Some New Markovian Models for Traffic and Performance Evaluation of Telecommunication Networks

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Abstract. A new queueing model is proposed, namely the $MM\sum_{k=1}^{K} CPP_k/GE/c/L\,G$-Queue, also known as the *Sigma* queue. A steady state solution for this queue in a computable form is presented, based on an analytic methodology applying certain non-trivial transformations on the balance equations, which turn out to be of the QBD-M (Quasi Simultaneous-Multiple-Bounded Births and Deaths) type. As a consequence, existing methods such as the spectral expansion method or the matrix-geometric method can be used to obtain the steady state probabilities. The utility of this queue is demonstrated through the performance analysis of an optical burst switching (OBS) node. Numerical results show the impact of the burstiness of arrival process on the performance of the OBS multiplexer.

1 Introduction

It is well known that the traffic in today's telecommunication systems often exhibits burstiness, that is batches of transmission units (e.g. packets) arriving together, with correlations among the inter-arrival times. Several models have been proposed to model such arrival, service processes and the queues and networks with these processes. These include the compound Poisson process (CPP) in which the inter-arrival times have generalized exponential (GE) probability distribution [4], the Markov modulated Poisson process (MMPP) and the self-similar traffic models such as the Fractional Brownian Motion (FBM) [15]. The CPP and the $\sum_{k=1}^{K} CPP_k$ traffic models often give a good representation of the burstiness (batch size distribution) of the traffic from one or more sources [9], but not the auto-correlations of the inter-arrival times observed in real traffic. Conversely, the MMPP models can capture the auto-correlations but not the burstiness [11,16]. The self-similar traffic models such as the FBM can represent both burstiness and auto-correlations, but they are analytically intractable in

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a queuing context. The arriving traffic to a node is often the superposition of traffic from a number of sources, which complicates the system analysis further.

In order to make the modeling capability vastly flexible and also to accommodate the superposition of multiple arrival streams, a new traffic and queuing model, the *Markov modulated* $\sum_{k=1}^{K} CPP_k/GE/c/L\,G$-queue is introduced in this paper. This is a homogeneous multi-server queue with c servers, GE service times and with the superposition of K independent positive and an independent negative[1] customer arrival streams each of which is a CPP, i.e. a Poisson point process with batch arrivals of geometrically distributed batch size. In other words, inter-arrival times of each of these $K+1$ arrival streams are also independent GE random variables. Also, all the $K+2$ GE distributions (K positive and 1 negative customer inter-arrival times, and the service time) are jointly modulated by a continuous time Markov chain (CTMC), also termed as the modulating process. The notational representation of this new queue is the $MM \sum_{k=1}^{K} CPP_k/GE/c/L\,G$-queue. It is also termed as the *Sigma* queue, for easier reference. Thus, the *Sigma* queue and its extensions can capture a large class of traffic and queuing models applicable to todays Internet in the Markovian framework.

We propose the $MM \sum_{k=1}^{K} CPP_k/GE/c/L\,G$-Queue and present a methodology for the solution of the new queue. The new methodology applies certain transformations to the balance equations to produce a computable form (of the QBD-M type). As a consequence, the steady state solution is possible by the spectral expansion method [3,12] or an appropriate extension of Naoumov's method [13] for QBD-M processes, based on the matrix-geometric solution [14].

The queuing model and its variants were successfully used to model [5,6] High-speed Downlink Packet Access (HSDPA [1]) terminal categories. In this paper, we demonstrate the utility of this new queue through the performance analysis of an optical burst switching (OBS) node. We revisit the problem of performance analysis of an OBS multiplexer that was attempted by Turner [18]. Briefly, the multiplexer assigns arriving bursts to channels in a link with c available data channels (known as wavelengths, in OBS terminology) and storage locations for $L-c$ bursts. An arriving burst is diverted to a storage location if all c data channels are in use when it arrives. A burst will be discarded when it arrives if all c channels are busy and all $L-c$ burst storage locations are being used by bursts that have not yet been assigned a channel. However, once a stored burst has been assigned to an output channel, its storage location becomes available for use by an arriving burst, since the stored burst will vacate space. Turner has proposed a birth-death process to analyze this problem [18]. However, Turner's model has some limitations like the assumption of exponential inter-burst arrival process and constant burst size. The use of the proposed new queue overcomes those limitations. Extensive numerical study is carried out with the use of the

[1] Negative customers remove (positive) customers from the queue and have been used to model random neural networks, task termination in speculative parallelism, faulty components in manufacturing systems and server breakdowns [8].

new model. The results quantitatively show the impact of the burstiness of the arrival process on the performance of an optical burst switching multiplexer. It is worth emphasizing that the performance analysis of an optical burst switching node is only one of many applications for the new generalized queuing model.

The rest of the paper is organized as follows. We propose the $MM\sum_{k=1}^{K} CPP_k/GE/c/L\,G$-queue in section 2. Next, we present a solution technique for the steady state joint probability distribution in section 3. We derive the departure size distribution in section 4. We show some numerical results in 5. The paper then concludes in section 6.

2 The $MM\sum_{k=1}^{K} CPP_k/GE/c/L\,G$-Queue

2.1 The Arrival Process

The arrival and service processes are modulated by the same continuous time, irreducible Markov phase process with N states. Let Q be the generator matrix of this process, given by

$$Q = \begin{bmatrix} -q_1 & q_{1,2} & \cdots & q_{1,N} \\ q_{2,1} & -q_2 & \cdots & q_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N,1} & q_{N,2} & \cdots & -q_N \end{bmatrix} ,$$

where $q_{i,k}(i \neq k)$ is the instantaneous transition rate from phase i to phase k, and

$$q_i = \sum_{j=1}^{N} q_{i,j} \qquad (i = 1, \ldots, N).$$

In the above, $q_{i,i} = 0$. Let $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ be the vector of equilibrium probabilities of the modulating phases. Then, $\mathbf{r}$ is uniquely determined by the equations:

$$\mathbf{r}Q = 0 \quad ; \quad \mathbf{r}\mathbf{e}_N = 1 \ ,$$

where $\mathbf{e}_N$ stands for the column vector with N elements, each of which is unity.

The arrival process of the customers (positive customers) ($MM\sum_{k=1}^{K} CPP_k$) is the superposition of K independent CPP arrival streams of packets (customers) in any given modulating phase. Also, in each modulating phase, there is a CPP arrivals of negative customers. The packets of different arrival streams are not distinguishable. The parameters of the GE inter-arrival time distribution of the k^{th} $(1 \leq k \leq K)$ positive customer arrival stream in phase i are $(\sigma_{i,k}, \theta_{i,k})$, and (ρ_i, δ_i) are those of the negative customers. That is, the inter-arrival time probability distribution function is $1 - (1 - \theta_{i,k})e^{-\sigma_{i,k}t}$, in phase i, for the k^{th} stream of positive customers and $1 - (1 - \delta_i)e^{-\rho_i t}$ for the negative customers. Thus, all the $K + 1$ arrival *point*-processes are Poisson, with batches arriving at each point having geometric size distribution. Specifically, the probability that a batch is of size s is $(1-\theta_{i,k})\theta_{i,k}^{s-1}$, in phase i, for the k^{th} stream of positive customers, and $(1-\delta_i)\delta_i^{s-1}$ for the negative customers.

Let $\sigma_{i,.}, \overline{\sigma_{i,.}}$ be the average arrival rate of customer batches and customers in phase i respectively. Let $\sigma, \overline{\sigma}$ be the overall average arrival rate of batches and customers respectively. Then, we get

$$\sigma_{i,.} = \sum_{k=1}^{K} \sigma_{i,k}, \qquad \overline{\sigma_{i,.}} = \sum_{k=1}^{K} \frac{\sigma_{i,k}}{(1-\theta_{i,k})}, \qquad \sigma = \sum_{i=1}^{N} \sigma_{i,.} r_i, \qquad \overline{\sigma} = \sum_{i=1}^{N} \overline{\sigma_{i,.}} r_i. \tag{1}$$

Because of the superposition of many CPP's, the overall arrivals in phase i can be considered as bulk-Poisson ($M^{[x]}$) with arrival rate $\sigma_{i,.}$ and with a batch size distribution $\{\pi_{i,l}\}$, that is, more general than mere geometric. The probability that the batch size is l ($\pi_{i,l}$) and the overall batch size distribution ($\pi_{.,l}$), can be given by

$$\pi_{i,l} = \sum_{k=1}^{K} \frac{\sigma_{i,k}}{\sigma_{i,.}} (1-\theta_{i,k})\theta_{i,k}^{l-1}, \quad \pi_{.,l} = \sum_{i=1}^{N} r_i \pi_{i,l}. \tag{2}$$

Clearly, by choosing K and other parameters appropriately, it may then be possible to approximate $\{\pi_{i,l}\}$ or $\{\pi_{.,l}\}$ to suit certain given classes of batch size distribution, however this is a matter for further research only.

2.2 The GE Multi-server

The service facility has c homogeneous servers in parallel, each with GE-distributed service times with parameters (μ_i, ϕ_i) in phase i. The service discipline is FCFS and each server serves at most one positive customer at any given time. Negative customers neither wait in the queue, nor are served. The operation of the GE server is similar to that described for the CPP arrival processes above. L denotes the queuing capacity, in all phases, including the customers in service, if any. L can be finite or infinite. We assume, when the number of customers is j and the arriving batch size of positive customers is greater than $L-j$ (assuming finite L), that only $L-j$ customers are taken in and the rest are rejected.

However, the batch size associated with a service completion is bounded by one more than the number of customers waiting to commence service at the departure instant. For queues of length $c \le j < L+1$ (including any customers in service), the maximum batch size at a departure instant is $j-c+1$, only one server being able to complete a service period at any one instant under the assumption of exponentially distributed batch-service times. Thus, the probability that a departing batch has size s is $(1-\phi_i)\phi_i^{s-1}$ for $1 \le s \le j-c$ and ϕ_i^{j-c} for $s = j-c+1$. In particular, when $j = c$, the departing batch has size 1 with probability one, and this is also the case for all $1 \le j \le c$ since each customer is already engaged by a server and there are then no customers waiting to commence service.

It is assumed that the first positive customer in a batch arriving at an instant when the queue length is less than c (so that at least one server is free) *never* skips service, i.e. always has an exponentially distributed service time. However, even without this assumption the methodology described in this paper is still applicable.

2.3 Negative Customer Semantics

A negative customer removes a positive customer in the queue, according to a specified *killing discipline*. We consider here a variant of the RCE killing discipline (removal of customers from the end of the queue), where the most recent positive arrival is removed, but which does *not* allow a customer actually in service to be removed: a negative customer that arrives when there are no positive customers waiting to start service has no effect. We may say that customers in service are immune to negative customers or that the service itself is *immune servicing*. Such a killing discipline is suitable for modeling of load balancing where work is transferred from overloaded queues but never work, that is, actually in progress.

When a batch of negative customers of size $l\,(1 \leq l < j-c)$ arrives, l positive customers are removed from the end of the queue leaving the remaining $j-l$ positive customers in the system. If $l \geq j-c \geq 1$, then $j-c$ positive customers are removed, leaving none waiting to commence service (queue length equal to c). If $j \leq c$, the negative arrivals have no effect.

$\overline{\rho_i}$, the average arrival rate of negative customers in phase i and $\overline{\rho}$, the overall average arrival rate of negative customers are given by

$$\overline{\rho_i} = \frac{\rho_i}{1-\delta_i} \quad ; \quad \overline{\rho} = \sum_{i=1}^{N} r_i \overline{\rho_i}. \tag{3}$$

2.4 Condition for Stability

When L is finite, the system is ergodic since the representing Markov process is irreducible. Otherwise, i.e. when $L = \infty$, the overall average departure rate increases with the queue length, and its maximum (the overall average departure rate when the queue length tends to ∞) can be determined as

$$\overline{\mu} = c \sum_{i=1}^{N} \frac{r_i \mu_i}{1-\phi_i}. \tag{4}$$

Hence, the necessary and sufficient condition for the existence of steady state probabilities is

$$\overline{\sigma} < \overline{\rho} + \overline{\mu}\ . \tag{5}$$

2.5 The Steady State Balance Equations

The state of the system at any time t can be specified completely by two integer-valued random variables, $I(t)$ and $J(t)$. $I(t)$ varies from 1 to N, representing the phase of the modulating Markov chain, and $0 \leq J(t) < L+1$ represents the number of positive customers in the system at time t, including any in service. The system is now modeled by a continuous time discrete state Markov process, $\overline{Y}$ (Y if L is infinite), on a rectangular lattice strip. Let $I(t)$, the phase, vary in the horizontal direction and $J(t)$, the queue length or *level*, in the vertical direction. We denote the steady state probabilities by $\{p_{i,j}\}$, where $p_{i,j} = \lim_{t\to\infty} Prob(I(t) = i, J(t) = j)$, and let $\mathbf{v}_j = (p_{1,j}, \ldots, p_{N,j})$.

The process $\overline{Y}$ evolves due to the following instantaneous transition rates:

(a) $q_{i,k}$ – purely lateral transition rate – from state (i,j) to state (k,j), for all $j \geq 0$ and $1 \leq i,k \leq N$ $(i \neq k)$, caused by a phase transition in the Markov chain governing the arrival phase process ($q_{i,i} = 0$);

(b) $B_{i,j,j+s}$ – s-step upward transition rate – from state (i,j) to state $(i,j+s)$, for all phases i, caused by a new batch arrival of size s positive customers. For a given j, s can be seen as bounded when L is finite and unbounded when L is infinite;

(c) $C_{i,j,j-s}$ – s-step downward transition rate – from state (i,j) to state $(i,j-s)$, $(j-s \geq c+1)$ for all phases i, caused by either a batch service completion of size s or a batch arrival of negative customers of size s;

(d) $C_{i,c+s,c}$ – s-step downward transition rate – from state $(i,c+s)$ to state (i,c), for all phases i, caused by a batch arrival of negative customers of size $\geq s$ or a batch service completion of size s $(1 \leq s \leq L-c)$;

(e) $C_{i,c-1+s,c-1}$ – s-step downward transition rate, from state $(i,c-1+s)$ to state $(i,c-1)$, for all phases i, caused by a batch departure of size s $(1 \leq s \leq L-c+1)$;

(f) $C_{i,j+1,j}$ – 1-step downward transition rate, from state $(i,j+1)$ to state (i,j), $(c \geq 2\,;\, 0 \leq j \leq c-2)$, for all phases i, caused by a single departure;

where

$$B_{i,j-s,j} = \sum_{k=1}^{K} (1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k}$$
$$(\forall i\ ;\ 0 \leq j-s \leq L-2\ ;\ j-s < j < L)\ ;$$

$$B_{i,j,L} = \sum_{k=1}^{K} \sum_{s=L-j}^{\infty} (1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} = \sum_{k=1}^{K} \theta_{i,k}^{L-j-1}\sigma_{i,k}$$
$$(\forall i\ ;\ j \leq L-1)\ ;$$

$$
\begin{aligned}
C_{i,j+s,j} &= (1-\phi_i)\phi_i^{s-1} c\mu_i + (1-\delta_i)\delta_i^{s-1}\rho_i \\
&\quad (\forall i \ ; \ c+1 \le j \le L-1 \ ; \ 1 \le s \le L-j) \ ; \\
&= (1-\phi_i)\phi_i^{s-1} c\mu_i + \delta_i^{s-1}\rho_i \\
&\quad (\forall i \ ; \ j = c \ ; \ 1 \le s \le L-c) \ ; \\
&= \phi_i^{s-1} c\mu_i \\
&\quad (\forall i \ ; \ j = c-1 \ ; \ 1 \le s \le L-c+1) \ ; \\
&= 0 \qquad (\forall i \ ; \ c \ge 2 \ ; \ 0 \le j \le c-2 \ ; \ s \ge 2) \ ; \\
&= (j+1)\mu_i \\
&\quad (\forall i \ ; \ c \ge 2 \ ; \ 0 \le j \le c-2 \ ; \ s = 1) \ .
\end{aligned}
$$

Define,

$$
\begin{aligned}
B_{j-s,j} &= \mathrm{Diag}\left[B_{1,j-s,j}, B_{2,j-s,j}, \ldots, B_{N,j-s,j}\right] \\
&\qquad\qquad\qquad\qquad (j-s < j \le L) \ ; \\
B_s &= B_{j-s,j} \qquad (j < L) \\
&= \mathrm{Diag}\left[\ldots, \sum_{k=1}^{K} \sigma_{i,k}(1-\theta_{i,k})\theta_{i,k}^{s-1}, \ldots\right] \ ; \\
\Sigma_k &= \mathrm{Diag}\left[\sigma_{1,k}, \sigma_{2,k}, \ldots, \sigma_{N,k}\right] \ (k = 1, 2, \ldots, K) \ ; \\
\Theta_k &= \mathrm{Diag}\left[\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{N,k}\right] \ (k = 1, 2, \ldots, K) \ ; \\
\Sigma &= \sum_{k=1}^{K} \Sigma_k \ ; \\
R &= \mathrm{Diag}\left[\rho_1, \rho_2, \ldots, \rho_N\right] \ ; \\
\Delta &= \mathrm{Diag}\left[\delta_1, \delta_2, \ldots, \delta_N\right] \ ; \\
M &= \mathrm{Diag}\left[\mu_1, \mu_2, \ldots, \mu_N\right] \ ; \\
\Phi &= \mathrm{Diag}\left[\phi_1, \phi_2, \ldots, \phi_N\right] \ ; \\
C_j &= jM \qquad (0 \le j \le c) \ ; \\
&= cM = C \qquad (j \ge c) \ ; \\
C_{j+s,j} &= \mathrm{Diag}\left[C_{1,j+s,j}, C_{2,j+s,j}, \ldots, C_{N,j+s,j}\right] \ ; \\
E &= \mathrm{Diag}(\mathbf{e}_N^{'}) \ .
\end{aligned}
$$

Then, we get,

$$
B_s = \sum_{k=1}^{K} \Theta_k^{s-1}(E-\Theta_k)\Sigma_k \ ;
$$

$$
B_1 = B = \sum_{k=1}^{K} (E-\Theta_k)\Sigma_k \ ;
$$

$$B_{L-s,L} = \sum_{k=1}^{K} \Theta_k^{s-1} \Sigma_k \; ;$$

$$\begin{aligned} C_{j+s,j} &= C(E-\Phi)\Phi^{s-1} + R(E-\Delta)\Delta^{s-1} \\ &\quad (c+1 \le j \le L-1 \; ; \; s = 1,2,\ldots,L-j) \; ; \\ &= C(E-\Phi)\Phi^{s-1} + R\Delta^{s-1} \\ &\quad (j = c \; ; \; s = 1,2,\ldots,L-c) \; ; \\ &= C\Phi^{s-1} \\ &\quad (j = c-1 \; ; \; s = 1,2,\ldots,L-c+1) \; ; \\ &= 0 \;\; (c \ge 2 \; ; \; 0 \le j \le c-2 \; ; \; s \ge 2) \; ; \\ &= C_{j+1} \;\; (c \ge 2 \; ; \; 0 \le j \le c-2 \; ; \; s = 1) \; . \end{aligned}$$

The steady state balance equations are

– for the L^{th} row or level:

$$\sum_{s=1}^{L} \mathbf{v}_{L-s} B_{L-s,L} + \mathbf{v}_L \left[Q - C - R \right] = 0 \; ; \tag{6}$$

– for the j^{th} row or level:

$$\sum_{s=1}^{j} \mathbf{v}_{j-s} B_s + \mathbf{v}_j \left[Q - \Sigma - C_j - R I_{j>c} \right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \qquad (0 \le j \le L-1) \; ; \tag{7}$$

– normalization

$$\sum_{j=0}^{L} \mathbf{v}_j \mathbf{e}_N = 1 \; . \tag{8}$$

Note that $I_{j>c} = 1$ if $j > c$ else 0, and $\mathbf{e}_N$ is a column vector of size N with all ones.

2.6 Performance Measures

Some performance measures can be derived as follows:

– Average number of bursts in the system

$$E(j) = \sum_{j=0}^{L} j \mathbf{v}_j \mathbf{e}. \tag{9}$$

– Burst loss probability

$$\sum_{i=1}^{N} \sum_{j=0}^{L} \sum_{l=L-j+1}^{\infty} p_{i,j} \pi_{i,l} \frac{(l-(L-j))\sigma_{i,\cdot}}{\overline{\sigma}}. \tag{10}$$

3 Solution Methodology and Technique

3.1 Transforming the Balance Equations

When $L < c + K + 4$, then the number of the states of the Markov process Y is not large and can be solved by traditional methods [17]. However, when L is large, computationally more efficient other methods are available.

In this Section the balance equations are essentially transformed into a computable form (of QBD-M type). The necessary mathematical transformations are based on profound theoretical analysis and proofs. Moreover, they are very convenient for (software or program) implementation.

Let the balance equations for level j be denoted by $<\mathbf{j}>$. Hence, $<\mathbf{0}>$, $<\mathbf{1}>$, ..., $<\mathbf{j}>$, ..., $<\mathbf{L}>$ are the balance equations for the levels $0, 1, \ldots, j, \ldots, L$ respectively. Substituting $B_{L-s,L} = \sum_{k=1}^{K} \Theta_k^{s-1} \Sigma_k$ and $B_s = \sum_{k=1}^{K} \Theta_{.k}^{s-1} (E - \Theta_k) \Sigma_k$ in (6, 7), we get the balance equations for level L and for all the other levels as:

$$<\mathbf{L}> : \quad \sum_{s=1}^{L} \sum_{k=1}^{K} \mathbf{v}_{L-s} \Theta_k^{s-1} \Sigma_k + \mathbf{v}_L \left[Q - C - R\right] = 0;$$

$$<\mathbf{L-1}> : \quad \sum_{s=1}^{L-1} \sum_{k=1}^{K} \mathbf{v}_{L-1-s} \Theta_k^{s-1} (E - \Theta_k) \Sigma_k + \mathbf{v}_{L-1} \left[Q - \Sigma - C_{L-1} - R\right] + \mathbf{v}_L C_{L,L-1} = 0;$$

$$<\mathbf{L-2}> : \quad \sum_{s=1}^{L-2} \sum_{k=1}^{K} \mathbf{v}_{L-2-s} \Theta_k^{s-1} (E - \Theta_k) \Sigma_k + \mathbf{v}_{L-2} \left[Q - \Sigma - C_{L-2} - R\right] + \sum_{s=1}^{2} \mathbf{v}_{L-2+s} C_{L-2+s,L-2} = 0;$$

$$<\mathbf{L-3}> : \quad \sum_{s=1}^{L-3} \sum_{k=1}^{K} \mathbf{v}_{L-3-s} \Theta_k^{s-1} (E - \Theta_k) \Sigma_k + \mathbf{v}_{L-3} \left[Q - \Sigma - C_{L-3} - R\right] + \sum_{s=1}^{3} \mathbf{v}_{L-3+s} C_{L-3+s,L-3} = 0;$$

$$\vdots$$

$$<\mathbf{j}> : \quad \sum_{s=1}^{j} \sum_{k=1}^{K} \mathbf{v}_{j-s} \Theta_k^{s-1} (E - \Theta_k) \Sigma_k \mathbf{v}_j \left[Q - \Sigma - C_j - R\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \;\; (j = L-4, L-5, \ldots, c+K+2);$$

$$< \mathbf{c+K+1} > \ : \ \sum_{s=1}^{c+K+1}\sum_{k=1}^{K}\mathbf{v}_{c+K+1-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k + \mathbf{v}_{c+K+1}\left[Q-\Sigma-C_{c+K+1}-R\right] + \sum_{s=1}^{L-c-K-1}\mathbf{v}_{c+K+1+s}C_{c+K+1+s,c+K+1} = 0;$$

$$\vdots$$

$$< \mathbf{j} > \ : \ \sum_{s=1}^{j}\sum_{k=1}^{K}\mathbf{v}_{j-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k + \mathbf{v}_j\left[Q-\Sigma-C_j-RI_{j>c}\right] + \textstyle\sum_{s=1}^{L-j}\mathbf{v}_{j+s}C_{j+s,j} = 0 \quad (j = c+K, c+K-1, \ldots, 0).$$

Define the functions, $F_{K,l}$, $(l = 1, 2, \ldots, K)$ as,

$$\begin{aligned} F_{K,l} &= \sum_{\substack{1\le k_1,k_2,\ldots,k_l\le K \\ k_1\neq k_2\neq\ldots\neq k_l}} \Theta_{k_1}\Theta_{k_2}\ldots\Theta_{k_l} \quad (l = 1, 2, \ldots, K) \\ &= E \quad \text{if } l = 0 \\ &= 0 \quad \text{if } l \le -1 \text{ or } l > K \end{aligned} \tag{11}$$

From the above, for example,

$$\begin{aligned} K = 1: &\quad F_{1,0} = E; F_{1,1} = \Theta_1 \\ K = 2: &\quad F_{2,0} = E; F_{2,1} = \Theta_1 + \Theta_2; F_{2,2} = \Theta_1\Theta_2 \\ K = 3: &\quad F_{3,0} = E; F_{3,1} = \Theta_1 + \Theta_2 + \Theta_3; \\ &\quad F_{3,2} = \Theta_1\Theta_2 + \Theta_2\Theta_3 + \Theta_3\Theta_1; F_{3,3} = \Theta_1\Theta_2\Theta_3. \end{aligned}$$

The required $F_{K,l}$'s can be computed easily using the following properties and the recursion which also can be considered as an alternate definition for $F_{K,l}$'s. Let the arrival streams be numbered as $1, 2, \ldots, K$ in any order. Define $F_{k,l}$ be the function in equation (11) when only the first k streams of customer arrivals are present and the rest are absent. Then, we get

$$\begin{aligned} &F_{k,0} = E \ , \ \ F_{k,k} = \prod_{n=1}^{k}\Theta_n \ \ (k = 1, 2, \ldots, K); \\ &F_{k,l} = 0 \ (k = 1, 2, \ldots, K;\ l < 0) \\ &F_{k,l} = 0 \ (k = 1, 2, \ldots, K;\ l > k) \end{aligned} \tag{12}$$

The recursion, then, is

$$\begin{aligned} F_{1,0} &= E \ ; \ \ F_{1,1} = \Theta_1 \ ; \\ F_{k,l} &= F_{k-1,l} + \Theta_k F_{k-1,l-1} \\ &\quad (2 \le k \le K \ , \ 1 \le l \le k-1) \ . \end{aligned} \tag{13}$$

Transformation 1. *Modify simultaneously the balance equations for levels $j\,(L-1 \geq j \geq K)$, by the transformation:*

$$< \mathbf{j} >^{(\mathbf{1})} \longleftarrow < \mathbf{j} > + \sum_{l=1}^{K} (-1)^l < \mathbf{j} - \mathbf{l} > F_{K,l} \qquad (K \leq j \leq L-1)$$

$$< \mathbf{j} >^{(\mathbf{1})} \longleftarrow < \mathbf{j} > \qquad (j = L \text{ or } j < K).$$

Apply the following transformation to the resulting equations.

Transformation 2. *Modify simultaneously the balance equations for levels $j\,(L-3 \geq j \geq c+K+1)$, by the transformation:*

$$< \mathbf{j} >^{(\mathbf{2})} \longleftarrow < \mathbf{j} >^{(\mathbf{1})} - < \mathbf{j} + \mathbf{1} >^{(\mathbf{1})} (\Phi + \Delta) + < \mathbf{j} + \mathbf{2} >^{(\mathbf{1})} \Phi\Delta \qquad (c+K+1 \leq j \leq L-3)$$

$$< \mathbf{j} >^{(\mathbf{2})} \longleftarrow < \mathbf{j} >^{(\mathbf{1})} \qquad (j > L-3 \text{ or } j < c+K+1).$$

With these above two transformations, the transformed balance equation, $< \mathbf{j} >^{(\mathbf{2})}$'s, for the rows $(c+K+1 \leq j \leq L-3)$, will be of the form:

$$\mathbf{v}_{j-K} Q_0 + \mathbf{v}_{j-K+1} Q_1 + \ldots + \mathbf{v}_{j+2} Q_{K+2} = 0 \qquad (j = L-3, L-4, \ldots, c+K+1), \tag{14}$$

where $Q_0, Q_1, \ldots, Q_{K+2}$ are $K+3$ number of j-independent matrices and can be easily derived algebraically from the system parameters by following the above mentioned transformation procedures.

Thus, the resulting equations (14) corresponding to the rows from $c+K+1$ to $L-3$, are of the same form as those of the QBD-M processes and hence have an efficient solution by several methods such as the spectral expansion method, Bini-Meini's method or the matrix-geometric method with folding or block size enlargement [10].

3.2 Spectral Expansion Solution of the Balance Equations

The set of equations (14) concerning the levels $c+K+1$ to $L-3$ have the coefficient matrices $Q_0, Q_1, \ldots, Q_{K+2}$ that are independent of j and hence have an efficient solution by the spectral expansion method [12]. These Q_l's $(l = 0, 1, \ldots, K+2)$ can be obtained quite easily following the computational procedure in Appendix A, A.1.

Define the matrix polynomials $Z(\lambda)$ and $\overline{Z}(\xi)$ as,

$$Z(\lambda) = Q_0 + Q_1\lambda + Q_2\lambda^2 + \ldots + Q_{K+2}\lambda^{K+2}, \tag{15}$$

$$\overline{Z}(\xi) = Q_{K+2} + Q_{K+1}\xi + Q_K\xi^2 + \ldots + Q_0\xi^{K+2}\,. \tag{16}$$

Then, the spectral expansion solution for $\mathbf{v}_j$ $(c+1 \leq j \leq L-1)$ is given by

$$\mathbf{v}_j = \sum_{l=1}^{KN} a_l \boldsymbol{\psi}_l \lambda_l^{j-c-1} + \sum_{l=1}^{2N} b_l \boldsymbol{\gamma}_l \xi_l^{L-1-j} \ (c+1 \leq j \leq L-1), \tag{17}$$

where λ_l $(l = 1, 2, \ldots, KN)$ are the KN eigenvalues of least absolute value of the matrix polynomial $Z(\lambda)$ and ξ_l $(l = 1, 2, \ldots, 2N)$ are the $2N$ eigenvalues of least absolute value of the matrix polynomial $\overline{Z}(\xi)$. $\boldsymbol{\psi}_l$ and $\boldsymbol{\gamma}_l$ are the left-eigenvectors of $Z(\lambda)$ and $\overline{Z}(\xi)$ respectively, corresponding to the eigenvalues λ_l and ξ_l respectively. a_l's and b_l's are arbitrary constants to be determined later.

It can be proved that the matrix $\sum_{l=0}^{K+2} Q_l$ is singular, so that $\lambda = 1$ is an eigenvalue on the unit-circle for both $Z(\lambda)$ and $\overline{Z}(\xi)$. If (5) is satisfied, the number of eigenvalues of $Z(\lambda)$ that are strictly within the unit circle is KN. If (5) is not satisfied, that number is $KN - 1$.

These and also certain other properties of these eigenvalues, Eigenvectors, also the relevant spectral analysis are dealt with (some of them are proved, others explained in detail) in [12,2]. Some of them are summarized below. Let the rank of Q_0 be $N - n_0$ and that of Q_{K+2} be $N - n_{K+2}$. Then,

(a) $Z(\lambda)$ would have $d = (K+2)N - n_{K+2}$ eigenvalues of which n_0 are zero eigenvalues (also referred to as null eigenvalues), whereas $\overline{Z}(\xi)$ would have n_{K+2} zero eigenvalues and $(K+2)N - n_0 - n_{K+2}$ non-zero eigenvalues.
(b) If $(\lambda \neq 0, \boldsymbol{\psi})$ is a non-zero eigenvalue-eigenvector pair of $Z(\lambda)$, then there exists a corresponding non-zero eigenvalue-eigenvector pair, $(\xi = \frac{1}{\lambda}, \boldsymbol{\gamma} = \boldsymbol{\psi})$ for $\overline{Z}(\xi)$. Thus, the non-zero eigenvalues of these two matrix polynomials are mutually reciprocal.
(c) The KN eigenvalues of least absolute value of $Z(\lambda)$ and the $2N$ eigenvalues of least absolute value of $\overline{Z}(\xi)$ lie either strictly inside, or on, their respective unit-circles, but not outside.
(d) There is no problem posed by multiple eigenvalues, i.e. independent eigenvectors having coincident eigenvalues, since each pair $(\lambda, \boldsymbol{\psi})$ is *distinct.*

If the unknowns a_l's and b_l's are determined in such a way that all the balance equations are satisfied, then the vectors $\mathbf{v}_j$ $(c+1 \leq j \leq L-1)$ can be computed from the steady state solution (17). Hence, the unknowns are the scalars, $a_1, a_2, \ldots, a_{K \cdot N}, b_1, b_2, \ldots, b_{2N}$, and the vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_c, \mathbf{v}_L$. These are totally, $(c+2)N + K \cdot N + 2N = (c+K+4)N$ scalar unknowns. In order to solve for them, we still have the transformed balance equations concerning the levels $0, 1, \ldots, c+K, L-2, L-1, L$ and also the equation (8). These are $(c+K+4)N+1$ linear simultaneous equations in the above $(c+K+4)N$ scalar unknowns. Of these equations only $(c+K+4)N$ equations (including equation (8)) are independent. Hence, all these unknowns can be solved for.

As far as the computational complexity is regarded the solution of $(c+K+4)N$ equations requires $O(c^3N^3 + K^3N^3)$ work and a generalized nonsymmetric eigenvalue-problem of dimension $(K+2)N$ is solved at the expense of $O(K^3N^3)$ amount of work. Therefore, the computational complexity of the proposed method is $O(c^3N^3 + K^3N^3)$.

3.3 System with Infinite Queuing Capacity

So far the analysis has been for the case of finite L. A corresponding analysis for the case of infinite queuing capacity yields the spectral expansion solution as follows:

$$\mathbf{v}_j = \sum_{l=1}^{KN} a_l \boldsymbol{\psi}_l \lambda_l^{j-c} \quad (j = c+1, c+2, \ldots) . \tag{18}$$

Here, we need only the KN relevant eigenvalues-eigenvectors of $Z(\lambda)$, and the KN a_k's (which exist as real or complex conjugate pairs) are to be computed. Notice that the equation (18) is the same as (17) when the limit $L \to \infty$ is taken. Notice also that the computation time for this case would be much less than that for finite L.

4 Notes on the Departure Process

4.1 Departure Batch Size Distribution

From Sect. 3.2, we have the solution for the steady state probabilities, $p_{i,j}$. The marginal probabilities, $p_{i.}$ and $p_{.j}$ are then defined as:

$$p_{i.} = \sum_{j=0}^{L} p_{i,j} \;\; ; p_{.j} = \sum_{i=0}^{L} p_{i,j} \; . \tag{19}$$

Now consider the system in the state (i, j), where $j > c$. Here, all the c servers are busy, with $j-c$ unattended positive customers in the queue. In this state, the departure rate associated with a batch size of s is $(1-\phi_i)\phi_i^{s-1} c\mu_i$ for $1 \le s \le j-c$ and for $s = j-c+1$. Hence, the average rate at which batches of size n, for $2 \le n \le L-c+1$, depart from the queue is

$$\nu_n = \sum_{i=1}^{N} \sum_{j=c+n}^{L} p_{i,j}(1-\phi_i)\phi_i^{n-1} c\mu_i + \sum_{i=1}^{N} p_{i,c+n-1}\phi_i^{n-1} c\mu_i \;\; (n = 2, \ldots, L-c+1). \tag{20}$$

The average rate of single departures is

$$\nu_1 = \sum_{i=1}^{N} \sum_{j=1}^{c} p_{i,j} j \mu_i + \sum_{i=1}^{N} \sum_{j=c+1}^{L} p_{i,j}(1-\phi_i)c\mu_i . \tag{21}$$

Thus, by the Law of Large Numbers for Markov chains, $\frac{\nu_n}{\sum_{s=1}^{L-c+1} \nu_s}$ is the equilibrium probability that the batch size is n $(n = 1, 2, \ldots, L-c+1)$.

The number of batch departures per unit time ν is

$$\nu = \sum_{s=1}^{L-c+1} \nu_s , \tag{22}$$

and the average departure rate of positive customers $\overline{\nu}$

$$\overline{\nu} = \sum_{s=1}^{L-c+1} s\nu_s. \tag{23}$$

Hence the loss rate of positive customers due to either overflow or being killed by negative customers is $\overline{\sigma} - \overline{\nu}$.

4.2 System at Departure Epochs

Let $\beta_{i,j}$ be the probability that the state of the system is (i,j) immediately after a (batch) departure epoch. Then, $\beta_{i,j}$ is proportional to the probability flux into state (i,j) *due to a departure*, i.e. $\beta_{i,j} \propto f_{i,j}$ where, for $1 \le i \le N$,

$$\begin{aligned} f_{i,j} &= \sum_{n=1}^{L-j} p_{i,j+n} c\mu_i (1-\phi_i)\phi_i^{n-1} \qquad (j \ge c) \\ f_{i,c-1} &= \sum_{n=1}^{L-c+1} p_{i,c-1+n} c\mu_i \phi_i^{n-1} \\ f_{i,j} &= p_{i,j+1}(j+1)\mu_i \qquad (0 \le j \le c-2). \end{aligned}$$

The normalisation constant is the reciprocal of the sum of all the $f_{i,j}$, i.e. the reciprocal of

$$\sum_{i=1}^{N} \left[\sum_{j=c}^{\infty} \sum_{n=1}^{L-j} p_{i,j+n} c\mu_i (1-\phi_i)\phi_i^{n-1} + \sum_{n=1}^{L-c+1} p_{i,c-1+n} c\mu_i \phi_i^{n-1} + \sum_{j=0}^{c-2} p_{i,j+1}(j+1)\mu_i \right].$$

5 Numerical Results

5.1 Case Study

In this section we present some numerical results to demonstrate the capabilities of the proposed model. We consider an optical burst switching[2] multiplexer that assigns arriving bursts to channels in a link with c available data channels (wavelengths) and storage locations for $L-c$ bursts. An arriving burst is diverted to a storage location if all c data channels are in use when it arrives. A burst will be discarded if it arrives when all c channels are busy and all $L-c$ burst storage locations are being used by bursts that have not yet been assigned a channel. However, once a stored burst has been assigned to an output channel, its storage location becomes available for use by an arriving burst, since the stored burst will vacate space at the same rate that the arriving burst occupies it. It is apparent

[2] Packet and burst switching have been proposed for optical networks because they are better to accommodate bursty traffic generated by IP applications. Optical burst switched (OBS) networks switch packets of constant or variable length while the payload data stays in the optical domain [18,19,21,7,18,20].

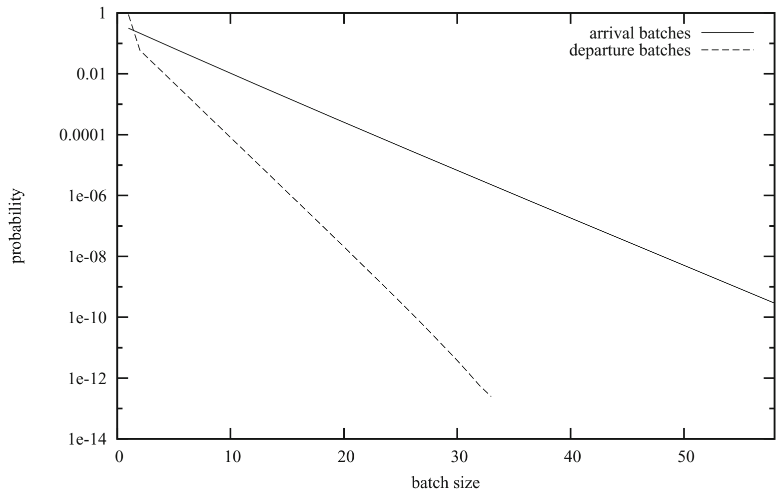

Fig. 1. Arrival and departure batch size distribution at load=0.5, $c = 32$, $L = 64$

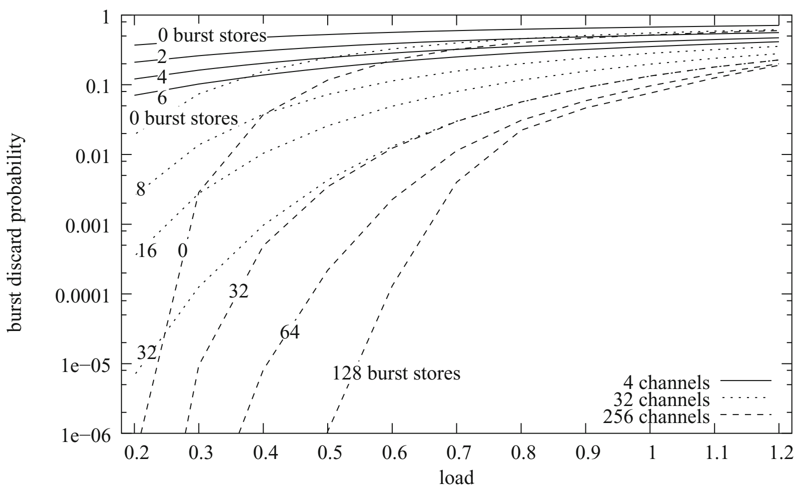

Fig. 2. Burst loss probability vs load and c

that an optical burst switching multiplexer can be modeled as a queue of c servers and a buffer of size $L - c$. In the literature, Turner has proposed one dimensional birth-death process to analyze this problem [18]. However, Turner's model has some limitations like the assumption of exponential burst arrival process and constant burst size. Our model overcomes these limitations.

The aim is to show the impact of the burstiness of the offered traffic on the performance of the multiplexer. To compare with Turner's model, we produced the similar figure as presented in [18]. However, the numerical results are obtained with different arrival and serving processes, and the assumption of no negative customers. The following system parameters were used:

$$\begin{aligned} & & \big[q_{i,j}\big] &= \begin{bmatrix} -0.2 & 0.2 \\ 0.9 & -0.9 \end{bmatrix}; \\ \big[\sigma_{i,k}\big] &= \begin{bmatrix} 1 & 2 \\ 2 & 2.5 \end{bmatrix}; & \big[\theta_{i,k}\big] &= \begin{bmatrix} 0.65 & 0.7 \\ 0.65 & 0.7 \end{bmatrix}; \\ \big[\phi_i\big] &= \big[5\ 5\big]; & \big[\mu_i\big] &= \big[0.5\ 0.5\big], \end{aligned}$$

where all $\sigma_{i,k}$ was scaled as appropriate to set the system load to the examined values and the number of positive sources (K) was chosen to be 2. The batch size distribution was chosen to be geometric (see Fig. 1).

The departure batch size, however, is a function of steady-state system behaviour, and thus all system parameters. Its distribution has a finite support set consisting of the states $\{1, 2, ..., L - c + 1\}$ as illustrated in Figure 1.

Figure 2 plots the burst discard probabilities for different channel numbers with different space available for burst storage. Space allocated for burst storage is printed on the lines in the figure. It can be observed that batch arrivals can be better handled by increasing the storage space (at the expense of some queuing delay) than by increasing the number of channels. The performance of 256 channels with no burst storage space is worse than that of 32 channels with a storage space for 8 bursts in our example for relative load values above 0.4.

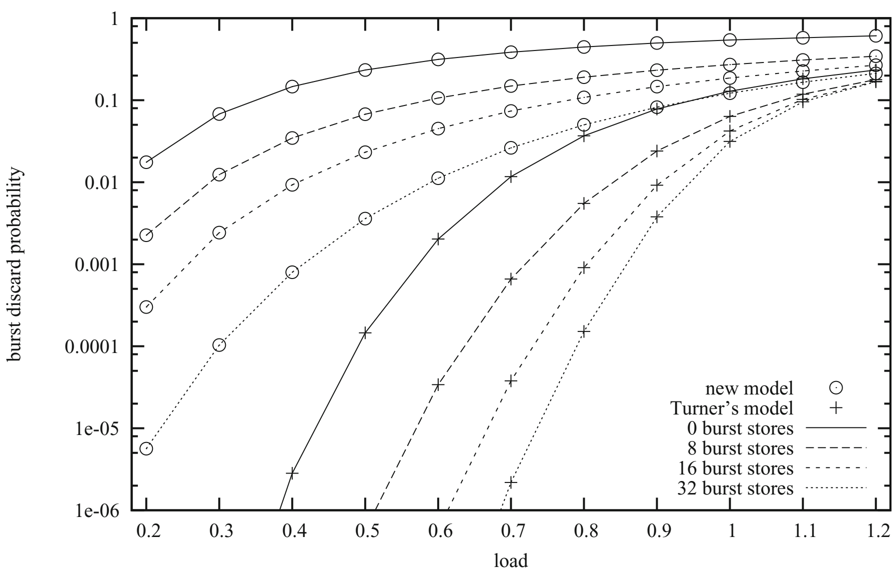

Fig. 3. Burst loss probability vs load: comparison with Turner's model ($c = 32$)

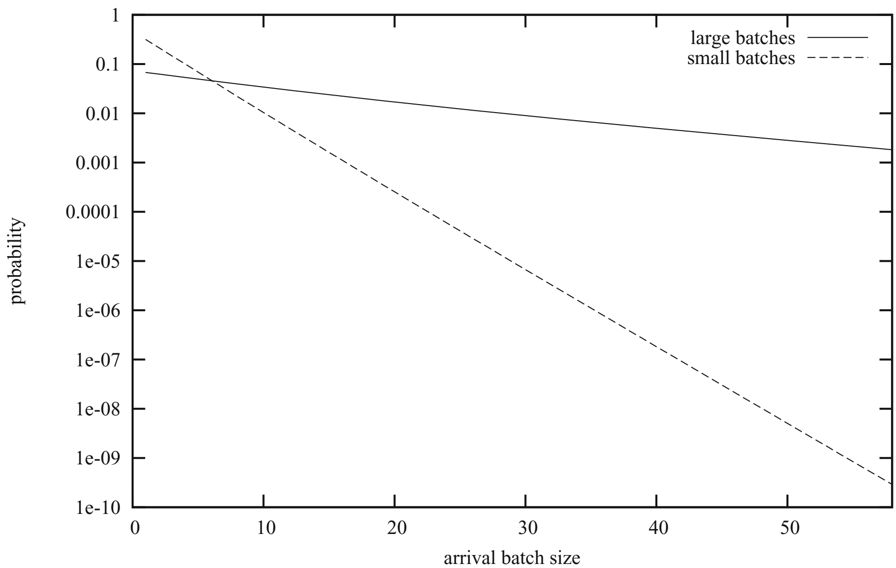

Fig. 4. Arrival batch distributions

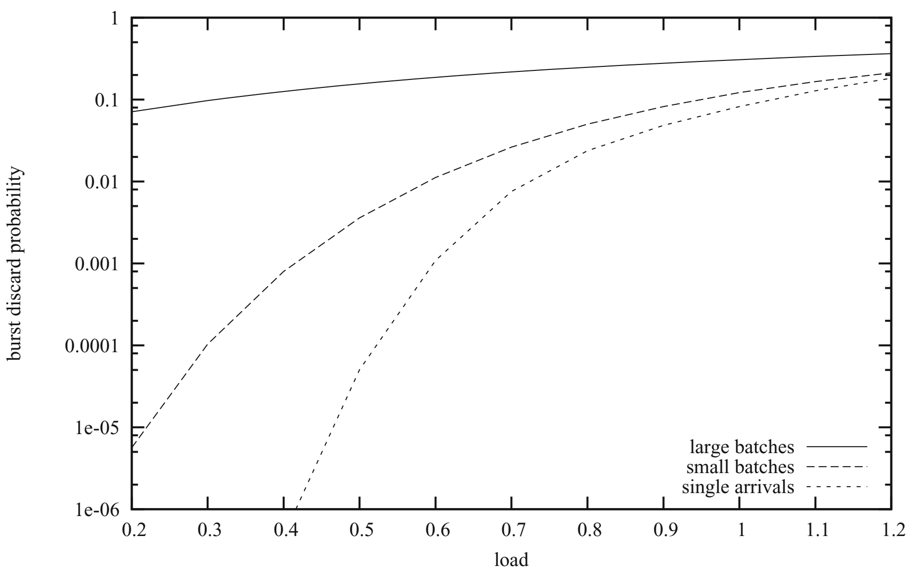

Fig. 5. Burst loss probability vs load: impact of arrival burstiness ($c = 32, L = 64$) with the arrival batch distributions in Fig. 4

Figure 3 depicts the results obtained with Turner's model and our model in the case of 32 channels. Turner's model shows that burst discard probability may be

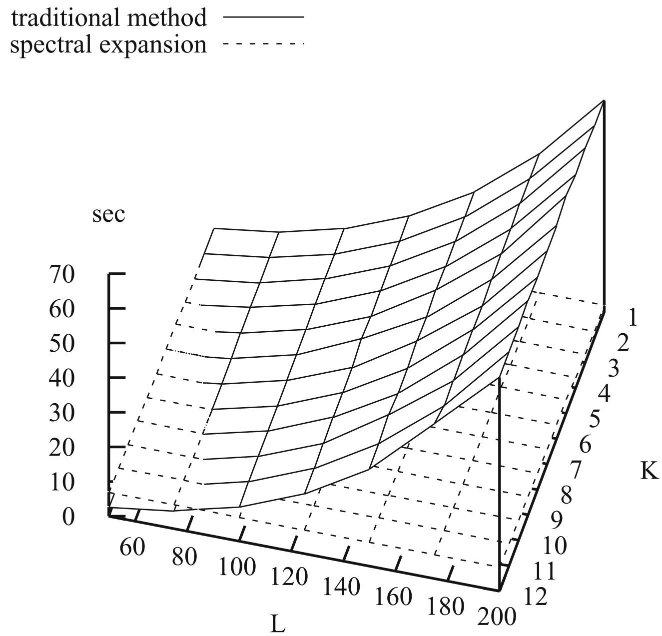

Fig. 6. Runtimes of different methods for different input parameters

kept under 10^{-6} for load values almost as high as 0.7 with a buffer of adequate size. However, batch arrivals make the situation worse (high utilization is very hard to be realized). The impact of burst arrival is further demonstrated in Figures 4 and 5, where different burst arrival distributions are used. The figures show that when the higher the probability of long batches is, the more significant performance loss is encountered at the same system for the same load values.

Figure 6 demonstrates the trade-off between the two solution methods. The implementations were run on a SUN Ultra 60 Workstation for different values of L and K, while c was always 3. The displayed times were averaged over five runs. It can be observed that the performance of the spectral expansion method overcomes the traditional method in a large ranges of parameters, moreover and it does not depends on the buffer size. Moreover, it takes even less time in the case of infinite buffer.

Further reduction in the computation complexity can be obtained for the new method based on the spectral expansion if we express the vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_c, \mathbf{v}_L$ as the functions of the unknowns a_l's and b_l's. At the present the implementation of this idea is in progress.

6 Conclusions

One of the research aims in the performance evaluation of telecommunications networks is to find analytically tractable queuing models with the capability of capturing the burstiness and correlation of the traffic. In this context, the first contribution of this paper is the derivation of the *exact* result for the equilibrium state probabilities of the $MM \sum_{k=1}^{K} CPP_k/GE/c/L\,G$-queue, which can capture the burstiness and correlation of the traffic and take into account environmentally sensitive service times. We also provide ideas to extend the modeling capability of the new queue.

The queuing model and its variants were successfully used to model [5,6] High-speed Downlink Packet Access HSDPA terminal categories. In this paper, we have applied the new queue for the performance analysis of multiplexers in optical burst switching networks.

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Appendix

A Obtaining the Matrices Q_l's

Consider any row j where $c+K+1 \leq j \leq L-3$. With Transformation 1, we get

$$
\begin{aligned}
<\mathbf{j}>^{(\mathbf{1})} &\longleftarrow <\mathbf{j}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j}-\mathbf{l}> F_{K,l} \\
<\mathbf{j}+\mathbf{1}>^{(\mathbf{1})} &\longleftarrow <\mathbf{j}+\mathbf{1}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j}+\mathbf{1}-\mathbf{l}> F_{K,l} \\
<\mathbf{j}+\mathbf{2}>^{(\mathbf{1})} &\longleftarrow <\mathbf{j}+\mathbf{2}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j}+\mathbf{2}-\mathbf{l}> F_{K,l}.
\end{aligned} \tag{24}
$$

Applying Transformation 2 to the j^{th} row, and substituting from the above (24) for $<\mathbf{j}+\mathbf{1}>, <\mathbf{j}+\mathbf{2}>$, we get

$$
\begin{aligned}
<\mathbf{j}>^{(\mathbf{2})} \longleftarrow\ & <\mathbf{j}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j}-\mathbf{l}> F_{K,l} \\
& - \left[<\mathbf{j}+\mathbf{1}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j}+\mathbf{1}-\mathbf{l}> F_{K,l} \right] (\Phi + \Delta) \\
& + \left[<\mathbf{j}+\mathbf{2}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j}+\mathbf{2}-\mathbf{l}> F_{K,l} \right] \Phi\Delta.
\end{aligned} \tag{25}
$$

Expanding and grouping the terms together, equation (25) can be written as

$$
<\mathbf{j}>^{(\mathbf{2})} \longleftarrow \sum_{l=-2}^{K} <\mathbf{j}-\mathbf{l}> G_{K,l}, \tag{26}
$$

where

$$
G_{K,l} = (-1)^l [F_{K,l} + (\Phi + \Delta) F_{K,l+1} + \Phi\Delta F_{K,l+2}] \ (l = -2, -1, \ldots, K). \tag{27}
$$

The balance equations $<\mathbf{j}+\mathbf{2}>$, $<\mathbf{j}+\mathbf{1}>$, $<\mathbf{j}>, \ldots,$ $<\mathbf{j}-\mathbf{l}>, \ldots,$ $<\mathbf{j}-\mathbf{K}>$, respectively are given by,

$$
\sum_{s=1}^{j+2} \sum_{k=1}^{K} \mathbf{v}_{j+2-s} \Theta_k^{s-1} (E - \Theta_k) \Sigma_k + \mathbf{v}_{j+2} [Q - \Sigma - C_{j+2} - R] \\
+ \sum_{s=1}^{L-j-2} \mathbf{v}_{j+2+s} C_{j+2+s,j+2} = 0 \ ;
$$

$$\sum_{s=1}^{j+1}\sum_{k=1}^{K}\mathbf{v}_{j+1-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_{j+1}\left[Q-\Sigma-C_{j+1}-R\right] + \sum_{s=1}^{L-j-1}\mathbf{v}_{j+1+s}C_{j+1+s,j+1}=0\ ;$$

$$\sum_{s=1}^{j}\sum_{k=1}^{K}\mathbf{v}_{j-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_{j}\left[Q-\Sigma-C_{j}-R\right] + \sum_{s=1}^{L-j}\mathbf{v}_{j+s}C_{j+s,j}=0\ ;$$

$$\vdots$$

$$\sum_{s=1}^{j-l}\sum_{k=1}^{K}\mathbf{v}_{j-l-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_{j-l}\left[Q-\Sigma-C_{j-l}-R\right] + \sum_{s=1}^{L-j+l}\mathbf{v}_{j-l+s}C_{j-l+s,j-l}=0\ ;$$

$$\vdots$$

$$\sum_{s=1}^{j-K}\sum_{k=1}^{K}\mathbf{v}_{j-K-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_{j-K}\left[Q-\Sigma-C_{j-K}-R\right] + \sum_{s=1}^{L-j+K}\mathbf{v}_{j-K+s}C_{j-K+s,j-K}=0\ ;$$

Substituting or applying the above to (26), for the coefficients (Q_{K-m}) of $\mathbf{v}_{j-m}$ in $<\mathbf{j}>^{(\mathbf{2})}$, we get

$$Q_{K-m}=\sum_{l=-2}^{m-1}\left[\sum_{n=1}^{K}\Theta_n^{m-l-1}(E-\Theta_n)\Sigma_n\right]G_{K,l}+\left[Q-\Sigma-C_{j-m}-R\right]G_{K,m} + \sum_{l=m+1}^{K}\left[C_{j-m,j-l}\right]G_{K,l}$$

$$(m=j-L,\ldots,-2,-1,0,\ldots,K,\ldots,j)\ . \tag{28}$$

Also, for $m=-2,-1,0,\ldots,K$, substituting $C_{j-m}=C$ and $C_{j-m,j-l}=C_{j-l+l-m,j-l}=C(E-\Phi)\Phi^{l-m-1}+R(E-\Delta)\Delta^{l-m-1}$ in (28), we get

$$Q_{K-m} = \sum_{l=-2}^{m-1} \left[\sum_{n=1}^{K} \Theta_n^{m-l-1} (E - \Theta_n) \Sigma_n \right] G_{K,l} + [Q - \Sigma - C - R] \, G_{K,m}$$
$$+ \sum_{l=m+1}^{K} \left[C(E - \Phi)\Phi^{l-m-1} + R(E - \Delta)\Delta^{l-m-1} \right] G_{K,l}$$
$$(m = -2, -1, 0, \ldots, K). \qquad (29)$$

Using the above, the required Q_l's can be computed easily as described in the subsection below. Notice the above Q_l's in equation (29) are j- independent.

The other coefficients, i.e. those of $\mathbf{v}_{j-K-1}, \mathbf{v}_{j-K-2}, \ldots, \mathbf{v}_0$ and of $\mathbf{v}_{j+3}, \mathbf{v}_{j+4}, \ldots$, can be shown to be zero. This is elaborated in the next section for a numerical value of K, and a general proof is given later on.

A.1 Computation

After obtaining $F_{K,l}$'s thus, $G_{K,l}\,(l = -2, -1, \ldots, K)$ can be computed from (27). Then, using them directly in (29), the required $Q_l\,(l = 0, 1, \ldots, K+2)$ can be computed.

An alternate way of computing the $G_{K,l}$'s is by the following properties and recursion which are obtained from (12,13,27)

$$G_{1,-2} = \Phi\Delta \; ; \; G_{1,-1} = -(\Phi + \Delta) - \Phi\Delta\Theta_1 \; ;$$
$$G_{1,0} = E + (\Phi + \Delta)\Theta_1 \; ; \; G_{1,1} = -\Theta_1 \; ;$$
$$G_{k,l} = 0 \; (l \leq -3) \; ; \; G_{k,l} = 0 \; (l \geq k+1) \; ;$$
$$G_{k,l} = G_{k-1,l} - \Theta_k G_{k-1,l-1} \; (2 \leq k \leq K \; , \; -2 \leq l \leq k). \qquad (30)$$

Another interesting property of $G_{k,l}$'s is

$$\sum_{l=-2}^{k} G_{k,l} = \sum_{l=-2}^{k} G_{k-1,l} - \Theta_k \sum_{l=-2}^{k} G_{k-1,l-1}$$
$$= \sum_{l=-2}^{k-1} G_{k-1,l} - \Theta_k \sum_{l-1=-2}^{k-1} G_{k-1,l-1}$$
$$(\text{since } G_{k-1,k} = G_{k-1,-3} = 0)$$
$$= \left[\sum_{l=-2}^{k-1} G_{k-1,l} \right] (E - \Theta_k). \qquad (31)$$

Also, from (30), we have $\sum_{l=-2}^{1} G_{1,l} = (E - \Phi)(E - \Delta)(E - \Theta_1)$. Hence, we get the following result combining the above results:

$$\sum_{l=-2}^{k} G_{k,l} = (E - \Phi)(E - \Delta) \prod_{n=1}^{k} (E - \Theta_n). \qquad (32)$$