we use a function mapping each issue into a person, $\lambda : M \to N$, called a *linkage*. The next axiom requires the existence of recognized personal spheres. However, it does not impose any specific condition regarding what form the recognition should take, except for a minimal "symmetry" condition, which says that collective choice functions should treat person i and i's issues (constituting i's personal sphere) symmetrically to any other person j and j's issues. Technically, when names of person i and all i's issues are switched simultaneously to names of person j and all j's issues, social decision should also be switched accordingly. Given a linkage $\lambda \in \Lambda$, for all $i \in N$, let us call elements in $\lambda^{-1}(i)$ person i's issues. Let $\pi : N \to N$ and $\delta : M \to M$ are permutations on N and on M such that for all $i \in N$, δ maps the set of person i's issues *onto* the set of person $\pi(i)$'s issues. Let ${}^{\delta}_{\pi}P$ be the matrix such that for all $i \in N$ and all $k \in M$, ${}^{\delta}_{\pi}P_{ik} \equiv P_{\pi(i)\delta(k)}$. Then person i and his issue k play the same role in ${}^{\delta}_{\pi}P$ as person $\pi(i)$ and his issue $\delta(k)$ do in P.

Symmetric Linkage There is a linkage $\lambda : M \to N$ such that for all permutations $\pi : N \to N$ and all permutations $\delta : M \to M$, if for all $i \in N$, δ maps the set of i's issues $\lambda^{-1}(i)$ *onto* the set of $\pi(i)$'s issues $\lambda^{-1}(\pi(i))$, then for all $k \in M$, $f_k\left({}^{\delta}_{\pi}P\right) = f_{\delta(k)}(P)$.

Symmetry holds in the model of group identification if the collective choice function satisfies symmetric linkage and the linkage is the identity function, which means that the qualification of i is recognized as i's personal sphere, as is natural in this model.

A condition on systems of powers that is necessary and sufficient for symmetric linkage is *horizontal equality*: for all pair of persons i and $j \in N$ with the same number of issues under W_1, that is, $|W_1^{-1}(i)| = |W_1^{-1}(j)|$, their powers are associated with the same consent-quotas function, that is, for all $k \in W_1^{-1}(i)$ and all $l \in W_1^{-1}(j)$, $W_2(k) = W_2(l)$ (Ju 2010, Proposition 4).[28] When $i = j$, this

[28] A linkage creates primitive differences among persons and among issues in this setting; except for this, all other aspects of the model give equal standing to all persons (they share the same set of potential opinion vectors) and to all issues. A linkage differentiates persons vertically depending on the number of issues one is associated with. Horizontal equality allows us to incorporate this vertical differentiation in systems of powers not harming equality too much among persons.

property says that person i's powers on two different issues are associated with the same consent-quotas function.

Adding symmetric linkage to issues monotonicity and issues independence provides a characterization of collective choice functions represented by a system of powers.

Theorem 4.6.1 (Ju 2010). *Let $\mathcal{D} \in \{\mathcal{V}_{Di}, \mathcal{V}_{Tri}\}$. A collective choice function on $\mathcal{D}$ satisfies issues monotonicity, issues independence, and symmetric linkage if and only if it is represented by a system of powers satisfying the ladder property and horizontal equality. Moreover, the system is unique up to the equivalence relation $\sim$.*

4.6.2 Group Identification

We now consider group identification problems, where $M = N$. Several recent studies on group identification introduced by Kasher and Rubinstein (1997) formulate principles of liberalism in this specific model and establish axiomatic characterizations of "liberal" collective choice function.

4.6.2.1 Liberalism and Axiomatic Characterizations

A system of powers W on the domain of dichotomous problems $\mathcal{V}_{Di}$ is *liberal* if $W_1(i) = i$ for all $i \in N$ and all powers are decisive. The *liberal* collective choice function on $\mathcal{V}_{Di}$ is represented by the liberal system of powers.

The next axiom incorporates the minimal sense of liberalism by requiring only that if someone qualifies (disqualifies) herself, then not everyone should be disqualified (qualified), in other words, there should be someone, possibly the same person, who is qualified (disqualified).

Semi-Liberal Principle: For all $V \in \mathcal{V}_{Di}$, if for some $i \in N$, $V_{ii} = 1$, then for some $j \in N$, $c_j(V) = 1$; if for some $i \in N$, $V_{ii} = -1$, then for some $j \in N$, $c_j(V) = -1$.

Sung and Dimitrov (2003) establish the following characterization of the liberal collective choice function.

Theorem 4.6.2 (Sung and Dimitrov 2003). *Assume $M = N$. A collective choice function on $\mathcal{V}_{Di}$ satisfies independence, symmetry, and semi-liberal principle if and only if it is the liberal function.*

This is a strengthening of the characterization by Kasher and Rubinstein (1997) where they impose monotonicity and unanimity as well as the three axioms above. Sung and Dimitrov (2003) show that these two additional axioms are redundant and that the three remaining axioms are logically independent.

Samet and Schmeidler (2003) propose the following two interesting axioms.[29] The first axiom says, in their words, that non-Hobbits' opinions about Hobbits do not matter in determining who are Hobbits.

Exclusive Self-Determination: If $V, V' \in \mathcal{D}$ are such that for all $i, j \in N$, $V_{ij} \neq V'_{ij}$ only if $c_i(V) = -1$ and $c_j(V) = 1$, then $c(V) = c(V')$.

The next axiom says that the two groups, of Hobbits and of qualifiers of Hobbits, should coincide.

Affirmative Self-Determination: For all $V \in \mathcal{D}$, $c(V) = c(V^t)$, where V^t is the transpose of V.

Imposing either one of the two self-determination axioms together with other axioms defined earlier, we have the following characterization of liberal choice function:

Theorem 4.6.3 (Samet and Schmeidler 2003). *Assume $M = N$. A collective choice function on $\mathcal{V}_{Di}$ satisfies monotonicity, independence, non-degeneracy and exclusive self-determination (or affirmative self-determination) if and only if it is the liberal function.*

Ju (2010) extends this result on the domain of trichotomous opinions $\mathcal{V}_{Tri}$. Çengelci and Sanver (2010) introduces the axiom of positive weak equal treatment property requiring that all persons should be qualified when everyone qualifies himself. Based on this axiom or a variant, they establish a characterization of the liberal choice function. Ju (2009a) weakens monotonicity and non-degeneracy in Theorem 4.6.3 and obtains an alternative characterization result.

For all $x, x' \in \{-1, 1\}^N$, let $x \wedge x' \equiv (\min\{x_i, x'_i\})_{i \in N}$ and $x \vee x' \equiv (\max\{x_i, x'_i\})_{i \in N}$. Similarly, for all $V, V' \in \mathcal{V}_{Di}$, let $V \wedge V' \equiv (\min\{V_{ij}, V'_{ij}\})_{i \in N, j \in N}$ and $V \vee V' \equiv (\max\{V_{ij}, V'_{ij}\})_{i \in N, j \in N}$. Miller (2008) considers an extended framework where a collective choice function is used to identify more than one groups. The key axiom in Miller (2008) pertains to the two methods of identifying a collective consisting of persons with feature a and feature b. One method is to identify the collective with feature a and the collective with feature b separately and take the intersection of the two groups. The other method is to identify the collective with feature a and feature b at once. The next axiom requires that both methods should yield the same group.

Meet Separability: For all $V, V' \in \mathcal{D}$, $c(V) \wedge c(V') = c(V \wedge V')$.

The same requirement for identifying a collective consisting of persons with feature *a or* feature b is as follows.

Join Separability: For all $V, V' \in \mathcal{D}$, $c(V) \vee c(V') = c(V \vee V')$.

Miller (2008) shows that the liberal function is the only collective choice function satisfying the two separability axioms as well as non-degeneracy and anonymity.

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[29] See Samet and Schmeidler (2003, pp. 222–224), for detailed discussion and motivation for the two axioms.

Theorem 4.6.4 (Miller 2008). *Assume $M = N$. A collective choice function on $\mathcal{V}_{Di}$ satisfies meet separability, join separability, non-degeneracy, and symmetry if and only if it is the liberal function.*

Miller (2008, Theorem 2.5) shows that collective choice functions satisfying the first three axioms depend only on a single vote and call them one-vote rules. From this result, Theorem 4.6.4 directly follows because only the liberal function among these one-vote rules can satisfy symmetry. Note that independence is not needed in this characterization result and in fact, it is implied by the four axioms.

4.6.2.2 Consent-Based Choice Functions

Samet and Schmeidler (2003) introduce a spectrum of choice functions connecting issue-wise majority function and the liberal function as two extreme functions of the family. A *consent-based choice function* on the domain of dichotomous opinions $\mathcal{V}_{Di}$ is represented by a system of powers W such that for all $i \in N$, $W_1(i) = i$ and $q_+ + q_- \leq n + 2$, where $(q_+, q_-) = W_2(k)$ for all $k \in N$.

Theorem 4.6.5 (Samet and Schmeidler 2003). *Assume $M = N$. A collective choice function on $\mathcal{V}_{Di}$ satisfies monotonicity, independence, and symmetry if and only if it is a consent-based choice function.*

A collective choice function c on $\mathcal{V}_{Di}$ satisfies *self-duality* if for all $V \in \mathcal{V}_{Di}$, $c(-V) = -c(V)$. Adding self-duality to the three axioms above, Samet and Schmeidler (2003, Theorem 2) characterize the subfamily of consent-based choice functions of which the consent quotas functions satisfy the following property: for all $i, j \in N$, $W_2(i) = W_2(j) = (q_+, q_-)$ and $q_+ = q_-$.

Note that self-dual consent-based choice functions have the same consent quotas for all persons. Allowing for different consent quotas across persons, a slightly larger family can be defined. This family is characterized by Çengelci and Sanver (2010, Theorem 4.1) with the set of four axioms, monotonicity, independence, self-duality and a weaker version of anonymity axiom. It should be noted that in this characterization, they do not impose symmetry, which plays a crucial role in Samet and Schmeidler (2003).

When $q_+ + q_- = n + 1$, parts (i) and (ii) of (4.20) are identical to the single condition that for all $V \in \mathcal{V}_{Di}$ and all $i \in N$, $c_i(V) = 1$ if and only if $||V_+^i|| \geq q_+$. Thus social decisions are made anonymously. Conversely, anonymous consent-based choice functions have consent quotas with $q_+ + q_- = n + 1$.

When there is an odd number of persons, the two conditions $q_+ = q_-$ and $q_+ + q_- = n + 1$ are satisfied only by the issue-wise majority function. Therefore, the issue-wise majority function is characterized by adding anonymity to the four axioms of monotonicity, independence, symmetry, and self-duality (Samet and Schmeidler 2003, p. 225). Replacing anonymity with neutrality, gives an alternative characterization of the issue-wise majority function.

4.6.3 Simple Preferences and the Paradox of Paretian Liberal

Compatibility of *Pareto efficiency* and existence of the so-called libertarian rights (decisive powers) is widely studied by a number of authors after Sen (1970a,b). We investigate Sen's paradox of Paretian liberal (Sen 1970a,b) in the current opinion aggregation framework by considering separable preference relations. We formulate Sen's liberal rights as a person's decisive power on a certain issue (Gibbard 1974). Note that each separable preference R_0 is associated with an opinion vector V_0, each positive (resp. negative or zero) component of V_0 representing the corresponding issue as a good (resp. a bad or a null). Obviously, there are a number of separable preference relations corresponding to a single opinion vector.

Sen's paradox holds on the separable preferences domain.[30] Sen's (1970a,b) *minimal liberalism* postulates that there should be at least two persons who have decisive powers. Assume that persons 1 and 2 are given the decisive powers on the first issue and the second issue respectively. Consider the following preference relations of the two persons. For person 1, the first issue is a bad and the second issue is a good. But person 1 cares so much about the second issue (person 2's issue) that he prefers the positive decision on this issue to the negative decision no matter what decisions are made on the other issues. For person 2, the second issue is a bad and the first issue is a good. But person 2 cares so much about the first issue (person 1's issue) that he prefers the positive decision on this issue to the negative decision no matter what decisions are made on the other issues. Then by the decisive powers of the two persons, decisions on the first and the second issues are both negative. But the two persons will be better off at any decision with positive components for both issues. This confirms that minimal liberalism and Pareto efficiency are incompatible on the separable preferences domain.

Preference relations in the above example are "meddlesome"' (Blau 1975). One may hope that without such relations, the paradox of Paretian liberal will not occur. Unfortunately, the paradox holds even in a substantially restricted environment where only trichotomous or dichotomous preference relations are admissible. Consider a *trichotomous preference relation* R_0 that is a separable preference relation represented by a function $U_0 \colon \{-1,1\}^M \to \mathbb{R}$ such that for each $x \in \{-1,1\}^M$, $U_0(x) = \sum_{k \in M: x_k = 1} V_{0k}$, where $V_0 \in \{-1,0,1\}^M$ is the opinion vector corresponding to R_0.[31] Let $\mathcal{A}^*_{Tri}$ be the family of all such trichotomous preference relations. Let $\mathcal{A}^*_{Di}$ be the subfamily of dichotomous preferences in $\mathcal{A}^*_{Tri}$.

Proposition 4.6.1 (Ju 2008). *When there are at least three persons, no Pareto efficient collective choice function on $\mathcal{A}^*_{Di}$ or $\mathcal{A}^*_{Tri}$ satisfies minimal liberalism.*

Proof. Suppose that persons 1 and 2 have the decisive powers respectively on issue 1 and issue 2. Consider the profile of dichotomous preference relations

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[30] This was originally proven by Gibbard (1974, Theorem 2).

[31] Equivalently, $U_0(x) = |\{k \in M : x_k = 1 \text{ and } P_{0k} = 1\}| - |\{k \in M : x_k = 1 \text{ and } P_{0k} = -1\}|$.

$(R_i)_{i \in N}$ given by the following opinion vectors: $V_1 \equiv (1, -1, -1, \ldots, -1)$, $V_2 \equiv (-1, 1, -1, \ldots, -1)$, and for all $i \in N \backslash \{1, 2\}$, $V_i \equiv (-1, \ldots, -1)$. Then by the decisive powers of persons 1 and 2, $c_1(R) = c_2(R) = 1$. If $c(\cdot)$ is *Pareto efficient*, for all $k \in M \backslash \{1, 2\}, c_k(R) = -1$. Thus $c(R) = (1, 1, -1, \ldots, -1)$. Note that this alternative is indifferent to $x \equiv (-1, \ldots, -1)$ for both person 1 and person 2 and x is preferred to $c(R)$ by all others. This contradicts *Pareto efficiency*. □

Note that unlike the previous paradox on the separable preferences domain, we need the assumption $n \geq 3$. The case with two persons ruled out by this assumption is very limited. In fact, the paradox does not apply in the two-person case (decisiveness is quite close to majority principle since one person's opinion accounts for 50%).

Collective choice functions that are represented by a system of powers do not satisfy minimal liberalism if no power is decisive. However they capture a somewhat weak sense of liberalism because they allow limited powers to individuals. Ju (2008) shows among these collective choice functions, there do exist Pareto efficient ones on $\mathcal{A}^*_{Tri}$. Issue-wise majority function is an example and all other Pareto efficient functions are very close to the issue-wise majority function. The only difference is when the number of voters in favor of an issue is the same as the number of voters against the issue, in which case the person who has the power on the issue dictates the social decision. Thus the exercise of a person's power is most limited. To be compatible with Pareto efficiency, exclusive powers that can be assigned to individuals should be limited so extremely that the resulting collective choices are very close to the issue-wise majority function where no individual has an exclusive power.

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