Stochastic Ordering of Semi-Markov Processes

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Abstract. In this tutorial we address the stochastic ordering of semi-Markov processes in the usual and level-crossing stochastic ordering senses. We highlight the sample-path approach for the comparison of semi-Markov processes and for the simulation of processes with a given distribution.

Keywords: Markov renewal processes, sample-path approach, semi-Markov processes, stochastic ordering.

1 Introduction

The desire to confront random quantities is probably as old as probability theory itself; in this line of reasoning, Bawa [4] traces the origins of stochastic ordering or dominance in the works of J. Bernoulli in 1713, *Ars Conjectardi*. However, it has been mainly in the last decades that stochastic ordering has progressively became to be recognized as an important tool in the area of applied stochastic processes, as illustrated, e.g, in the bibliographies of Bawa [4], Levy [31], and Mosler and Scarsini [39]. The most popular approaches used to establish stochastic ordering results are: coupling constructions (Lindvall [35]; and Thorisson [50]), sample-path approaches (El-Taha and Stidham [13]; and Stoyan [48]) and some pure analytic results (Kijima [26]; and Shaked and Shanthikumar [44]).

The rich history of applications of stochastic ordering is also made clear, e.g., in Shaked and Shanthikumar [44], which specifically expands on the applications of stochastic ordering in the areas of statistical inference, risk theory, economics, biology, scheduling, operations research, queueing theory, and reliability theory, and reinforced by Arnold [1], Cabral Morais [38], van Doorn [52], Joe [23], Kijima and Ohnishi [27], Lindvall [35], Marshall and Olkin [36], Müller and Stoyan [40], Stoyan [48], Szekli [49], Thorisson [50], and Tong [51]. A reflection of the relevance of stochastic ordering in applications is the significative number of fairly recent books on stochastic processes that have included as chapters or parts of the book stochastic ordering concepts and results, e.g., Baccelli and Brémaud [3], Kijima [26], Kulkarni [29], Last and Brandt [30], and Ross [43].

Semi-Markov processes (SMPs) and the related Markov renewal processes (MRPs) have a well established theory (see, e.g., Çinlar [9], Kulkarni [29], and Limnios and Oprişan [33]) and have many applications (see, e.g., Asmussen [2];

D. Kouvatsos (Ed.): Next Generation Internet, LNCS 5233, pp. 393–422, 2011.

Disney and Kiessler [11]; and Janssen and Limnios [22]). The latter fact is easily understood if we note that SMPs and MRPs are basically in a one-to-one correspondence and, moreover, SMPs generalize both discrete-time Markov chains (DTMCs) and continuous-time Markov chains (CTMCs), whereas MRPs generalize renewal processes and the so called Markovian arrival processes. Thus, the stochastic ordering of SMPs has broad impact in applied stochastic processes.

The analysis of SMPs and MRPs started to be developed in the 1950's by important probabilists, namely: Levy [32], Smith [45,46], Pyke [41,42], Feller [14], and Çinlar [7,8]. However, explicit references to the stochastic comparison of SMPs appeared only several years later, with the work of Sonderman [47] in the usual (in distribution) stochastic ordering sense.

In the tutorial we will review the literature on the stochastic ordering of SMPs in the usual stochastic ordering sense, as well as in the level-crossing stochastic ordering sense, recently proposed by A. Irle and J. Gani [21] and investigated by its proponents and the authors of this tutorial. A process X is said to be smaller than Y in the usual sense if there are copies $\hat{X}$ and $\hat{Y}$ (i.e., processes with the same distributions as the original ones) of the processes X and Y defined on a common probability space such that their trajectories are ordered in the almost sure sense. Similarly, a process X is said to be smaller in level-crossing than Y if it takes X stochastically longer than Y to exceed any given level. As illustrated by Irle and Gani [21], the level-crossing ordering of stochastic processes in the usual sense is (strictly) weaker than the usual stochastic ordering of the processes.

We will start in Section 2 with the presentation of the definition of MRPs and SMPs and present in Section 3 procedures to simulate such processes. Then, in sections 4 and 5, we briefly review the main results on the stochastic comparison of SMPs in the usual and in the level-crossing stochastic ordering senses, respectively. We will follow the sample-path approach, which is useful for simulating pairs of stochastically ordered processes, and will address only SMPs with totally ordered state spaces.

2 Markov Renewal and Semi-Markov Processes

In this section, we provide the definitions of a Markov Renewal process (MRP) and of a semi-Markov process (SMP). We relate these two types of processes and give their characterizations in terms of their associated: (transition distribution) kernel, embedded (transition) kernel, and (failure) rate kernel. We start with the definition of MRP [cf., e.g., [9] or [29]].

2.1 Markov Renewal Processes

Definition 1. *We say that a bivariate process* $(Z, S) = (Z_n, S_n)_{n\in\mathbb{N}}$ *is a MRP with (countable) phase space* I *and kernel* $Q = (Q_t)_{t\in\mathbb{R}_+}$*, where* $Q_t = (Q_{ij}(t))_{i,j\in I}$ *is a family of sub-distribution functions such that* $\sum_{j\in I} Q_{ij}(t)$ *is a distribution function, for all* $i \in I$*, if it is a Markov process on* $I \times \mathbb{R}_+$ *such that* $S_0 = 0$ *and*

$$Q_{ij}(t) = \mathbf{P}\,(\,Z_{n+1} = j\,,\; S_{n+1} - S_n \leq t|\, Z_n = i, S_n = s)$$

for all $n \in \mathbb{N}$, $i, j \in I$ *and* $s, t \in \mathbb{R}_+$.

MRPs are used, e.g., in the modelling of arrival processes to queueing networks, where S models the network arrival epochs and Z models the influence of environmental factors in the structure of the interarrival times. In this context, $Q_{ij}(t)$ denotes the probability that, given that after an arrival the process is in phase i, the next arrival will put the process in phase j and will take place within t time units. From the definition of MRP it follows that this last event does not depend on the last arrival epoch, i.e.,

$$Q_{ij}(t) = \mathbf{P}\,(\,Z_{n+1} = j\,,\; S_{n+1} - S_n \leq t|\, Z_n = i)\,.$$

Embedded kernel characterization. Another natural characterization of a MRP is through its embedded kernel, which separates the embedded transition probabilities from the distributions of the holding times in states between transitions. From the definition of MRP, it follows that if (Z, S) is a MRP with kernel Q, then Z is a discrete time Markov chain (DTMC) with one-step transition probability matrix $P = Q(\infty)$ with

$$p_{ij} = Q_{ij}(\infty) = \mathbf{P}\,(\,Z_{n+1} = j|\, Z_n = i)$$

denoting the probability that if the previous phase transition leads the process to phase i the phase process will next move to phase j.

On the other hand, conditional to the next phase being j, i.e., given that the process makes a transition from phase i to phase j, then the amount of time the process stays in phase i before moving to phase j has distribution function

$$F_{(i,j)}(t) = \mathbf{P}\,(\,S_{n+1} - S_n \leq t|\, Z_n = i, Z_{n+1} = j) = \frac{Q_{ij}(t)}{Q_{ij}(\infty)}$$

where, by convention, we let $F_{(i,j)}(t) = 1$, for all $t \in \mathbb{R}_+$, whenever $p_{ij} = 0$. It thus follows that

$$Q_{ij}(t) = p_{ij}\, F_{(i,j)}(t)\;, \quad t \in \mathbb{R}_+$$

for all $i, j \in I$, and we say that the MRP (Z, S) has embedded kernel (P, F), where $P = (p_{ij})_{i,j\in I}$ is a stochastic transition probability matrix and $F = (F_{(i,j)})_{i,j\in I}$ is a matrix of distribution functions of nonnegative random variables such that if $p_{ij} = 0$, then $F_{(i,j)}(t) = 1$, $t \in \mathbb{R}_+$. Thus, a MRP is completely characterized by its embedded kernel.

Rate kernel characterization. Alternatively to the previous characterizations, a MRP may also be characterized via its failure rate kernel. Consider a MRP W with phase space I and kernel Q such that the sub-distributions $Q_{ij}(t)$ are absolutely continuous. Then, $q = (q_t)_{t\in\mathbb{R}_+}$ with $q_t = (q_{ij}(t))_{i,j\in I}$ such that

$$q_{ij}(t) = \frac{\mathrm{d}Q_{ij}(t)}{\mathrm{d}t}$$

is called the density kernel of W, and $q_i(t) = \sum_{j\in I} q_{ij}(t)$ denotes the density of the time needed for a transition from phase i to take place. Moreover, letting

$$r_{ij}(t) = \frac{q_{ij}(t)}{1 - \sum_{l\in I} Q_{il}(t)}$$

then $R = (R_t)_{t\in\mathbb{R}_+}$, with $R_t = (r_{ij}(t))_{i,j\in I}$, is called the failure rate kernel of W and

$$r_i(t) = \sum_{j\in I} r_{ij}(t) = \frac{q_i(t)}{1 - \sum_{l\in I} Q_{il}(t)}$$

denotes the failure rate at time t of the time needed for a transition from phase i to take place.

In this case, as $r_i(t)$ characterizes $q_i(t)$ through $q_i(t) = r_i(t)\ \exp\left\{-\int_0^t r_i(s)\,\mathrm{d}s\right\}$ [cf., e.g., [24]], it immediately follows that $r_{ij}(t)$ characterizes $q_{ij}(t)$ through

$$q_{ij}(t) = r_{ij}(t)\ \exp\left\{-\int_0^t r_i(s)\,\mathrm{d}s\right\}, \quad i,j\in I,\ t\in\mathbb{R}_+. \tag{1}$$

Thus, the MRP W is completely characterized by its failure rate kernel.

2.2 Semi-Markov Processes

We now introduce the definition of a SMP in terms of its usual characterizations.

Definition 2. *A process $W = (W_t)_{t\in\mathbb{R}_+}$ is a SMP with countable state space I and (admitting) kernel Q (embedded kernel (P,F); failure rate kernel R) if*

$$W_t = Z_n\,, \quad S_n \le t < S_{n+1} \tag{2}$$

for some MRP (Z,S) with phase space I and kernel Q (embedded kernel (P,F); failure rate kernel R).

The most common description of the evolution of an SMP is through its embedded kernel. An SMP with embedded kernel (P,F) and initial probability distribution vector p evolves as follows. The process starts in phase i with probability p_i, and afterwards changes from phase to phase according to the transition probability matrix P. It moves to phase k after entering phase j, with probability p_{jk}, independently of previous phase changes. After deciding the next phase to visit, say k, from phase j, the process stays in phase j before making the transition to phase k a random holding time, independent of previous holding times in phases and phase transitions, having distribution function $F_{(j,k)}(t)$. If the SMP has kernel Q, then $p_{jk} = Q_{jk}(\infty)$ and $F_{(j,k)}(t) = Q_{jk}(t)/p_{jk}$ case $p_{jk} > 0$.

We end the section noting that given a SMP W with state space I, the process $(Z_n, S_n)_{n\in\mathbb{N}}$ with

$$(Z_0,S_0) = (W_0,0) \quad \text{and} \quad \begin{cases} S_{n+1} = \inf\{t \ge S_n : W_t \ne W_{t^-}\} \\ Z_{n+1} = W_{S_{n+1}} \end{cases}$$

for $n \in \mathbb{N}$, is a MRP with phase space I. In particular, if Q $((P,F);\, R)$ denotes the kernel (embedded kernel; failure rate kernel) of $(Z_n, S_n)_{n\in\mathbb{N}}$, then W admits the kernel (embedded kernel; failure rate kernel) Q $((P,F);\, R)$, called the natural kernel of W.

Conversely, if (Z,S) is a MRP with phase space I and kernel Q (embedded kernel (P,F); failure rate kernel R), then the process W with

$$W_t = Z_n \ , \quad S_n \leq t < S_{n+1}$$

is a SMP with state space I and kernel Q (embedded kernel (P,F); failure rate kernel R).

3 Simulation of Semi-Markov Processes

Having introduced the usual characterizations of a SMP, we proceed to describe a procedure to simulate (generate) a SMP with countable totally ordered state space I, order isomorphic to a subset of integers, and a given parametrization. For that, let F^{-1} denote the generalized inverse function of a distribution function F, i.e.,

$$F^{-1}(u) = \inf\{t : F(t) \geq u\}, \qquad \text{for } u \in [0,1]$$

with the convention that $\inf \emptyset = +\infty$. Moreover, to simplify the writing and avoid extra notation for the associated distribution functions, if p denotes a probability vector and Z denotes a random variable or distribution, like the exponential distribution with rate λ, Exp (λ), then we let p^{-1} and Z^{-1} denote the generalized inverse functions of the distribution function associated to p and Z, respectively.

The generalized inverse function is in the base of the standard method to simulate copies of random variables with prescribed distributions. In fact, if U is a uniform random variable on $(0,1)$, Unif$(0,1)$, and F is an arbitrary distribution function, then $F^{-1}(U)$ is a random variable with distribution function F, i.e.,

$$U \sim \text{Unif}(0,1) \Longrightarrow F^{-1}(U) \sim F. \tag{3}$$

At this point, it is important to note that, given two distribution functions F and G, then

$$F(t) \geq G(t), \quad t \in \mathbb{R} \Longrightarrow F^{-1}(u) \leq G^{-1}(u), \quad u \in (0,1). \tag{4}$$

This fact ([49], Lemma C) is of paramount importance in the simulation of stochastic ordered random variables and processes.

We proceed to address the simulation of a SMP from its embedded kernel.

3.1 Simulation via Embedded Kernel

To simulate a SMP W with state space I and embedded kernel (P,F) it suffices to simulate a MRP (Z,S) with phase space I and embedded kernel (P,F) and then obtain $W = (W_t)_{t\in\mathbb{R}}$ from

$$W_t = Z_n \ , \quad S_n \leq t < S_{n+1}.$$

In turn, to simulate a MRP (Z, S) with state space I and embedded kernel (P, F), it suffices to simulate a DTMC Z with state space I and a sequence $S = (S_n)_{n\in\mathbb{N}}$ in such a way that Z has associated transition probability matrix P and, conditional to $Z_n = i$ and $Z_{n+1} = j$, the time interval between the n-th and the $n+1$-th phase transitions $H_{n+1} = S_{n+1} - S_n$ has distribution $F_{(i,j)}(\cdot)$ and is independent of $(Z_k, S_k)_{k<n}$. In fact, this procedure leads to a sequence $(Z, S) = (Z_n, S_n)_{n\in\mathbb{N}}$ for which

$$\begin{aligned} Q_{ij}(t) &= \mathbf{P}\left(Z_{n+1} = j,\, S_{n+1} - S_n \leq t | \, Z_n = i\right) \\ &= \mathbf{P}\left(Z_{n+1} = j | \, Z_n = i\right) \mathbf{P}\left(S_{n+1} - S_n \leq t | \, Z_n = i,\, Z_{n+1} = j\right) \\ &= p_{ij}\, F_{(i,j)}(t) \end{aligned}$$

that is, to a MRP with embedded kernel (P, F).

For that, let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ be two sequences of independent uniform random variables on $(0, 1)$, defined on independent probability spaces $\Lambda_1 = (\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ and $\Lambda_2 = (\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, respectively, and construct (Z, S) on the product space $\Lambda = \Lambda_1 \times \Lambda_2$ as follows.

For $\omega = (\omega_1, \omega_2) \in \Omega$, use $(U_n(\omega_1))_{n\in\mathbb{N}}$ to construct $Z(\omega_1)$ on Λ_1 from

$$\begin{aligned} Z_0(\omega_1) &= p^{-1}(U_0(\omega_1)) \\ Z_{n+1}(\omega_1) &= [p_{Z_n(\omega_1)\cdot}]^{-1}(U_{n+1}(\omega_1)), \quad n \in \mathbb{N} \end{aligned}$$

where p denotes the initial phase probability vector. At the same time, use the sequences $(Z_n(\omega_1))_{n\in\mathbb{N}}$ and $(V_n(\omega_2))_{n\in\mathbb{N}_+}$ to generate the time intervals between state transitions $H(\omega) = (H_n(\omega))_{n\in\mathbb{N}_+}$, making

$$H_{n+1}(\omega) = \left[F_{(Z_n, Z_{n+1})(\omega_1)}\right]^{-1}(V_{n+1}(\omega_2)), \quad n \in \mathbb{N}. \tag{5}$$

Finally, obtain the renewal sequence $S(\omega)$ by setting $S_0(\omega) = 0$ and

$$S_{n+1}(\omega) = S_n(\omega) + H_{n+1}(\omega), \quad n \in \mathbb{N}.$$

By construction and in view of (3):

- Z_0 has probability vector p.
- $Z_{n+1}|Z_n = i$ has probability vector $p_{i\cdot}$, for $n \in \mathbb{N}$.
- $[H_{n+1}|Z_n = i, Z_{n+1} = j]$ has distribution function $F_{(i,j)}(\cdot)$, for $n \in \mathbb{N}$.

As in addition, for each $n \in \mathbb{N}$, (U_{n+1}, V_{n+1}) is independent of $\{U_0, (U_m, V_m)_{m\leq n}\}$ it follows that, given Z_n, $(Z_{n+1}, S_{n+1} - S_n)$ is independent of $(Z_k, S_k)_{k<n}$ and, thus, the process (Z, S) is a MRP with embedded kernel (P, F).

The previous procedure leads to the algorithm presented in Fig. 1 for the simulation of an SMP with initial phase probability vector p and embedded kernel (P, F).

Input: Independent sequences of independent Unif$(0,1)$ random variables $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ and $N \in \mathbb{N}_+$
$Z_0 := p^{-1}(U_0), \quad S_0 := 0$
for $n = 0, 1, \ldots, N-1$ **do**
$Z_{n+1} := [p_{Z_n\cdot}]^{-1}(U_{n+1})$
$S_{n+1} = S_n + [F_{(Z_n, Z_{n+1})}]^{-1}(V_{n+1})$
end for
Output: $W_t := Z_n$ for $S_n \le t < S_{n+1},\ 0 \le n < N$

Fig. 1. Simulation of an SMP with initial phase probability vector p and embedded kernel (P, F)

3.2 Simulation via Rate Kernel

In a similar manner, to simulate a SMP W with failure rate kernel R it suffices to simulate a MRP (Z, S) with failure rate kernel R and then obtain W from

$$W_t = Z_n\ , \quad S_n \le t < S_{n+1}.$$

The simulation of a MRP (Z, S) with failure rate kernel R is fairly different from the simulation based on the embedded kernel. In this case, it is generated a Poisson process with rate modulated by the state of the process and, conditional to the fact that after the last Markov renewal epoch the MRP moved or stayed in phase i, it is taken into account the failure rate in phase i to decide if the next Poisson arrival epoch will make part or not of the random sequence S. Then, if so, it is decided what the next phase will be with a procedure which assures that, if an event takes place t units of time after the transition instant to phase i then the next phase will be j with probability $\frac{r_{ij}(t)}{\lambda_i}$, where λ_i denotes the Poisson uniformization rate in phase i. The following lemma will be useful for such a construction.

Lemma 1. *Let J be an ordered set, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, i be an element of J, $\beta = (\beta_j)_{j\in J}$ be a sub-stochastic vector, and U be a uniform random variable on $(0,1)$. Then, the random variable*

$$\text{Failure}(i, \beta, U) = \begin{cases} 1 & U \notin \left]\sum_{k\le i}\beta_k, 1 - \sum_{k>i}\beta_k\right] \\ 0 & \textit{otherwise} \end{cases} \tag{6}$$

is a Bernoulli random variable with parameter $\sum_{k\in J}\beta_k$. In addition, if we let $\beta^{(i)}$ denote the probability vector obtained from β making

$$\beta_j^{(i)} = \begin{cases} \beta_j & j \ne i \\ 1 - \sum_{l\ne i}\beta_l & j = i \end{cases}, \quad j \in J \tag{7}$$

and let $F_{\beta^{(i)}}(\cdot)$ be its associated distribution function, then the random variable

$$\text{NewState}(i, \beta, U) = F^{-1}_{\beta^{(i)}}(U) \tag{8}$$

takes values on J and has probability function $\beta^{(i)}$.

To simulate a MRP (Z, S) with initial phase distribution p and failure rate kernel R, with bounded failure transition rate from each phase of the MRP, as before, let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ be two sequences of independent uniform random variables on $(0, 1)$, defined on independent probability spaces $\Lambda_1 = (\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ and $\Lambda_2 = (\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, and construct (Z, S) on the product space $\Lambda = \Lambda_1 \times \Lambda_2$ as next described.

The sequence $(V_n)_{n\in\mathbb{N}}$ along with the consecutive phases of the phase process Z are used to simulate on Λ a sequence of arrival epochs of a modulated Poisson process $(T_m)_{m\in\mathbb{N}}$ with rate vector $\lambda = (\lambda_i)_{i\in I}$ such that

$$\lambda_i \geq \sup_t r_i(t)$$

where λ_i denotes the modulated Poisson rate in phase i, so that the MRP and the associated Poisson process are inter-dependent. On the other hand, the sequence $(U_n)_{n\in\mathbb{N}}$ along with the Failure procedure, defined in Lemma 1, are used to decide whether or not these potential renewal epochs correspond to effective failure time instants and should be included as real renewal epochs. In case the answer is affirmative, the NewState procedure, defined in Lemma 1, is used to generate the phases associated to the Poisson arrival epochs.

Specifically, for $\omega = (\omega_1, \omega_2) \in \Omega$, generate the initial phase and time from

$$Z_0(\omega) = p^{-1}(U_0(\omega_1)) \quad \text{and} \quad S_0(\omega) = 0$$

and, let $\hat{Z}_0(\omega) = Z_0(\omega)$ and $T_0(\omega) = 0$. Then, starting with $n = 0$, proceed recursively for $m \in \mathbb{N}_+$ as follows, where at the end of the cycle m will denote the index of the epoch of the uniformizing Poisson process corresponding to S_{n+1}. Generate new arrival epochs of the uniformizing Poisson process, letting

$$T_m(\omega) = T_{m-1}(\omega) + \left[\text{Exp}(\lambda_{Z_n(\omega)})\right]^{-1}(V_m(\omega_2))$$

until

$$\text{Failure}(Z_n(\omega), r_{Z_n(\omega)\cdot}(T_m(\omega) - S_n(\omega))/\lambda_{Z_n(\omega)}, U_m(\omega_1)) = 1,$$

in which case we consider that a new phase change takes place $T_m(\omega) - S_n(\omega)$ instants after the previous renewal epoch $S_n(\omega)$.

In this case, add the time $T_m(\omega)$ to the Markov renewal time sequence, and determine the new phase $Z_{n+1}(\omega)$ of the Markov renewal phase sequence, i.e.,

$$\begin{aligned} S_{n+1}(\omega) &= T_m(\omega) \\ Z_{n+1}(\omega) &= \text{NewState}(Z_n(\omega), r_{Z_n(\omega)\cdot}(T_m(\omega) - S_n(\omega))/\lambda_{Z_n(\omega)}, U_m(\omega_1)) \end{aligned}$$

and, finally, increment n by one unit.

As shown in [15], the previous procedure guarantees that the generated process is a MRP with failure rate kernel R. This follows since, in view of Lemma 1,

Input: Independent sequences $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ of independent Unif$(0,1)$ random variables, a set of nonnegative numbers $\lambda = (\lambda_i)$ such that $\lambda_i \geq \sup_t r_i(t)$, and $N \in \mathbb{N}_+$
$Z_0 := p^{-1}(U_0)$
$S_0 := 0,\ T_0 := 0$
$n := 0,\ m := 0$
while $(n < N)$ **do**
  do
    $m := m + 1$
    $T_m := T_{m-1} + [\text{Exp}(\lambda_{Z_n})]^{-1}(V_m)$
  until $(\text{Failure}(Z_n, r_{Z_n\cdot}(T_m - S_n)/\lambda_{Z_n}, U_m) = 1)$
  $S_{n+1} := T_m$
  $Z_{n+1} := \text{NewState}(Z_n, r_{Z_n\cdot}(S_{n+1} - S_n)/\lambda_{Z_n}, U_m)$
  $n := n + 1$
end while
Output: $W_t := Z_n$ for $S_n \leq t < S_{n+1},\ 0 \leq n < N$

Fig. 2. Simulation of an SMP with initial phase probability vector p and rate kernel R

conditional to the fact that an event of the Poisson process takes place $t = T_m - S_n$ units of time after the last transition epoch, S_n, at which the MRP is in phase $Z_n = i$, then a failure occurs at that instant with probability $r_i(t)/\lambda_i$, in which case $(Z_{n+1}, S_{n+1}) = (j, T_m) = (j, S_n + t)$ with probability $r_{ij}(t)/r_i(t)$.

The presented procedure leads to the algorithm presented in Fig. 2 for the simulation of a MRP with initial phase probability vector p and rate kernel R.

4 Usual Stochastic Ordering of Semi-Markov Processes

In this section we present the main results for the comparability of two SMPs in the usual stochastic ordering sense. For this purpose, we start with the definition of stochastic ordering of random vectors and stochastic processes, in the usual stochastic ordering sense (c.f., e.g., [44] or [40]).

Definition 3. *Given two real-valued random vectors $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$ whose components take values on an ordered set J, we say that X is stochastically smaller than Y in the usual sense, written $X \leq_{st} Y$, if and only if*

$$\mathbf{P}(X \in U) \leq \mathbf{P}(Y \in U), \quad \textit{for all increasing sets}^1\ U \textit{ in } J^n.$$

Roughly speaking, we say that a random vector X is stochastically smaller than a random vector Y, in the usual sense, if X is less likely than Y to take large values, where by large we mean values in any increasing set.

[1] Given an ordered set J, $U \subseteq J^n$ is called an increasing set if $x \in U$ and $x \leq y$ implies that $y \in U$, with $\leq$ denoting the componentwise ordering for vectors.

For the particular case of two real-valued random variables X and Y, the previous definition specializes into

$$X \leq_{st} Y \iff \mathbf{P}(X \geq x) \leq \mathbf{P}(Y \geq x), \quad x \in \mathbb{R}$$

as the upper sets of $\mathbb{R}$ are the intervals of the form $[u, \infty)$ or (u, ∞), $u \in \mathbb{R}$.

For the sake of simplicity, throughout the paper, order relation symbols will be applied indistinctively to compare random variables or their associated distribution functions, i.e., $X \leq_{st} Y$ is equivalent to $F^X \leq_{st} F^Y$, for random variables X and Y, with F^X and F^Y denoting the distribution functions of X and Y, respectively.

An important property of the usual stochastic order is that it is closed under convolutions [40]. That is, given sequences of independent random variables $\{X_m, 1 \leq m \leq n\}$ and $\{Y_m, 1 \leq m \leq n\}$, for a positive integer n, then

$$[X_m \leq_{st} Y_m, 1 \leq m \leq n] \Longrightarrow \sum_{m=1}^{n} X_m \leq_{st} \sum_{m=1}^{n} Y_m. \tag{9}$$

For finite measure vectors, i.e., nonnegative vectors with finite sum of their entries, we have the following definition.

Definition 4. *Given two finite measure vectors $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$ with indices on a countable ordered set I, then we say that a is smaller than b in the usual ordering sense, written $a \leq_{st} b$, if*

$$\sum_{j \geq k} a_j \leq \sum_{j \geq k} b_j, \quad k \in I. \tag{10}$$

In this case, if a and b are probability vectors we say that a is stochastically smaller than b, in the usual sense.

If X and Y are discrete random variables with support in the same ordered set I, with respective probability vectors p^X and p^Y, then $X \leq_{st} Y \iff p^X \leq_{st} p^Y$.

The usual stochastic ordering of two stochastic processes establishes the stochastic ordering of all their finite dimensional distributions, as follows.

Definition 5. *Given two stochastic processes $X = (X(t))_{t \in \mathbb{R}_+}$ and $Y = (Y(t))_{t \in \mathbb{R}_+}$ with common partially ordered state space $(I, \leq)$, then the process X is said to be stochastically smaller than Y in the usual stochastic ordering sense, written $X \leq_{st} Y$, if and only if*

$$(X(t_1), X(t_2), \ldots, X(t_n)) \leq_{st} (Y(t_1), Y(t_2), \ldots, Y(t_n))$$

for all $n \in \mathbb{N}_+$ and $t_1, t_2, \ldots, t_n \in \mathbb{R}_+$.

Alternative characterizations of the usual stochastic ordering, useful to establish stochastic ordering results in various applications, have been proposed in the literature [cf., e.g., [25,26,40]].

Theorem 1. *Given two stochastic processes X and Y with common partially ordered state space $(I, \leq)$, the following conditions are equivalent to $X \leq_{st} Y$:*

(i) For all $n \geq 1$, $t_1, t_2, \ldots, t_n \in \mathbb{R}_+$, and non-decreasing real function f,

$$E[f(X(t_1), X(t_2), \ldots, X(t_n))] \leq E[f(Y(t_1), Y(t_2), \ldots, Y(t_n))].$$

(ii) There exists $\hat{X} =_{st} X$ and $\hat{Y} =_{st} Y$ defined on a common probability space such that

$$\mathbf{P}(\hat{X}(t) \leq \hat{Y}(t), \text{ for all } t \geq 0) = 1.$$

(iii) There exists a coupling $(\hat{X}, \hat{Y})$ of X and Y with support on $\{(x, y) \in E \times E : x \leq y\}$.

As will be seen further, characterization *(ii)*, establishing that the usual stochastic order of two stochastic processes is equivalent to the pathwise comparability of some equivalent versions of these processes, was a key tool in the derivation of the main results presented throughout this tutorial paper.

Sufficient conditions for the stochastic ordering in the usual sense of two SMPs were established by Sonderman [47].

Theorem 2 (Sonderman [47, Theorem 3.2]). *For $W = X, Y$, let W be a SMP with ordered phase space I, order-isomorphic to some subset of $\mathbb{Z}$, initial probability vector p^W and failure rate kernel $R^W = (R^W(t))_{t \in \mathbb{R}_+}$, with $R^W(t) = (r_{ij}^W(t))_{i,j \in I}$. Then, $X \leq_{st} Y$ if the initial phase distributions and the failure rates satisfy*

$$p^X \leq_{st} p^Y$$

and, for all $s, t \in \mathbb{R}_+$ and $i \leq j$,

$$\sum_{k \leq n} r_{ik}^X(s) \geq \sum_{k \leq n} r_{jk}^Y(t), \ n < i, \tag{11}$$

and

$$\sum_{k \geq n} r_{ik}^X(s) \leq \sum_{k \geq n} r_{jk}^Y(t), \ n > j. \tag{12}$$

A proof of the previous result, specially tailored for the simulation of pathwise ordered equivalent versions of the involved SMPs, can be found in [15]. There, under the conditions of Theorem 2, copies of the SMPs to be compared with ordered sample-paths are constructed in a common probability space. Namely, a coupling $(X^\star, Y^\star)$ of (X, Y) such that $X^\star \leq Y^\star$ is constructed in the following way.

Let $(U_n)_{n \in \mathbb{N}_+}$ and $(V_n)_{n \in \mathbb{N}_+}$ denote two sequences of independent uniform random variables on $(0, 1)$, defined on independent probability spaces $\Lambda_1 = (\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ and $\Lambda_2 = (\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, respectively. Construct the processes $X^\star$ and $Y^\star$ on the product probability space $(\Omega, \mathcal{F}, \mathbf{P}) = \Lambda_1 \times \Lambda_2$, simulating two MRPs (Z^X, S^X) and (Z^Y, S^Y) with failure rate kernel R^X and R^Y in such a way that the construction of $X^\star$ and $Y^\star$ from

$$W_t^\star = Z_n^W, \quad S_n^W \le t < S_{n+1}^W, \quad W = X, Y \tag{13}$$

leads to $X_t^\star(\omega) \le Y_t^\star(\omega)$, for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

For that, generate the potential transitions epochs on both MRPs through a common doubly stochastic Poisson process with rate modulated by the phases of the two processes. Namely, whenever $X^\star$ is in phase i and $Y^\star$ is in phase j, use for uniformization rate a value λ_{ij} such that

$$\lambda_{ij} \ge 2 \sup_t \max\{r_i^X(t), r_j^Y(t)\}. \tag{14}$$

Specifically, for $\omega = (\omega_1, \omega_2) \in \Omega$, generate the initial phases and times from

$$Z_0^W(\omega) = [p^W]^{-1}(U_0(\omega_1)) \quad \text{and} \quad S_0^W(\omega) = 0, \quad W = X, Y. \tag{15}$$

Then, starting with $T_0(\omega) = 0$, $\hat{Z}_0^W(\omega) = Z_0^W(\omega)$ and $n_W = 0$, $W = X, Y$, proceed recursively for $m \in \mathbb{N}_+$ as follows. First let

$$\lambda^\star = \lambda_{Z_{n_X}^X(\omega)\, Z_{n_Y}^Y(\omega)}$$

denote the uniformization rate for the phase vector $(Z_{n_X}^X(\omega), Z_{n_Y}^Y(\omega))$ and let

$$\begin{aligned}
T_m(\omega) &= T_{m-1}(\omega) + [\mathrm{Exp}(\lambda^\star)]^{-1}\,(V_m(\omega_2)) \\
\hat{Z}_m^X(\omega) &= \mathrm{NewState}(Z_{n_X}^X(\omega), r_{Z_{n_X}^X(\omega)\cdot}^X(T_m(\omega) - S_{n_X}^X(\omega))/\lambda^\star, U_m(\omega_1)) \\
\hat{Z}_m^Y(\omega) &= \mathrm{NewState}(Z_{n_Y}^Y(\omega), r_{Z_{n_Y}^Y(\omega)\cdot}^Y(T_m(\omega) - S_{n_Y}^Y(\omega))/\lambda^\star, U_m(\omega_1)).
\end{aligned}$$

In sequence, for each $W = X, Y$ for which

$$\mathrm{Failure}(Z_{n_W}^W(\omega), r_{Z_{n_W}^W(\omega)\cdot}^W(T_m(\omega) - S_{n_W}^W(\omega))/\lambda^\star, U_m(\omega_1)) = 1 \tag{16}$$

i.e., for which the time $T_m(\omega)$ is a transition epoch of Z^W, include the time $T_m(\omega)$ and the phase $\hat{Z}_m^W(\omega)$ as a new pair of the sequence (Z^W, S^W) by making

$$S_{n_W+1}^W(\omega) = T_m(\omega) \quad \text{and} \quad Z_{n_W+1}^W(\omega) = \hat{Z}_m^W(\omega)$$

and let $n_W = n_W + 1$.

Finally, for $W = X, Y$, construct the SMP $W^\star$ from

$$W_t^\star(\omega) = \hat{Z}_m^W(\omega), \quad T_m(\omega) \le t < T_{m+1}(\omega) \tag{17}$$

for $m \in \mathbb{N}$, which is equivalent to generate $W^\star$ from

$$W_t^\star(\omega) = Z_n^W(\omega), \quad S_n^W(\omega) \le t < S_{n+1}^W(\omega)$$

for $n \in \mathbb{N}$.

By construction, for $W = X, Y$ the MRP (Z^W, S^W) has failure rate kernel R^W and thus the generated SMP $W^\star$ is such that $W^\star =_{st} W$. In addition, as for each pair of states (i, j) such taht $\hat{Z}^X_m(\omega) = i \leq j = \hat{Z}^Y_m(\omega)$ the uniformization rates are selected with the guaranty that $\sum_{k\neq i} \frac{r^X_{ik}(s)}{\lambda_{ij}} \leq 0.5$ and $\sum_{k\neq j} \frac{r^Y_{jk}(t)}{\lambda_{ij}} \leq 0.5$, from conditions (11)-(12) it follows that

$$\text{NewState}(i, r^X_{i\cdot}(s)/\lambda_{ij}, u) \leq \text{NewState}(j, r^Y_{j\cdot}(t)/\lambda_{ij}, u)$$

for $s, t \in \mathbb{R}_+$ and $u \in (0, 1)$. Thus, the generation of the next transitions from a common uniform generator guarantee that $\hat{Z}^X_{m+1}(\omega) \leq \hat{Z}^Y_{m+1}(\omega)$, for all $\omega \in \Omega$. As such, by induction on m and in view of (17), the proposed construction leads to two SMPs $X^\star$ and $Y^\star$ such that $X^\star_t(\omega) \leq Y^\star_t(\omega)$, for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

The decribed procedure leads to the algorithm of Fig. 3 to simulate, in a common probability space, two *st*-ordered SMPs X and Y with initial probability vectors p^X and p^Y, and failure rate kernels R^X and R^Y, respectively, under the conditions of Theorem 2.

Input: Independent sequences of independent Unif$(0, 1)$ random variables $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$, a nonnegative matrix $\lambda = (\lambda_{ij})_{i,j\in I}$ such that $\lambda_{ij} \geq 2 \sup_t \max\{r^X_i(t), r^Y_j(t)\}$, and a positive value TMAX

$n_X := n_Y := 0;$ $\quad T_0 := 0,\ \ m := 0$

$Z^X_0 := [p^X]^{-1}(U_0);$ $\quad Z^Y_0 := [p^Y]^{-1}(U_0)$

while $\left(\min\{S^X_{n_X}, S^Y_{n_Y}\} < \text{TMAX}\right)$ **do**

  $\lambda^\star = \lambda_{Z^X_{n_X}\, Z^Y_{n_Y}}$

  do

    $m := m + 1$

    $T_m := T_{m-1} + [\text{Exp}(\lambda^\star)]^{-1}(V_m)$

  until $(\text{Failure}(Z^X_{n_X}, r^X_{Z^X_{n_X}\cdot}(T_m - S^X_{n_X})/\lambda^\star, U_m) = 1$ or

    $\text{Failure}(Z^Y_{n_Y}, r^Y_{Z^Y_{n_Y}\cdot}(T_m - S^Y_{n_Y})/\lambda^\star, U_m) = 1)$

  for $W = X, Y$ **do**

    if $(\text{Failure}(Z^W_{n_W}, r^W_{Z^W_{n_W}\cdot}(T_m - S^W_{n_W})/\lambda^\star, U_m) = 1)$ **then**

      $Z^W_{n_W+1} := \text{NewState}(Z^W_{n_W}, r^W_{Z^W_{n_W}\cdot}(T_m - S^W_{n_W})/\lambda^\star, U_m)$

      $S^W_{n_W+1} := T_m$

      $n_W := n_W + 1$

    end if

  end for

end while

Output: $X_t := Z^X_l$ for $S^X_l \leq t < S^X_{l+1},\ 0 \leq l < n_X$

$Y_t := Z^Y_l$ for $S^Y_l \leq t < S^Y_{l+1},\ 0 \leq l < n_Y$

Fig. 3. Simulation of two *st*-ordered CTMCs, under the conditions of Theorem 2

4.1 Usual Stochastic Ordering of CTMCs

For the particular case of two CTMCs, Sonderman's result specialize into the earlier derived Kirstein's [28] sufficient conditions for the stochastic ordering in the usual sense of two CTMCs.

Corollary 1 (Kirstein [28]). *Let X and Y be CTMCs with ordered state space I, order isomorphic to a subset of $\mathbb{Z}$, initial probability vectors p^X and p^Y, and (infinitesimal) generator matrices Q^X and Q^Y, respectively. Then $X \leq_{st} Y$ provided that*

$$p^X \leq_{st} p^Y \tag{18}$$

and

$$\sum_{m \geq n} q^X_{im} \leq \sum_{m \geq n} q^Y_{jm}, \quad \text{for all } i \leq j \text{ and } (n \leq i \text{ or } n > j). \tag{19}$$

In fact, for CTMCs, the failure rate at time t of the time needed for a transition from phase i to phase j to take place does not depend on t, i.e., $r_{ij}(t) = r_{ij}$, for all $t \in \mathbb{R}_+$. As a consequence, Sonderman's conditions (11)-(12) are equivalent to conditions (19) since

$$q^W_{ij} = \begin{cases} r^W_{ij} & j \neq i \\ -\sum_{l \neq i} r^W_{il} & j = i \end{cases}, \quad W = X, Y.$$

Under the conditions of Corollary 1, the simulation of *st*-ordered uniformizable CTMCs can be done in a simpler manner. In fact, it suffice to uniformize both

Input: Independent sequences of independent Unif$(0,1)$ random variables $(U_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}_+}$, and a positive value TMAX

$\lambda := \sup_{i\in I} 2\{q^X_i, q^Y_i, 1\}$

$P^{\bar{X}} := \mathbf{I} + \dfrac{Q^X}{\lambda}; \qquad P^{\bar{Y}} := \mathbf{I} + \dfrac{Q^Y}{\lambda}$

$\bar{X}_0 := [p^X]^{-1}(U_0); \qquad \bar{Y}_0 := [p^Y]^{-1}(U_0)$

$T_0 := 0; \qquad n := 0$

while $(T_n <$ TMAX$)$ **do**

  $\bar{X}_{n+1} := [p^{\bar{X}}_{\bar{X}_n\cdot}]^{-1}(U_{n+1})$

  $\bar{Y}_{n+1} := [p^{\bar{Y}}_{\bar{Y}_n\cdot}]^{-1}(U_{n+1})$

  $T_{n+1} := T_n + [\mathrm{Exp}(\lambda)]^{-1}(A_{n+1})$

  $n := n + 1$

end while

Output: $X^\star_t := \bar{X}_l$ for $T_l \leq t < T_{l+1}$, $0 \leq l < n$

$Y^\star_t := \bar{Y}_l$ for $T_l \leq t < T_{l+1}$, $0 \leq l < n$

Fig. 4. Simulation of two *st*-ordered CTMCs, under the conditions of Corollary 1

chains at a common uniformization rate, and then simulate transitions in both chains from a common sequence of independent uniform generators and, at same time, use another independent sequence of independent uniform generators to simulate the holding times in states before transitions on both chains from another common generator, as presented in the algorithm of Figure 4, where **I** denotes de identity matrix of an appropriate dimension.

The strongest generalization of such result was achieved by [37], who provides the characterization of the usual ordering of CTMCs with partially ordered state spaces in terms of conditions on their infinitesimal transition rates to upper sets.

5 Level-Crossing Ordering of Semi-Markov Processes

In this section we will focus on the comparability of SMPs in the level-crossing ordering sense, which compares stochastic processes in terms of the times they take to reach or exceed high levels. Specifically, a process X is said to be stochastic smaller in level-crossing than Y if it takes X stochastically longer to reach or exceed any given level than it does Y.

The analysis of this stochastic ordering for processes with common ordered state spaces, order isomorphic to a subset of integers, was pioneered by A. Irle and J. Gani motivated by problems of comparing random times for the detection of words. As remarked by these authors, the usual stochastic ordering was too strong to be used in the envisaged context. As such, in their pioneering work [21], times for detection of words were modelled as first passage times to up-cross levels in skip-free to the right DTMCs[2] and were directly compared in the usual stochastic ordering sense. Specifically, [21, Theorem 4.1] shows that, for two skip-free to the right DTMCs with common ordered state space, the ordering in distribution of their transition probabilities for any common initial state (which does not guarantee the usual stochastic ordering of the respective DTMCs) implies the level-crossing ordering of the DTMCs.

Imposing extra stochastic ordering conditions on the holding times in states before transitions Irle [20] established sufficient conditions for the level-crossing ordering of skip-free to the right SMPs, paying particular attention to the ordering of uniformizable CTMCs and birth-and-death processes, along with Wiener processes. These results were later improved by Ferreira and Pacheco [16,17] for skip-free to the right DTMCs, SMPs and CTMCs with common ordered state spaces. The lc-ordering analysis had further developments in [18] for general (i.e., non-skip-free to the right) DTMCs, SMPs and CTMCs with totally ordered state spaces.

Hereafter we give an overview of the main results derived in these papers for the level-crossing of two SMPs. For the sake of simplicity, the results will be presented only in terms of the level-crossing ordering of stochastic processes in

[2] We recall that a trajectory of a stochastic process with ordered state space I, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, is said to be *skip-free to the right* if it does not have jumps up more than one level and the stochastic process itself is *skip-free to the right* if its trajectories are almost surely skip-free to the right.

the usual sense. As such, in the following, we just refer to level-crossing ordering instead of level-crossing ordering in the usual sense. Nevertheless, we note that the results derived in [18] are valid in a general framework in which the comparison of the passage times to up-cross levels may be made using, aside the usual stochastic ordering, any integral stochastic order relation for positive variables closed for convolution, which includes many important cases, such as the Laplace transform and the increasing concave order [40,44].

5.1 Preliminaries

Let Γ be either the set of natural numbers $\mathbb{N}$, positive integers $\mathbb{N}_+$ or real nonnegative numbers $\mathbb{R}_+$. Given a set I, order isomorphic to a bounded or unbounded interval of $\mathbb{Z}$, and $y \in I$, we let $\bar{I} = I \setminus \{\sup I\}$, where $\sup I$ is the supremum of set I, $I^A = I \cap A$ denote the restriction of I to states in A, and $I^{\leq y} = I^{(-\infty,y]}$ denote the restriction of I to states smaller or equal to y.

Moreover, if $W = (W_t)_{t\in\Gamma}$ is a stochastic process with state space I, we let S_y^W denote the hitting time of the set of values greater or equal to y, i.e.,

$$S_y^W = \inf\{t \in \Gamma : W_t \geq y\} = \inf\{t \in \Gamma : W_t \in I^{\geq y}\}$$

where $\inf \emptyset = +\infty$. Finally, to introduce the definition of level-crossing ordering of stochastic processes, we let $S_{x,y}^W$ denote the hitting time of the set of values greater or equal to y when departing from state x, i.e.,

$$S_{x,y}^W = [\inf\{t \in \Gamma : W_t \geq y\} | W_0 = x].$$

Definition 6. *Let $X = (X_t)_{t\in\Gamma}$ and $Y = (Y_t)_{t\in\Gamma}$ be stochastic processes with ordered state space I. Then, the process X is said to be smaller in level-crossing than Y, denoted $X \leq_{lc} Y$, if, for any common initial state x, $S_{x,y}^Y \leq_{st} S_{x,y}^X$, for all $y \in I$, i.e.,*

$$X \leq_{lc} Y \iff S_{x,y}^Y \leq_{st} S_{x,y}^X, \text{ for all } x, y \in I.$$

The next result, provided in [17, Theorem 1], asserts that stochastic processes are stochastically monotone increasing in the level-crossing ordering sense with respect to time-clock speed, i.e., if the time-clock speed of a process is increased, then the resulting process is faster in level-crossing than the original process.

Theorem 3 (Ferreira and Pacheco [17, Theorem 1]). *Given a stochastic process $X = (X_t)_{t\in\mathbb{R}_+}$ with ordered state space, the α-parameterized family of processes $\{X^{(\alpha)}, \alpha > 0\}$ where $X_t^{(\alpha)} = X_{\alpha t}$, for $t \in \mathbb{R}_+$, denote the time-clock speed change of X by factor α, is stochastically increasing in the level-crossing ordering sense, i.e., $X^{(\alpha_1)} \leq_{lc} X^{(\alpha_2)}$, for all $\alpha_1 \leq \alpha_2$.*

Given a stochastic process $W = (W_t)_{t\in\Gamma}$ with ordered state space I, we let $W^{\leq y}$, $y \in I$, denote the process W restricted to the state space $I^{\leq y}$ in such a way that

state y is made absorbing and all states of W greater or equal to y are collapsed into state y, namely,

$$W_t^{\leq y} = \begin{cases} W_t & t < S_y^W \\ y & t \geq S_y^W \end{cases}.$$

Note that if W is a SMP with ordered state space I and $y \in I$, then $W^{\leq y}$ is also a SMP whose parameters are easily derived from the parameters of the original process. With this notation, we consider the following definition.

Definition 7. *Let $\triangle$ denote a property and W be a stochastic process with ordered state space I. Then, W has the lower-$\triangle$ property if and only if the process $W^{\leq x}$ has the $\triangle$ property, for all $x \in I$.*

In the next two subsections, we present sufficient conditions and algorithms to simulate level-crossing ordered SMPs, treating separately the cases where we impose the condition of one of the processes involved in the comparison being skip-free to the right and the case where we do not.

5.2 Level-Crossing Ordering of Skip-Free to the Right SMPs

As mentioned in the introduction, the pioneering result for the level-crossing ordering of SMPs was provided in [20] and established sufficient conditions for the level-crossing ordering of skip-free to the right SMPs, with the random times to up-cross levels being compared in the usual stochastic ordering sense. Specifically, using the characterization of an SMP via its embedded kernel, which separates the embedded transition probabilities from the distributions of the holding times in states between transitions, ([20], Theorem 2.1) establishes that the level-crossing ordering of two skip-free to the right SMPs follows from the ordering in distribution of their transition probabilities from common states, and from the reversed order of the holding times in common states before the processes make transitions.

Theorem 4 (Irle [20, Theorem 2.1]). *Let $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (Y_t)_{t\in\mathbb{R}_+}$ be two lower-regular skip-free to the right SMPs with ordered state space I, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, and embedded kernel (P^X, F^X) and (P^Y, F^Y), respectively. Then $X \leq_{lc} Y$ provided that*

$$p_{i\cdot}^X \leq_{st} p_{i\cdot}^Y, \quad i \in I$$

and

$$F_{(a,b)}^X \geq_{st} F_{(c,d)}^Y, \quad a, b, c, d \in I.$$

By means of a sample-path based coupling approach [34], this result was later improved in [16] by removing the stochastic ordering conditions involving the transition probabilities from the highest state (if it exists), removing the lower-regularity and the skip-free to the right properties of the faster of the two processes (in level-crossing), and relaxing the conditions on the comparison of the

times between transitions in X and Y (namely that $F^X_{(a,b)} \geq_{st} F^Y_{(c,d)}$ for all $a, b, c, d \in I$) to $F^X_{(a,b)} \geq_{st} F^Y_{(a,c)}$ for all $a \in \bar{I}$ and $b, c \in I$, such that $b \leq c$.

Theorem 5 (Ferreira and Pacheco [16, Theorem 4.1]). *Let $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (Y_t)_{t\in\mathbb{R}_+}$ be two SMPs with ordered state space I, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, and embedded kernels (P^X, F^X) and (P^Y, F^Y), respectively, such that*

$$p^X_{i\cdot} \leq_{st} p^Y_{i\cdot}, \quad i \in \bar{I} \tag{20}$$

and

$$F^X_{(a,b)} \geq_{st} F^Y_{(a,c)} \tag{21}$$

holds simultaneously for all $a \in \bar{I}$ and $b, c \in I$, with $b \leq c$, such that $p^X_{ab}\, p^Y_{ac} > 0$.

If the processes X is skip-free to the right and lower-regular, then $X \leq_{lc} Y$.

To prove this result, the authors showed that, under the conditions of Theorem 5, we may construct, in a common probability space, copies of the SMPs to be compared with level-crossing ordered sample-paths, i.e., to construct

$$X^\star =_{st} X, \quad Y^\star =_{st} Y, \qquad \text{such that} \qquad X^\star \leq_{lc} Y^\star.$$

For that, using two independent sequences of independent uniform random variables, $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$, the authors begin to simulate two DTMCs, $\hat{X}$ and $\hat{Y}$, (with common initial state) with transition probability matrices P^X and P^Y, respectively, such that $\hat{X} \leq_{lc} \hat{Y}$. The main idea of the proof consists in simulating $\hat{Y}$, the faster of the two DTMCs in level-crossing, in advance and to simulate transitions in both chains from a common uniform generator only when the slower of the two DTMCs, $\hat{X}$, reaches successively each one of the states visited by $\hat{Y}$. Accordingly, they propose to first generate $\hat{Y}$ using the standard procedure to simulate a DTMC from a sequence of independent uniform random variables, $(U_n)_{n\in N}$. Then, to simulate $\hat{X}$ based on the generated sample path of $\hat{Y}$ and the skip-free to the right property of X: (a) whenever $\hat{X}$ reaches the next state on the sample path of $\hat{Y}$, the next transition in $\hat{X}$ is simulated using the generator previously used to simulate the transition from the next state in $\hat{Y}$; and (b) any other transition in $\hat{X}$ is simulated from an independent sequence of independent uniform random variables, $(V_n)_{n\in\mathbb{N}}$.

From the construction, as the embedded DTMCs at transition epochs of X and Y satisfy the conditions (20), it readily follows that, whenever $\hat{X}$ reaches the next state on the sample path of $\hat{Y}$, the next transition will put $\hat{X}$ in a smaller state than the next state visited by $\hat{Y}$

$$\hat{X}_n = \hat{Y}_m \Longrightarrow \hat{X}_{n+1} = \left[p^X_{\hat{X}_n\cdot}\right]^{-1}(p) \leq \hat{Y}_{m+1} = \left[p^Y_{\hat{Y}_m\cdot}\right]^{-1}(p), \quad p \in (0,1).$$

As a consequence, $\hat{X}$ will need at least as many transitions as $\hat{Y}$ to reach a state greater or equal to any given desired state, so that $\hat{X} \leq_{lc} \hat{Y}$.

At the same time, they propose to use two other independent sequences of independent uniform random variables, $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$, to simulate the holding times in states before transitions on both processes (say $(H_n^{X^\star})_{n\in\mathbb{N}_+}$ and $(H_n^{Y^\star})_{n\in\mathbb{N}_+}$) in the following way: (a) the holding times in states given the next state visited for the two SMPs are computed from a common generator, the sequence $(A_n)_{n\in\mathbb{N}}$, whenever the corresponding transitions in $\hat{X}$ and $\hat{Y}$ are simulated from a common uniform generator, the sequence $(U_n)_{n\in\mathbb{N}}$; and (b) the holding times in any other state (of $X^\star$) are simulated from the sequence $(B_n)_{n\in\mathbb{N}}$, i.e., the holding times in states given the next state visited in $X^\star$ are computed from the sequence $(B_n)_{n\in\mathbb{N}}$ whenever the corresponding transitions in $\hat{X}$ are simulated from the sequence $(V_n)_{n\in\mathbb{N}}$. Finally, the SMPs $X^\star$ and $Y^\star$ are obtained by letting, for $W = X, Y$,

$$W_t^\star = \hat{W}_n, \text{ for } S_n^{W^\star} \leq t < S_{n+1}^{W^\star},$$

where $S_0^{W^\star} = 0$ and $S_{n+1}^{W^\star} = S_n^{W^\star} + H_{n+1}^{W^\star}$.

By construction, $X^\star =_{st} X$, $Y^\star =_{st} Y$ and the lc-ordered DTMCs $\hat{X}$ and $\hat{Y}$ are constructed in such a way that the sequence of states visited by $\hat{X}$ until it reaches a state greater or equal to any desired given state includes the sequence of states visited by $\hat{Y}$ to reach the same set of states. As the holding times of $X^\star$ and $Y^\star$ in states when $X^\star$ reaches successively the states visited by $Y^\star$ are simulated from a common uniform generator, from (21), we conclude that $X^\star$ spends at least as much time as $Y^\star$ in each of the states visited by $Y^\star$ before reaching the desired set of states. As the usual stochastic order is closed for convolution (9), it follows that $S_l^{X^\star} \geq S_l^{Y^\star}$, for all $l \in I$, i.e., $X^\star \leq_{lc} Y^\star$.

Based in this procedure, the algorithm of Figure 5 simulates, under the conditions of Theorem 5, two lc-ordered SMPs, $X^\star$ and $Y^\star$, with common initial probability vector p, and embedded kernels (P^X, F^X) and (P^Y, F^Y) respectively.

As noted by the authors ([16], Theorem 4.2), the conditions (21) in Theorem 5, on the stochastic ordering of the holding times between transitions, can be further relaxed to

$$F^X_{(a,b)} \oplus F^X_{(b,b+1)} \oplus F^X_{(b+1,b+2)} \oplus \cdots \oplus F^X_{(\min(c,a+1)-1,\min(c,a+1))} \geq_{st} F^Y_{(a,c)}$$

for all $a \in \bar{I}$ and $b, c \in I$, with $b \leq c$, such that $P^X_{ab}\left(\prod_{k=b+1}^{\min(c,a+1)} P^X_{k-1,k}\right) P^Y_{ac} > 0$, with $\oplus$ denoting convolution. However, such a relaxation is paid at the cost of obtaining conditions that are much more difficult to check.

5.3 Level-Crossing Ordering of General SMPs

The level-crossing ordering of general (i.e., non-skip-free to the right) SMPs was addressed in [18]. Under stronger stochastic ordering conditions on the transition probabilities departing from certain states and on the holding times in states between transitions, the authors asserted the following result.

Input: Independent sequences of independent Unif$(0,1)$ random variables $(U_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}}$, $(A_n)_{n\in\mathbb{N}_+}$ and $(B_n)_{n\in\mathbb{N}_+}$, and a positive value TMAX

$\hat{X}_0 := \hat{Y}_0 := p^{-1}(U_0)$
$S_0^{X^\star} := S_0^{Y^\star} := 0$
$k := 0$
while $(S_k^{Y^\star} < \text{TMAX})$ **do**
  $k := k + 1$
  $\hat{Y}_k := [p^Y_{\hat{Y}_{k-1}\cdot}]^{-1}(U_k)$
  $S_k^{Y^\star} := S_{k-1}^{Y^\star} + [F^Y_{(\hat{Y}_{k-1},\hat{Y}_k)}]^{-1}(A_k)$
end while
$n := 0$
$m := 0$
while $(S_n^{X^\star} \leq \text{TMAX})$ **do**
  $m := m + 1$
  $n := n + 1$
  $\hat{X}_n := [p^X_{\hat{X}_{n-1}\cdot}]^{-1}(U_m)$
  $S_n^{X^\star} := S_{n-1}^{X^\star} + [F^X_{(\hat{X}_{n-1},\hat{X}_n)}]^{-1}(A_m)$
  while $\left(\hat{X}_n < \hat{Y}_m \quad \& \quad S_n^{X^\star} < \text{TMAX}\right)$ **do**
    $n := n + 1$
    $\hat{X}_n := [p^X_{\hat{X}_{n-1}\cdot}]^{-1}(V_n)$
    $S_n^{X^\star} := S_{n-1}^{X^\star} + [F^X_{(\hat{X}_{n-1},\hat{X}_n)}]^{-1}(B_n)$
  end while
end while
Output: $X_t^\star := \hat{X}_l$ for $S_l^{X^\star} \leq t < S_{l+1}^{X^\star}$, $0 \leq l < n$
$Y_t^\star := \hat{Y}_l$ for $S_l^{Y^\star} \leq t < S_{l+1}^{Y^\star}$, $0 \leq l < k$

Fig. 5. Simulation of two *lc*-ordered SMPs under the conditions of Theorem 5

Theorem 6. *Let $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (Y_t)_{t\in\mathbb{R}_+}$ be two SMPs with ordered state space I, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, and embedded kernel (P^X, F^X) and (P^Y, F^Y), respectively, such that X is lower-regular. Then $X \leq_{lc} Y$, if*

$$p^X_{x\cdot} \leq_{st} p^Y_{y\cdot}, \quad \text{for all } x, y \in \bar{I}, \quad x \leq y \tag{22}$$

and

$$F^X_{(a,b)} \geq_{st} F^Y_{(c,d)} \tag{23}$$

for all $a, c \in \bar{I}$ and $b, d \in I$, with $a \leq c$, $b \leq d$, and $p^X_{ab}\, p^Y_{cd} > 0$.

This result was proved using a sample-path based coupling approach. The authors showed how to simulate, under conditions (22)-(23) of Theorem 6, SMPs $X^\star$ and $Y^\star$, departing from the same state, such that $X^\star =_{st} X$, $Y^\star =_{st} Y$, and $X^\star \leq_{lc} Y^\star$.

The simulation in this case is quite simple as the DTMCs embedded at transition epochs and the holding times in states before transitions are generated in a

synchronized manner. The copies of the DTMCs embedded at transition epochs, $\hat{X}$ and $\hat{Y}$, are simulated from an independent sequence of independent uniform random variables, $(U_n)_{n\in\mathbb{N}_+}$, making $\hat{W}_n = [P^W_{\hat{W}_n\cdot}]^{-1}(U_n)$ for $W = X, Y$. Simulating the transitions on both processes from a common uniform generator, the conditions (22) guarantee that, before $\hat{Y}$ reaches the highest level: transitions on $\hat{X}$ departing from states smaller or equal to the ones from which $\hat{Y}$ departs on the same instant always put $\hat{X}$ in states smaller or equal than the ones for which $\hat{Y}$ makes the transition, i.e.,

$$\hat{X}_n \leq \hat{Y}_n \Longrightarrow \hat{X}_{n+1} = [P^X_{\hat{X}_n\cdot}]^{-1}(U_n) \leq [P^Y_{\hat{Y}_n\cdot}]^{-1}(U_n) = \hat{Y}_{n+1}.$$

Thus, when the DTMCs start from a common level, this procedure leads to $\hat{X}_n \leq \hat{Y}_n$, for all $n \leq \inf\{m \in \mathbb{N} : \hat{Y}_m = \sup I\}$, and consequently to $\hat{X} \leq_{lc} \hat{Y}$.

At the same time, the sequences of holding times in states between transitions for both processes, say $(H^{X^\star}_n)_{n\in\mathbb{N}_+}$ and $(H^{Y^\star}_n)_{n\in\mathbb{N}_+}$, are simulated from an independent sequence of independent uniform random variables, $(A_n)_{n\in\mathbb{N}_+}$, making $H^{W^\star}_n = [F^W_{(\hat{W}_{n-1},\hat{W}_n)}]^{-1}(A_n)$, for $W = X, Y$. Finally, the SMPs $X^\star$ and $Y^\star$ are obtained by letting, for $W = X, Y$,

$$W^\star_t = \hat{W}_n, \text{ for } S^{W^\star}_n \leq t < S^{W^\star}_{n+1},$$

where $S^{W^\star}_0 = 0$ and $S^{W^\star}_{n+1} = S^{W^\star}_n + H^{W^\star}_{n+1}$.

By construction, $X^\star =_{st} X$ and $Y^\star =_{st} Y$. In addition, before $\hat{Y}$ reaches the higher state, the embedded DTMCs $\hat{X}_n$ and $\hat{Y}_n$ are strictly ordered, then $\hat{X}$ will need at least as many transitions as $\hat{Y}$ to reach a state greater or equal to any given state. Since, in addition, the holding times in states before transitions are simulated in both processes from a common uniform generator, then, from (4) and (23), before $\hat{Y}$ reaches the highest state: $X^\star$ spends in each sucessive state at least as much time as $Y^\star$. Thus, as the usual stochastic order is closed under convolution (9), we necessarily have $S^{X^\star}_l \geq_{st} S^{Y^\star}_l$, for all $l \in I$, i.e., $X^\star \leq_{lc} Y^\star$. Based on the described procedure, the algorithm of Figure 6 simulates two lc-ordered SMPs $X^\star$ and $Y^\star$, with common initial probability vector p, under the conditions of Theorem 6.

For SMPs with equal transitions probabilities other than the supremum of the state space, we can relax the conditions (23) of Theorem 6, on the times between state transitions, to conditions involving transitions between the same state in both processes, establishing that an increase of the times between state transitions of an SMP in the usual stochastic ordering sense gives rise to an increase of the associated (upper) level-crossing times in the same sense, as next stated.

Corollary 2 (Ferreira and Pacheco [18, Corollary 3]). *Let $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (Y_t)_{t\in\mathbb{R}_+}$ be two SMPs with ordered state space I, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, and embedded kernels (P^X, F^X) and (P^Y, F^Y), respectively, such that X is lower-regular. Then $X \leq_{lc} Y$ if*

$$p^X_{i\cdot} =_{st} p^Y_{i\cdot} \quad \text{and } F^X_{(i,j)} \geq_{st} F^Y_{(i,j)} \tag{24}$$

for all $i \in \bar{I}$, and for all $i \in \bar{I}$ and $j \in I$ such that $p^X_{ij} > 0$, respectively.

This result extends, to non-skip-free to the right SMPs, results for skip-free to the right SMPs with common embedded transition probability matrices, with respect to: the Laplace transform and the mean value order, in [10], and the usual stochastic order, in [20].

Input: Independent sequences of independent Unif$(0,1)$ random variables $(U_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$, and a positive value TMAX
$\hat{X}_0 := \hat{Y}_0 := p^{-1}(U_0)$
$S_0^{X^\star} := S_0^{Y^\star} := 0$
$n := 0$
while $\left(\min\{S_n^{X^\star}, S_n^{Y^\star}\} < \text{TMAX}\right)$ **do**
  $n := n + 1$
  $\hat{X}_n := [p^X_{\hat{X}_{n-1}\cdot}]^{-1}(U_n)$
  $\hat{Y}_n := [p^Y_{\hat{Y}_{n-1}\cdot}]^{-1}(U_n)$
  $S_n^{X^\star} := S_{n-1}^{X^\star} + [F^X_{(\hat{X}_{n-1},\hat{X}_n)}]^{-1}(A_n)$
  $S_n^{Y^\star} := S_{n-1}^{Y^\star} + [F^Y_{(\hat{Y}_{n-1},\hat{Y}_n)}]^{-1}(A_n)$
end while
Output: $X^\star_t := \hat{X}_k$ for $S_k^{X^\star} \leq t < S_{k+1}^{X^\star}$, $0 \leq k < n$
$Y^\star_t := \hat{Y}_k$ for $S_k^{Y^\star} \leq t < S_{k+1}^{Y^\star}$, $0 \leq k < n$

Fig. 6. Simulation of two *lc*-ordered SMPs under the conditions of Theorem 6

5.4 Level-Crossing Ordering of CTMCs

A CTMC W with ordered state space I and generator matrix $Q^W = (q^W_{ij})_{i,j\in I}$, whose corresponding transition rate from state i is $q^W_i = -q^W_{ii} = \sum_{j\neq i} q^W_{ij}$, may be interpreted as an SMP with one-step embedded transition probability matrix $P^W = (p^W_{ij})_{i,j\in I}$, where

$$p^W_{ij} = \begin{cases} (1-\delta_{ij})\frac{q^W_{ij}}{q^W_i} & q^W_i > 0 \\ \delta_{ij} & q^W_i = 0 \end{cases}$$

with δ denoting the Kronecker delta function, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, and holding times in state i exponentially distributed with rate q_i, regardless of the state visited at the next transition.

In view of the previous, the translation of Theorem 5 and Theorem 6 for the level-crossing ordering of two CTMCs goes as follows.

Corollary 3. *Let X and Y be CTMCs with state space I, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, vectors q^X and q^Y of transition rates from states, and embedded transition probability matrices P^X and P^Y, respectively. Then $X \leq_{lc} Y$ if either*

(i) *X is lower-regular, and*

$$q_i^X \le q_j^Y \quad \textit{and} \quad p_{i\cdot}^X \le_{st} p_{j\cdot}^Y, \quad \textit{for all } i,j \in \bar{I} \textit{ with } i \le j. \tag{25}$$

(ii) *X is skip-free to the right and lower-regular, and*

$$q_i^X \le q_i^Y \quad \textit{and} \quad p_{i\cdot}^X \le_{st} p_{i\cdot}^Y, \quad \textit{for all } i \in \bar{I}. \tag{26}$$

As next stated, these results were further improved in [16,18] by means of an adequate modulated adaptive uniformization of the CTMCs.

Theorem 7 (Ferreira and Pacheco ([16, Theorem 5.1], [18, Theorem 5])). *Let X and Y be CTMCs with state space I, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, and generator matrices Q^X and Q^Y, respectively. Then $X \le_{lc} Y$ if either*

(i) *X and Y are lower-regular and there exists a matrix $\bar{\beta} = (\beta_{i,j})_{i,j\in\bar{I}}$, with entries in $(0,1]$, such that*

$$\sum_{n\ge m} q_{in}^X \le \beta_{i,j} \sum_{n\ge m} q_{jn}^Y, \textit{ for all } i,j \in \bar{I} \textit{ such that } i \le j \textit{ and } (m \le i \textit{ or } m > j). \tag{27}$$

(ii) *X and Y are lower-regular, X is skip-free to the right, and there exists a vector $\bar{\alpha} = (\alpha_i)_{i\in\bar{I}}$, with entries in $(0,1]$, such that*

$$\sum_{n\ge m} q_{in}^X \le \alpha_i \sum_{n\ge m} q_{in}^Y, \quad \textit{for all } i \in \bar{I} \textit{ and } m \in I. \tag{28}$$

Figure 7 presents two algorithms for the simulation of level crossing ordered CTMCs: one for general CTMCs satisfying (i), and the other for the case in which the slower CTMC is skip-free to the right and the CTMCs satisfy (ii). Specifically, the algorithm presented on the left-hand side [right-hand side] of Figure 7 simulates, under the conditions (i) [(ii)], two CTMCs X and Y such that $X \le_{lc} Y$. Once again, these algorithms are proposed based on a sample-path based coupling proofs of these results, provided in [16,18], of which we next give a brief sketch.

The first construction uses two dependent modulated Poisson uniformization processes with rates modulated by the states of the two processes at appropriately chosen times, and generates independently the transitions in both processes from a common generator sequence. Namely, if at the time of occurrence of the nth event of the modulated Poisson uniformization process associated to X (Y) the process X (Y) goes to state i (j), then, the amounts of time X and Y stay in states i and j until the next events take place in the modulated Poisson uniformization processes associated to X and Y are generated from a common generator and have exponential distributions with rates λ_{ij}^X and λ_{ij}^Y, respectively, such that λ_{ij}^Y is greater or equal to $2\max\left\{q_j^Y, q_i^X/\beta_{ij}, 1\right\}$ and $\lambda_{ij}^X = \beta_{ij}\lambda_{ij}^Y$, so that

$$0 < \lambda_{ij}^X = \beta_{ij}\lambda_{ij}^Y \le \lambda_{ij}^Y, \quad \text{for all } i \in \bar{I} \wedge j \in I \text{ such that } i \le j. \tag{29}$$

Moreover, the probability vectors of the state the processes X and Y go to after those events occur are

$$\hat{p}_{i\cdot}^{(X,j)} = \mathbf{e}_i + \frac{q_{i\cdot}^X}{\lambda_{ij}^X} \quad \text{and} \quad \hat{p}_{j\cdot}^{(Y,i)} = \mathbf{e}_j + \frac{q_{j\cdot}^Y}{\lambda_{ij}^Y}$$

respectively. Since, under conditions (i), we have $\lambda_{ij}^X \geq \lambda_{ij}^Y$ and $\hat{p}_{i\cdot}^{(X,j)} \leq_{st} \hat{p}_{j\cdot}^{(Y,i)}$, whenever $i \leq j < \sup I$, the conditions of Theorem 6 are satisfied, and thus the procedure described guarantees that $X \leq_{lc} Y$.

If, in addition, the slower CTMC is skip-free to the right, then instead of comparing the upper sums of different rows of the generator matrices of the two CTMCs, as (27) imposes, we may compare only the upper sums for the same rows of the generator matrices of the two CTMCs, as stated in (28). In this case, the procedure to simulate lc-ordered CTMCs is presented on the right-hand side of Figure 7 and is based on two uniformizing Poisson processes with state dependent rates modulated by the states of the process itself, such that, whenever X (Y) is in state i (j) the uniformizing Poisson process of X (Y) has rate $\beta_i\lambda_i$ (λ_j), where

$$\lambda_k = \max\{q_k^Y, q_k^X/\beta_k, 1\}. \tag{30}$$

The amounts of time X and Y stay in states i and j until the next events of the corresponding modulated Poisson uniformization processes take place are generated from a common generator and have exponential distributions with rates $\beta_i\lambda_i$ and λ_j, respectively, and the probability vectors of the state the processes X and Y go to after those events occur are

$$\hat{p}_{i\cdot}^X = \mathbf{e}_i + \frac{q_{i\cdot}^X}{\beta_i\lambda_i} \quad \text{and} \quad \hat{p}_{j\cdot}^Y = \mathbf{e}_j + \frac{q_{j\cdot}^Y}{\lambda_j}.$$

Since, under conditions (ii), we have $\beta_i\lambda_i \leq \lambda_i$ and $\hat{p}_{i\cdot}^X \leq_{st} \hat{p}_{i\cdot}^Y$, for all $i \in \bar{I}$, i.e., the conditions of Theorem 5 are satisfied, it follows that $X \leq_{lc} Y$.

We end this section noting that, uniformizing the CTMCs X and Y with two possibly different constant (i.e., non state-dependent) uniformization rates, say α^X and α^Y, respectively, such that $\alpha = \alpha^X/\alpha^Y \leq 1$, then conditions (28) specialize into

$$\exists\, \alpha \in (0,1] : \sum_{m \geq n} q_{im}^X \leq \alpha \sum_{m \geq n} q_{im}^Y, \quad \text{for all } i \in \bar{I} \text{ and } n \in I.$$

This result was first achieved in ([17], Theorem 3.1) by observing that the level-crossing ordering is stochastically monotone increasing with respect to time clock speed-ups, and constitutes itself a first generalization of ([20], Corollary 2.1) which establishes the same conclusion for two lower-uniformizable skip-free to the right CTMCs with the constant α taking the value one.

It is important to note also that the sufficient conditions for the level-crossing ordering of two general CTMCs are weaker but related to Kirstein's (19) sufficient conditions for the stochastic ordering of CTMCs in the usual sense. In fact, if such conditions are valid for $\beta_{i,j} = 1$, then the CTMCs, departing from a common state, will be also ordered in the usual sense.

Input: independent sequences of independent Unif$(0,1)$ random variables $(U_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}_+}$, the matrix $(\beta_{ij})_{i,j\in I}$, and a positive value TMAX

for $(i,j\in I)$ **do**

$$\lambda_{ij} := 2\max\{q_i^Y, q_i^X/\beta_{ij}, 1\}$$
$$\hat{p}_{i\cdot}^{(X,j)} := \mathbf{e}_i + \frac{q_{i\cdot}^X}{\beta_{ij}\lambda_{ij}}$$
$$\hat{p}_{j\cdot}^{(i,Y)} := \mathbf{e}_j + \frac{q_{j\cdot}^Y}{\lambda_{ij}}$$

end for

$Z_0^X := Z_0^Y := \mathbf{p}^{-1}(U_0)$
$S_0^X := S_0^Y := 0$
$n := 0$

while $\left(\min\{S_n^X, S_n^Y\} < \text{TMAX}\right)$
$n := n+1$
$$Z_n^X := \left[\hat{P}^{(X,Z_{n-1}^Y)}_{Z_{n-1}^X\cdot}\right]^{-1}(U_n)$$
$$Z_n^Y := \left[\hat{P}^{(Y,Z_{n-1}^X)}_{Z_{n-1}^Y\cdot}\right]^{-1}(U_n)$$
$$S_n^X := S_{n-1}^X + \left[\text{Exp}\left(\beta_{Z_{n-1}^X Z_{n-1}^Y}\lambda_{Z_{n-1}^X Z_{n-1}^Y}\right)\right]^{-1}(A_n)$$
$$S_n^Y := S_{n-1}^Y + \left[\text{Exp}\left(\lambda_{Z_{n-1}^X Z_{n-1}^Y}\right)\right]^{-1}(A_n)$$
end while

Output:
$X_t := Z_k^X$ for $S_k^X \le t < S_{k+1}^X$, $0 \le k < n$
$Y_t := Z_k^Y$ for $S_k^Y \le t < S_{k+1}^Y$, $0 \le k < n$

Input: independent sequences of independent Unif$(0,1)$ random variables $(U_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}_+}$, $(A_n)_{n\in\mathbb{N}_+}$ and $(B_n)_{n\in\mathbb{N}_+}$, the vector $(\beta_i)_{i\in I}$, and a value TMAX

for $(i\in I)$ **do**

$$\lambda_i := \max\{q_i^Y, q_i^X/\beta_i\}$$
$$\hat{p}_{i\cdot}^X := \mathbf{e}_i + \frac{q_{i\cdot}^X}{\beta_i\lambda_i}$$
$$\hat{p}_{i\cdot}^Y := \mathbf{e}_i + \frac{q_{i\cdot}^Y}{\lambda_i}$$

end for
$Z_0^X := Z_0^Y := \mathbf{p}^{-1}(U_0)$
$S_0^X := S_0^Y := 0$
$k := 0$

while $(S_k^Y < \text{TMAX})$

$k := k+1$
$$Z_k^Y := \left[\hat{P}^Y_{Z_{k-1}^Y\cdot}\right]^{-1}(U_k)$$
$$S_k^Y := S_{k-1}^Y + \left[\text{Exp}(\lambda_{Z_{k-1}^Y})\right]^{-1}(A_k)$$

end while

$S_0^X := 0,\ m := 0,\ n := 0$
while $(S_n^X < \text{TMAX})$
$m := m+1$
$n := n+1$
$$Z_n^X := \left[\hat{P}^X_{Z_{n-1}^X\cdot}\right]^{-1}(U_m)$$
$$S_n^X := S_{n-1}^X + \left[\text{Exp}(\beta_{Z_{n-1}^X}\lambda_{Z_{n-1}^X})\right]^{-1}(A_m)$$
while $\left(Z_n^X < Z_m^Y \quad \& \quad S_n^X < \text{TMAX}\right)$
$n := n+1$
$$Z_n^X := \left[\hat{P}^X_{Z_{n-1}^X\cdot}\right]^{-1}(V_n)$$
$$S_n^X := S_{n-1}^X + \left[\text{Exp}(\beta_{Z_{n-1}^X}\lambda_{Z_{n-1}^X})\right]^{-1}(B_n)$$
end while
end while
Output:
$X_t := Z_l^X$ for $S_l^X \le t < S_{l+1}^X$, $0 \le l < n$
$Y_t := Z_l^Y$ for $S_l^Y \le t < S_{l+1}^Y$, $0 \le l < k$

Fig. 7. Algorithm for the simulation of two *lc*-ordered CTMCs with initial probability vector $\mathbf{p}$, under conditions: (27), on the left-hand side; and, (28), on the right-hand side

5.5 Some Applications

As an illustration of the applicability of the results presented in the previous section, we apply those results to Poisson shock models and birth-and-death processes (with possible catastrophes) to derive sets of sufficient conditions for the level-crossing ordering of such processes.

Level-crossing ordering of two birth-and-death processes with catastrophes. Let I be a subset of $\mathbb{N}$, $\lambda = (\lambda_i)_{i\in I}$, $\mu = (\mu_i)_{i\in I}$ and $\beta = (\beta_i)_{i\in I}$ be nonnegative vectors such that $\lambda_{\sup I} = 0$ if I is bounded above and $\mu_{\inf I} = \beta_{\inf I} = 0$, and $C = (c_{ij})_{i,j\in I}$ be a lower-triangular stochastic matrix.

A $(I, \lambda, \mu, \beta, C)$ birth-and-death process with catastrophes (BDC process) is a skip-free to the right CTMC with state space I and generator matrix Q, where

$$q_{ij} = \beta_i c_{ij} + \mu_i \delta_{j,i-1} + \lambda_i \delta_{j,i+1}, \quad i \neq j. \tag{31}$$

In such processes, the nonnegative parameters λ_i, μ_i and β_i are interpreted as the birth, death and catastrophe rates of the process in state i. In addition, the matrix C is seen as the catastrophe probability matrix with c_{ij} denoting the probability that the state resulting from a catastrophe taking place in state i is j.

A direct application of Theorem 7 to BDC processes leads to the following set of sufficient conditions for their level-crossing ordering.

Theorem 8. *For $W = X, Y$, let W be an $(I, \lambda^W, \mu^W, \beta^W, C^W)$ BDC process. Then:*

(i) *$X \leq_{lc} Y$ provided that, for some vector $\bar{\alpha} = (\alpha_i)_{i\in\bar{I}}$ with entries in $(0, 1]$, the following conditions hold*

$$\lambda_i^X \leq \alpha_i \lambda_i^Y \ \wedge\ \mu_i^X \geq \alpha_i\, \mu_i^Y \ \wedge\ \beta_i^X \geq \alpha_i\, \beta_i^Y, \quad \textit{for all } i \in \bar{I} \tag{32}$$

$$c_{i\cdot}^X \leq_{st} c_{i\cdot}^Y, \quad \textit{for all } i \in \bar{I}. \tag{33}$$

(ii) *$X \leq_{st} Y$ provided that the following conditions hold*

$$\lambda_j^X \leq \lambda_j^Y \ \wedge\ \mu_i^X \geq \mu_m^Y \ \wedge\ \beta_i^X \geq \beta_m^Y, \quad \textit{for all } j \textit{ and } i \leq m \tag{34}$$

$$c_{i\cdot}^X \leq_{st} c_{j\cdot}^Y, \quad \textit{for all } i, j \in \bar{I} \textit{ such that } i \leq j. \tag{35}$$

Important types of catastrophe families are described, e.g., in [5] and [12]. These include: Binomial (p), $0 \leq p \leq 1$; Geometric (p), $0 \leq p \leq 1$; Uniform; Deterministic(f), where $f = (f_i)_{i\in I}$ is a vector such that $f_i \leq i$, for all $i \in I$; and Total. Some details on the catastrophe probability matrices associated to each of these types of catastrophe families are given in Table 1. In the following we use the denotation of the type of catastrophe indistinctly of the associated catastrophe probability matrix; thus we write, e.g., $C = \text{Binomial}(p)$ whenever C is the catastrophe probability matrix of Binomial(p) catastrophes.

Table 1. Important types of catastrophe families

Type of catastrophe	$c_{ij} \quad (0 \le j \le i)$	$\sum_{j=0}^{k} c_{ij} \quad (0 \le k \le i)$
Binomial (p), $p \in [0,1]$	$\binom{i}{j} p^{i-j}(1-p)^j$	$\sum_{j=0}^{k} \binom{i}{j} p^{i-j}(1-p)^j$
Geometric (p), $p \in [0,1]$	$p^i \delta_{j0} + (1-p)p^{i-j}\mathbf{1}_{\{j>0\}}$	p^{i-k}
Uniform	$1/(i+1)$	$(k+1)/(i+1)$
Deterministic(f), $0 \le f_i \le i$	δ_{jf_i}	$\mathbf{1}_{\{k \ge f_i\}}$
Total	δ_{j0}	1

Table 2. Some ordering relations associated to catastrophe probability matrices

C^X	C^Y	$c^X_{i\cdot} \le_{st} c^Y_{i\cdot}$	$C^X \le_K C^Y$
Binomial (p_1)	Binomial (p_2)	$p_1 \ge p_2$	$p_1 \ge p_2$
Geometric (p_1)	Geometric (p_2)	$p_1 \ge p_2$	$p_1 \ge p_2$
Binomial (p_1)	Geometric (p_2)	$p_1 \ge p_2$	$p_1 \ge p_2$
Uniform	Uniform	yes	yes
Deterministic(f^X)	Deterministic(f^Y)	$f^X_i \le f^Y_i$	$f^X \le f^Y$ and $f^X \uparrow$
Total	arbitrary	yes	yes

Table 2 presents some situations where the ordering relations (33) and (35) involving catastrophe probability matrices hold, which are relevant for the use of Theorem 8. Note that, in particular, binomial and geometric catastrophes [[5] and [12]] are stochastically decreasing in the parameter and total catastrophes [6] are the smallest catastrophes, in sense of both (33) and (35).

Nothe that, Theorem 8 (i) implies that BDC processes stochastically increase in the level-crossing ordering sense as the catastrophe distribution in each state increases stochastically in the usual sense. That is: BDC processes, X and Y, that share the birth, death and catastrophe rates but have different catastrophe probability matrices, C^X and C^Y, satisfy $X \le_{lc} Y$ provided that $c^X_{i\cdot} \le_{st} c^Y_{i\cdot}$ for all $i \in \bar{I}$. Thus, e.g., BDC processes with binomial catastrophes stochastically decrease in level-crossing with the catastrophe probability. Note also that if two BDC processes, X and Y, have the same catastrophe probability matrix, it suffices to show that (32) holds to conclude that $X \le_{lc} Y$.

Level-crossing ordering of two birth-and-death processes. Particular consequences of Theorem 8 follow for the level-crossing ordering of birth-and-death processes (BD processes), i.e., BDC processes $(I, \lambda, \mu, \beta, C)$ with null vector β (which we denote by (I, λ, μ, C) BD processes), thus turning irrelevant the form of the catastrophe probability matrix C.

Corollary 4. *For $W = X, Y$, let W be an (I, λ^W, μ^W) BD process. If*

$$\lambda^X_i \le \alpha_i \lambda^Y_i \quad \wedge \quad \mu^X_i \ge \alpha_i \mu^Y_i, \quad \textit{for all } i \in \bar{I} \tag{36}$$

for some vector $\bar{\alpha} = (\alpha_i)_{i\in\bar{I}}$ *with entries in* $(0,1]$ *, then* $X \leq_{lc} Y$*. Moreover, if* X *and* Y *are irreducible, with conditions (36) holding for* $\alpha_i = \alpha$*,* $i \in \bar{I}$*, for some constant* $\alpha \in (0,1)$*, then the same conclusion is obtained if (36) is replaced by*

$$\sup_{i\neq \sup I} \frac{\lambda_i^X}{\lambda_i^Y} \leq \alpha \leq \inf_{i\neq \inf I, \sup I} \frac{\mu_i^X}{\mu_i^Y}. \tag{37}$$

We note that for irreducible BD processes both Kirstein's conditions (19), for the *st*-ordering, and Irle's conditions, for the *lc*-ordering and derived in [20], given respectively by

$$\lambda_i^X \leq \lambda_i^Y \quad \text{and} \quad \mu_i^X \geq \mu_i^Y, \quad i \in \bar{I}$$

and

$$\lambda_i^X + \mu_i^X \leq \lambda_i^Y + \mu_i^Y \quad \text{and} \quad \frac{\mu_i^X}{\lambda_i^X} \geq \frac{\mu_i^Y}{\lambda_i^Y}, \quad i \in \bar{I},$$

are not equivalent to (36).

As the number of customers in a $M/M/s/c$ system [see, e.g., [19]] with arrival rate η and death rate γ, where the system capacity c may be either finite or infinite, can be seen as a BD processes on $\mathbb{N}$ having birth rates $\lambda_i = \eta\, \mathbf{1}_{\{0\leq i\leq c-1\}}$ and death rates $\mu_i = \gamma \min(i, s)$, Corollary 4 applies directly to derive sufficient conditions for the level-crossing ordering of two $M/M/s/c$ systems, as follows.

Corollary 5 (Ferreira and Pacheco ([17, Corollary 4.2])). *For* $W = X, Y$*, let* W *denote the number of customers in an* $M/M/s^W/\, c^W$ *system with arrival rate* λ^W *and service rate* μ^W*. If* $c^X \leq c^Y$ *and*

$$\frac{\lambda^X}{\lambda^Y} \leq \alpha \leq \frac{\mu^X}{\mu^Y} \min\left(1, \frac{s^X}{s^Y}\right) \tag{38}$$

for some $\alpha \in (0,1]$*, then* $X \leq_{lc} Y$*.*

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