Chapter 8
Option Hedging in Continuous Time

Here we review some applications to mathematical finance of the tools introduced in the previous chapters. We construct a market model with jumps in which exponential normal martingales are used to model random prices. We obtain pricing and hedging formulas for contingent claims, extending the classical Black-Scholes theory to other complete markets with jumps.

8.1 Market Model

Let $(M_t)_{t\in\mathbb{R}_+}$ be a martingale having the chaos representation property of Definition 2.8.1 and angle bracket given by $d\langle M, M\rangle_t = \alpha_t^2 dt$. By a modification of Proposition 2.10.2, $(M_t)_{t\in[0,T]}$ satisfies the structure equation

$$d[M, M]_t = \alpha_t^2 dt + \phi_t dM_t.$$

When $(\phi_t)_{t\in[0,T]}$ is deterministic, $(M_t)_{t\in[0,T]}$ is alternatively a Brownian motion or a compensated Poisson martingale, depending on the vanishing of $(\phi_t)_{t\in[0,T]}$.
Let $r : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\sigma : \mathbb{R}_+ \longrightarrow (0,\infty)$ be deterministic non negative bounded functions. We assume that $1 + \sigma_t\phi_t > 0$, $t \in [0, T]$. Let $(A_t)_{t\in\mathbb{R}_+}$ denote the price of the riskless asset, given by

$$\frac{dA_t}{A_t} = r_t dt, \qquad A_0 = 1, \qquad t \in \mathbb{R}_+, \tag{8.1.1}$$

i.e.

$$A_t = A_0 \exp\left(\int_0^t r_s ds\right), \qquad t \in \mathbb{R}_+.$$

For $t > 0$, let $(S^x_{t,u})_{u\in[t,T]}$ be the price process with risk-neutral dynamics given by

$$dS^x_{t,u} = r_t S^x_{t,u} du + \sigma_u S^x_{t,u^-} dM_u, \quad u \in [t, T], \quad S^x_{t,t} = x,$$

N. Privault, *Stochastic Analysis in Discrete and Continuous Settings*,
Lecture Notes in Mathematics 1982, DOI 10.1007/978-3-642-02380-4_8,

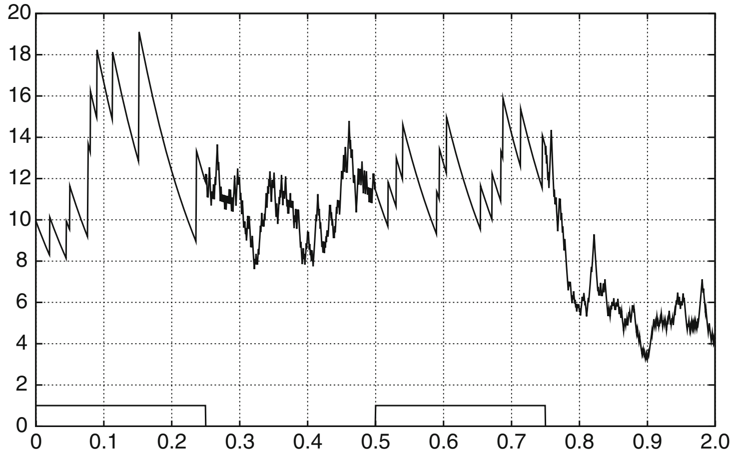

Fig. 8.1 Sample trajectory of $(S_t)_{t\in[0,T]}$

cf. Relation 2.13.5. Recall that when $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic we have

$$S^x_{t,T} = x\exp\left(\int_t^T \sigma_u\alpha_u i_u dB_u + \int_t^T (r_u - \phi_u\lambda_u\sigma_u - \frac{1}{2}i_u\sigma_u^2\alpha_u^2)du\right) \times \prod_{k=1+N_t}^{k=N_T}(1+\sigma_{T_k}\phi_{T_k}), \tag{8.1.2}$$

$0 \le t \le T$, with $S_t = S^1_{0,t}$, $t \in [0,T]$. Figure 8.1 shows a sample path of $(S_t)_{t\in[0,T]}$ when the function $(i_t)_{t\in[0,T]}$ takes values in $\{0,1\}$, with $S_0 = 10$, $\sigma_t = 10$, and $\alpha_t = 1$, $t \in [0,T]$.
Let η_t and ζ_t be the numbers of units invested at time t, respectively in the assets $(S_t)_{t\in\mathbb{R}_+}$ and $(A_t)_{t\in\mathbb{R}_+}$. The value of the portfolio V_t at time t is given by

$$V_t = \zeta_t A_t + \eta_t S_t, \qquad t \in \mathbb{R}_+. \tag{8.1.3}$$

Definition 8.1.1. *The portfolio V_t is said to be self-financing if*

$$dV_t = \zeta_t dA_t + \eta_t dS_t. \tag{8.1.4}$$

The self-financing condition can be written as

$$A_t d\zeta_t + S_t d\eta_t = 0, \qquad 0 \le t \le T$$

under the approximation $d\langle S_t, \eta_t\rangle \simeq 0$.

Let also

$$\tilde{V}_t = V_t \exp\left(-\int_0^t r_s ds\right) \quad \text{and} \quad \tilde{S}_t = S_t \exp\left(-\int_0^t r_s ds\right)$$

denote respectively the discounted portfolio price and underlying asset price.

Lemma 8.1.2. *The following statements are equivalent:*

i) the portfolio V_t is self-financing,
ii) we have

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \sigma_u \eta_u \tilde{S}_u dM_u, \qquad t \in \mathbb{R}_+, \tag{8.1.5}$$

iii) we have

$$V_t = V_0 \exp\left(\int_0^t r_u du\right) + \int_0^t \sigma_u \eta_u \exp\left(\int_u^t r_u du\right) S_u dM_u, \tag{8.1.6}$$

$t \in \mathbb{R}_+$.

Proof. First, note that (8.1.5) is clearly equivalent to (8.1.6). Next, the self-financing condition (8.1.4) shows that

$$\begin{aligned} dV_t &= \zeta_t dA_t + \eta_t dS_t \\ &= \zeta_t A_t r_t dt + \eta_t r_t S_t dt + \sigma_t \eta_t S_t dM_t \\ &= r_t V_t dt + \sigma_t \eta_t S_t dM_t, \end{aligned}$$

$t \in \mathbb{R}_+$, hence

$$\begin{aligned} d\tilde{V}_t &= d\left(\exp\left(-\int_0^t r_s ds\right) V_t\right) \\ &= -r_t \exp\left(-\int_0^t r_s ds\right) V_t dt + \exp\left(-\int_0^t r_s ds\right) dV_t \\ &= \exp\left(-\int_0^t r_s ds\right) \sigma_t \eta_t S_t dM_t, \qquad t \in \mathbb{R}_+, \end{aligned}$$

i.e. (8.1.5) holds. Conversely, if (8.1.5) is satisfied we have

$$\begin{aligned} dV_t &= d(A_t \tilde{V}_t) \\ &= \tilde{V}_t dA_t + A_t d\tilde{V}_t \\ &= \tilde{V}_t A_t r_t dt + \sigma_t \eta_t S_t dM_t \\ &= V_t r_t dt + \sigma_t \eta_t S_t dM_t \\ &= \zeta_t A_t r_t dt + \eta_t S_t r_t dt + \sigma_t \eta_t S_t dM_t \\ &= \zeta_t dA_t + \eta_t dS_t, \end{aligned}$$

hence the portfolio is self-financing. □

8.2 Hedging by the Clark Formula

In the next proposition we compute a self-financing hedging strategy leading to an arbitrary square-integrable random variable F, using the Clark formula Proposition 4.2.3.

Proposition 8.2.1. *Given $F \in L^2(\Omega)$, let*

$$\eta_t = \frac{\exp\left(-\int_t^T r_s ds\right)}{\sigma_t S_t} \mathbb{E}[D_t F | \mathcal{F}_t], \tag{8.2.1}$$

$$\zeta_t = \frac{\exp\left(-\int_t^T r_u du\right) \mathbb{E}[F|\mathcal{F}_t] - \eta_t S_t}{A_t}, \qquad t \in [0,T]. \tag{8.2.2}$$

Then the portfolio $(\eta_t, \zeta_t)_{t\in[0,T]}$ is self-financing and yields a hedging strategy leading to F, i.e. letting

$$V_t = \zeta_t A_t + \eta_t S_t, \qquad 0 \le t \le T,$$

we have

$$V_t = \exp\left(-\int_t^T r_u du\right) \mathbb{E}[F|\mathcal{F}_t], \tag{8.2.3}$$

$0 \le t \le T$. *In particular we have $V_T = F$ and*

$$V_0 = \exp\left(-\int_0^T r_u du\right) \mathbb{E}[F].$$

Proof. Applying (8.2.2) at $t = 0$ we get

$$\mathbb{E}[F] \exp\left(-\int_0^T r_u du\right) = V_0,$$

hence from (8.2.2), the definition (8.2.1) of η_t and the Clark formula we obtain

$$\begin{aligned}
V_t &= \zeta_t A_t + \eta_t S_t \\
&= \exp\left(-\int_t^T r_u du\right) \mathbb{E}[F|\mathcal{F}_t] \\
&= \exp\left(-\int_t^T r_u du\right) \left(\mathbb{E}[F] + \int_0^t \mathbb{E}[D_u F|\mathcal{F}_u] dM_u\right) \\
&= V_0 \exp\left(\int_0^t r_u du\right) + \exp\left(-\int_t^T r_u du\right) \int_0^t \mathbb{E}[D_u F|\mathcal{F}_u] dM_u \\
&= V_0 \exp\left(\int_0^t r_u du\right) + \int_0^t \eta_u \sigma_u S_u \exp\left(\int_u^t r_s ds\right) dM_u, \qquad 0 \le t \le T,
\end{aligned}$$

and from Lemma 8.1.2 this also implies that the portfolio $(\eta_t, \zeta_t)_{t\in[0,T]}$ is self-financing. □

The above proposition shows that there always exists a hedging strategy starting from

$$V_0 = \mathbb{E}[F] \exp\left(-\int_0^T r_u du\right).$$

Conversely, since there exists a hedging strategy leading to

$$\tilde{V}_T = F \exp\left(-\int_0^T r_u du\right),$$

then by (8.1.5), $(\tilde{V}_t)_{t\in[0,T]}$ is necessarily a martingale with initial value

$$\tilde{V}_0 = \mathbb{E}[\tilde{V}_T] = \mathbb{E}[F] \exp\left(-\int_0^T r_u du\right).$$

We now consider the hedging of European call option with payoff $F = (S_T - K)^+$ using the Clark formula in the setting of deterministic structure equations. In this case the next proposition allows us to compute the hedging strategy appearing in (8.2.1).

Proposition 8.2.2. *Assume that $\phi_t \geq 0$, $t \in [0,T]$. Then for $0 \leq t \leq T$ we have*

$$\mathbb{E}[D_t(S_T - K)^+ | \mathcal{F}_t] = \mathbb{E}\Big[i_t\sigma_t S^x_{t,T} \mathbf{1}_{[K,\infty)}(S^x_{t,T}) + \frac{j_t}{\phi_t}(\sigma_t\phi_t S^x_{t,T} - (K - S^x_{t,T})^+)\mathbf{1}_{[\frac{K}{1+\sigma_t},\infty)}(S^x_{t,T})\Big]_{x=S_t}.$$

Proof. By Lemma 4.6.2, using Definition 4.6.1 and Relation (4.6.4) we have, for any $F \in \mathcal{S}$,

$$D_t F = D^B_t F + \frac{j_t}{\phi_t}(T^\phi_t F - F), \quad t \in [0,T]. \tag{8.2.4}$$

We have $T^\phi_t S_T = (1 + \sigma_t\phi_t)S_T$, $t \in [0,T]$, and the chain rule $D^B f(F) = f'(F)D^B F$, cf. Relation (5.2.1), holds for $F \in \mathcal{S}$ and $f \in \mathcal{C}^2_b(\mathbb{R})$. Since $\mathcal{S}$ is an algebra for deterministic $(\phi_t)_{t\in[0,T]}$, we may approach $x \mapsto (x-K)^+$ by polynomials on compact intervals and proceed e.g. as in [97], p. 5-13. By dominated convergence, $F = (S_T - K)^+ \in \mathrm{Dom}\,(D)$ and (8.2.4) becomes

$$D_t(S_T - K)^+ = i_t\sigma_t S_T \mathbf{1}_{[K,\infty)}(S_T) + \frac{j_t}{\phi_t}((1+\sigma_t\phi_t)S_T - K)^+ - (S_T - K)^+),$$

$0 \leq t \leq T$. The Markov property of $(S_t)_{t\in[0,T]}$ implies

$$\mathbb{E}\left[D^B_t(S_T - K)^+ | \mathcal{F}_t\right] = i_t\sigma_t\, \mathbb{E}\left[S^x_{t,T}\mathbf{1}_{[K,\infty)}(S^x_{t,T})\right]_{x=S_t},$$

and

$$\begin{aligned}
&\frac{j_t}{\phi_t}\,\mathbb{E}[(T_t^\phi S_T - K)^+ - (S_T-K)^+|\mathcal{F}_t]\\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[((1+\sigma_t\phi_t)S_{t,T}^x - K)^+ - (S_{t,T}^x - K)^+\right]_{x=S_t}\\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[((1+\sigma_t\phi_t)S_{t,T}^x - K)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x)\right]_{x=S_t}\\
&\quad -\frac{j_t}{\phi_t}\,\mathbb{E}\left[(S_{t,T}^x - K)^+\mathbf{1}_{[K,\infty)}(S_{t,T}^x)\right]_{x=S_t}\\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[\sigma_t\phi_t S_{t,T}^x\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x) + (S_{t,T}^x - K)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},K]}(S_{t,T}^x)\right]_{x=S_t}\\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[\sigma_t\phi_t S_{t,T}^x\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x) - (K - S_{t,T}^x)^+\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty]}(S_{t,T}^x)\right]_{x=S_t}\\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[(\sigma_t\phi_t S_{t,T}^x - (K - S_{t,T}^x)^+)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x)\right]_{x=S_t}.
\end{aligned}$$

□

If $(\phi_t)_{t\in[0,T]}$ is not constrained to be positive then

$$\begin{aligned}
&\mathbb{E}[D_t(S_T-K)^+|\mathcal{F}_t] = i_t\sigma_t\,\mathbb{E}\left[S_{t,T}^x\mathbf{1}_{[K,\infty)}(S_{t,T}^x)\right]_{x=S_t}\\
&\quad+\frac{j_t}{\phi_t}\,\mathbb{E}\left[\sigma_t\phi_t S_{t,T}^x\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x) + (S_{t,T}^x - K)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},K]}(S_{t,T}^x)\right]_{x=S_t},
\end{aligned}$$

with the convention $\mathbf{1}_{[b,a]} = -\mathbf{1}_{[a,b]}$, $0 \le a < b \le T$. Proposition 8.2.2 can also be proved using Lemma 3.7.2 and the Itô formula (2.12.4).

In the sequel we assume that $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic and

$$dM_t = i_t dB_t + \phi_t(dN_t - \lambda_t dt), \qquad t\in\mathbb{R}_+, \qquad M_0 = 0,$$

as in Relation (2.10.4).

Next we compute

$$\exp\left(-\int_0^T r_s ds\right)\mathbb{E}\left[(S_T-K)^+\right]$$

in terms of the Black-Scholes function

$$\mathrm{BS}(x,T;r,\sigma^2;K) = \mathrm{e}^{-rT}\,\mathbb{E}[(x\mathrm{e}^{rT-\sigma^2T/2+\sigma W_T} - K)^+],$$

where W_T is a centered Gaussian random variable with variance T.

Proposition 8.2.3. *The expectation*

$$\exp\left(-\int_0^T r_s ds\right) \mathbb{E}\left[(S_T - K)^+\right]$$

can be computed as

$$\begin{aligned}
&\exp\left(-\int_0^T r_s ds\right) \mathbb{E}\left[(S_T - K)^+\right] \\
&= \exp\left(-\Gamma_0(T)\right) \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^T \cdots \int_0^T \\
&\quad BS\left(S_0 \exp\left(-\int_0^T \phi_s \gamma_s \sigma_s ds\right) \prod_{i=1}^{i=k} \left(1 + \sigma_{t_i} \phi_{t_i}\right), T; R_T, \frac{\Gamma_0(T)}{T}; K\right) \\
&\quad \gamma_{t_1} \cdots \gamma_{t_k} dt_1 \cdots dt_k.
\end{aligned}$$

Proof. Similarly to Proposition 3.7.3 we have

$$\mathbb{E}\left[e^{-TR_T}(S_T - K)^+\right] = \sum_{k=0}^{\infty} \mathbb{E}\left[e^{-TR_T}(S_T - K)^+ | N_T = k\right] \mathbb{P}(N_T = k),$$

with

$$\mathbb{P}(N_T = k) = \exp(-\Gamma_0(T)) \frac{(\Gamma_0(T))^k}{k!}, \qquad k \in \mathbb{N}.$$

Conditionally to $\{N_T = k\}$, the jump times $(T_1, \ldots, T_k)$ have the law

$$\frac{k!}{(\Gamma_0(T))^k} 1_{\{0<t_1<\cdots<t_k<T\}} \gamma_{t_1} \cdots \gamma_{t_k} dt_1 \cdots dt_k,$$

since the process $(N_{\Gamma_0^{-1}(T)_t})_{t\in\mathbb{R}_+}$ is a standard Poisson process. Hence, conditionally to

$$\{N(\Gamma_0^{-1}(\Gamma_0(T))) = k\} = \{N_T = k\},$$

its jump times $(\Gamma_0(T_1), \ldots, \Gamma_0(T_k))$ have a uniform law on $[0, \Gamma_0(T)]^k$. We then use the fact that $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ are also independent under $\mathbb{P}$ since $(r_t)_{t\in\mathbb{R}_+}$ is deterministic, and the identity in law

$$S_T \overset{law}{=} S_0 X_T \exp\left(-\int_0^T \phi_s \lambda_s \sigma_s ds\right) \prod_{k=1}^{k=N_T} \left(1 + \sigma_{T_k} \phi_{T_k}\right),$$

where

$$X_T = \exp\left(TR_T - \Gamma_0(T)/2 + \left(\frac{\Gamma_0(T)}{T} \right)^{1/2} W_T \right),$$

and W_T is independent of N. □

8.3 Black-Scholes PDE

As in the standard Black-Scholes model, it is possible to determine the hedging strategy in terms of the Delta of the price in the case $(r_t)_{t\in\mathbb{R}_+}$ is deterministic.
Let the function $C(t,x)$ be defined by

$$\begin{aligned} C(t,S_t) &= V_t \\ &= \exp\left(-\int_t^T r_u du \right) \mathbb{E}[(S_T - K)^+ \mid \mathcal{F}_t] \\ &= \exp\left(-\int_t^T r_u du \right) \mathbb{E}[(S_T - K)^+ \mid S_t], \qquad t \in \mathbb{R}_+. \end{aligned}$$

cf. (8.2.3). An application of the Itô formula leads to

$$\begin{aligned} dC(t,S_t) = &\left(\frac{\partial C}{\partial t} + r_t S_t \frac{\partial C}{\partial x} + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} i_t \alpha_t^2 S_t^2 \sigma_t^2 + \lambda_t \Theta C \right)(t,S_t)dt \\ &+ S_t \sigma_t \frac{\partial C}{\partial x}(t,S_t) dM_t + \left(C(t, S_t(1+\sigma_t \phi_t)) - C(t,S_t) \right)(dN_t - \lambda_t dt) \end{aligned} \tag{8.3.1}$$

where

$$\Theta C(t,S_t) = C(t, S_t(1+\sigma_t\phi_t)) - C(t,S_t) - \frac{\partial C}{\partial x}(t,S_t) S_t \sigma_t \phi_t.$$

The process

$$\begin{aligned} \widetilde{C}_t &:= C(t,S_t) \exp\left(-\int_0^t r_s ds \right) \\ &= \exp\left(-\int_0^T r_u du \right) \mathbb{E}[(S_T - K)^+ \mid \mathcal{F}_t] \\ &= \tilde{V}_t \end{aligned}$$

is a martingale from Propositions 2.5.8 and 8.2.1-(ii), with

$$\begin{aligned} d\widetilde{C}_t &= \eta_t d\widetilde{S}_t \\ &= \sigma_t \eta_t S_t dM_t \\ &= i_t \sigma_t \eta_t S_t dB_t + \sigma_t \phi_t \eta_t S_t (dN_t - \lambda_t dt), \end{aligned} \tag{8.3.2}$$

from Lemma 8.1.2. Therefore, by identification of (8.3.1) and (8.3.2),

$$\begin{cases} r_t C(t, S_t) = \left(\dfrac{\partial C}{\partial t} + r_t S_t \dfrac{\partial C}{\partial x} + \dfrac{1}{2} i_t \alpha_t^2 S_t^2 \sigma_t^2 \dfrac{\partial^2 C}{\partial x^2} + \lambda_t \Theta C \right) (t, S_t), \\ \\ \eta_t \sigma_t S_t dM_t = S_t \sigma_t \dfrac{\partial C}{\partial x}(t, S_t) dM_t + (C(t, S_t(1 + \sigma_t \phi_t)) - C(t, S_t)) (dN_t - \lambda_t dt) . \end{cases}$$

Therefore, by identification of the Brownian and Poisson parts,

$$\begin{cases} i_t \eta_t S_t \sigma_t = i_t S_t \sigma_t \dfrac{\partial C}{\partial x}(t, S_t) \\ \\ j_t \eta_t S_t \sigma_t \phi_t = C(t, S_t(1 + \sigma_t \phi_t)) - C(t, S_t). \end{cases} \tag{8.3.3}$$

The term $\Theta C(t, S_t)$ vanishes on the set

$$\{t \in \mathbb{R}_+ \ : \ \phi_t = 0\} = \{t \ : \ i(t) = 1\}.$$

Therefore, (8.3.3) reduces to

$$\eta_t = \frac{\partial C}{\partial x}(t, S_t),$$

i.e. the process $(\eta_t)_{t \in \mathbb{R}_+}$ is equal to the usual Delta (8.3) on $\{t \in \mathbb{R}_+ \ : \ i_t = 1\}$, and to

$$\eta_t = \frac{C(t, S_t(1 + \phi_t \sigma_t)) - C(t, S_t)}{S_t \phi_t \sigma_t}$$

on the set $\{t \in \mathbb{R}_+ \ : \ i_t = 0\}$.

Proposition 8.3.1. *The Black-Scholes PDE for the price of a European call option is written as*

$$\frac{\partial C}{\partial t}(t, x) + r_t x \frac{\partial C}{\partial x}(t, x) + \frac{1}{2} \alpha_t^2 x^2 \sigma_t^2 \frac{\partial^2 C}{\partial x^2}(t, x) = r_t C(t, x),$$

on $\{t : \phi_t = 0\}$*, and as*

$$\frac{\partial C}{\partial t}(t, x) + r_t x \frac{\partial C}{\partial x}(t, x) + \lambda_t \Theta C(t, x) = r_t C(t, x),$$

on the set $\{t \in \mathbb{R}_+ \ : \ \phi_t \neq 0\}$*, under the terminal condition* $C(T, x) = (x - K)^+$*.*

8.4 Asian Options and Deterministic Structure

The price at time t of an Asian option is defined as

$$\mathbb{E}\left[e^{-\int_t^T r_s ds} \left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \Big| \mathcal{F}_t \right].$$

The next proposition provides a replicating hedging strategy for Asian options in the case of a deterministic structure equation. Following [74], page 91, and [13], we define the auxiliary process

$$Y_t = \frac{1}{S_t} \left(\frac{1}{T} \int_0^t S_u du - K \right), \quad t \in [0, T]. \tag{8.4.1}$$

Proposition 8.4.1. *There exists a measurable function $\tilde{C}$ on $\mathbb{R}_+ \times \mathbb{R}$ such that $\tilde{C}(t, \cdot)$ is $\mathcal{C}^1$ for all $t \in \mathbb{R}_+$, and*

$$S_t \tilde{C}(t, Y_t) = \mathbb{E}\left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \Big| \mathcal{F}_t \right].$$

Moreover, the replicating portfolio for an Asian option with payoff

$$\left(\frac{1}{T} \int_0^T S_u du - K \right)^+$$

is given by (8.1.3) and

$$\begin{aligned} \eta_t = {} & \frac{1}{\sigma_t} e^{-\int_t^T r_s ds} \Big(\tilde{C}(t, Y_t)\sigma_t \\ & + (1 + \sigma_t \phi_t) \left(\frac{j_t}{\phi_t} \left(\tilde{C}\left(t, \frac{Y_t}{1 + \sigma_t \phi_t} \right) - \tilde{C}(t, Y_t) \right) - i_t \sigma_t Y_t \partial_2 \tilde{C}(t, Y_t) \right) \Big). \end{aligned} \tag{8.4.2}$$

Proof. With the above notation, the price at time t of the Asian option becomes

$$\mathbb{E}\left[e^{-\int_t^T r_s ds} S_T (Y_T)^+ \Big| \mathcal{F}_t \right].$$

For $0 \le s \le t \le T$, we have

$$d(S_t Y_t) = \frac{1}{T} d\left(\int_0^t S_u du - K \right) = \frac{S_t}{T} dt,$$

hence

$$\frac{S_t Y_t}{S_s} = Y_s + \frac{1}{T}\int_s^t \frac{S_u}{S_s} du.$$

Since S_u/S_t is independent of S_t by (8.1.2), we have, for any sufficiently integrable payoff function H,

$$\begin{aligned}\mathbb{E}\left[H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right] &= \mathbb{E}\left[H\left(S_t Y_t + \frac{1}{T}\int_t^T S_u du\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[H\left(xy + \frac{x}{T}\int_t^T \frac{S_u}{S_t} du\right)\right]_{y=Y_t,\ x=S_t}.\end{aligned}$$

Let $C \in \mathcal{C}_b^2(\mathbb{R}_+ \times \mathbb{R}^2)$ be defined as

$$C(t,x,y) = \mathbb{E}\left[H\left(xy + \frac{x}{T}\int_t^T \frac{S_u}{S_t} du\right)\right],$$

i.e.

$$C(t,S_t,Y_t) = \mathbb{E}\left[H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right].$$

When $H(x) = \max(x,0)$, since for any $t \in [0,T]$, S_t is positive and $\mathcal{F}_t$-measurable, and S_u/S_t is independent of $\mathcal{F}_t$, $u \geq t$, we have:

$$\begin{aligned}\mathbb{E}\left[H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right] &= \mathbb{E}\left[S_T (Y_T)^+\Big|\mathcal{F}_t\right] \\ &= S_t\, \mathbb{E}\left[\left(Y_T \frac{S_T}{S_t}\right)^+\Big|\mathcal{F}_t\right] \\ &= S_t\, \mathbb{E}\left[\left(Y_t + \frac{1}{T}\int_t^T \frac{S_u}{S_t} du\right)^+\Big|\mathcal{F}_t\right] \\ &= S_t\, \mathbb{E}\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t} du\right)^+\right]_{y=Y_t} \\ &= S_t \tilde{C}(t,Y_t),\end{aligned}$$

with

$$\tilde{C}(t,y) = \mathbb{E}\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t} du\right)^+\right].$$

We now proceed as in [13], which deals with the sum of a Brownian motion and a Poisson process. From the expression of $1/S_t$ given by (8.1.2) we have

$$d\left(\frac{1}{S_t}\right) = \frac{1}{S_{t^-}}\left(\left(-r_t + \frac{\alpha_t^2\sigma_t^2}{1+\sigma_t\phi_t}\right)dt - \frac{\sigma_t}{1+\sigma_t\phi_t}dM_t\right),$$

hence by (2.12.4), Itô's formula and the definition (8.4.1) of Y_t, we have

$$dY_t = Y_t\left(-r_t + \frac{\alpha_t^2\sigma_t^2}{1+\sigma_t\phi_t}\right)dt + \frac{1}{T}dt - \frac{Y_{t^-}\sigma_t}{1+\sigma_t\phi_t}dM_t.$$

Assuming that $H \in \mathcal{C}_b^2(\mathbb{R})$ and applying Lemma 3.7.2 we get

$$\begin{aligned}\mathbb{E}\left[D_t H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right] &= L_t C(t, S_t, Y_t)\\ &= i_t\left(\sigma_t S_{t^-}\partial_2 C(t, S_t, Y_t) - \frac{Y_t\sigma_t}{1+\sigma_t\phi_t}\partial_3 C(t, S_t, Y_t)\right)\\ &\quad + \frac{j_t}{\phi_t}\left(C\left(t, S_{t^-} + \sigma_t S_{t^-}, Y_{t^-} - \frac{Y_t\sigma_t}{1+\sigma_t\phi_t}\right) - C(t, S_{t^-}, Y_{t^-})\right),\end{aligned} \tag{8.4.3}$$

where L_t is given by (2.12.5). Next, given a family $(H_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_b^2$ functions, such that $|H_n(x)| \leq x^+$ and $|H_n'(x)| \leq 2$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, and converging pointwise to $x \to x^+$, by dominated convergence (8.4.3) holds for $C(t, x, y) = x\tilde{C}(t, y)$ and we obtain:

$$\begin{aligned}\mathbb{E}\left[D_t\left(\frac{1}{T}\int_0^T S_u du - K\right)^+\Bigg|\mathcal{F}_t\right] &= i_t\tilde{C}(t, Y_t)\sigma_t S_t\\ &+ S_t\left(\frac{j_t}{\phi_t}\left(\tilde{C}\left(t, \frac{Y_t}{1+\sigma_t\phi_t}\right) - \tilde{C}(t, Y_t)\right) - i_t\sigma_t Y_t\partial_2\tilde{C}(t, Y_t)\right)\\ &+ S_t\sigma_t\phi_t\left(\frac{j_t}{\phi_t}\left(\tilde{C}\left(t, \frac{Y_t}{1+\sigma_t\phi_t}\right) - \tilde{C}(t, Y_t)\right) - i_t\sigma_t Y_t\partial_2\tilde{C}(t, Y_t)\right).\end{aligned}$$

□

As a particular case we consider the Brownian motion model, i.e. $\phi_t = 0$, for all $t \in [0, T]$, so $i_t = 1$, $j_t = 0$ for all $t \in [0, T]$, and we are in the Brownian motion model. In this case we have

$$\begin{aligned}\eta_t &= \mathrm{e}^{-\int_t^T r_s ds}\left(-Y_t\partial_2\tilde{C}(t, Y_t) + \tilde{C}(t, Y_t)\right)\\ &= \mathrm{e}^{-\int_t^T r_s ds}\left(S_t\frac{\partial}{\partial x}\tilde{C}\left(t, \frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du - K\right)\right)_{|x=S_t} + \tilde{C}(t, Y_t)\right)\\ &= \frac{\partial}{\partial x}\left(x\mathrm{e}^{-\int_t^T r_s ds}\tilde{C}\left(t, \frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du - K\right)\right)\right)_{|x=S_t}, \quad t \in [0, T],\end{aligned}$$

which can be denoted informally as a partial derivative with respect to S_t.

8.5 Notes and References

See e.g. [74] and [135] for standard references on stochastic finance, and [97] for a presentation of the Malliavin calculus applied to continuous markets. The use of normal martingales in financial modelling has been first considered in [35]. The material on Asian options is based on [70] and [12]. Hedging strategies for Lookback options have been computed in [15] using the Clark-Ocone formula.