Diffusion Approximation as a Modelling Tool

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Abstract. Diffusion theory is already a vast domain of knowledge. This tutorial lecture does not cover all results; it presents in a coherent way an approach we have adopted and used in analysis of a series of models concerning evolution of some traffic control mechanisms in computer, especially ATM, networks. Diffusion approximation is presented from engineer's point of view, stressing its utility and commenting numerical problems of its implementation. Diffusion approximation is a method to model the behavior of a single queueing station or a network of stations. It allows one to include in the model general sevice times, general (also correlated) input streams and to investigate transient states, which, in presence of bursty streams (e.g. of multimedia transfers) in modern networks, are of interest.

Keywords: diffusion approximation, queueing models, performance evaluation, transient analysis.

1 Single G/G/1 Station

1.1 Preliminaries

Let $A(x)$, $B(x)$ denote the interarrival and service time distributions at a service station. The distributions are general but not specified, the method requires only their two first moments. The means are: $E[A] = 1/\lambda$, $E[B] = 1/\mu$ and variances are $\text{Var}[A] = \sigma_A^2$, $\text{Var}[B] = \sigma_B^2$. Denote also squared coefficients of variation $C_A^2 = \sigma_A^2\lambda^2$, $C_B^2 = \sigma_B^2\mu^2$. $N(t)$ represents the number of customers present in the system at time t.

Define

$$\tau_k = \sum_{i=1}^{K} a_i,$$

where a_i are time intervals between arrivals. We assume that they are independent and indentically distributed random variables, hence, according to the central limit theorem, distribution of a variable

$$\frac{T_k - k\,\lambda}{\sigma_A\sqrt{k}}$$

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tends to standard normal distribution as $k \to \infty$:

$$\lim_{k\to\infty} P[\frac{T_k - k\,\lambda}{\sigma_A \sqrt{k}} \le x] = \Phi(x), \quad \text{where} \quad \Phi(x) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\, d\xi.$$

hence for a large k: $P[\tau_k \le x\sigma_A\sqrt{k} + k/\lambda] \approx \Phi(x)$. Denote $t = x\sigma_A\sqrt{k} + k/\lambda$, or $k = t\lambda - x\sigma_A\sqrt{k}\lambda$ and for large values of k, $k \approx t\lambda$ or $\sqrt{k} \approx \sqrt{t\lambda}$. Denote by $H(t)$ the number of customers arriving to the station during a time t; note that $P[H(t) \ge k] = P[\tau_k \le t]$, hence

$$\begin{aligned}\Phi(x) &\approx P[\tau_k \le x\sigma_A\sqrt{k} + k/\lambda] = P[H(t) \ge k] = \\ &= P[H(t) \ge t\lambda - x\sigma_A\sqrt{t\lambda}\,\lambda] = P\left[\frac{H(t) - t\lambda}{\sigma_A\sqrt{t\lambda}\,\lambda} \ge -x\right]\end{aligned}$$

As for the normal distribution $\Phi(x) = 1 - \Phi(-x)$, then $P[\xi \ge -x] = P[\xi \le x]$, and

$$P\left[\frac{H(t) - t\lambda}{\sigma_A\sqrt{t\lambda}\,\lambda} \le x\right] \approx \Phi(x)\,,$$

that means that the number of customers arriving at the interval of length t (sufficiently long to assure large k) may be approximated by the normal distribution with mean λt and variance $\sigma_A^2\lambda^3 t$. Similarly, the number of customers served in this time is approximately normally distributed with mean μt and variance $\sigma_B^2\mu^3 t$, provided that the server is busy all the time. Consequently, the changes of $N(t)$ within interval $[0, t]$, $N(t) - N(0)$, have approximately normal distribution with mean $(\lambda - \mu)t$ and variance $(\sigma_A^2\lambda^3 + \sigma_B^2\mu^3)t$.

Diffusion approximation [54,55] replaces the process $N(t)$ by a continuous diffusion process $X(t)$ the incremental changes of wich $dX(t) = X(t+dt) - X(t)$ are normally distributed with the mean βdt and variance αdt, where β, α are coefficients of the diffusion equation

$$\frac{\partial f(x,t;x_0)}{\partial t} = \frac{\alpha}{2}\frac{\partial^2 f(x,t;x_0)}{\partial x^2} - \beta\frac{\partial f(x,t;x_0)}{\partial x} \tag{1}$$

which defines the conditional pdf $f(x,t;x_0) = P[x \le X(t) < x + dx \mid X(0) = x_0]$ of $X(t)$. The both processes $X(t)$ and $N(t)$ have normally distributed changes; the choice $\beta = \lambda - \mu$, $\alpha = \sigma_A^2\lambda^3 + \sigma_B^2\mu^3 = C_A^2\lambda + C_B^2\mu$ ensures the same ratio of time-growth of mean and variance of these distributions.

More formal justification of diffusion approximation is in limit theorems for $G/G/1$ system given by Iglehart and Whitte [42,43]. If $\hat{N}_n$ is a series of random variables deriven from $N(t)$:

$$\hat{N}_n = \frac{N(nt) - (\lambda - \mu)nt}{(\sigma_A^2\lambda^3 + \sigma_B^2\mu^3)\sqrt{n}}\,,$$

then the series is weakly convergent (in the sense of distribution) to ξ where $\xi(t)$ is a standard Wiener process (i.e. diffusion process with $\beta = 0$ i $\alpha = 1$) provided that $\varrho > 1$, that means if the system is overloaded and has no equilibrium state. In the case of $\varrho = 1$ the series $\hat{N}_n$ is convergent to ξ_R. The $\xi_R(t)$ pocess is $\xi(t)$ process limited to half-axis $x > 0$:

$$\xi_R(t) = \xi(t) - \inf\left[\xi(u), \ \ 0 \le u \le t\right].$$

Service station with $\varrho \ge 1$ does not attein steady-state the number of customers is linearily growing, with fluctuatins around this deterministic trend. For service stations in equilibrium, with $\varrho < 1$, there is no similar theorems and we should rely on heuristic confirmation of the utility of this approximation.

The process $N(t)$ is never negative, hence $X(t)$ should be also restrained to $x \ge 0$. A simple solution is to put a *reflecting barrier* at $x = 0$, [48,49]. In this case

$$\int_0^\infty f(x,t;x_0)dx = 1\,, \quad \text{and} \quad \frac{\partial}{\partial t}\int_0^\infty f(x,t;x_0)dx = \int_0^\infty \frac{\partial f(x,t;x_0)}{\partial t}dx = 0\,;$$

replacing the integrated function with the right side of the diffusion equation we get the boundary condition corresponding to the reflecting barrier at zero:

$$\lim_{x\to 0}\left[\frac{\alpha}{2}\frac{\partial f(x,t;x_0)}{\partial x} - \beta f(x,t;x_0)\right] = 0\,. \tag{2}$$

The solution of Eq. (1) with conditions (2) is [48]

$$f(x,t;x_0) = \frac{\partial}{\partial x}\left[\Phi\left(\frac{x - x_0 - \beta t}{\alpha t}\right) - \mathrm{e}^{\frac{2\beta x}{\alpha}}\,\Phi\left(\frac{x + x_0 + \beta t}{\alpha t}\right)\right] \tag{3}$$

If the station is not overloaded, $\varrho < 1$ ($\beta < 0$), then steady-state exists. The density function does not depend on time: $\lim_{t\to\infty} f(x,t;x_0) = f(x)$, and partial differential equation (1) becoms an ordinary one:

$$0 = \frac{\alpha}{2}\frac{d^2 f(x)}{dx^2} - \beta\frac{d\,f(x)}{dx} \qquad \text{with solution} \qquad f(x) = -\frac{2\beta}{\alpha}\mathrm{e}^{\frac{2\beta x}{\alpha}}\,. \tag{4}$$

This solution approximates the queue at $G/G/1$ system:

$$p(n,t;n_0) \approx f(n,t;n_0),$$

and at steady-state $p(n) \approx f(n)$; one can also choose e.g. $p(0) \approx \int_0^{0.5} f(x)dx$, $p(n) \approx \int_{n-0.5}^{n+0.5} f(x)dx$, $n = 1, 2, \ldots$, [48].

The reflecting barrier excludes the stay at zero: the process is immediately reflected; Eqs. (3),(4) hold for $x > 0$ and $f(0) = 0$. Therefore this version of diffusion with reflecting barrier is a heavy-load approximation: it gives reasonable results if the utilisation ϱ of the investigated station is close to 1, i.e. probability $p(0)$ of the empty system is negligable. The errors are larger for small values of x as the mechanism of reflecting barrier does not correspond to the behaviour of a service station; some improvement may be achieved by renormalisation or [48] by shifting the barrier to $x = -C_B^2$ (for $C_B^2 \le 1$), [29].

This inconvenience may be removed by introduction of another limit condition at $x = 0$: *a barrier with instantaneous (elementary) jumps* [32]. When the diffusion process comes to $x = 0$, it remains there for a time exponentially distributed with a parameter λ_0 and then it returns to $x = 1$. The time when the process is at $x = 0$ corresponds to the idle time of the system. The choice of $\lambda_0 = \lambda$ is exact for Poisson input stream and approximate otherwise. Diffusion equation becomes

$$\begin{aligned} \frac{\partial f(x,t;x_0)}{\partial t} &= \frac{\alpha}{2}\frac{\partial^2 f(x,t;x_0)}{\partial x^2} - \beta\frac{\partial f(x,t;x_0)}{\partial x} + \lambda p_0(t)\delta(x-1)\ , \\ \frac{dp_0(t)}{dt} &= \lim_{x\to 0}\ [\frac{\alpha}{2}\frac{\partial f(x,t;x_0)}{\partial x} - \beta f(x,t;x_0)] - \lambda p_0(t)\ , \end{aligned} \quad (5)$$

where $p_0(t) = P[X(t) = 0]$. The term $\lambda p_0(t)\delta(x-1)$ gives the probability density that the process is started at point $x = 1$ at the moment t because of the jump from the barrier. The second equation makes balance of the $p_0(t)$: the term $\lim_{x\to 0}\ [\frac{\alpha}{2}\frac{\partial f(x,t;x_0)}{\partial x} - \beta f(x,t;x_0)]$ gives the probability flow *into* the barrier and the term $\lambda p_0(t)$ represents the probability flow *out* of the barrier.

1.2 Steady State Solution

In stationary state, when $\lim_{t\to\infty} p_0(t) = p_0$, $\lim_{t\to\infty} f(x,t;x_0) = f(x)$, Eq.(5) becomes ordinary differential and its solution, if $\varrho = \lambda/\mu \neq 1$, may be expressed as:

$$f(x) = \begin{cases} \dfrac{\lambda p_0}{-\beta}(1 - e^{zx}) & \text{for} \quad 0 < x \le 1, \\ \dfrac{\lambda p_0}{-\beta}(e^{-z} - 1)e^{zx} & \text{for} \quad x \ge 1\ , \qquad z = \frac{2\beta}{\alpha}\ . \end{cases} \quad (6)$$

We get p_0 from normalisation: $p_0 = 1 - \varrho$, i.e. the exact result. The mean queue length

$$\begin{aligned} E[N] \approx \int_0^\infty xf(x)dx &= \frac{\lambda p_0}{-\beta}\left(\int_0^1 x(1-e^{zx})dx + \int_1^\infty x(e^{-z}-1)e^{zx}dx\right) = \\ &= \frac{\lambda p_0}{-\beta}\left(0.5 - \frac{1}{z}\right) = \left[0.5 + \frac{C_A^2\varrho + C_B^2}{2(1-\varrho)}\right]\varrho\ . \end{aligned} \quad (7)$$

if we take $p(n) = \int_{n-1}^{n} f(x)dx\ , \quad n = 1, 2, 3, \ldots$. then

$$E[N] = \left[1 + \frac{C_A^2\varrho + C_B^2}{2(1-\varrho)}\right]\varrho\ . \quad (8)$$

The solution (8) gives better results then (7) for small values of C_A^2, C_B^2 and small ϱ. The first discussion of errors, which are growing with C_A^2, C_B^2, was presented in [57].

1.3 Transient Solution

Consider a diffusion process with an absorbing barrier (absorbing barrier means that the process is finished when it attains the barrier) at $x = 0$, started at $t = 0$ from $x = x_0$. Its probability density function $\phi(x,t;x_0)$ has the following form, see e.g. [3]

$$\phi(x,t;x_0) = \frac{e^{\frac{\beta}{\alpha}(x-x_0) - \frac{\beta^2}{2\alpha}t}}{\sqrt{2\Pi\alpha t}}\left[e^{-\frac{(x-x_0)^2}{2\alpha t}} - e^{-\frac{(x+x_0)^2}{2\alpha t}}\right]\ . \quad (9)$$

The density function of first passage time from $x = x_0$ to $x = 0$ is

$$\gamma_{x_0,0}(t) = \lim_{x\to 0}[\frac{\alpha}{2}\frac{\partial}{\partial x}\phi(x,t;x_0) - \beta\phi(x,t;x_0)] = \frac{x_0}{\sqrt{2\Pi\alpha t^3}}e^{-\frac{(\beta t+1)^2}{2\alpha t}} . \tag{10}$$

Suppose that the process starts at $t = 0$ at a point x with density $\psi(x)$ and evry time when it comes to the barrier it stays there for a time given by a density function $l_0(x)$ and then reappears at $x = 1$. The total stream $\gamma_0(t)$ of mass probability that enters the barrier is

$$\gamma_0(t) = p_0(0)\delta(t) + [1 - p_0(0)]\gamma_{\psi,0}(t) + \int_0^t g_1(\tau)\gamma_{1,0}(t-\tau)d\tau \tag{11}$$

where

$$\gamma_{\psi,0}(t) = \int_0^\infty \gamma_{\xi,0}(t)\psi(\xi)d\xi , \qquad g_1(\tau) = \int_0^\tau \gamma_0(t)l_0(\tau - t)dt . \tag{12}$$

The density function of the diffusion process with instantaneous returns is

$$f(x,t;x_0) = \phi(x,t;\psi) + \int_0^t g_1(\tau)\phi(x,t-\tau;1)d\tau . \tag{13}$$

When Laplace transforms of these equations are considered, we have

$$\begin{aligned} \bar\gamma_0(s) &= p_0(0) + [1 - p_0(0)]\bar\gamma_{\psi,0}(s) + \bar g_1(s)\bar\gamma_{1,0}(s) , \\ \bar g_1(s) &= \bar\gamma_0(s)\bar l_0(s) \end{aligned} \tag{14}$$

where

$$\bar\gamma_{x_0,0}(s) = e^{-x_0\frac{\beta + A(s)}{\alpha}} , \qquad \bar\gamma_{\psi,0}(s) = \int_0^\infty \bar\gamma_{\xi,0}(s)\psi(\xi)d\xi ,$$

and then

$$\bar g_1(s) = \Big[p_0(0) + [1 - p_0(0)]\bar\gamma_{\psi,0}(s)\Big]\frac{\bar l_0(s)}{1 - \bar l_0(s)\bar\gamma_{1,0}(s)} . \tag{15}$$

Equation (13) in terms of Laplace transform becomes

$$\bar f(x,s;x_0) = \bar\phi(x,s;\psi) + \bar g_1(s)\bar\phi(x,s;1) ,$$

where

$$\bar\phi(x,s;x_0) = \frac{\mathrm{e}^{\frac{\beta(x-x_0)}{\alpha}}}{A(s)}\left[\mathrm{e}^{-|x-x_0|\frac{A(s)}{\alpha}} - \mathrm{e}^{-|x+x_0|\frac{A(s)}{\alpha}}\right] , \tag{16}$$

$$\bar\phi(x,s;\psi) = \int_0^\infty \bar\phi(x,s;\xi)\psi(\xi)d\xi , \qquad A(s) = \sqrt{\beta^2 + 2\alpha s} . \tag{17}$$

This approach was proposed in [7]. The inverse transforms of these functions could only be numerical and they may be obtained with the use of an inversion algorithm; we have used for this purpose the Stehfest's algorithm [59]. In this algorithm a function $f(t)$ is obtained from its transform $\bar{f}(s)$ for any fixed argument t as

$$f(t) = \frac{\ln 2}{2} \sum_{i=1}^{N} V_i \, \bar{f}\left(\frac{\ln 2}{t} i\right) , \tag{18}$$

where

$$V_i = (-1)^{N/2+i} \sum_{k=\lfloor \frac{i+1}{2} \rfloor}^{\min(i,N/2)} \frac{k^{N/2+1}(2k)!}{(N/2-k)!k!(k-1)!(i-k)!(2k-i)!} . \tag{19}$$

N is an even integer end depends on a computer precision; we used $N = 12 - 40$.

Fig. 1a presents, for a certain t, a comparison of diffusion and exact results, known in case of $M/M/1$ station and expressed by a series of Bessel functions, e.g. [47]. If the diffusion results are below a certain level, the values of the diffusion density are automatically set to zero because of numerical errors of the applied Laplace inversion. The above transient solution of $G/G/1$ model assumes that parameters of the model are constant. If they are changing we should define the time periods where they are constant and solve diffusion equation within this intervals separately, transient solution obtained at the end of one serves as the initial condition for the next interval – see Fig. 1b. The curves "diffusion1", "diffusion2" correspond to mean queues computed with the use of $p(n,t) = f(n,t)$ and formulas (8).

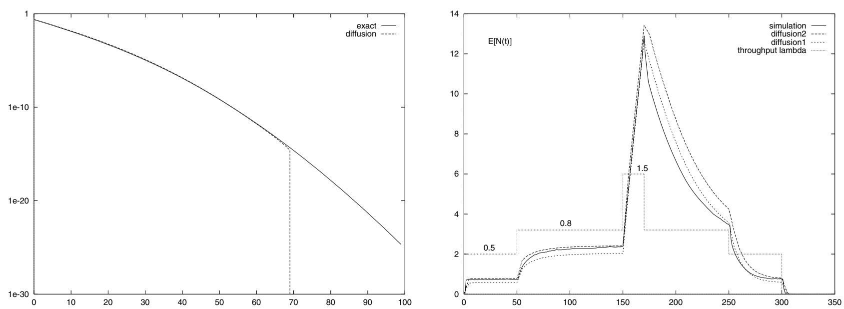

Fig. 1. (a) Exact distribution of M/M/1 queue, expressed by Bessel functions, and its diffusion approximation; $t = 70$, $\lambda = 0.75$, $\mu = 1$, queue is empty at the beginning; (b) The mean queue length following the changes of traffic intensity; diffusion approximation and simulation results; service time constant and equal 1 time unit; Poisson input traffic: M/D/1

1.4 Drawbacks of Stehfest Algorithm

The main drawback of Stehfest algorithm is a very large range of the values taken by the coefficients V_n in Eqs. (18) and (19) for even relatively small N. For instance, they are approximately in the range 10^0–10^{15} for $N = 20$. Fig. 3 presents the error functions shown in log–log–scale, for the shifted Heaviside (unit step) function $f(t) = \mathbf{1}(t-1)$.

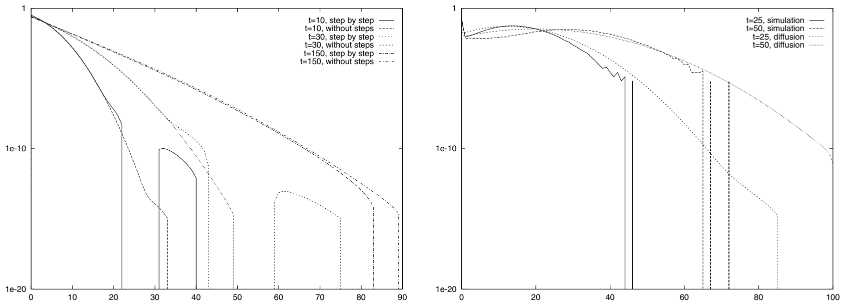

Fig. 2. Left: Comparison of diffusion approximations of the queue distribution at M/M/1 station having parameters $\lambda = 0.75$ and $\mu = 1$; the queue is empty at the beginning; results are obtained at $t = 10, 30, 150$ (a) with the initial condition fixed at $t = 0$ for all t and (b) computed separately for each interval of length $\Delta = 1$ with initial conditions determined at the end of previous interval. Right: Comparison of diffusion and simulation estimations of the queue distribution for $t = 25$, $t = 50$; the source has two phases with intensities $\lambda_1 = 1.6$, $\lambda_2 = 0.4$; the queue is empty at the beginning, computations are done within intervals $\Delta = 1$, simulation has 150 000 repetitions.

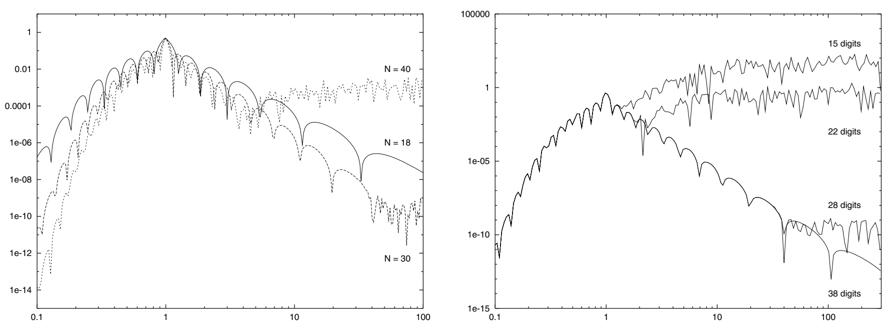

Fig. 3. Dependence of error function on value of N in Stehfest formula and on precision of numbers used for calculations

Its Laplace transform e^{-s}/s is inverted with Stehfest algorithm and compared with the original function. Because of a discontinuity in $t = 1$, the error in this point must be equal to 0.5. Fig. 3a shows the absolute error with respect to the summation limit N of Eq. (18). It is visible that for small values of N the error is relatively small and it achieves the theoretical value of 0.5 for $t = 1$. We can also notice that for $t < 1$ the absolute error is decreasing with the increase of N. The most important fact is that big values of both N and t cause totally erroneous output of the algorithm. Fig. 3b shows the influence of the used precision on the error function for the same example. A relatively high value of $N = 30$ is chosen to show the advantage of using larger mantissa. The increase of N makes narrower the interval of accurate calculations for t. The use of greater precision practically doesn't influence error function in usable range of t.

The errors of inversion algorithm are especially visible when the time axis is composed of small intervals and at each interval the density function is obtained from the results of previous interval (hence the errors in all former intervals influence the current results), see Fig. 2a. The transient queue distributions of the same service station but with the input stream of *on-off* type, given by diffusion and simulation are compared in Fig. 2b. The *on* and *off* phases have exponential distribution and their mean values are equal 100 time units. Although the simulation results were obtained by averaging 150 000 realizations of the random process, the tail of distribution where the probabilities have small values, was not accessible to simulation.

2 G/G/1/N Station

In the case of a queueue limited to N positions, the second barrier of the same type as at $x = 0$ is placed at $x = N$. The model equations become [32]

$$\begin{aligned}
\frac{\partial f(x,t;x_0)}{\partial t} &= \frac{\alpha}{2}\frac{\partial^2 f(x,t;x_0)}{\partial x^2} - \beta\frac{\partial f(x,t;x_0)}{\partial x} + \\
&\quad +\lambda_0 p_0(t)\delta(x-1) + \lambda_N p_N(t)\delta(x-N+1)\ , \\
\frac{dp_0(t)}{dt} &= \lim_{x\to 0}\,[\frac{\alpha}{2}\frac{\partial f(x,t;x_0)}{\partial x} - \beta f(x,t;x_0)] - \lambda_0 p_0(t)\ , \\
\frac{dp_N(t)}{dt} &= \lim_{x\to N}\,[-\frac{\alpha}{2}\frac{\partial f(x,t;x_0)}{\partial x} + \beta f(x,t;x_0)] - \lambda_N p_N(t)\ ,
\end{aligned} \tag{20}$$

where $\delta(x)$ is Dirac delta function.

2.1 Steady State

In stationary state, when $\lim_{t\to\infty} p_0(t) = p_0$, $\lim_{t\to\infty} p_N(t) = p_N$, $\lim_{t\to\infty} f(x,t;x_0) = f(x)$, Eqs.(20) become ordinary differential ones and their solution, if $\varrho = \lambda/\mu \neq 1$, may be expressed as:

$$f(x) = \begin{cases} \dfrac{\lambda p_0}{-\beta}(1-e^{zx}) & \text{for} \quad 0 < x \le 1\ , \\ \dfrac{\lambda p_0}{-\beta}(e^{-z}-1)e^{zx} & \text{for} \quad 1 \le x \le N-1\ , \\ \dfrac{\mu p_N}{-\beta}(e^{z(x-N)}-1) & \text{for} \quad N-1 \le x < N\ , \end{cases} \tag{21}$$

where $z = \frac{2\beta}{\alpha}$ and p_0, p_N are determined through normalization

$$p_0 = \lim_{t\to\infty} p_0(t) = \{1 + \varrho e^{z(N-1)} + \frac{\varrho}{1-\varrho}[1-e^{z(N-1)}]\}^{-1}\ , \tag{22}$$

$$p_N = \lim_{t\to\infty} p_N(t) = \varrho p_0 e^{z(N-1)}\ . \tag{23}$$

The steady-state solution does not depend on the distributions of the sojourn times in boundaries but only on their first moments.

Classes. We follow [33]. When the input stream λ is composed of K classes of customers and $\lambda = \sum_{k=1}^{K} \lambda^{(k)}$ (all parameters concerning class k have an upper index with brackets) then the joint service time pdf is defined as

$$b(x) = \sum_{k=1}^{K} \frac{\lambda^{(k)}}{\lambda} b^{(k)}(x) ,$$

hence

$$\frac{1}{\mu} = \sum_{k=1}^{K} \frac{\lambda^{(k)}}{\lambda} \frac{1}{\mu^{(k)}} , \qquad \text{and} \qquad C_B^2 = \mu^2 \sum_{k=1}^{K} \frac{\lambda_{(k)}}{\lambda} \frac{1}{\mu_{(k)}^2} ({C_B^{(k)}}^2 + 1) - 1 . \tag{24}$$

We assume that the input streams of different class customers are mutually independent, the number of class k customers that arrived within sufficiently long time period is normally distributed with variance $\lambda^{(k)} {C_A^{(k)}}^2$; the sum of independent randomly distributed variables has also normal distribution with variance which is the sum of composant variances, hence

$$C_A^2 = \sum_{k=1}^{K} \frac{\lambda^{(k)}}{\lambda} {C_A^{(k)}}^2 . \tag{25}$$

The above parameters yield α, β of the diffusion equation; function $f(x)$ approximates the distribution $p(n)$ of customers of all classes present in the queue: $p(n) \approx f(n)$ and the probability that there are $n^{(k)}$ customers of class k is

$$p_k(n^{(k)}) = \sum_{n=n^{(k)}}^{N} \left[p(n) \binom{n}{n^{(k)}} \left(\frac{\lambda^{(k)}}{\lambda} \right)^{n^{(k)}} \left(1 - \frac{\lambda^{(k)}}{\lambda} \right)^{n-n^{(k)}} \right] , \quad k = 1, \ldots, K . \tag{26}$$

2.2 G/G/1/N, Transient Solution

The approach presented for $G/G/1$ station may be also used in case of two barriers with instantaneous returns, [7]. Consider a diffusion process with two absorbing barriers at $x = 0$ and $x = N$, started at $t = 0$ from $x = x_0$. Its probability density function $\phi(x, t; x_0)$ has the following form cf. [3]

$$\phi(x, t; x_0) = \begin{cases} \delta(x - x_0) & \text{for } t = 0 \\ \frac{1}{\sqrt{2\Pi \alpha t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left[\frac{\beta x'_n}{\alpha} - \frac{(x - x_0 - x'_n - \beta t)^2}{2\alpha t} \right] \right. & \\ \qquad \left. - \exp\left[\frac{\beta x''_n}{\alpha} - \frac{(x - x_0 - x''_n - \beta t)^2}{2\alpha t} \right] \right\} & \text{for } t > 0 , \end{cases} \tag{27}$$

where $x'_n = 2nN$, $x''_n = -2x_0 - x'_n$.

If the initial condition is defined by a function $\psi(x)$, $x \in (0, N)$, $\lim_{x\to 0}\psi(x) = \lim_{x\to N}\psi(x) = 0$, then the pdf of the process has the form $\phi(x, t; \psi) = \int_0^N \phi(x, t; \xi)\psi(\xi)d\xi$. The Laplace transform of $\phi(x, t; x_0)$ can be expressed as

$$\bar{\phi}(x, s; x_0) = \frac{\exp[\frac{\beta(x-x_0)}{\alpha}]}{A(s)} \sum_{n=-\infty}^{\infty} \left\{ \exp\left[-\frac{|x - x_0 - x'_n|}{\alpha} A(s)\right] - \exp\left[-\frac{|x - x_0 - x''_n|}{\alpha} A(s)\right] \right\} , \tag{28}$$

where $A(s) = \sqrt{\beta^2 + 2\alpha s}$. For computational efficiency we rearranged the Eq. (28) to the form

$$\bar{\phi}(x, s; x_0) = \frac{\exp[\frac{\beta(x-x_0)}{\alpha}]}{A(s)} \Big\{ \mathbf{1}_{(x\geq x_0)} \left[\exp\left(-\frac{xA(s)}{\alpha}\right) 2\sinh\left(\frac{x_0 A(s)}{\alpha}\right)\right] + \mathbf{1}_{(x_0<x)} \left[\exp\left(-\frac{x_0 A(s)}{\alpha}\right) 2\sinh\left(\frac{xA(s)}{\alpha}\right)\right] - 2\sinh\left(\frac{xA(s)}{\alpha}\right) 2\sinh\left(\frac{x_0 A(s)}{\alpha}\right) \times \sum_{n=1}^{\infty} \exp\left(-2nN\frac{A(s)}{\alpha}\right) \Big\} \tag{29}$$

Similarly, $\bar{\phi}(x, s; \psi) = \int_0^N \bar{\phi}(x, s; \xi)\psi(\xi)d\xi$.

The probability density function $f(x, t; \psi)$ of the diffusion process with elementary returns is composed of the function $\phi(x, t; \psi)$ which represents the influence of the initial conditions and of a spectrum of functions $\phi(x, t-\tau; 1)$, $\phi(x, t-\tau; N-1)$ which are pd functions of diffusion processes with absorbing barriers at $x = 0$ and $x = N$, started at time $\tau < t$ at points $x = 1$ and $x = N - 1$ with densities $g_1(\tau)$ and $g_{N-1}(\tau)$:

$$f(x, t; \psi) = \phi(x, t; \psi) + \int_0^t g_1(\tau)\phi(x, t-\tau; 1)d\tau + \int_0^t g_{N-1}(\tau)\phi(x, t-\tau; N-1)d\tau \ . \tag{30}$$

Densities $\gamma_0(t)$, $\gamma_N(t)$ of probability that at time t the process enters to $x = 0$ or $x = N$ are

$$\gamma_0(t) = p_0(0)\delta(t) + [1 - p_0(0) - p_N(0)]\gamma_{\psi,0}(t) + \int_0^t g_1(\tau)\gamma_{1,0}(t-\tau)d\tau + \int_0^t g_{N-1}(\tau)\gamma_{N-1,0}(t-\tau)d\tau \ ,$$

$$\gamma_N(t) = p_N(0)\delta(t) + [1 - p_0(0) - p_N(0)]\gamma_{\psi,N}(t) + \int_0^t g_1(\tau)\gamma_{1,N}(t-\tau)d\tau + \int_0^t g_{N-1}(\tau)\gamma_{N-1,N}(t-\tau)d\tau \ , \tag{31}$$

where $\gamma_{1,0}(t)$, $\gamma_{1,N}(t)$, $\gamma_{N-1,0}(t)$, $\gamma_{N-1,N}(t)$ are densities of the first passage time between corresponding points, e.g.

$$\gamma_{1,0}(t) = \lim_{x\to 0}[\frac{\alpha}{2}\frac{\partial\phi(x, t; 1)}{\partial x} - \beta\phi(x, t; 1)] \ . \tag{32}$$

For absorbing barriers

$$\lim_{x\to 0} \phi(x,t;x_0) = \lim_{x\to N} \phi(x,t;x_0) = 0 \,,$$

hence $\gamma_{1,0}(t) = \lim_{x\to 0} \frac{\alpha}{2}\frac{\partial\phi(x,t;1)}{\partial x}$. The functions $\gamma_{\psi,0}(t)$, $\gamma_{\psi,N}(t)$ denote densities of probabilities that the initial process, started at $t=0$ at the point ξ with density $\psi(\xi)$ will end at time t by entering respectively $x=0$ or $x=N$.

Finally, we may express $g_1(t)$ and $g_N(t)$ with the use of functions $\gamma_0(t)$ and $\gamma_N(t)$:

$$g_1(\tau) = \int_0^\tau \gamma_0(t) l_0(\tau - t) dt \,, \qquad g_{N-1}(\tau) = \int_0^\tau \gamma_N(t) l_N(\tau - t) dt \,, \tag{33}$$

where $l_0(x)$, $l_N(x)$ are the densities of sojourn times in $x=0$ and $x=N$; the distributions of these times are not restricted to exponantial ones as it is in Eq. (20). Laplace transforms of Eqs. (31,33) give us $\bar g_1(s)$ and $\bar g_{N-1}(s)$:

$$\bar g_1(s) = \left\{ p_0(0) + \bar\gamma_{\psi,0}(s) \ + [p_N(0) + \bar\gamma_{\psi,N}(s)] \frac{\bar l_N(s)\bar\gamma_{N-1,0}(s)}{1 - \bar l_N(s)\bar\gamma_{N-1,N}(s)} \right\} \cdot$$
$$\cdot \frac{\bar l_0(s)}{1 - \bar l_0(s)\bar\gamma_{1,0}(s)} \left[1 - \frac{\bar l_0(s)\bar\gamma_{1,N}(s)}{1 - \bar l_0(s)\bar\gamma_{1,0}(s)} \frac{\bar l_N(s)\bar\gamma_{N-1,0}(s)}{1 - \bar l_N(s)\bar\gamma_{N-1,N}(s)} \right]^{-1} \,,$$

$$\bar g_{N-1}(s) = \frac{\bar l_N(s)}{1 - \bar l_N(s)\bar\gamma_{N-1,N}(s)} \ [p_N(0) + \bar\gamma_{\psi,N}(s) + \bar g_1(s)\bar\gamma_{1,N}(s)]$$

and the Laplace transform of the density function $f(x,t;\psi)$ is obtained as

$$\bar f(x,s;\psi) = \bar\phi(x,s;\psi) + \bar g_1(s)\, \bar\phi(x,s;1) + \bar g_{N-1}(s)\, \bar\phi(x,s;N-1) \,. \tag{34}$$

Probabilities that at the moment t the process has the value $x=0$ or $x=N$ are

$$\bar p_0(s) = \frac{1}{s} \left[\bar\gamma_0(s) - \bar g_1(s) \right] , \qquad \bar p_N(s) = \frac{1}{s} \left[\bar\gamma_N(s) - \bar g_{N-1}(s) \right] . \tag{35}$$

3 State-Dependent Diffusion Parameters, G/G/N/N Transient Model

In the case of $G/G/N/N$ model, the value of the diffusion process corresponds to the number of active channels. The output stream depends on the number of occupied channels, hence the diffusion parameters depend also on the value of the diffusion process: $\alpha = \alpha(x,t)$, $\beta = \beta(x,t)$.

The diffusion interval $x \in [0,N]$ is divided into subintervals of unitary length and the parameters are kept constant within these subintervals. For each time- and space-subinterval with constant parameters, transient diffusion is obtained. The equations for space-intervals are solved together with balance equations for probability flows between neighbouring intervals. The results are obtained in the form of Laplace transforms of density functions of the investigated diffusion process and then inverted numerically.

If $n-1<x<n$, it is supposed that n channels are busy, hence we choose

$$\alpha(x,t)=\lambda(t)C_A^2(t)+n\mu C_B^2, \qquad \beta(x,t)=\lambda(t)-n\mu \quad \text{for} \quad n-1<x<n. \tag{36}$$

Jumps from $x=N$ to $x=N-1$ are performed with density μ.

In transient state we should balance the probability flows between neighbouring intervals with different diffusion parameters. We put imaginary barriers at the borders of these intervals and suppose that the diffusion process entering the barrier at n, $n=1,2,\dots N-1$, from its left side (the process is growing) is absorbed and immediately reappears at $x=n+\varepsilon$. Similarily, the process which is diminishing and enters the barrier from its right side reappears at its other side at $x=n-\varepsilon$.

The density functions for the intervals are as follows:

$$\begin{aligned}
f_1(x,t;\psi_1) &= \phi_1(x,t;\psi_1)+\int_0^t g_1(\tau)\phi_1(x,t-\tau;1)d\tau + \\
&\quad + \int_0^t g_{1-\varepsilon}(\tau)\phi_1(x,t-\tau;1-\varepsilon)d\tau\,, \\
f_n(x,t;\psi_n) &= \phi_n(x,t;\psi_n)+\int_0^t g_{n-1+\varepsilon}(\tau)\phi_n(x,t-\tau;n-1+\varepsilon)d\tau + \\
&\quad + \int_0^t g_{n-\varepsilon}(\tau)\phi_n(x,t-\tau;n-\varepsilon)d\tau, \qquad n=2,\dots N-1, \\
f_N(x,t;\psi_N) &= \phi_N(x,t;\psi_N)+\int_0^t g_{N-1+\varepsilon}(\tau)\phi_N(x,t-\tau;N-1+\varepsilon)d\tau + \\
&\quad + \int_0^t g_{N-1}(\tau)\phi_N(x,t-\tau;N-1)d\tau
\end{aligned} \tag{37}$$

and the probability mass flows entering the barriers are:

$$\gamma_n^R(t)=g_{n-\varepsilon}(t), \qquad \gamma_n^L(t)=g_{n+\varepsilon}(t), \qquad n=1,\dots,N-1 \tag{38}$$

and $g_1(t)$, $g_{N-1}(t)$ are the same as in $G/G/1/N$ model, Eq. (33). The densities $\gamma_{N_i}^R(t)$, $\gamma_{N_i}^L(t)$ are obtained in the similar way as in $G/G/1/N$, see Eq. (31):

$$\begin{aligned}
\gamma_0(t) &= p_0(0)\delta(t)+\gamma_{\psi_1,0}(t)+\int_0^t g_1(\tau)\gamma_{1,0}(t-\tau)d\tau + \\
&\quad + \int_0^t g_{1-\varepsilon}(\tau)\gamma_{1-\varepsilon,0}(t-\tau)d\tau\,, \\
\gamma_1^L(t) &= \gamma_{\psi_1,1}(t)+\int_0^t g_1(\tau)\gamma_{1+\varepsilon,1}(t-\tau)d\tau + \\
&\quad + \int_0^t g_{2-\varepsilon}(\tau)\gamma_{2-\varepsilon,1}(t-\tau)d\tau\,, \\
\gamma_1^R(t) &= \gamma_{\psi_2,1}(t)+\int_0^t g_{1+\varepsilon}(\tau)\gamma_{1+\varepsilon,1}(t-\tau)d\tau +
\end{aligned}$$

$$+\int_0^t g_{2-\varepsilon}(\tau)\gamma_{2-\varepsilon,1}(t-\tau)d\tau \ ,$$

$$\gamma_n^L(t) = \gamma_{\psi_n,n}(t) + \int_0^t g_{n-1+\varepsilon}(\tau)\gamma_{n-1+\varepsilon,n}(t-\tau)d\tau +$$

$$+\int_0^t g_{n-\varepsilon}(\tau)\gamma_{n-\varepsilon,n}(t-\tau)d\tau \ ,$$

$$\gamma_n^R(t) = \gamma_{\psi_{n+1},n}(t) + \int_0^t g_{n+\varepsilon}(\tau)\gamma_{n+\varepsilon,n}(t-\tau)d\tau +$$

$$+\int_0^t g_{n+1-\varepsilon}(\tau)\gamma_{n+1-\varepsilon,n}(t-\tau)d\tau \ , \qquad n = 2, \ldots N-1$$

$$\gamma_N(t) = p_N(0)\delta(t) + \gamma_{\psi_N,N}(t) + \int_0^t g_{N-1+\varepsilon}(\tau)\gamma_{N-1+\varepsilon,N}(t-\tau)d\tau +$$

$$+\int_0^t g_{N-1}(\tau)\gamma_{N-1,N}(t-\tau)d\tau \ , \qquad (39)$$

where $\gamma_{i,j}(t)$ are the densities of the first passage time between points i, j and are obtained as in $G/G/1/N$ model, Eq.(32). This system of equations is subject to Laplace transformation and once again the Laplace transforms $\bar{f}_n(x, s; \psi_n)$ are obtained numerically, for a series of s values needed by the inversion algorithm for a specified t.

4 Open Network of $G/G/1$, G/G/1/N Queues, Steady State

The open networks of $G/G/1$ queues were studied in [33]. Let M be the number of stations and suppose at the beginning that there is one class of customers. The throughput of station i is, as usual, obtained from traffic equations

$$\lambda_i = \lambda_{0i} + \sum_{j=1}^{M} \lambda_j r_{ji} \ , \qquad i = 1, \ldots, M, \qquad (40)$$

where r_{ji} is routing probability between station j and station i; λ_{0i} is external flow of customers coming from outside of network.

Second moment of interarrival time distribution is obtained from two systems of equations; the first defines C_{Di}^2 as a function of C_{Ai}^2 and C_{Bi}^2; the second defines C_{Aj}^2 as another function of $C_{D1}^2, \ldots, C_{DM}^2$:

1) The formula (41) is exact for $M/G/1$, $M/G/1/N$ stations and is approximate in the case of non-Poisson input [2]

$$d_i(t) = \varrho_i b_i(t) + (1-\varrho_i)a_i(t) * b_i(t) \ , \qquad i = 1, \ldots, M, \qquad (41)$$

where * denotes the convolution operation. From (41) we get

$$C_{Di}^2 = \varrho_i^2 C_{Bi}^2 + C_{Ai}^2(1-\varrho_i) + \varrho_i(1-\varrho_i) \ . \qquad (42)$$

2) Customers leaving station i according to the distribution $D_i(x)$ choose station j with probability r_{ij}: intervals between customers passing this way has pdf $d_{ij}(x)$

$$d_{ij}(x) = d_i(x) r_{ij} + d_i(x) * d_i(x)(1 - r_{ij}) r_{ij} + d_i(x) * d_i(x) * d_i(x)(1 - r_{ij})^2 r_{ij} + \cdots \tag{43}$$

or, after Laplace transform,

$$\bar{d}_{ij}(s) = \bar{d}_i(s) r_{ij} + \bar{d}_i(s)^2 (1 - r_{ij}) r_{ij} + \bar{d}_i(s)^3 (1 - r_{ij})^2 r_{ij} + \cdots = \frac{r_{ij} \bar{d}_i(s)}{1 - (1 - r_{ij}) \bar{d}_i(s)},$$

hence

$$E[D_{ij}] = \frac{1}{\lambda_i r_{ij}}, \qquad C^2_{Dij} = r_{ij}(C^2_{Di} - 1) + 1 . \tag{44}$$

$E[D_{ij}]$, C^2_{Dij} refer to interdeparture times; the number of customers passing from station i to j in a time interval t has approximately normal distribution with mean $\lambda_i r_{ij} t$ and variation $C^2_{Dij} \lambda_i r_{ij} t$. The sum of streams entering station j has normal distribution with mean

$$\lambda_j t = [\sum_{i=1}^{M} \lambda_i r_{ij} + \lambda_{0j}]\, t \qquad \text{and variance} \qquad \sigma^2_{Aj} t = \{\sum_{i=1}^{M} C^2_{Dij} \lambda_i r_{ij} + C^2_{0j} \lambda_{0j}\} t ,$$

hence

$$C^2_{Aj} = \frac{1}{\lambda_j} \sum_{i=1}^{M} r_{ij} \lambda_i [(C^2_{Di} - 1) r_{ij} + 1] + \frac{C^2_{0j} \lambda_{0j}}{\lambda_j} . \tag{45}$$

Parameters λ_{0j}, C^2_{0j} represent the external stream of customers. For K classes od customers with routing probabilities $r^{(k)}_{ij}$ (let us assume for simplicity that the customers do not change their classes) we have

$$\lambda^{(k)}_i = \lambda^{(k)}_{0i} + \sum_{j=1}^{M} \lambda^{(k)}_j r^{(k)}_{ji} , \qquad i = 1, \ldots, M; \;\; k = 1, \ldots, K, \tag{46}$$

and

$$C^2_{Di} = \lambda_i \sum_{k=1}^{K} \frac{\lambda^{(k)}_i}{{\mu^{(k)}_i}^2} [{C^{(k)}_{Bi}}^2 + 1] + 2\varrho_i (1 - \varrho_i) + (C^2_{Ai} + 1)(1 - \varrho_i) - 1 . \tag{47}$$

A customer in the stream leaving station i belongs to class k with probability $\lambda^{(k)}_i / \lambda_i$ and we can determine ${C^{(k)}_{Di}}^2$ in the similar way as it has been done in Eqs. (43-44), replacing r_{ij} by $\lambda^{(k)}_i / \lambda_i$:

$${C^{(k)}_{Di}}^2 = \frac{\lambda^{(k)}_i}{\lambda_i} (C^2_{Di} - 1) + 1 ; \tag{48}$$

then

$$C^2_{Aj} = \frac{1}{\lambda_j} \sum_{l=1}^{K} \sum_{k=1}^{K} r^{(k)}_{ij} \lambda_i \left[\left(\frac{\lambda^{(k)}_i}{\lambda_i} (C^2_{Di} - 1) \right) r^{(k)}_{ij} + 1 \right] + \sum_{k=1}^{K} \frac{{C^{(k)}_{0j}}^2 \lambda^{(k)}_{0j}}{\lambda_j} . \tag{49}$$

Eqs. (42), (45) or (47), (49) form a linear system of equations and allow us to determine C^2_{Ai} and, in consequence, parameters β_i, α_i for each station.

5 Open Network of G/G/1, G/G/1/N Queues, One Class, Transient Solution

In the case of one class of customers, the time axis is divided into small intervals (equal e.g to the smallest mean service time) and at the beginning of each interval the equations (40),(42),(45) are used to determine the input parameters of each station based on the values of $\varrho_i(t)$ obtained at the end of the precedent interval, [26]. A software tool was prepared [15,16] and the examples below, see Fig. 4, are computed with its use.

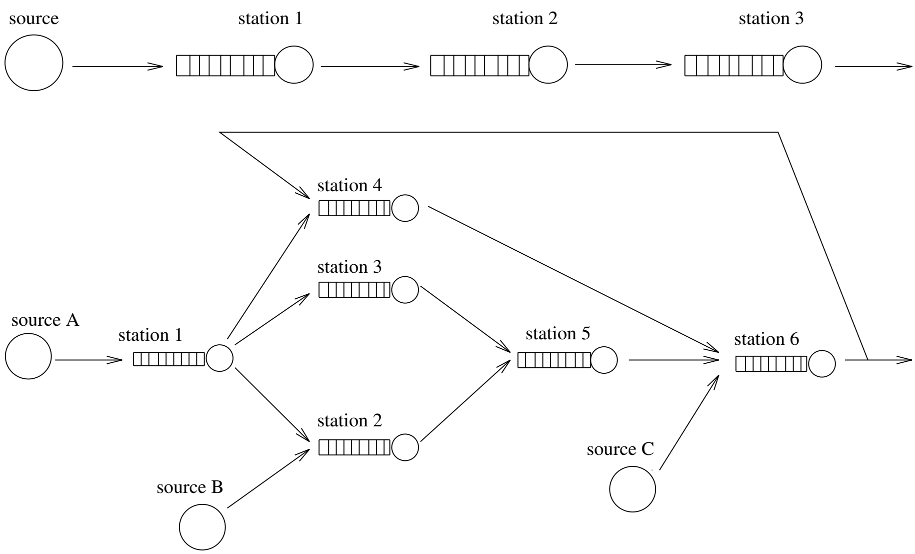

Fig. 4. Models 1 and 2

Model 1. The network is composed of the source and three stations in tandem. The source parameters are: $\lambda = 0.1\ t \in [0, 10]$, $\lambda = 4.0\ t \in [10, 20]$.
Parameters of all stations are the same: $N_i = 10$, $\mu_i = 2$, $C^2_{Bi} = 1$, $i = 1, 2, 3$.

Fig. 5a presents mean queue lengths of stations in Model 1 as a function of time. Diffusion approximation is compared with simulation.

Model 2, its topologu is in Fig. 4. The characteristics of three sources and of one station are changing with time in the following pattern:
source A: $\lambda_A = 0.1$ for $t \in [0, 10]$, $\lambda_A = 4.0$ for $t \in [10, 21]$, $\lambda_A = 0.1$ for $t \in [21, 40]$,
source B: $\lambda_B = 0.1$ for $t \in [0, 11]$, $\lambda B = 4.0$ for $t \in [11, 20]$, $\lambda_B = 0.1$ for $t \in [20, 40]$,

source C: $\lambda_C = 0.1$ for $t \in [0, 15]$, $\lambda_C = 2.0$ for $t \in [15, 22]$, $\lambda_C = 4.0$ for $t \in [22, 30]$, $\lambda_C = 2.0$ for $t \in [30, 31]$, $\lambda_C = 0.1$ for $t \in [31, 40]$.
Station 6: $\mu_6 = 2$ for $t \in [10, 15]$ and $t \in [31, 40]$; $\mu_6 = 4$ for $t \in [15, 31]$.

Other parameters are constant: $N_1 = N_4 = 10$, $N_3 = 5$, $N_2 = N_6 = 20$, $\mu_1 = \cdots = \mu_5 = 2$. Routing probabilities are: $r_{12} = r_{13} = r_{14} = 1/3$, $r_{64} = 0.8$. Initial state: $N_1(0) = 5$, $N_1(0) = 5$, $N_2(0) = 10$, $N_3(0) = 10$, $N_4(0) = 5$, $N_5(0) = 5$, $N_6(0) = 10$. The results in the form of mean queue lengths are presented and compared with simulation in Figs. 5, 6.

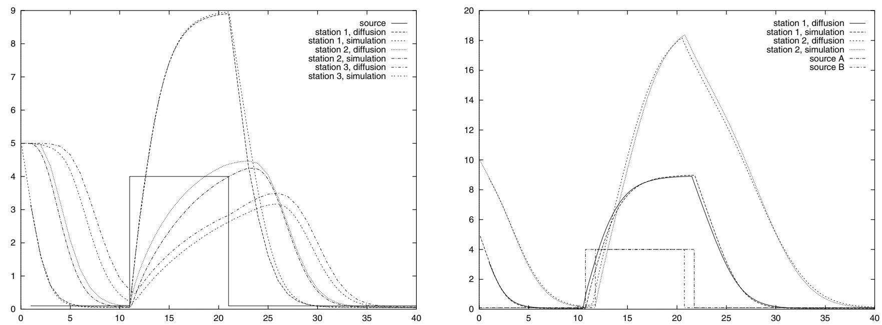

Fig. 5. (a) Model 1: mean queue lengths of station1, station2 and station3 as a function of time — diffusion and simulation (100 000 repetitions) results; the source intensity $\lambda(t)$ is indicated. (b) Model 2: Mean queue lengths of station1 and station2 as a function of time — diffusion and simulation (100 000 repetitions) results; the source intensities $\lambda_A(t)$, $\lambda_B(t)$ are indicated.

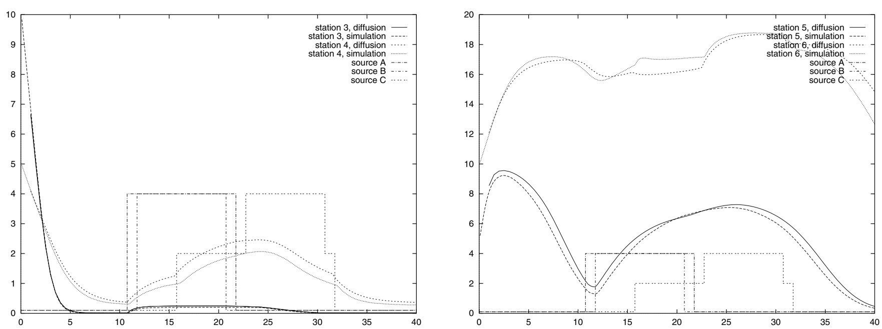

Fig. 6. Model 2: mean queue lengths of station3 and station4 (a) and of station5 and station6 (b)

6 Leaky Bucket Model

In the *leaky bucket* scheme, the cells, before entering the network, must obtain a token. Tokens are generated at constant rate and stocked in a buffer of finite capacity B. If there is a token available, an arriving cell consumes it and leaves the bucket. If not, it

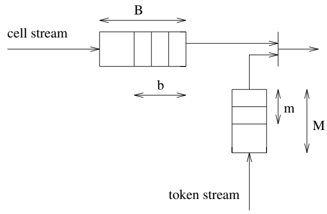

Fig. 7. Leaky bucket scheme

waits for the token in the cell buffer. The capacity of this buffer is also limited to M, Fig. 7. Tokens and cells arriving to full buffers are lost. The diffusion process $X(t)$ is defined on the interval $x \in [0, N = B + M]$ where B is the capacity of cell buffer and M is the capacity of token buffer, [12]. The current value of the process is defined as $x = b - m + M$, b and m being the current contents of the buffers.

Let us suppose that the cell interarrival time distribution has the mean $1/\lambda_c$ and squared coefficient of variation $C_{A_c}^2$. The tokens are generated with constant rate λ_t, hence $C_{A_t}^2 = 0$. Arrival of a cell increases the value of the process and arrival of a token decreases it, therefore we choose the parameters of the diffusion process as:

$$\beta = \lambda_c - \lambda_t , \qquad \alpha = \lambda_c C_{A_c}^2 .$$

The process evolves between two barriers placed at $x = 0$ and at $x = M + B$; $x = 0$ represents the state where the whole token buffer is occupied and arriving tokens are lost; $x = M + B$ represents the state where the token buffer is empty and the cell buffer is full: arriving cells are lost.

The sojourn time at $x = M + B$ corresponds to the residual token interarrival time, i.e. the time between the moment when the cell buffer becomes full and the moment of the next token arrival. In [37] the density of holding time at the upper barrier of $M/D/1/N$ diffusion model was obtained; we follow this approach and assume that the density function $l_{M+B}(x)$ is

$$l_{M+B}(x) = \frac{\lambda_c e^{\lambda_c x}}{e^{\lambda_c/\lambda_t} - 1}. \tag{50}$$

The cell loss ratio $L(t)$ may be bounded by expression [37]

$$L(t) \le p_N(t) Pr[Arr(t, t + 1/\lambda_t) \ge 1]$$

where $Arr(t, t + 1/\lambda_t)$ is the number of cell arrivals during interval $[t, t + 1/\lambda_t]$.

If the cell stream is Poisson, the pdf $l_0(x)$ of the sojourn time at $x = 0$ is defined by the density of cell interarrival time; otherwise we take this density as an approximation of $l_0(x)$. Note that the sojourn times in boundaries are defined here by the densities $l_0(t)$, $l_N(t)$ and *are not restricted to exponential distributions*.

The values $x > M$ of the process correspond to states where cells are waiting for tokens, the value $x - M$ determines in this case the number of cells in the buffer; $x < M$ means that there are tokens waiting for cells and the value $M - x$ corresponds to the number of tokens in the buffer. Probability of b cells in the buffer at time

t is defined by $f(M+b,t)$; probability of the empty cell buffer is given by $p_t(t) = p_0(t) + \int_0^M f(x,t)dx$. Probability of m tokens in the buffer is given by $f(M-m,t)$ and probability of empty token buffer is determined by $\int_M^{M+B} f(x,t)dx + p_N(t)$ where $p_0(t) = Pr[X(t)=0], p_N(t) = Pr[X(t)=N]$.

The service time is constant, hence the density function of the cell waiting time for tokens (response time of leaky bucket) may be estimated as $r(x,t) = \lambda_t f(\lambda_t x + M, t)$.

Hence, using G/G/1/N model we obtain transient $f(x,t;\psi)$ and steady-state $f(x)$ distributions of the diffusion process for $0 \le x \le M+B$. This gives us the distribution of the number of tokens and cells in the leaky bucket, the response time distribution, the loss probabilities, the properties of the output stream, etc. The capacities of cell and token buffers may be null, so we are able to consider a number of leaky bucket variants.

The output process of the leaky bucket is the same as the cell input process provided, with probability $p_t(t)$, that there are tokens available and it is the same as token input process with probability $[1 - p_T 9t)]$ that tokens are not available; at the time moment t the pdf $d(x)$ of interdeparture times in the output stream is

$$d(x,t) = p_t(t)a(x,t) + [1 - p_t(x,t)]\delta(x - \frac{1}{\lambda_t}) \, , \tag{51}$$

where $a(x,t)$ is the time-dependent pdf of cell interarrival times distribution. Eq. (51) gives us the mean value and squared coefficient of interdeparture times distribution, i.e. whole information needed to incorporate one or multiple leaky-bucket stations (for example a cascade of leaky-buckets with different parameters) in the diffusion queueing network model of $G/G/1$ or $G/G/1/N$ stations. The principles of the latter model were given in [33].

Numerical example

At $t=0$ the cell buffer is empty and the token buffer contains $M(0)$ tokens. The tokens are generated regularly each time-unit. The cell arrival stream is Poisson; the mean interarrival time is 0.5 time-unit for $0 \le t < 100$ and 1.5 time units for $t \ge 100$, i.e. there is a traffic wave exceeding the accorded level during the first 100 units and then the traffic goes down below this level.

The buffer capacities are $B = M = 100$. Figure 8a displays the diffusion and simulation results concerning the output stream of leaky bucket for the initial number of tokens $M(0) = 0$, 50 and 100. The output dynamics given by simulation and by difusion model are very similar. Simulation results are obtained as a mean of 100 000 independent runs. If there is no tokens at the beginning, the cell stream is immediately cut to the level of token intensity (one cell per time unit), the excess of cells is stocked in the cell buffer and transmitted later, when $t > 100$ and input rate becomes smaller. If there are tokens in the token buffer, a part (for $M(0) = 50$) or almost totality (for $M(0) = 100$) of the traffic wave may enter into the network.

Figure 8b presents the comparison of mean number of cells in the cell buffer as a function of time, for different initial content of the token buffer $M(0) = 0$, 50 and 100, obtained by diffusion and simulation models. In Figure 9 the distributions of cell buffer contents obtained by simulation and by diffusion are presented for several moments, including $t = 100$, i.e. the end of high traffic period, when the congestion is the biggest. We see that although the the buffer is full is important (≈ 0.4). Note that we could mean

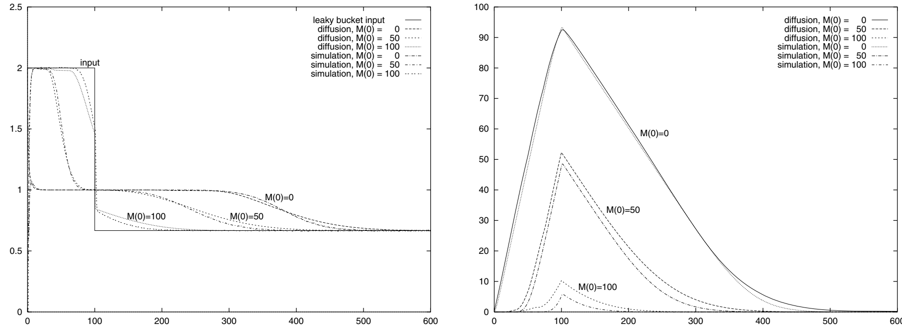

Fig. 8. (a) The input and output of leaky bucket as a function of time — the stream intensities for the initial number of tokens $M(0) = 0$, 50 and 100; diffusion and simulation results (b) Mean number of cells in the cell buffer as a function of time, $M(0) = 0$, 50 and 100; diffusion and simulation results

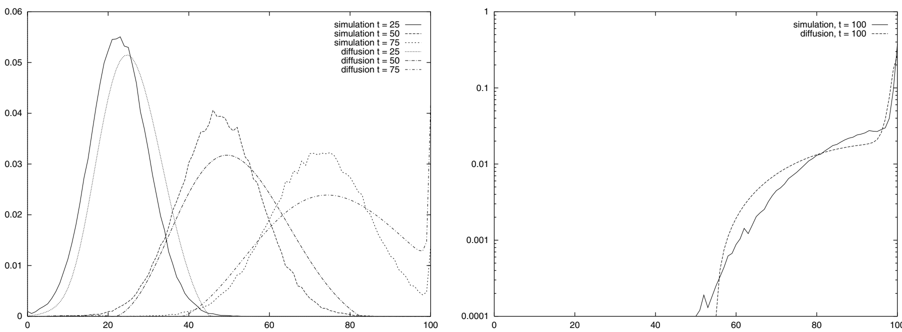

Fig. 9. Density of the number of cells during high source activity period, $t = 25, 50, 75$ (a), $t = 100$ (b); $M(0) = 0$; diffusion and simulation results

queue length is below the buffer capacity, the probability that not obtain this result with the use of fluid approximation even if the mean number of cells in the buffer predicted by diffusion and fluid approximations were similar.

7 Multiclass FIFO Queues, Output Streams Dynamics

Consider a queueing network model representing a computer network. Customers (packets of fixed size) are grouped into classes. Each class represents a connection between two points of the network and its description includes the features of the source and the itinerary across the network. Customer queues at servers correspond to the queues of cells waiting at a node to be sent further. At a queue exit the class of a customer should be known to determine its routing. In steady state queuing models this probability, for a certain class k, is given by the ratio of the throughput $\lambda^{(k)}$ of this class customers passing the node to the total througput λ of this node : $\lambda^{(k)}/\lambda$, where $\lambda = \sum_{k=1}^{K} \lambda^{(k)}$ and

K is the number of classes passing across the node. In transient state, the throughputs are a function of time and the probability $\lambda^{(k)}(t)/\lambda(t)$ as well. Morover, the flows at the entrance and the exit are not the same: $\lambda^{(k)}_{i,\,in} \neq \lambda^{(k)}_{i,\,out}$. The composition of output flow reflects the previous compositions at the entrance delayed by the response time of the queue.

To solve this problem, we choose the constant service time as the time unit, divide the time axis into intervals of unitary length and assume that the flow parameters are constant during that interval, e.g. $\lambda^{(k)}(\theta)$ denotes the class k flow at station i during an interval θ, $\theta = 1, 2, \ldots$

The input stream $\sum_k \lambda^{(k)}_{in}(\theta)$ reaches the output of the queue with a delay corresponding to the queue length in the buffer at the arrival time. The part $p(n,\theta)\sum_k \lambda^{(k)}_{i,\,in}(\theta)$ of this submitted load cannot be served and the corresponding flow cannot appear at the output before the time $t = \theta + n + 1$. So, taking into account these delays, the unfinished work ready to be processed at the time t in the station which is initially empty can be expressed as:

$$U^{(k)}(t) = \sum_{n=1}^{N} \lambda^{(k)}_{in}(t-n) \cdot p(n-1, t-n) \quad \text{for} \quad k = 1, \ldots, K.$$

A similar formulation may be easily derived for initially nonempty queue knowing its composition at $t = 0$. Remark that some accumulation periods (of high or quickly increasing load) may yield that ready work at t exceeds the server capacity: $\sum_k U^{(k)}(t) > 1$. Such a phenomenon introduces some additional delay in the transfer of the input stream to the output.

To compute the output throughput $\lambda^{(k)}_{out}(t)$, we find first the *ready time* ϑ_1 of the cells at the head of the queue. This is equal to the smallest value of θ such that:

$$\sum_{\tau=0}^{\theta} \sum_k U^{(k)}(\tau) \geq \sum_{\tau=0}^{t-1} \sum_k \lambda^{(k)}_{out}(\tau). \tag{52}$$

Then, we determine the smallest ϑ_2 for which

$$\sum_{\tau=0}^{\vartheta_2} \sum_k U^{(k)}(\tau) - \sum_{\tau=0}^{t-1} \sum_k \lambda^{(k)}_{out}(\tau) \geq 1 - p(0,t).$$

The output throughput $\lambda^{(k)}_{out}(t)$ is obtained as

$$\lambda^{(k)}_{out}(t) = \sum_{\theta=\vartheta_1}^{\vartheta_2} w_\theta \cdot U^{(k)}(\theta) \tag{53}$$

where w_{ϑ_1} represents the percentage of $U^{(k)}(\vartheta_1)$ that has not been sent yet, $w_\theta = 1$ for $\theta \neq \vartheta_1$ and ϑ_2 , and w_{ϑ_2} is chosen such that $\sum_{\theta=\vartheta_1}^{\vartheta_2} w_\theta \cdot U(\theta) = 1 - p(0,t)$.

Numerical example. Consider a switch having 3 input streams end one output being the sum of the three input streams. The output queue has the capacity of 100 packets and is analysed with the use of $G/D/1/100$ multiclass diffusion model and $M/D/1$ multiclass fluid approximation model. The service time is constant, $\mu = 1$. The input streams are Poisson with parameters $\lambda^{(k)}(t)$ chosen as:

time t	0 – 10	10 – 20	20 – 30	30 – 40	40 – 80	> 80
$\lambda_{in}^{(1)}(t)$	0.1	0.8	0.8	0.1	0.1	0
$\lambda_{in}^{(2)}(t)$	0.2	0.2	0.2	0.2	0.2	0
$\lambda_{in}^{(3)}(t)$	0	0	0.5	0.5	0	0

Figures 10 – 12 present the resulting output stream (queue is empty at the beginning). These results, obtained with the use of diffusion approximation and fluid flow approximation, are compared with simulation results which represent the mean of 400 000 independent runs and practically may be considered as exact. The differencies between input and output streams are clearly visible for the whole stream as well as for each class separately. They justify the need of the presented approach in the transient analysis of

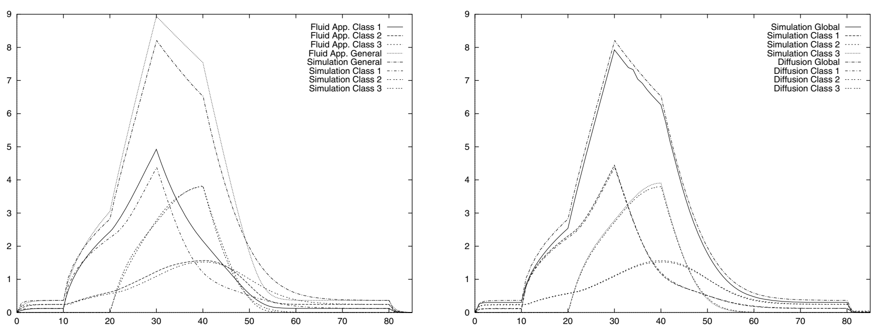

Fig. 10. The mean queue length: global and per classes as a function of time – simulation and fluid approximation results (a), simulation and diffusion approximation results (b)

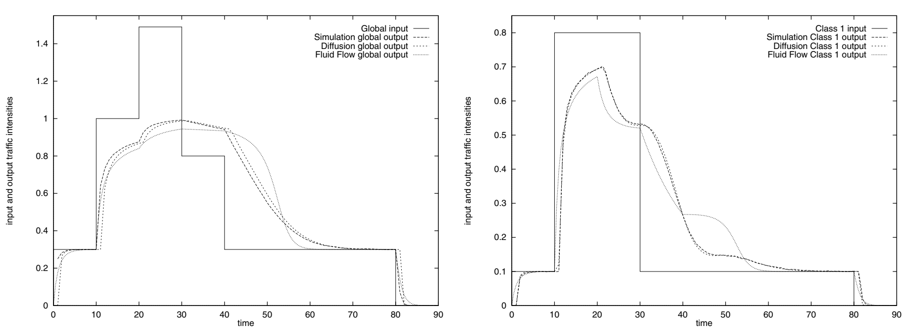

Fig. 11. The intensities $\lambda_{in}(t)$, $\lambda_{out}(t)$ of global input and output flows (a) and for class 1 traffic (b); the output flow is obtained by simulation, diffusion and fluid flow approximations

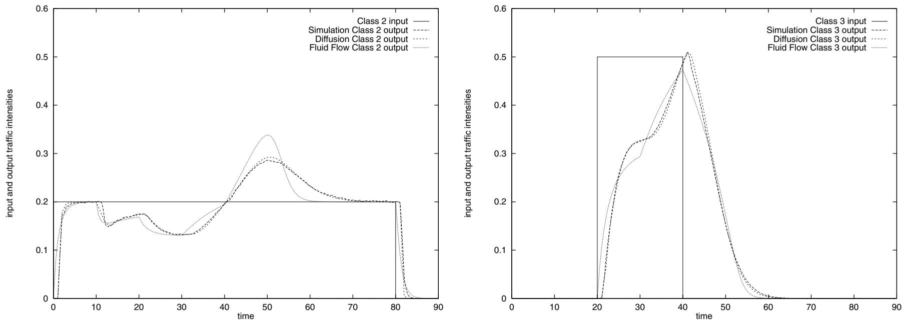

Fig. 12. The intensities of $\lambda_{in}^{(2)}(t)$, $\lambda_{out}^{(2)}(t)$ input and output flows for class 2 traffic (a) and for class 3 traffic (b); the output flow is obtained by simulation, diffusion and fluid flow approximations

networks: the impact the switch queue on the flow dynamics is important and cannot be neglected. In all considered cases the results of diffusion approximation are very close to simulation and clearly better that those obtained with the fluid flow approximation.

8 Switch with Threshold (Partial Buffer Sharing) Algorithm

In a node with *partial buffer sharing policy* the diffusion process represents the content of the cell buffer. The process is determined on the interval $x \in [0, N]$ where N is the buffer capacity, [12]. When the number of cells is equal or greater than the threshold N_1 ($N_1 < N$), only priority cells are admitted and ordinary ones are lost. Diffusion process represents the number cells of both classes, hence its parameters depend on their input and service parameters which are different for $x \leq N_1$ and $x > N_1$:

$$\beta(x) = \begin{cases} \beta_1 = \lambda^{(1)} + \lambda^{(2)} - \mu & \text{for } 0 < x \leq N_1 \,, \\ \beta_2 = \lambda^{(1)} - \mu & \text{for } N_1 < x < N \end{cases} \tag{54}$$

and

$$\alpha(x) = \begin{cases} \alpha_1 = \lambda^{(1)} {C_A^{(1)}}^2 + \lambda^{(2)} {C_A^{(2)}}^2 + \mu C_B^2 & \text{for } 0 < x \leq N_1 \,, \\ \alpha_2 = \lambda^{(1)} {C_A^{(1)}}^2 + \mu C_B^2 & \text{for } N_1 < x < N. \end{cases} \tag{55}$$

We assume constant service time, hence $C_B^2 = 0$. Once again we use the Eq. 50 to represent the sojourn time in the barrier at $x = N$ and to determine μ_N as the inverse of the mean sojourn time.

Steady state solution. Let $f_1(x)$ and $f_2(x)$ denote the pdf function of the diffusion process in intervals $x \in (0, N_1]$ and $x \in [N_1, N)$. We suppose that

· $\lim_{x\to 0}\ f_1(x,t;x_0) = \lim_{x\to N}\ f_2(x,t;x_0) = 0$,
· $f_1(x)$ and $f_2(x)$ functions have the same value at the point N_1: $f_1(N_1) = f_2(N_1)$,
· there is no probability mass flow within the interval $x \in (1, N-1)$:

$$\frac{\alpha_n}{2}\,\frac{d f_n(x)}{d x} - \beta_n f_n(x) = 0 \,, \qquad x \in (1, N_1), n = 1 \quad \text{and} \quad x \in (N_1, N-1), n = 2,$$

and we obtain the solution of diffusion equations:

$$f_1(x) = \begin{cases} \dfrac{[\lambda^{(1)} + \lambda^{(2)}]p_0}{-\beta_1}\,(1 - e^{z_1 x}) & \text{for } 0 < x \le 1 \,, \\ \dfrac{[\lambda^{(1)} + \lambda^{(2)}]p_0}{-\beta_1}(1 - e^{z_1})e^{z_1(x-1)} & \text{for } 1 \le x \le N_1 \,, \end{cases}$$

$$f_2(x) = \begin{cases} f_1(N_1)e^{z_2(x-N_1)} & \text{for } N_1 \le x \le N-1 \,, \\ \dfrac{\mu p_N}{\beta_2}\left[1 - e^{z_2(x-N)}\right] & \text{for } N-1 \le x < N \,, \end{cases} \tag{56}$$

where $z_n = \dfrac{2\beta_n}{\alpha_n}$, $n = 1, 2$. Probabilities p_0, p_N are obtained with the use of normalization condition. The loss ratio $L^{(1)}$ is expressed by the probability p_N, the loss ratio $L^{(2)}$ is determined by the probability $P[x > N_1] = \int_{N_1}^{N} f_2(x)dx + p_N$.

Numerical example. Fig. 13 presents in linear and logarithmic scale the steady-sate distribution given by Eqs. (56) of the number of cells present in a station. The buffer length is $N = 100$, the threshold value is $N_1 = 50$. Some of the values are compared with simulation histograms which we were able to obtain only for relatively large values of probabilities.

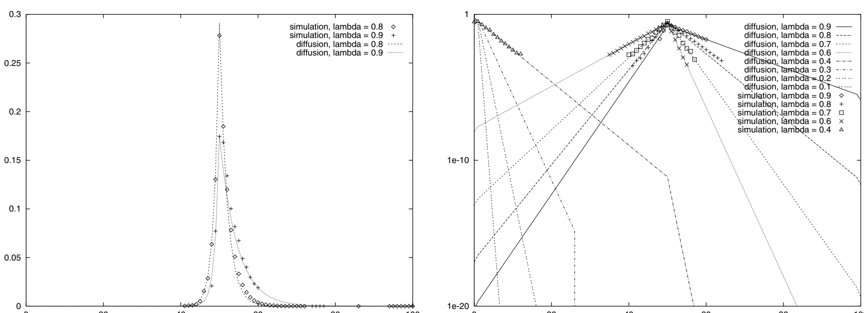

Fig. 13. Steady state distribution of the number of cells for traffic densities $\lambda^{(1)} = \lambda^{(2)} = \lambda = 0.9, 0.9$; diffusion and simulation results, normal (a) and logarithmic (b) scale

Transient solution. The transient solution is obtained with technique presented earlier for $G/G/N/N$ model. It makes use of the balance equations for probability flows crossing the barrier situated at the boundary between the intervals with different diffusion coefficients, i.e. at $x = N_1$. Let us consider two separate diffusion processes $X_1(t)$, $X_2(t)$:

$X_1(t)$ is defined on the interval $x \in (0, N_1)$. At $x = 0$ there is a barrier with sojourn times defined by a pdf $l_0(t)$ and instantaneous returns to the point $x = 1$. At $x = N_1$ an absorbing barrier is placed. Denote by $\gamma^{L}_{N1}(t)$ the pdf that the process enters the absorbing barrier at $x = N_1$. The process is reinitiated at $x = N_1 - \varepsilon$ with a density $g_{N_1-\varepsilon}(t)$.

$X_2(t)$ is defined on the interval $x \in (N_1, N)$. It is limited by an absorbing barrier at $x = N_1$ and by a barrier with instantaneous returns at $x = N$. The sojourn time at this barrier is defined by a pdf $l_N(t)$ and the returns are performed to $x = N - 1$. The process is reinitiated at $x = N_1 + \varepsilon$ with a density $g_{N_1+\varepsilon}(t)$. Denote by $\gamma^R_{N1}(t)$ the pdf that the process $X_2(t)$ enters the absorbing barrier at $x = N_1$.
The interaction between two processes is given by equations

$$g_{N_1+\varepsilon}(t) = \gamma^L_{N_1}(t) \quad \text{and} \quad g_{N_1-\varepsilon}(t) = \gamma^R_{N_1}(t),$$

i.e. the probability density that one process enters to its absorbing barrier is equal to the density of reinitialization of the other process in the vicinity of the barrier.

Equations (31) and (33) form a set of eight equations with eight unknown functions. When we transform these equations with the use of Laplace transform, the convolutions of density functions become products of transforms and we have a set of linear equations where the unknown variables are: $\bar{g}_1(s)$, $\bar{g}_{N_1-\varepsilon}(s)$, $\bar{g}_{N_1+\varepsilon}(s)$, $\bar{g}_{N-1}(s)$, $\bar{\gamma}_0(s)$, $\bar{\gamma}_N(s)$, $\bar{\gamma}_{N_1-\varepsilon}(s)$, $\bar{\gamma}_{N_1+\varepsilon}(s)$. They may be expressed by all other functions, that means $\bar{\gamma}_{\psi_1,0}(s)$, $\bar{\gamma}_{\psi_1,N_1}(s)$, $\bar{\gamma}_{1,0}(s)$, $\bar{\gamma}_{1,N_1}(s)$, $\bar{\gamma}_{N_1-\varepsilon,0}(s)$, $\bar{\gamma}_{N_1-\varepsilon,N_1}(s)$, $\bar{\gamma}_{\psi_2,N1}(s)$, $\bar{\gamma}_{\psi_2,N}(s)$, $\bar{\gamma}_{N_1+\varepsilon,N_1}(s)$, $\bar{\gamma}_{N_1+\varepsilon,N}(s)$, $\bar{\gamma}_{N-1,N_1}(s)$, $\bar{\gamma}_{N-1,N}(s)$ which are already determined by equations of type (32). This way we obtain the functions $\bar{g}_1(s)$, $\bar{g}_{N_1-\varepsilon}(s)$, $\bar{g}_{N_1+\varepsilon}(s)$, $\bar{g}_{N-1}(s)$ and use them in the pdfs (37). The time-domain originals $f_1(x, t; \psi_1)$, $f_2(x, t; \psi_2)$ are obtained numerically [59] from their transforms. The density of the whole process is

$$f(x, t; \psi) = \begin{cases} f_1(x, t; \psi_1) \text{ for } 0 < x < N_1\,, \\ f_2(x, t; \psi_2) \text{ for } N_1 < x < N\,. \end{cases}$$

To see the evolution of the number of cells belonging to a class, we have to consider the composition of input and output streams.

Numerical example. Let us suppose that at the beginning the buffer is empty and during the interval $t \in [0, 100]$ the input stream of priority cells has ratio $\lambda^{(1)} = 2$ cells per time unit and the one of low priority cells $\lambda^{(2)} = 1$; for $t > 100$ the ratio of high priority cells is $\lambda^{(1)} = 0.6667$, the ratio of low priority cells does not change. The service time is constant and equal one time unit. The buffer length is $N = 100$, the value of threshold varies between $N_1 = 50$ and $N_1 = 90$. The value of ε in Eqs. (37)–(39) was chosen $\varepsilon = 0.1$. In Fig. 14 the distributions of the number of cells at the buffer obtained by simulation and by diffusion model for chosen time moments are compared. Diffusion and simulation results are placed in separate figures to preserve their legibility. The shape of curves given by two models is very similar. At the end of second period ($t = 400, 500, 600$) the steady state distribution is attained.

Fig. 15 displays the mean number of cells in the buffer as a function of time. During the first 100 time units the congestion is clearly visible, the buffer quickly becomes saturated; during the second period the queue is also overcrowded, probability that the threshold is exceeded is near 0.7 but owing to the buffer sharing policy the probability that the buffer is inaccessible for priority cells remains negligible – Fig. 16.

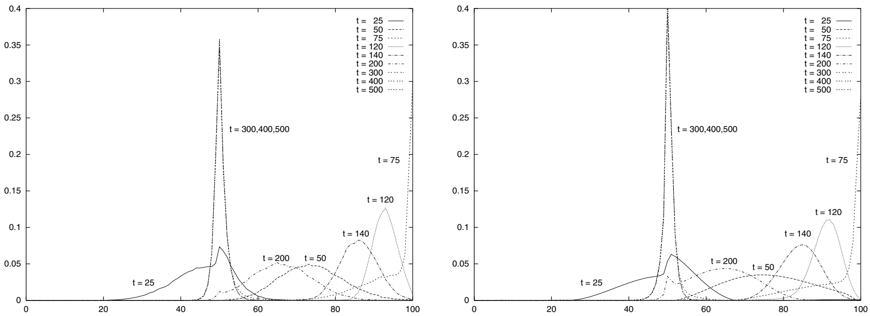

Fig. 14. Distribution of the number of all cells in the buffer for several time moments $t = 25 - 500$; buffer size $N = 100$, threshold $N_1 = 50$; simulation (a) and diffusion (b) results

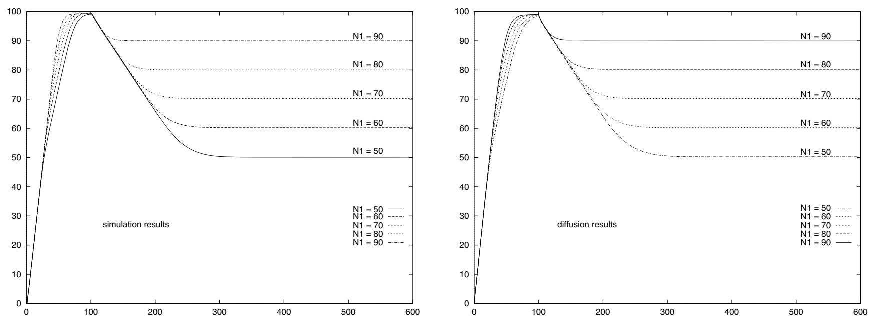

Fig. 15. Mean number of cells as a function of time, parametrized by the value $N_1 = 50, 60, 70, 80, 90$ of the threshold, simulation (a) and diffusion (b) results

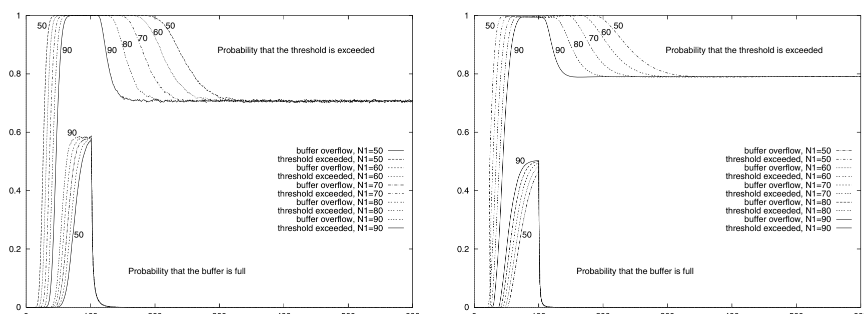

Fig. 16. Probability that the buffer of length $N = 100$ is full (priority cells are lost) and that the threshold is exceeded (ordinary cells are lost) as a function of time, parametrized by the threshold value $N_1 = 50, 60, 70, 80, 90$; simulation (a) and diffusion (b) results

The threshold value N_1 is a parameter of displayed curves. If N_1 increases the mean values of low priority cells increases (they have more space in the buffer, hence less of them is rejected) and the number of priority cells increases too (as there is more class 2 cells in the queue, class 1 cells wait longer).

Fig. 17 displays the mean number of high and low priority cells given by the approach we have described above and compared with simulation results. We see that the steady state mean value of class 2 cells is underestimated (because of overestimation of class 2 losses by diffusion approximation seen in Fig. 16) but the dynamics of class 2 cells vanishing from the queue during heavy saturation periods is well captured.

Some numerical problems were encountered when computing expressions of $\phi(x, t, \psi)$ and $\gamma_0(t)$ for very small values of $\lambda^{(1)}_{\text{eff}}(t)$, $\mu^{(2)}_{\text{eff}}(t)$ and forced us to very careful programming.

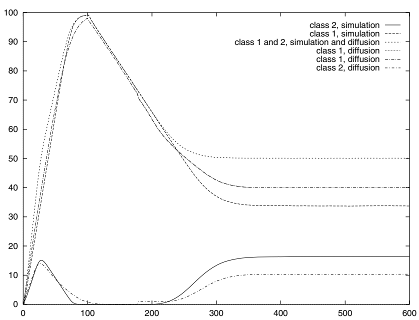

Fig. 17. Mean value of class 1 and class 2 cells as a function of time; $N = 100$, $N_1 = 50$

9 Conclusions

We describe how the diffusion approximation formalism is applied to study transient and behavior of G/G/1 and G/G/1/N network of queueus and we present some other models related to congestion control. The way we switch from one model to another demonstrates the flexibility of the method. Also the introduction of self-similar traffic is possible: as we change the diffusion parameters each small time-interval, we can modulate them to reflect self-similarity and long-term correlation of the traffic. Some other applications may be considered: recently we have used the diffusion approximation to estimate transfer times inside a sensor network [22], to study priority queues [24], the work of call centers [23], and to investigate the stability of TCP connections with IP routers having AQM queues [21]. Also the application of diffusion approximation to model wireless networks based on IEEE 802.11 standard gives promising results, [25].

Therefore we consider the diffusion approximation as a very convenient tool in the analysis of transient states queueing models in performance evaluation of computer and communication networks.

Acknowledgements

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References

1. Atmaca, T., Czachórski, T., Pekergin, F.: A Diffusion Model of the Dynamic Effects of Closed-Loop Feedback Control Mechanisms in ATM Networks. In: 3rd IFIP Workshop on Performance Modelling and Evaluation of ATM Networks, Ilkley, UK, July 4-7 (1995); rozszerzona wersja w. Archiwum Informatyki Teoretycznej i Stosowanej (1), 41–56 (1999)
2. Burke, P.J.: The Output of a Queueing System. Operations Research 4(6), 699–704
3. Cox, R.P., Miller, H.D.: The Theory of Stochastic Processes. Chapman and Hall, London (1965)
4. Czachórski, T.: A Multiqueue Approximate Computer Performance Model with Priority Scheduling and System Overhead. Podstawy Sterowania 10(3), 223–240 (1980)
5. Czachórski, T.: A diffusion process with instantaneous jumps back and some its applications. Archiwum Informatyki Teoretycznej i Stosowanej 20(1-2), 27–46
6. Czachórski, T., Fourneau, J.M., Pekergin, F.: Diffusion Model of the Push-Out Buffer Management Policy. In: IEEE INFOCOM 1992, The Conference on Computer Communications, Florence (1992)
7. Czachórski, T.: A method to solve diffusion equation with instantaneous return processes acting as boundary conditions. Bulletin of Polish Academy of Sciences, Technical Sciences 41(4) (1993)
8. Czachórski, T., Fourneau, J.M., Pekergin, F.: Diffusion model of an ATM Network Node. Bulletin of Polish Academy of Sciences, Technical Sciences 41(4) (1993)
9. Czachórski, T., Fourneau, J.M., Pekergin, F.: Diffusion Models to Study Nonstationary Traffic and Cell Loss in ATM Networks. In: ACM 2nd Workshop on ATM Networks, Bradford (July 1994)
10. Czachórski, T., Fourneau, J.M., Kloul, L.: Diffusion Approximation to Study the Flow Synchronization in ATM Networks. In: ACM 3rd Workshop on ATM Networks, Bradford (July 1995)
11. Czachórski, T., Fourneau, J.M., Pekergin, F.: The dynamics of cell flow and cell losses in ATM networks. Bulletin of Polish Academy of Sciences, Technical Sciences 43(4) (1995)
12. Czachórski, T., Pekergin, F.: Diffusion Models of Leaky Bucket and Partial Buffer Sharing Policy: A Transient Analysis. In: 4th IFIP Workshop on Performance Modelling and Evaluation of ATM Networks, Ilkley (1996); also in: Kouvatsos, D.: ATM Networks, Performance Modelling and Analysis. Chapman and Hall, London (1997)
13. Czachórski, T., Atmaca, T., Fourneau, J.-M., Kloul, L., Pekergin, F.: Switch queues – diffusion models (in polisch). Zeszyty Naukowe Politechniki /Sl/askiej, seria Informatyka (32) (1997)
14. Czachórski, T., Pekergin, F.: Transient diffusion analysis of cell losses and ATM multiplexer behaviour under correlated traffic. In: 5th IFIP Workshop on Performance Modelling and Evaluation of ATM Networks, Ilkley, UK, July 21-23 (1997)
15. Czachórski, T., Pastuszka, M., Pekergin, F.: A tool to model network transient states with the use of diffusion approximation. In: Performance Tools 1998, Palma de Mallorca, Hiszpania, wrzesie/n (1998)
16. Czachórski, T., Jedrus, S., Pastuszka, M., Pekergin, F.: Diffusion approximation and its numerical problems in implementation of computer network models. Archiwum Informatyki Teoretycznej i Stosowanej (1), 41–56 (1999)

17. Czachórski, T., Fourneau, J.-M., Kloul, L.: Diffusion Method Applied to a Handoff Queueing Scheme. Archiwum Informatyki Teoretycznej i Stosowanej (1), 41–56 (1999)
18. Czachórski, T., Pekergin, F.: Probabilistic Routing for Time-dependent Traffic: Analysis with the Diffusion and Fluid Approximations. In: IFIP ATM Workshop, Antwerp (1999)
19. Czachórski, T., Pekergin, F.: Modelling the time-dependent flows of virtual connections in ATM networks. Bulletin of Polish Academy of Sciences, Techical Sciences 48(4), 619–628 (2000)
20. Czachórski, T., Fourneau, J.-M., Jędruś, S., Pekergin, F.: Transient State Analysis in Cellular Networks: the use of Diffusion Approximation. In: QNETS 2000, Ilkley (2000)
21. Czachórski, T., Grochla, K., Pekergin, F.: Stability and dynamics of TCP-NCR(DCR) protocol in presence of UDP flows. In: García-Vidal, J., Cerdà-Alabern, L. (eds.) Euro-NGI 2007. LNCS, vol. 4396, pp. 241–254. Springer, Heidelberg (2007)
22. Czachórski, T., Grochla, K., Pekergin, F.: Un modèle d'approximation de diffusion pour la distribution du temps d'acheminement des paquets dans les réseaux de senseurs. In: Proc. of CFIP 2008, Les Arcs, Mars 25-28 (2008), Proceedings, edition electronique, http://hal.archives-ouvertes.fr/CFIP2008
23. Czachórski, T., Fourneau, J.-M., Nycz, T., Pekergin, F.: Diffusion approximation model of multiserver stations with losses. In: Proc. of Third International Workshop on Practical Applications of Stochastic Modelling PASM 2008, Palma de Mallorca, September 23 (2008); To appear also as an issue of Elsevier's ENTCS (Electronic Notes in Theoretical Computer Science)
24. Czachórski, T., Nycz, T., Pekergin, F.: Transient states of priority queues – a diffusion approximation study. In: Proc. of The Fifth Advanced International Conference on Telecommunications AICT 2009, Venice, Mestre, Italy, May 24-28 (2009)
25. Czachórski, T., Grochla, K., Nycz, T., Pekergin, F.: A diffusion approximation model for wireless networks based on IEEE 802.11 standard submitted to COMCOM Special Journal Issue on Heterogeneous Networks: Traffic Engineering and Performance Evaluation of Computer Communications
26. Duda, A.: Diffusion Approximations for Time-Dependent Queueing Systems. IEEE J. on Selected Areas in Communications SAC-4(6) (September 1986)
27. Feller, W.: The parabolic differential equations and the associated semigroups of transformations. Annales Mathematicae 55, 468–519 (1952)
28. Feller, W.: Diffusion processes in one dimension. Transactions of American Mathematical Society 77, 1–31 (1954)
29. Filipiak, J., Pach, A.R.: Selection of Coefficients for a Diffusion-Equation Model of Multi-Server Queue. In: PERFORMANCE 1984, Proc. of The 10th International Symposium on Computer Performance. North Holland, Amsterdam (1984)
30. Gaver, D.P.: Observing stochastic processes, and approximate transform inversion. Operations Research 14(3), 444–459 (1966)
31. Gaver, D.P.: Diffusion Approximations and Models for Certain Congestion Problems. Journal of Applied Probability 5, 607–623 (1968)
32. Gelenbe, E.: On Approximate Computer Systems Models. J. ACM 22(2) (1975)
33. Gelenbe, E., Pujolle, G.: The Behaviour of a Single Queue in a General Queueing Network. Acta Informatica 7(fasc. 2), 123–136 (1976)
34. Gelenbe, E.: A non-Markovian diffusion model and its application to the approximation of queueing system behaviour, IRIA Rapport de Recherche no. 158 (1976)
35. Gelenbe, E.: Probabilistic models of computer systems. Part II. Acta Informatica 12, 285–303 (1979)
36. Gelenbe, E., Labetoulle, J., Marie, R., Metivier, M., Pujolle, G., Stewart, W.: Réseaux de files d'attente – modélisation et traitement numérique, Editions Hommes et Techniques, Paris (1980)

37. Gelenbe, E., Mang, X., Feng, Y.: A diffusion cell loss estimate for ATM with multiclass bursty traffic. In: Kouvatsos, D.D. (ed.) Performance Modelling and Evaluation of ATM Networks, vol. 2. Chapman and Hall, London (1996)
38. Gelenbe, E., Mang, X., Önvural, R.: Diffusion based statistical call admission control in ATM. Performance Evaluation 27-28, 411–436 (1996)
39. Halachmi, B., Franta, W.R.: A Diffusion Approximation to the Multi-Server Queue. Management Sci. 24(5), 522–529 (1978)
40. Heffes, H., Lucantoni, D.M.: A Markov modulated characterization of packetized voice and data traffic and related statistical multiplexer parformance. IEEE J. SAC SAC-4(6), 856–867 (1986)
41. Heyman, D.P.: An Approximation for the Busy Period of the $M/G/1$ Queue Using a Diffusion Model. J. of Applied Probility 11, 159–169 (1974)
42. Iglehart, D., Whitt, W.: Multiple Channel Queues in Heavy Traffic, Part I-III. Advances in Applied Probability 2, 150–177, 355–369 (1970)
43. Iglehart, D.: Weak Convergence in Queueing Theory. Advances in Applied Probability 5, 570–594 (1973)
44. Jouaber, B., Atmaca, T., Pastuszka, M., Czachórski, T.: Modelling the Sliding window Mechanism. In: The IEEE International Conference on Communications, ICC 1998, Atlanta, Georgia, USA, czerwiec 7-11, pp. 1749–1753 (1998)
45. Jouaber, B., Atmaca, T., Pastuszka, M., Czachórski, T.: A multi-barrier diffusion model to study performances of a packet-to-cell interface, art. S48.5, Session: Special applications in ATM Network Management. In: International Conference on Telecommunications ICT 1998, Porto Carras, Greece, czerwiec 22-25 (1998)
46. Kimura, T.: Diffusion Approximation for an M/G/m Queue. Operations Research 31(2), 304–321 (1983)
47. Kleinrock, L.: Queueing Systems. Theory, vol. I. Computer Applications, vol. II. Wiley, New York (1975, 1976)
48. Kobayashi, H.: Application of the diffusion approximation to queueing networks, Part 1: Equilibrium queue distributions. J. ACM 21(2), 316–328 (1974); Part 2: Nonequilibrium distributions and applications to queueing modeling. J. ACM 21(3), 459–469 (1974)
49. Kobayashi, H.: Modeling and Analysis: An Introduction to System Performance Evaluation Methodology. Addison Wesley, Reading (1978)
50. Kobayashi, H., Ren, Q.: A Diffusion Approximation Analysis of an ATM Statistical Multiplexer with Multiple Types of Traffic, Part I: Equilibrium State Solutions. In: Proc. of IEEE International Conf. on Communications, ICC 1993, Geneva, Switzerland, May 23-26, pp. 1047–1053 (1993)
51. Kulkarni, L.A.: Transient behaviour of queueing systems with correlated traffic. Performance Evaluation 27-28, 117–146 (1996)
52. Lee, D.-S., Li, S.-Q.: Transient analysis of multi-sever queues with Markov-modulated Poisson arrivals and overload control. Performance Evaluation 16, 49–66 (1992)
53. Maglaris, B., Anastassiou, D., Sen, P., Karlsson, G., Rubins, J.: Performance models of statistical multiplexing in packet video communications. IEEE Trans. on Communications 36(7), 834–844 (1988)
54. Newell, G.F.: Queues with time-dependent rates, Part I: The transition through saturation. J. Appl. Prob. 5, 436-451 (1968); Part II: The maximum queue and return to equilibrium, 579–590 (1968); Part III: A mild rush hour, 591–606 (1968)
55. Newell, G.F.: Applications of Queueing Theory. Chapman and Hall, London (1971)
56. Pastuszka, M.: Modelling transient states in computer networks with the use of diffusion approximation (in polish), Ph.D. Thesis, Silesian Technical University (Politechnika Śląska), Gliwice (1999)

57. Reiser, M., Kobayashi, H.: Accuracy of the Diffusion Approximation for Some Queueing Systems. IBM J. of Res. Develop. 18, 110–124 (1974)
58. Sharma, S., Tipper, D.: Approximate models for the Study of Nonstationary Queues and Their Applications to Communication Networks. In: Proc. of IEEE International Conf. on Communications, ICC 1993, Geneva, Switzerland, May 23-26, pp. 352–358 (1993)
59. Stehfest, H.: Algorithm 368: Numeric inversion of Laplace transform. Comm. of ACM 13(1), 47–49 (1970)
60. Veillon, F.: Algorithm 486: Numerical Inversion of Laplace Transform. Comm. of ACM 17(10), 587–589 (1974); also: Veillon, F.: Quelques méthodes nouvelles pour le calcul numérique de la transformé inverse de Laplace, Th. Univ. de Grenoble (1972)
61. Zwingler, D.: Handbook of Differential Equations, pp. 623–627. Academic Press, Boston (1989)