Pairwise Well-Formed Scales and a Bestiary of Animals on the Hexagonal Lattice

Jon Wild

Department of Music Research, Schulich School of Music,
McGill University, Montreal, Canada
wild@music.mcgill.ca

Abstract. Some pitch-class collections may be represented as subsets of a two-dimensional lattice or generalised *Tonnetz*. Whereas a well-formed scale of cardinality *n* is formed as a simple interval chain, and thus defined unambiguously by the size of its generating interval, there are a great number of inequivalent ways of forming connected *n*-subsets of the two-dimensional lattice defined by a given pair of basis intervals. Only very few of these connected subsets or *lattice animals* ever turn out to correspond to collections that possess the pairwise well-formed property. Pwwf scales are found to correspond to members of a small family of lattice animals that is independent of the generators at the basis of the lattice. Finally a method is shown for constructing a pair of generators that will yield any given heptatonic pwwf scale; the method is easily extended to other cardinalities.

Well-Formed Scales

Carey and Clampitt [1] introduced the concept of a *well-formed scale* to the music-theoretical community; these collections had earlier been investigated by Erv Wilson [2] under the name "Moments of Symmetry". For the purposes of the present paper an important characteristic of a well-formed scale can be expressed as follows: it is a scale with exactly two step sizes, whose "tokenised" cyclic interval list (e.g. *aaabaaabaab*) has the property that each token is maximally evenly distributed among the other tokens. Well-formed scales are *generated*; that is, they are formed by the iteration of a single generating interval, with the resulting pitches collapsed into an octave. The properties of such collections have been extensively researched in recent music-theoretical literature; see for example [3].

Pairwise Well-Formed Scales

Clampitt [4] introduced a generalisation of the well-formed scale: the *pairwise well-formed scale*. In his elegant study Clampitt demonstrates a number of interesting structural and transformational features of pwwf scales and locates examples from world musics as well as from 20th-century Western music. A simple characterisation of these scales that serves our current purposes is as follows: a pwwf scale has exactly three sizes of step (I will call such scales *three-stepped*), and its token list has the

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property that each token is maximally evenly distributed among the others. Only odd cardinalities of pwwf scale are possible, and the multiplicity of each step size must be coprime to the scale cardinality. Clampitt gives the example of the diatonic scale in Zarlino's syntonic tuning [5], whose scale steps enjoy the following frequency ratios relative to C: 1:1, 9:8, 5:4, 4:3, 3:2, 5:3, 15:8, 2:1. Setting the step sizes *a*=9:8, *b*=10:9, *c*=16:15 we obtain the token list *abcabac*, which is easily verified to have the relevant property: the *a*'s are as evenly distributed as three items could be among seven; the *b*'s are as evenly distributed as two items could be; likewise for the *c*'s.

The Syntonic Diatonic as a Generated Collection

Zarlino's syntonic diatonic may be characterised as generated by two intervals, for example 3:2 and 5:4. This is easily seen in a *Tonnetz* representation as in Figure 1. The horizontal arrows show intervals of a perfect fifth, and the vertical arrows a major third.[1]

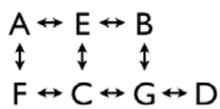

Fig. 1.

But whereas a singly-generated scale of *n* notes is unambiguously defined by the size of its generator, a doubly-generated scale like this one needs aspects of its *Tonnetz* geometry to be specified before it is uniquely determined. To illustrate, two other scales of cardinality 7 generated by the same pair of intervals are shown in Figure 2; these others are not pairwise well-formed and while the first at least is three-stepped, the second has more than three step sizes.

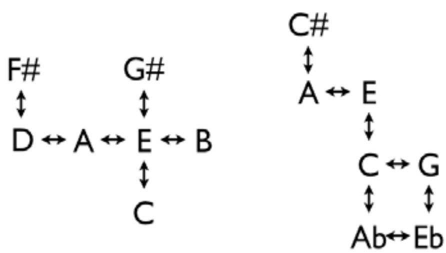

Fig. 2.

Not all doubly-generated scales, then, are three-stepped, much less pairwise well-formed. Later on we shall be interested in the converse question: are all pairwise well-formed scales doubly-generated?

[1] This pair is not the only choice of generators; we could just as easily have chosen the pair (3:2, 16:15), in which case the corresponding diagram would have the upper row offset by one position towards the left.

An Alternative Lattice Representation

The syntonic diatonic is often represented on a triangular lattice in order to emphasise the pure triads that characterise this scale. On such a lattice the connections by minor third are evident, as well as those by perfect fifth and major third; this third axis however does not represent an independent lattice dimension, because the minor third is the difference between the two generating intervals. It is well-known that the triangular lattice is dual to the hexagonal lattice: the nodes of the former become the cells of the latter; the edges of the former become connected hexagonal faces of the latter. In this manner the syntonic diatonic is represented as in Figure 3.

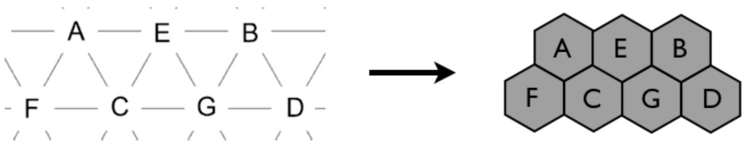

Fig. 3.

Animals on the Hexagonal Lattice—The Heptatonic Case

The first question to consider is: how many different connected configurations of *n* notes can we construct on this lattice? We shall start by considering *n*=7. We shall need to count as distinct those configurations that are related by symmetry operations on the hexagonal lattice (except for translation)—this is because rotating or reflecting amounts to permuting or inverting the generating intervals, which results in distinct scales. These configurations are analogous to *polyominoes* on a square grid, which have played an important role in recreational mathematics. Somewhat less studied are the extensions of polyominoes to other lattices such as the hexagonal one here; there are various names for these arrangements, like "polyhexes" but here I prefer the more whimsical name mathematicians sometimes use: *lattice animals*.[2] The number of *n*-animals on a hexagonal lattice increases surprisingly quickly. As with polyominoes and other polyforms, no formula is known for enumerating them directly; the animals must be explicitly generated by an algorithm whose runtime and memory requirements increase exponentially with *n*. The computation becomes prohibitively expensive when *n* reaches somewhere in the 30s. For *n* varying from 3 to 9 we obtain the following sequence for the number of *n*-animals on the hexagonal lattice: 11, 44, 186, 814, 3652, 16689, 77359. These numbers thus represent the quantity of distinct *n*-note scales that can be formed by concatenating generators of two given sizes in all possible arrangements.[3] A small number of the 3652 hepta-animals are depicted in Figure 4 to convey the variety of forms that are possible. Note that all connections are

[2] When we count rotations and reflections as distinct, we are enumerating *fixed* as opposed to *free* animals.

[3] These numbers include the animals formed by iterating only one of the two generators, along each of the three axes of the lattice. If we wished to exclude these "one-dimensional" animals, we would subtract 3 from these numbers.

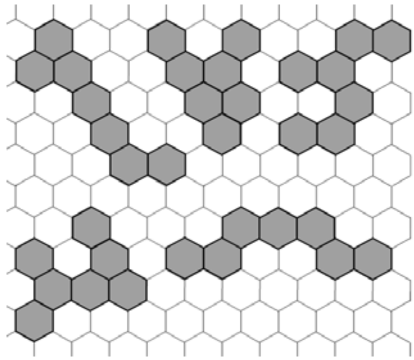

Fig. 4.

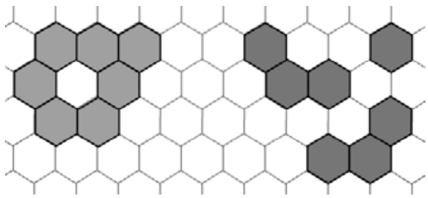

Fig. 5.

face-connections—in other words it is not enough for each note of a scale to be representable as some sum of the generators; the notes must form a connected subset (not necessarily *simply* connected, as the first animal in Figure 5 shows). A disconnected quasi-animal such as the second of Figure 5 is not enumerated here; of course if we relaxed the connectivity constraints to allow Figure 5 we would have an infinite number of scales to consider. Still, the large size of our bestiary of connected lattice animals would initially appear to be a confounding factor in the study of the relationship between animals and pwwf scales.

Three-Stepped Animals on the *Tonnetz* of Fifths and Thirds

The author wrote a computer program that generates all animals of a given size and evaluates the scale corresponding to each. Of the 3649 scales of 7 notes that use both generators (and remember we're dealing only with the *Tonnetz* generated by the pair (3:2, 5:4) for now), just 63 of them are restricted to three step sizes.

Pwwf Animals on the *Tonnetz* of Fifths and Thirds

Of the 63 three-stepped scales generated above, a third of them, 21, have the pairwise well-formed property. The corresponding 21 animals are shown in Figure 6; they are all row-convex for all three orientations of "row" (i.e. they have no gaps in the chains of generators, looked at from whatever angle) and they are "nice" in other ways developed below. As well as the syntonic diatonic mentioned above, we find an

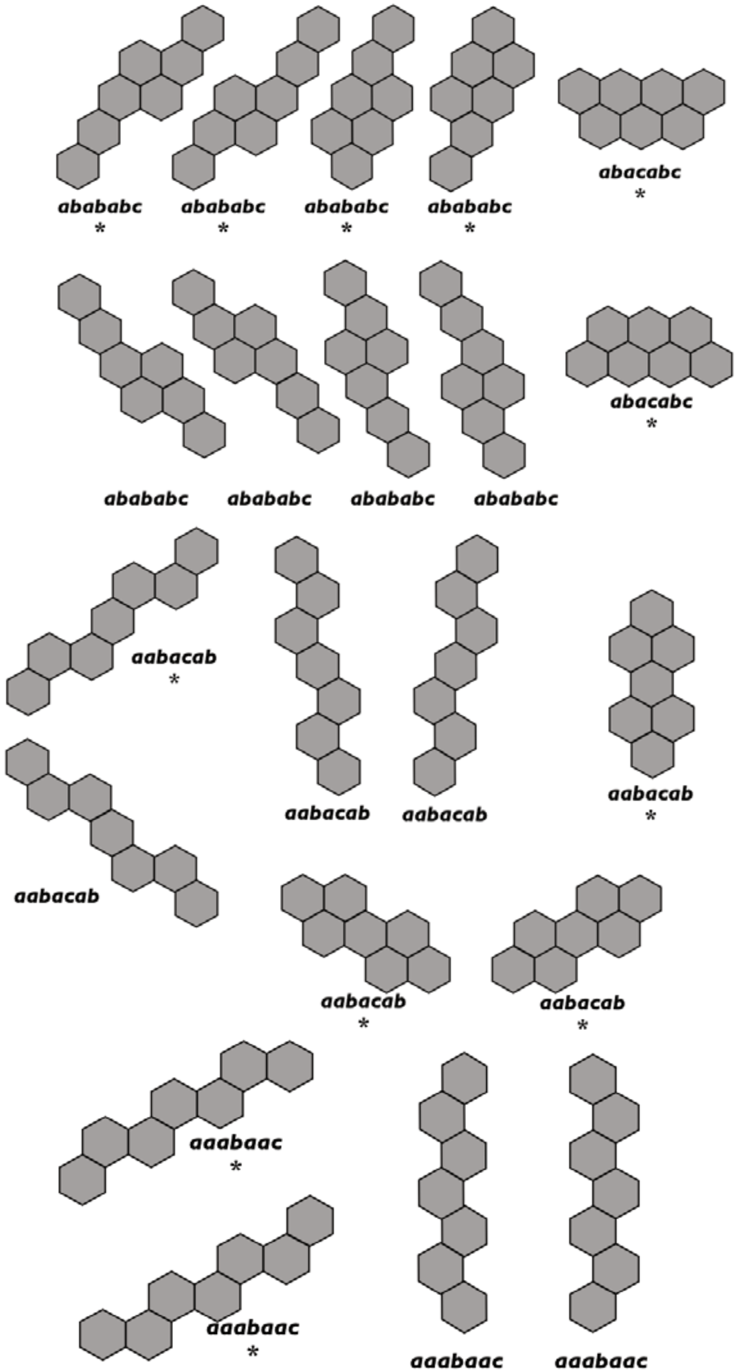
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Fig. 6.

alternative diatonic tuning (this one has also previously been noted by Clampitt as pwwf) where the note D is a syntonic comma lower than in Zarlino's scale; this variant appears in the 16th-century treatise by Fogliano [6] which slightly predates Zarlino's. A third pwwf tuning of a diatonic set, where G, E and B appear a syntonic comma lower than in the Fogliano tuning, is one I have not come across before. It contains four justly-tuned triads, one fewer than the other two diatonic tunings. Its mirror image (reflected in a vertical axis) is not diatonic; it is a tuning for the Hungarian gypsy scale which Clampitt [4] has previously identified as pwwf. Two other pwwf scales on this lattice contain four justly tuned triads; all the scales mentioned in this paragraph are shown labelled with note names in Figure 7.

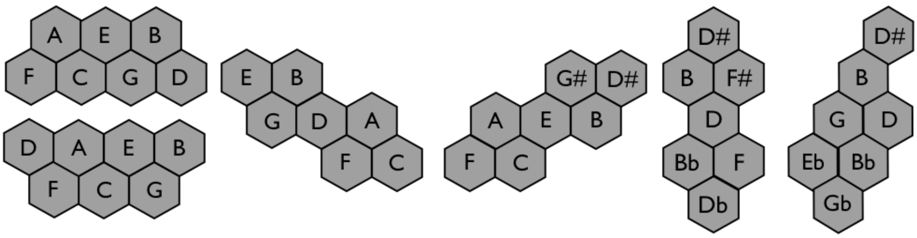

Fig. 7.

In Figure 6 a list of step size tokens was associated with each scale; Clampitt [7] has recently identified this kind of list with the *words* of mathematical word theory. Each word has been put in a sort of prime form: among all the mappings of tokens to step sizes, and all the rotations of the word, the one with lowest lexical position (i.e. soonest in alphabetical order) has been selected as representative of the whole word class. Reducing to these primeform words shows that the 21 pwwf scales on this lattice belong to just four classes: *aaabaac*, *aabacab*, *abababc*, and *abacabc*. These in fact exhaust the pwwf words possible for scales of cardinality 7.

In none of the 21 pwwf animals are all three of the axis intervals—the perfect fifth, major third and minor third—required to construct the scale.[4] This means each of the animals is generated by only two intervals—but they may be *any* two. If we are more strict about the generators, and only enumerate animals that require exclusively the perfect fifth and major third, we will eliminate all those that have "unsupported left-leaning segments", and find there are 12 animals remaining in our bestiary; they are the ones marked with an asterisk in Figure 6, which I call the "strict list" for the generator pair (3:2, 5:4). The unmarked animals also turn out to be generated by just two intervals—but those two intervals are the fifth and *minor* third; or the major and minor thirds. The corresponding scales would appear in the strict lists for the lattices formed by those pairs of generators.[5]

[4] To illustrate, an example of a (non-pwwf) collection that *does* require all three intervals is shown in Figure 8.

[5] An alternative formulation for the results of this paper would restrict itself to the strict lists. In that formulation a square rather than hexagonal grid would be appropriate, where scales are represented by polyominoes. This eliminates the possibility of animals that require connections along the third axis of the hexagonal grid.

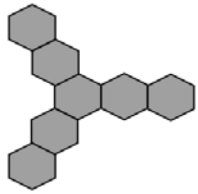

Fig. 8.

Pwwf Animals on Other Lattices

Once we begin to experiment with lattices generated by different intervals, a remarkable fact emerges empirically. Scales exhibiting pairwise well-formedness correspond

Table 1.

Pair of generators	Three-stepped heptatonic scales on this lattice	Pairwise well-formed scales on this lattice	PWWF scales, strict list **
3:2, 5:4	63	21	12
3:2, 7:4	105	44	13
5:4, 7:4	91	42	14
3:2, 11:8	83	22	16
5:4, 11:8	93	23	16
7:4, 11:8	62	19 *	14
3:2, 13:8	95	26	12
5:4, 13:8	72	26	17
7:4, 13:8	115	22	12
11:8, 13:8	70	46	14

*: The {7:4, 11:8} lattice includes one PWWF scale that is *singly*—not doubly—generated. That is, the seven-note chain formed by stacking the interval 11:7 has the pwwf property. The resulting scale has the form *aabacab*. Clampitt has previously remarked that some of these symmetrical pwwf scales may be generated by a single interval. Since a scale formed by iterating a single generator can have at most three step sizes, and since these scales can be pwwf (as in this example) or not (as in every other simply generated scale on any of the lattices in Table 1), such one-dimensional animals that "live" on a single row do not have a place in the healthy/unhealthy opposition scheme for two-dimensional lattice animals. It is trivial to find a lattice where this scale appears also as a two-dimensional animal: one generator is 11:7, and the other is 14641:2401, or 11:7 stacked four times.

**: In the "strict list" column only those scales are enumerated that can be built using the given generators literally, rather than using their combination or difference. This is a stricter requirement than that the collections form connected portions of the resulting lattices—although any collection that forms a connected portion of a lattice is strictly generated in this sense on *some* lattice.

again and again to a small group of recognisable animals from among the more than three thousand candidates, no matter the size of the generators of the lattice. First, a few statistics on the number of three-stepped and pwwf heptatonic scales for various combinations of generators are presented in Table 1.[6] As shown, different generators result in different counts.

As in the case of the lattice of Fifths and Thirds, pwwf animals on other lattices also only ever require two generators out of the three axis intervals. In fact we can make a stronger statement: pwwf animals only ever require two **directed** generators and it is possible to start from one cell and cumulatively generate all others using a pair of directed arrows, as shown in the first animal of Figure 9. Here the animal requires exclusively arrows to the right and arrows at an angle of 60 degrees (measured counterclockwise from horizontal). The second animal shown also does not require any arrows that point along the third axis—but it **does** need arrows pointing in both directions on the horizontal axis in order to reach everywhere from an initial cell, so it cannot be constructed using only two directed generators. The third animal shown requires only two directed generators, but cannot be generated from a single starting cell.[7]

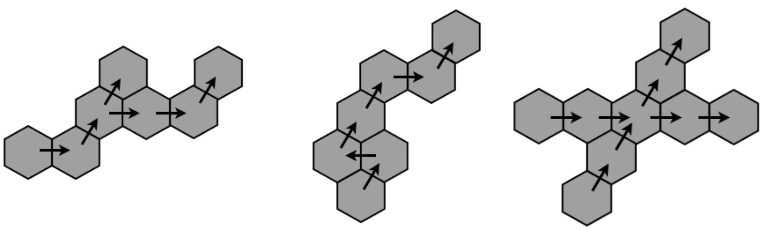

Fig. 9.

Healthy Animals

I designate as *healthy* those two-dimensional animals that, on some lattice, yield pairwise well-formed collections. Thus for example all the animals shown in Figure 6, that yield pwwf scales on the lattice of fifths and thirds, are healthy. Being healthy is not a guarantee that the corresponding scale is pwwf on *every* lattice. But the empirical evidence suggests that being healthy **and three-stepped** *is* sufficient. In other words if one of the healthy animals turns out to have three step sizes in its scale on a certain lattice, then those three step sizes will always be ordered as one of the pwwf words. And conversely, any three-stepped scale corresponding to an unhealthy animal is guaranteed *not* to be pwwf. At the time of submission these results have been

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[6] The generators entabulated here happen to be intervals from the harmonic series; any generators may of course be used, with the caveat that degenerate scales may result when it is possible to form identical sums of the two generators in more than one way.

[7] The ways in which the second and third animals shown in Figure 9 fail to meet the condition are in fact interchangeable: the middle animal could be redrawn to only require two directed generators if we had two "starting" cells; and the third animal could be redrawn with a single starting cell if we allowed one of the generating intervals to be used both upwards and downwards.

experimentally verified for a large number of generator-pairs, but the statement remains a conjecture for now.

Healthy heptatonic animals fall into what I identify as three families, shown in Figure 10. **Type I** animals (upper left of Figure 10) have two parallel chains of generators, of lengths 3 and 4, along any of the three axes. **Type II** animals (right-hand side) have three parallel chains of generators, of lengths 2, 3 and 2—in fact there are three different ways of seeing a Type II animal as three parallel chains of generators of lengths 2, 3 and 2, as shown in Figure 11. The **Type III** animal (lower left) has four parallel chains of generators, of lengths 2, 2, 2, and 1. Figure 12 shows there are two ways of seeing a Type III animal as four parallel chains of generators of lengths 2, 2, 2, and 1.

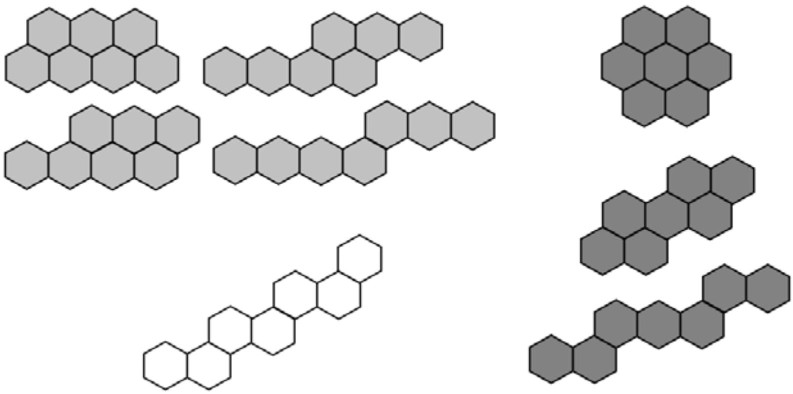

Fig. 10.

Fig. 11.

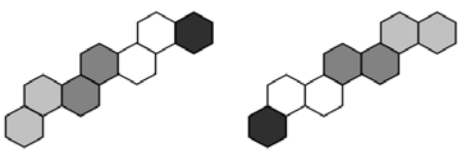

Fig. 12.

Several symmetries obtain: if a given Type I or Type III animal is pwwf on a given lattice, then so is its rotation by 180 degrees. Type II animals are rotationally symmetrical at 180 degrees; each Type II animal, if pwwf on a given lattice, will be accompanied by its mirror reflection if it is distinct.

While all the examples shown in Figure 10 were positioned so their generator chains run along the horizontal axis, any of these animals can be transformed into any of its rotations or reflections by permuting and/or inverting the generators used—the different orientations, if they are distinct, will represent the same scale on different lattices. Counting all the distinct rotations and reflections of the eight free animals shown in Figure 10 we find a total of 58 healthy animals as an upper limit for the number of heptatonic pwwf scales on any given lattice.

Type I animals support the pwwf words *aaabaac*, *abababc* and *abacabc*. Type II animals support exclusively the word *aabacab* (and I emphasise that this is true independent of the lattice's basis intervals). Type III animals, like Type I animals, support the words *aaabaac*, *abababc* and *abacabc*. In fact we see in Figure 13 how Type III animals may be decomposed into parallel chains of the same generator, of lengths 4 and 3; by substitution of generators, then, they will correspond to Type I animals on a different lattice.

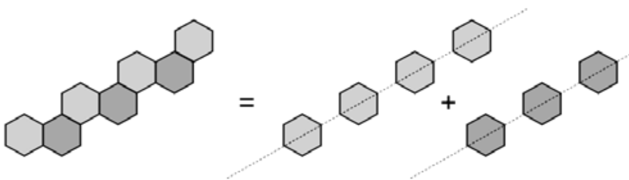

Fig. 13.

Further, it is easy to see how the four varieties of Type I animals are equivalent to one another under a substitution of generator, as are the three varieties of Type II.

We can conclude that if we are allowed to specify the lattice, we can construct every heptatonic pwwf scale discovered so far using only two animals: Type Ia for words *aaabaac*, *abababc* and *abacabc*, and Type IIa for the word *aabacab*.

It must not be imagined that the "unhealthy" heptatonic animals are uniformly misshapen. As the gallery of all non-pwwf three-stepped animals on the lattice of fifths and thirds in Figure 14 shows, many animals that support three-stepped scales that are *not* pairwise well-formed are nonetheless symmetrical, or present double chains of generators of other lengths than 4+3.

Arbitrary Heptatonic Pwwf Scales

Earlier we wondered whether *all* pwwf scales were doubly generated, just as all well-formed scales are generated by iterating a single interval. We could rephrase the question as follows: given the step-sizes *a, b* and *c* and a particular pwwf word, can we select a pair of generators such that the scale sought appears on that lattice? It turns out that the answer is yes: we can use a scale whose generators we already know, and reverse-engineer the required generators from the scalar mapping. For example, let us say we wish to construct a scale of the form *abacabc*, with step sizes *a*=150.7 cents, *b*=65.0 cents, and *c* therefore equal to 381.5 cents. We know the syntonic diatonic exhibits the same word; here that scale is shown on the lattice of fifths and thirds, where

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. . . . . . . . . o o
 . . . . . . . o o .
. . . . . . . o o . .

aaabacb. . . . o o .
. . . . . . o o . . .
 . . . . . o . . . .
. . . . o o . . . . .
```

```
aabaabc . . o o o . .
 . . . . . o o . . .
. . . . o o . . . . .

aaabcab. . . . o o .
. . . . . . . o . . .
 . . . . . o o . . .
. . . . o o . . . . .

aabaabc . o o . . . .
 . . . . . . o o . .
. . . . . . . o o o .

aabaacb . . . o o . .
 . . . . . o o . . .
. . . . o o o . . . .

aabcabc. . o o . . .
. . . . o o o o o . .

aabcbcb . . o . o . .
 . . . . . . o o o .
. . . . . . . o . o .

ababcac . . . . o .
 . . . . . . o o o o
. . . . . . . o . o .

aabcbcb . . . . o . o
 . . . . . . . o o o
. . . . . . . o . o
```

Fig. 14.

the numbers appearing on the nodes of the lattice indicate the scalar position of the corresponding pitch, when the scale is rotated to match the primeform word *abacabc* (i.e. in Lydian mode):

```
abacabc; a=203.9, b=182.4, c=111.7

. . . . . . . . . . . . . .

 . . . . 3 7 4 . . . . . .

. . . . 1 5 2 6 . . . . . .
```

The step sizes of the scale we seek to generate are completely unrelated to those of the syntonic diatonic, but we can use the pattern traced out by the scalar ordering to calculate new generators. The first scale step, 1-2, is comprised of two horizontal generators—this is apparent in the segment 1-(5)-2 on the lower row. Since the interval between degrees 1 and 2 (i.e., token *a*) is bissected by another scale degree on the lattice, we define GEN1 = (a+1200)/2. The other generator, ascending to the right on the lattice, connects scale degrees 1 and 3 (or 5 and 7, or 2 and 4). The word *abacabc* tells us that each of these generic thirds is the sum of step sizes *a* and *b*. So we define GEN2 = (a+b). Substituting the desired step size values a=150.7 and b=65.0 we obtain GEN1 = 694.05 and GEN2 = 213.8.

Likewise, by examining the mapping between scalar order and lattice arrangement for representative scales of each of the other heptatonic pwwf words, we can obtain suitable expressions for their generators as a function of their step sizes:[8]

abacabc:	GEN1 = (a+*1200*)/2;	GEN2 = (a+b);	animal is Type Ia.
abababc :	GEN1 = a+b;	GEN2 = b ;	animal is Type Ia.
aaabaac:	GEN1 = a;	GEN2 = 3a+b;	animal is Type Ia.
aabacab:	GEN1 = 2a+b+c;	GEN2 = –a;	animal is Type IIa.

Since the four words account for all heptatonic pwwf scales, it follows that *every* heptatonic pwwf scale is doubly generated.[9]

Pwwf Scales of Other Cardinalities

We find similar results for other cardinalities of scale: a given n-animal is either healthy or unhealthy, and the families of healthy animals are analogous to the heptatonic ones. The non-singular pwwf scales can be constructed as near-equal pairs of generator chains (for example, of lengths 5 and 4 when the scale cardinality is 9), corresponding to Types I and III. But we find no analogues to Type II animals for non-heptatonic scales: as Clampitt [4] has shown, the heptatonic pattern *aabacab* is unique among all pwwf words in having three different multiplicities of step sizes.

There are considerably fewer pwwf collections of cardinality 9 than for cardinality 7 on a given lattice, especially when considered relative to the hugely expanded num-

[8] In each case these are not the only expressions that would work.

[9] Some pwwf scales have an additional interpretation, as generated by a single interval. See the footnote to Table 1.

ber of candidate animals. The lattice of fifths and thirds, for example, only possesses four pwwf scales of cardinality 9 among over 77,000 candidate animals. A search for scales of cardinality 5 yields 20 pwwf scales among 186 animals on the same lattice. For n=5 and n=9 I have been able to use the above method for arbitrary pwwf scales to find formulae for the generator pairs as a function of the desired step sizes. I conjecture this will be possible for higher odd cardinalities, too.

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