Chapter 7
Local Gradients on the Poisson Space

We study a class of local gradient operators on Poisson space that have the derivation property. This allows us to give another example of a gradient operator that satisfies the hypotheses of Chapter 3, this time for a discontinuous process. In particular we obtain an anticipative extension of the compensated Poisson stochastic integral and other expressions for the Clark predictable representation formula. The fact that the gradient operator satisfies the chain rule of derivation has important consequences for deviation inequalities, computation of chaos expansions, characterizations of Poisson measures, and sensitivity analysis. It also leads to the definition of an infinite dimensional geometry under Poisson measures.

7.1 Intrinsic Gradient on Configuration Spaces

Let X be a Riemannian manifold with volume element σ, cf. e.g. [14]. We denote by T_xX the tangent space at $x \in X$, and let

$$TX = \bigcup_{x\in X} T_xX$$

denote the tangent bundle to X. Assume we are given a differential operator L defined on $\mathcal{C}_c^1(X)$ with adjoint L^*, satisfying the duality relation

$$\langle Lu, V\rangle_{L^2(X,\sigma;TX)} = \langle u, L^*V\rangle_{L^2(X,\sigma)}, \qquad u \in \mathcal{C}_c^1(X), \quad V \in \mathcal{C}_c^1(X, TX).$$

In the sequel, L will be mainly chosen equal to the gradient ∇^X on X. We work on the Poisson probability space $(\Omega^X, \mathcal{F}^X, \pi_\sigma^X)$ introduced in Definition 6.1.2.

Definition 7.1.1. *Given Λ a compact subset of X, we let $\mathcal{S}$ denote the set of functionals F of the form*

$$F(\omega) = f_0\mathbf{1}_{\{\omega(\Lambda)=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{\omega(\Lambda)=n\}} f_n(x_1, \dots, x_n), \tag{7.1.1}$$

N. Privault, *Stochastic Analysis in Discrete and Continuous Settings*,
Lecture Notes in Mathematics 1982, DOI 10.1007/978-3-642-02380-4_7,

where $f_n \in \mathcal{C}_c^1(\Lambda^n)$ *is symmetric in* n *variables,* $n \geq 1$*, with the notation*

$$\omega \cap \Lambda = \{x_1, \ldots, x_n\}$$

when $\omega(\Lambda) = n$*,* $\omega \in \Omega^X$*.*

In the next definition the differential operator L on X is "lifted" to a differential operator $\hat{D}^L$ on Ω^X.

Definition 7.1.2. *The intrinsic gradient* $\hat{D}^L$ *is defined on* $F \in \mathcal{S}$ *of the form* (7.1.1) *as*

$$\hat{D}_x^L F(\omega) = \sum_{n=1}^{\infty} \mathbf{1}_{\{\omega(\Lambda)=n\}} \sum_{i=1}^{n} L_{x_i} f_n(x_1, \ldots, x_n) \mathbf{1}_{\{x_i\}}(x), \quad \omega(dx) - a.e.,$$

$\omega \in \Omega^X$.

In other words if $\omega(\Lambda) = n$ and $\omega \cap \Lambda = \{x_1, \ldots, x_n\}$ we have

$$\hat{D}_x^L F = \begin{cases} L_{x_i} f_n(x_1, \ldots, x_n), & \text{if } x = x_i \text{ for some } i \in \{1, \ldots, n\}, \\ \\ 0, & \text{if } x \notin \{x_1, \ldots, x_n\}. \end{cases}$$

Let $\mathcal{I}$ denote the space of functionals of the form

$$\mathcal{I} = \left\{ f\left(\int_X \varphi_1(x)\omega(dx), \ldots, \int_X \varphi_n(x)\omega(dx) \right), \right. \\ \left. \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c^\infty(X),\ f \in \mathcal{C}_b^\infty(\mathbb{R}^n),\ n \in \mathbb{N} \right\},$$

and

$$\mathcal{U} = \left\{ \sum_{i=1}^{n} F_i u_i \ :\ u_1, \ldots, u_n \in \mathcal{C}_c^\infty(X),\ F_1, \ldots, F_n \in \mathcal{I},\ n \geq 1 \right\},$$

Note that for $F \in \mathcal{I}$ of the form

$$F = f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega \right), \qquad \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c^\infty(X),$$

we have

$$\hat{D}_x^L F(\omega) = \sum_{i=1}^{n} \partial_i f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega \right) L_x \varphi_i(x), \qquad x \in \omega.$$

The following result is the integration by parts formula satisfied by $\hat{D}^L$.

Proposition 7.1.3. *We have for* $F \in \mathcal{I}$ *and* $V \in \mathcal{C}_c^1(X, TX)$*:*

$$\mathbb{E}\left[\langle \hat{D}^L F, V\rangle_{L^2(X,d\omega;TX)}\right] = \mathbb{E}\left[F \int_X L^* V(x)\omega(dx)\right]$$

Proof. We have

$$\begin{aligned}
&\mathbb{E}\left[\langle \hat{D}^L F, V\rangle_{L^2(X,d\omega;TX)}\right] = \mathbb{E}\left[\sum_{x\in\omega}\langle \hat{D}^L_x F, V(x)\rangle_{TX}\right] \\
&= \sum_{n=1}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{\omega(\Lambda)=n\}} \sum_{i=1}^{n}\langle \hat{D}^L_{x_i} F, V(x_i)\rangle_{TX}\right] \\
&= \mathrm{e}^{-\sigma(\Lambda)} \sum_{n=1}^{\infty} \frac{\sigma(\Lambda)^n}{n!} \\
&\quad \sum_{i=1}^{n} \int_\Lambda \cdots \int_\Lambda \langle L_{x_i} f_n(x_1,\ldots,x_n), V(x_i)\rangle_{TX} \frac{\sigma(dx_1)}{\sigma(\Lambda)} \cdots \frac{\sigma(dx_n)}{\sigma(\Lambda)} \\
&= \mathrm{e}^{-\sigma(\Lambda)} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=1}^{n} \int_\Lambda \cdots \int_\Lambda f_n(x_1,\ldots,x_n) L^*_{x_i} V(x_i) \sigma(dx_1)\cdots\sigma(dx_n) \\
&= \mathrm{e}^{-\sigma(\Lambda)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_\Lambda \cdots \int_\Lambda f_n(x_1,\ldots,x_n) \sum_{i=1}^{n} L^*_{x_i} V(x_i) \sigma(dx_1)\cdots\sigma(dx_n) \\
&= \mathbb{E}\left[F \int_X L^* V(x)\omega(dx)\right].
\end{aligned}$$

□

In particular when $L = \nabla^X$ is the gradient on X we write $\hat{D}$ instead of $\hat{D}^X$ and obtain the following integration by parts formula:

$$\mathbb{E}\left[\langle \hat{D} F, V\rangle_{L^2(X,d\omega;TX)}\right] = \mathbb{E}\left[F \int_X \operatorname{div}^X V(x)\omega(dx)\right], \tag{7.1.2}$$

provided ∇^X and div^X satisfy the duality relation

$$\langle \nabla^X u, V\rangle_{L^2(X,\sigma;TX)} = \langle u, \operatorname{div}^X V\rangle_{L^2(X,\sigma)},$$

$u \in \mathcal{C}_c^1(X)$, $V \in \mathcal{C}_c^1(X, TX)$.

The next result provides a relation between the gradient ∇^X on X and its lifting $\hat{D}$ on Ω, using the operators of Definition 6.4.5.

Lemma 7.1.4. *For $F \in \mathcal{I}$ we have*

$$\hat{D}_x F(\omega) = \varepsilon_x^- \nabla^X \varepsilon_x^+ F(\omega) \quad \text{on} \qquad \{(\omega, x) \in \Omega^x \times X \ : \ x \in \omega\}. \tag{7.1.3}$$

Proof. Let

$$F = f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega\right), \qquad x \in X, \quad \omega \in \Omega^X,$$

and assume that $x \in \omega$. We have

$$\begin{aligned}
\hat{D}_x F(\omega) &= \sum_{i=1}^n \partial_i f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega\right) \nabla^X \varphi_i(x) \\
&= \sum_{i=1}^n \partial_i f\left(\varphi_1(x) + \int_X \varphi_1 d(\omega \backslash x), \ldots, \varphi_n(x) + \int_X \varphi_n d(\omega \backslash x)\right) \nabla^X \varphi_i(x) \\
&= \nabla^X f\left(\varphi_1(x) + \int_X \varphi_1 d(\omega \backslash x), \ldots, \varphi_n(x) + \int_X \varphi_n d(\omega \backslash x)\right) \\
&= \nabla^X \varepsilon_x^+ f\left(\int_X \varphi_1 d(\omega \backslash x), \ldots, \int_X \varphi_n d(\omega \backslash x)\right) \\
&= \left(\nabla^X \varepsilon_x^+ F\right)(\omega \backslash \{x\}) \\
&= \varepsilon_x^- \nabla^X \varepsilon_x^+ F(\omega).
\end{aligned}$$

□

The next proposition uses the operator δ^X defined in Definition 6.4.1.

Proposition 7.1.5. *For $V \in \mathcal{C}_c^\infty(X; TX)$ and $F \in \mathcal{I}$ we have*

$$\begin{aligned}
&\langle \hat{D} F(\omega), V \rangle_{L^2(X, d\omega; TX)} \\
&\qquad = \langle \nabla^X DF(\omega), V \rangle_{L^2(X, \sigma; TX)} + \delta^X(\langle \nabla^X DF, V \rangle_{TX})(\omega).
\end{aligned} \tag{7.1.4}$$

Proof. This identity follows from the relation

$$\hat{D}_x F(\omega) = (\nabla_x^X D_x F)(\omega \backslash \{x\}), \qquad x \in \omega,$$

and the application to $u = \langle \nabla^X DF, V \rangle_{TX}$ of the relation

$$\delta^X(u) = \int_X u(x, \omega \backslash \{x\}) \omega(dx) - \int_X u(x, \omega) \sigma(dx),$$

cf. Relation (6.5.2) in Proposition 6.5.2. □

In addition, for $F, G \in \mathcal{I}$ we have the isometry

$$\langle \hat{D} F, \hat{D} G \rangle_{L^2_\omega(TX)} = \langle \varepsilon^- \nabla^X \varepsilon^+ F, \varepsilon^- \nabla^X \varepsilon^+ G \rangle_{L^2_\omega(TX)}, \tag{7.1.5}$$

$\omega \in \Omega^X$, as an application of Relation (7.1.3) that holds $\omega(dx)$-a.e. for fixed $\omega \in \Omega^X$.
Similarly from (7.1.5) and Proposition 6.5.2 we have the relation

$$\langle \hat{D}F, \hat{D}G \rangle_{L^2_\omega(TX)} = \delta^X \left(\langle \nabla^X DF, \nabla^X DG \rangle_{TX} \right) + \langle \nabla^X DF, \nabla^X DG \rangle_{L^2_\sigma(TX)}, \tag{7.1.6}$$

$\omega \in \Omega^X$, $F, G \in \mathcal{I}$. Taking expectations on both sides in (7.1.4) using Relation (6.4.5), we recover Relation (7.1.2) in a different way:

$$\begin{aligned} \mathbb{E}[\langle \hat{D}F(\omega), V \rangle_{L^2(X, d\omega; TX)}] &= \mathbb{E}[\langle \nabla^X DF, V \rangle_{L^2(X, \sigma; TX)}] \\ &= \mathbb{E}[F \delta^X (\operatorname{div}^X V)], \end{aligned}$$

$V \in \mathcal{C}_c^\infty(X; TX)$, $F \in \mathcal{I}$.

Definition 7.1.6. *Let $\hat{\delta}_{\pi_\sigma}$ denote the adjoint of $\hat{D}$ under π_σ, defined as*

$$\mathbb{E}_{\pi_\sigma} \left[F \hat{\delta}_{\pi_\sigma}(G) \right] = \mathbb{E}_{\pi_\sigma} \left[\langle \hat{D}F, \hat{D}G \rangle_{L^2_\omega(TX)} \right],$$

on $G \in \mathcal{I}$ such that

$$\mathcal{I} \ni F \mapsto \mathbb{E}_{\pi_\sigma} \left[\langle \hat{D}F, \hat{D}G \rangle_{L^2_\omega(TX)} \right]$$

extends to a bounded operator on $L^2(\Omega^X, \pi_\sigma)$.

We close this section with a remark on integration by parts characterization of Poisson measures, cf. Section 6.6, using the local gradient operator instead of the finite difference operator. We now assume that $\operatorname{div}_\sigma^X$ is defined on $\nabla^X f$ for all $f \in \mathcal{C}_c^\infty(X)$, with

$$\int_X g(x) \operatorname{div}_\sigma^X \nabla^X f(x) \sigma(dx) = \int_X \langle \nabla^X g(x), \nabla^X f(x) \rangle_{T_x X} \sigma(dx),$$

$f, g \in \mathcal{C}_c^1(X)$.

As a corollary of our pointwise lifting of gradients we obtain in particular a characterization of the Poisson measure. Let

$$H_\sigma^X = \operatorname{div}_\sigma^X \nabla^X$$

denote the Laplace-Beltrami operator on X.

Corollary 7.1.7. *The isometry relation*

$$\mathbb{E}_\pi \left[\langle \hat{D}F, \hat{D}G \rangle_{L^2_\omega(TX)} \right] = \mathbb{E}_\pi \left[\langle \nabla^X DF, \nabla^X DG \rangle_{L^2_\sigma(TX)} \right], \tag{7.1.7}$$

$F, G \in \mathcal{I}$, holds under the Poisson measure π_σ with intensity σ. Moreover, under the condition

$$\mathcal{C}_c^\infty(X) = \{H_\sigma^X f \ : \ f \in \mathcal{C}_c^\infty(X)\},$$

Relation (7.1.7) entails $\pi = \pi_\sigma$.

Proof.

i) Relations (6.4.5) and (7.1.6) show that (7.1.7) holds when $\pi = \pi_\sigma$.
ii) If (7.1.7) is satisfied, then taking $F = I_n(u^{\otimes n})$ and $G = I_1(h)$, $h, u \in \mathcal{C}_c^\infty(X)$, Relation (7.1.6) implies

$$\begin{aligned} \mathbb{E}_\pi\left[\delta((H_\sigma^X h) u I_{n-1}(u^{\otimes(n-1)}))\right] &= \mathbb{E}_\pi\left[\delta\left(\langle \nabla^X DF, \nabla^X h\rangle_{TX}\right)\right] \\ &= 0, \qquad n \geq 1, \end{aligned}$$

hence $\pi = \pi_\sigma$ from Corollary 6.6.3.

□

We close this section with a study of the intrinsic gradient $\hat{D}$ when $X = \mathbb{R}_+$. Recall that the jump times of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are denoted by $(T_k)_{k\geq 1}$, with $T_0 = 0$, cf. Section 2.3. In the next definition, all $\mathcal{C}^\infty$ functions on

$$\Delta_d = \{(t_1, \ldots, t_d) \in \mathbb{R}_+^d \ : \ 0 \leq t_1 < \cdots < t_d\}$$

are extended by continuity to the closure of Δ_d.

Definition 7.1.8. *Let $\mathcal{S}$ denote the set of smooth random functionals F of the form*

$$F = f(T_1, \ldots, T_d), \qquad f \in \mathcal{C}_b^1(\mathbb{R}_+^d), \quad d \geq 1. \tag{7.1.8}$$

We have

$$\hat{D}_t F = \sum_{k=1}^d \mathbf{1}_{\{T_k\}}(t) \partial_k f(T_1, \ldots, T_d), \qquad dN_t - a.e.,$$

with $F = f(T_1, \ldots, T_d)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, where $\partial_k f$ is the partial derivative of f with respect to its k-th variable, $1 \leq k \leq d$.

Lemma 7.1.9. *Let $F \in \mathcal{S}$ and $h \in \mathcal{C}_b^1(\mathbb{R}_+)$ with $h(0) = 0$. We have the integration by parts formula*

$$\mathbb{E}\left[\langle \hat{D}F, h\rangle_{L^2(\mathbb{R}_+, d\omega)}\right] = -\,\mathbb{E}\left[F\left(\sum_{k=1}^d h'(T_k) - \int_0^{T_d} h'(t)dt\right)\right].$$

Proof. By integration by parts on Δ_d using Relation (2.3.4) we have, for $F \in \mathcal{S}$ of the form (7.1.8),

$$\mathbb{E}[\langle \hat{D}F, h\rangle_{L^2(\mathbb{R}_+,dN_t)}] = \sum_{k=1}^{d} \int_0^\infty \int_0^{t_d} \cdots \int_0^{t_2} \mathrm{e}^{-t_d} h(t_k)\partial_k f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$= \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_1)\partial_1 f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$+ \sum_{k=2}^{d} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} h(t_k)\frac{\partial}{\partial t_k} \int_0^{t_k} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$- \sum_{k=2}^{d} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+1}} h(t_k)$$

$$\int_0^{t_k} \int_0^{t_{k-2}} \cdot \int_0^{t_2} f(t_1,\ldots,t_{k-2},t_k,t_k,\ldots,t_d)dt_1 \cdot \widehat{dt_{k-1}} \cdot dt_d$$

$$= - \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h'(t_1)f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$+ \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_3} h(t_2)f(t_2,t_2,\ldots,t_d)dt_2\cdots dt_d$$

$$- \sum_{k=2}^{d} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+1}} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$+ \int_0^\infty \mathrm{e}^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$+ \sum_{k=2}^{d-1} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+2}} h(t_{k+1})$$

$$\int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdot \int_0^{t_2} f(t_1,.,t_{k-1},t_{k+1},t_{k+1},.,t_d)dt_1 \cdot \widehat{dt_k} \cdot dt_d$$

$$- \sum_{k=2}^{d} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+1}} h(t_k) \int_0^{t_k} \int_0^{t_{k-2}} \cdot \int_0^{t_2} f(t_1,.,t_{k-2},t_k,t_k,.,t_d)dt_1 \cdot dt_d$$

$$= - \sum_{k=1}^{d} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$+ \int_0^\infty \mathrm{e}^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d$$

$$= - \mathbb{E}\left[F\left(\sum_{k=1}^{d} h'(T_k) - \int_0^{T_d} h'(t)dt\right)\right],$$

where $\widehat{dt_k}$ denotes the absence of dt_k in the multiple integrals with respect to $dt_1\cdots dt_d$. □

As a consequence we have the following corollary which directly involves the compensated Poisson stochastic integral.

Corollary 7.1.10. *Let $F \in \mathcal{S}$ and $h \in \mathcal{C}_b^1(\mathbb{R}_+)$ with $h(0) = 0$. We have the integration by parts formula*

$$\mathbb{E}[\langle \hat{D}F, h\rangle_{L^2(\mathbb{R}_+,d\omega)}] = -\,\mathbb{E}\left[F\int_0^\infty h'(t)d(N_t - t)\right]. \tag{7.1.9}$$

Proof. From Lemma 7.1.9 it suffices to notice that if $k > d$,

$$\begin{aligned}
\mathbb{E}[Fh'(T_k)] &= \int_0^\infty \mathrm{e}^{-t_k} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_k \\
&= \int_0^\infty \mathrm{e}^{-t_k} h(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_k \\
&\quad - \int_0^\infty \mathrm{e}^{-t_{k-1}} h(t_{k-1}) \int_0^{t_{k-1}} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_{k-1} \\
&= \mathbb{E}[F(h(T_k) - h(T_{k-1}))] \\
&= \mathbb{E}\left[F\int_{T_{k-1}}^{T_k} h'(t)dt\right],
\end{aligned}$$

in other terms the discrete-time process

$$\left(\sum_{k=1}^n h'(T_k) - \int_0^{T_k} h'(t)dt\right)_{k\geq 1} = \left(\int_0^{T_k} h'(t)d(N_t - t)\right)_{k\geq 1}$$

is a martingale. □

Alternatively we may also use the strong Markov property to show directly that

$$\mathbb{E}\left[F\left(\sum_{k=d+1}^\infty h'(T_k) - \int_{T_{d+1}}^\infty h'(s)ds\right)\right] = 0.$$

By linearity the adjoint $\hat{\delta}$ of $\hat{D}$ is defined on simple processes $u \in \mathcal{U}$ of the form $u = hG$, $G \in \mathcal{S}$, $h \in \mathcal{C}^1(\mathbb{R}_+)$, from the relation

$$\hat{\delta}(hG) = -G\int_0^\infty h'(t)d(N_t - t) + \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+,dN_t)}.$$

Relation (7.1.9) implies immediately the following duality relation.

Proposition 7.1.11. *For $F \in \mathcal{S}$ and $h \in \mathcal{C}_c^1(\mathbb{R}_+)$ we have :*

$$\mathbb{E}\left[\langle \hat{D}F, hG\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] = \mathbb{E}\left[F\hat{\delta}(hG)\right].$$

Proof. We have

$$\begin{aligned}\mathbb{E}\left[\langle \hat{D}F, hG\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] &= \mathbb{E}\left[\langle \hat{D}(FG), h\rangle_{L^2(\mathbb{R}_+,dN_t)} - F\langle \hat{D}G, h\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] \\ &= \mathbb{E}\left[F(G\hat{\delta}(h) - \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+,dN_t)})\right] \\ &= -\mathbb{E}\left[F\left(G\int_0^\infty h'(t)d(N_t - t) + \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+,dN_t)}\right)\right] \\ &= \mathbb{E}\left[F\hat{\delta}(hG)\right].\end{aligned}$$

□

7.2 Damped Gradient on the Half Line

In this section we construct an example of a gradient which, has the derivation property and, unlike $\hat{D}$, satisfies the duality Assumption 3.1.1 and the Clark formula Assumption 3.2.1 of Section 3.1. Recall that the jump times of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are denoted by $(T_k)_{k\geq 1}$, with $T_0 = 0$, cf. Section 2.3.

Let

$$r(t,s) = -(s \vee t), \qquad s,t \in \mathbb{R}_+,$$

denote the Green function associated to equation

$$\begin{cases} \mathcal{L}f := -f'', & f \in \mathcal{C}^\infty([0,\infty)) \\ f'(0) = f'(\infty) = 0. \end{cases}$$

In other terms, given $g \in \mathcal{C}^\infty([0,\infty))$, the solution of

$$g(t) = -f''(t), \qquad f'(0) = f'(\infty) = 0,$$

is given by

$$f(t) = \int_0^\infty r(t,s)g(s)ds, \qquad t \in \mathbb{R}_+.$$

Let also

$$\begin{aligned} r^{(1)}(t,s) &= \frac{\partial r}{\partial t}(t,s) \\ &= -\mathbf{1}_{]-\infty,t]}(s), \qquad s,t \in \mathbb{R}_+, \end{aligned}$$

i.e.

$$f(t) = \int_0^\infty r^{(1)}(t,s)g(s)ds$$
$$= -\int_0^t g(s)ds, \qquad t \in \mathbb{R}_+, \tag{7.2.1}$$

is the solution of

$$\begin{cases} f' = -g, \\ f(0) = 0. \end{cases}$$

Let $\mathcal{S}$ denote the space of functionals of the form

$$\mathcal{I} = \left\{ F = f(T_1, \ldots, T_d) \ : \ f \in \mathcal{C}_b^1(\mathbb{R}^d), \ d \geq 1 \right\},$$

and let

$$\mathcal{U} = \left\{ \sum_{i=1}^n F_i u_i \ : \ u_1, \ldots, u_n \in \mathcal{C}_c(\mathbb{R}_+), \ F_1, \ldots, F_n \in \mathcal{S}, \ n \geq 1 \right\}.$$

Definition 7.2.1. *Given $F \in \mathcal{S}$ of the form $F = f(T_1, \ldots, T_d)$, we let*

$$\tilde{D}_s F = -\sum_{k=1}^d \mathbf{1}_{[0,T_k]}(s) \partial_k f(T_1, \ldots, T_d).$$

Note that we have

$$\tilde{D}_s F = \sum_{k=1}^d r^{(1)}(T_k, s) \partial_k f(T_1, \ldots, T_d)$$
$$= \int_0^\infty r^{(1)}(t,s) \hat{D}_t F dN_t.$$

From Proposition 2.3.6 we have the following lemma.

Lemma 7.2.2. *For F of the form $F = f(T_1, \ldots, T_n)$ we have*

$$\mathbb{E}[\tilde{D}_t F | \mathcal{F}_t] = -\sum_{N_t < k \leq n} \mathbb{E}[\partial_k f(T_1, \ldots, T_n) | \mathcal{F}_t]$$
$$= -\sum_{N_t < k \leq n} \int_t^\infty \mathrm{e}^{-(s_n - t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}}$$
$$\partial_k f(T_1, \ldots, T_{N_t}, s_{N_t+1}, \ldots, s_n) ds_{N_t+1} \cdots ds_n.$$

According to Definition 3.2.2, $I\!D([a,\infty))$, $a > 0$, denotes the completion of $\mathcal{S}$ under the norm

$$\|F\|_{I\!D([a,\infty))} = \|F\|_{L^2(\Omega)} + \left(\mathbb{E}\left[\int_a^\infty |\tilde{D}_t F|^2 dt \right] \right)^{1/2},$$

i.e. $(\tilde{D}_t F)_{t\in[a,\infty)}$ is defined in $L^2(\Omega \times [a,\infty))$ for $F \in I\!D([a,\infty))$. Clearly, the stability Assumption 3.2.10 is satisfied by $\tilde{D}$ since

$$\mathbf{1}_{[0,T_k]}(t) = \mathbf{1}_{\{N_t<k\}}$$

is $\mathcal{F}_t$-measurable, $t \in \mathbb{R}_+$, $k \in \mathbb{N}$. Hence the following lemma holds as a consequence of Proposition 3.2.11. For completeness we provide an independent direct proof.

Lemma 7.2.3. *Let $T > 0$. For any $\mathcal{F}_T$-measurable random variable $F \in L^2(\Omega)$ we have $F \in I\!D_{[T,\infty)}$ and*

$$\tilde{D}_t F = 0, \qquad t \geq T.$$

Proof. In case $F = f(T_1, \ldots, T_n)$ with $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, F does not depend on the future of the Poisson process after T, it does not depend on the k-th jump time T_k if $T_k > T$, i.e.

$$\partial_i f(T_1, \ldots, T_n) = 0 \quad \text{for} \quad T_k > T, \qquad 1 \leq k \leq i \leq n.$$

This implies

$$\partial_i f(T_1, \ldots, T_n)\mathbf{1}_{[0,T_i]}(t) = 0 \quad t \geq T \quad i = 1, \ldots, n,$$

and

$$\tilde{D}_t F = -\sum_{i=1}^n \partial_i f(T_1, \ldots, T_n)\mathbf{1}_{[0,T_i]}(t) = 0 \quad t \geq T.$$

Hence $\tilde{D}_t F = 0$, $t \geq T$. □

Proposition 7.2.4. *We have for $F \in \mathcal{S}$ and $u \in \mathcal{C}_c(\mathbb{R}_+)$:*

$$\mathbb{E}[\langle \tilde{D}F, u\rangle_{L^2(\mathbb{R}_+,dt)}] = \mathbb{E}\left[F \int_0^\infty u(t)(dN_t - dt) \right]. \tag{7.2.2}$$

Proof. We have, using (7.2.1),

$$\mathbb{E}\left[\langle \tilde{D}F, u\rangle_{L^2(\mathbb{R}_+,dt)} \right] = \mathbb{E}\left[\int_0^\infty \int_0^\infty r^{(1)}(s,t)\hat{D}_s F u(t) dN_s dt \right]$$

$$= -\mathbb{E}\left[\left\langle \hat{D}.F, \int_0^{\cdot} u(t)dt \right\rangle_{L^2(\mathbb{R}_+,dN_t)}\right]$$
$$= \mathbb{E}\left[F \int_0^{\infty} u(t)d(N_t - t)\right],$$

from Corollary 7.1.10. □

The above proposition can also be proved by finite dimensional integration by parts on jump times conditionally to the value of N_T, see Proposition 7.3.3 below.

The divergence operator defined next is the adjoint of $\tilde{D}$.

Definition 7.2.5. *We define $\tilde{\delta}$ on $\mathcal{U}$ by*

$$\tilde{\delta}(hG) = G \int_0^{\infty} h(t)(dN_t - dt) - \langle h, \tilde{D}G\rangle_{L^2(\mathbb{R}_+)},$$

$G \in \mathcal{S}$, $h \in L^2(\mathbb{R}_+)$.

The closable adjoint

$$\tilde{\delta} : L^2(\Omega \times [0,1]) \longrightarrow L^2(\Omega)$$

of $\tilde{D}$ is another example of a Skorokhod type integral on the Poisson space. Using this definition we obtain the following integration by parts formula which shows that the duality Assumption 3.1.1 is satisfies by $\tilde{D}$ and $\tilde{\delta}$.

Proposition 7.2.6. *The divergence operator*

$$\tilde{\delta} : L^2(\Omega \times \mathbb{R}_+) \longrightarrow L^2(\Omega)$$

is the adjoint of the gradient operator

$$\tilde{D} : L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+),$$

i.e. we have

$$\mathbb{E}\left[F\tilde{\delta}(u)\right] = \mathbb{E}\left[\langle \tilde{D}F, u\rangle_{L^2(\mathbb{R}_+)}\right], \qquad F \in \mathcal{S}, \quad u \in \mathcal{U}. \tag{7.2.3}$$

Proof. It suffices to note that Proposition 7.2.4 implies

$$\mathbb{E}[\langle \tilde{D}F, hG\rangle_{L^2(\mathbb{R}_+,dt)}] = \mathbb{E}\left[\langle \tilde{D}(FG), h\rangle_{L^2(\mathbb{R}_+,dt)} - F\langle \tilde{D}G, h\rangle_{L^2(\mathbb{R}_+,dt)}\right]$$
$$= \mathbb{E}\left[F\left(G\int_0^{\infty} h(t)d(N_t - t) - \langle h, \tilde{D}G\rangle_{L^2(\mathbb{R}_+,dt)}\right)\right], \tag{7.2.4}$$

for $F, G \in \mathcal{S}$. □

As a consequence, the duality Assumption 3.1.1 of Section 3 is satisfied by $\tilde{D}$ and $\tilde{\delta}$ and from Proposition 3.1.2 we deduce that $\tilde{D}$ and $\tilde{\delta}$ are closable. Recall that from Proposition 6.4.9, the finite difference operator

$$D_t F = \mathbf{1}_{\{N_t<n\}}(f(T_1,\ldots,T_{N_t},t,T_{N_t+1},\ldots,T_{n-1}) - f(T_1,\ldots,T_n)),$$

$t \in \mathbb{R}_+$, $F = f(T_1,\ldots,T_n)$, defined in Chapter 6 satisfies the Clark formula Assumption 3.2.1, i.e. by Proposition 4.2.3 applied to $\phi_t = 1$, $t \in \mathbb{R}_+$, we have

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F \mid \mathcal{F}_t] d(N_t - t), \tag{7.2.5}$$

$F \in L^2(\Omega)$.

On the other hand, the gradient $\tilde{D}$ has the derivation property and for this reason it can be easier to manipulate than the finite difference operator D in recursive computations. Its drawback is that its domain is smaller than that of D, due to the differentiability conditions it imposes on random functionals. In the next proposition we show that the adapted projections of $(D_t F)_{t\in\mathbb{R}_+}$ and $(\tilde{D}_t F)_{t\in\mathbb{R}_+}$ coincide, cf. e.g. Proposition 20 of [102], by a direct computation of conditional expectations.

Proposition 7.2.7. *The adapted projections of $\tilde{D}$ and D coincide, i.e.*

$$\mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] = \mathbb{E}[D_t F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+.$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[\tilde{D}_t F|\mathcal{F}_t] &= -\sum_{k=1}^n \mathbf{1}_{[0,T_k]}(t)\, \mathbb{E}[\partial_k f(T_1,\ldots,T_n)|\mathcal{F}_t] \\
&= -\sum_{N_t<k\le n} \mathbb{E}[\partial_k f(T_1,\ldots,T_n)|\mathcal{F}_t] \\
&= -\sum_{N_t<k\le n} \\
&\quad \int_0^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} \partial_k f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) ds_{N_t+1}\cdots ds_n \\
&= -\sum_{k=N_t+2}^n \int_t^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \\
&\quad \cdots \frac{\partial}{\partial s_k} \int_t^{s_k} \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) ds_{N_t+1}\cdots ds_n \\
&\quad + \sum_{k=N_t+2}^n \int_t^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \\
&\quad \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},.,s_{k-2},s_k,s_k,s_{k+1},.,s_n) \\
&\quad ds_{N_t+1}\cdots \widehat{ds_{k-1}} \cdots ds_n
\end{aligned}$$

$$-\mathbf{1}_{\{n>N_t\}} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} \partial_{N_t+1} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$= -\mathbf{1}_{\{N_t<n-1\}} \int_t^\infty e^{-(s_n-t)} \frac{\partial}{\partial s_n} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$-\sum_{k=N_t+2}^{n-1} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots$$
$$\frac{\partial}{\partial s_k} \int_t^{s_k} \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) ds_{N_t+1}\cdots ds_n$$
$$+\sum_{k=N_t+2}^{n} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}}$$
$$f(T_1,\ldots,T_{N_t},s_{N_t+1},.,s_{k-2},s_k,s_k,s_{k+1},.,s_n) ds_{N_t+1}\cdots \widehat{ds_{k-1}}\cdots ds_n$$
$$-\mathbf{1}_{\{n>N_t\}} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} \partial_{N_t+1} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$= -\mathbf{1}_{\{N_t<n-1\}} \int_t^\infty e^{-(s_n-t)} \frac{\partial}{\partial s_n} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$-\sum_{k=N_t+2}^{n-1} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}}$$
$$f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_{k-1},s_{k+1},s_{k+1},\ldots,s_n) ds_{N_t+1}\cdots \widehat{ds_k}\cdots ds_n$$
$$+\sum_{k=N_t+2}^{n} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}}$$
$$f(T_1,\ldots,T_{N_t},s_{N_t+1},.,s_{k-2},s_k,s_k,s_{k+1},.,s_n) ds_{N_t+1}\cdots \widehat{ds_{k-1}}\cdots ds_n$$
$$-\mathbf{1}_{\{n>N_t\}} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} \partial_{N_t+1} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$= -\mathbf{1}_{\{N_t<n-1\}} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$+\mathbf{1}_{\{N_t<n-1\}} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+2},s_{N_t+2},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$-\mathbf{1}_{\{n>N_t\}} \int_t^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} \partial_{N_t+1} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$= -\mathbf{1}_{\{N_t<n-1\}} \int_0^\infty e^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$

$$
\begin{aligned}
&+\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+2},s_{N_t+2},\ldots,s_n)\\
&ds_{N_t+1}\cdots ds_n\\
&-\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+3}} f(T_1,\ldots,T_{N_t},s_{N_t+2},s_{N_t+2},\ldots,s_n)\\
&ds_{N_t+2}\cdots ds_n\\
&+\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+3}} f(T_1,\ldots,T_{N_t},t,s_{N_t+2},\ldots,s_n)\\
&ds_{N_t+2}\cdots ds_n\\
&-\mathbf{1}_{\{n=N_t+1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)} f(T_1,\ldots,T_{n-1},s_n)ds_n\\
&+\mathbf{1}_{\{n=N_t+1\}} f(T_1,\ldots,T_{n-1},t)\\
= &-\mathbf{1}_{\{n>N_t\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)\\
&ds_{N_t+1}\cdots ds_n\\
&+\mathbf{1}_{\{n<N_t\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+3}} f(T_1,\ldots,T_{N_t},t,s_{N_t+2},\ldots,s_n)\\
&ds_{N_t+2}\cdots ds_n\\
= &\ \mathbb{E}[D_tF|\mathcal{F}_t],
\end{aligned}
$$

from Lemma 6.4.10. □

As a consequence of Proposition 7.2.7 we also have

$$
\mathbb{E}[\tilde{D}_tF \mid \mathcal{F}_a] = \mathbb{E}[D_tF \mid \mathcal{F}_a], \qquad 0\le a\le t. \tag{7.2.6}
$$

For functions of a single jump time, by Relation (2.3.6) we simply have

$$
\begin{aligned}
\mathbb{E}[\tilde{D}_tf(T_n)|\mathcal{F}_t] &= -\mathbf{1}_{\{N_t<n\}}(t)\,\mathbb{E}[f'(T_n)|\mathcal{F}_t]\\
&= -\mathbf{1}_{\{N_t<n\}}\left(\mathbf{1}_{\{N_t\ge n\}}f'(T_n)+\int_t^\infty f'(x)p_{n-1-N_t}(x-t)dx\right)\\
&= -\int_t^\infty f'(x)p_{n-1-N_t}(x-t)dx\\
&= f(t)p_{n-1-N_t}(0)+\int_t^\infty f(x)p'_{n-1-N_t}(x-t)dx\\
&= f(t)\mathbf{1}_{\{T_{n-1}<t<T_n\}}+\int_t^\infty f(x)p'_{n-1-N_t}(x-t)dx,
\end{aligned}
$$

which coincides with

$$
\begin{aligned}
&\mathbb{E}[D_tf(T_n)|\mathcal{F}_t]\\
&= \mathbb{E}[\mathbf{1}_{\{N_t<n-1\}}(f(T_{n-1})-f(T_n))+\mathbf{1}_{\{N_t=n-1\}}(f(t)-f(T_n))|\mathcal{F}_t]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}[(\mathbf{1}_{\{T_{n-1}>t\}} f(T_{n-1}) + \mathbf{1}_{\{T_{n-1}<t<T_n\}} f(t) - \mathbf{1}_{\{T_n>t\}} f(T_n)) | \mathcal{F}_t] \\
&= \mathbf{1}_{\{T_{n-1}<t<T_n\}} f(t) + \mathbb{E}[(\mathbf{1}_{\{T_{n-1}>t\}} f(T_{n-1}) - \mathbf{1}_{\{T_n>t\}} f(T_n)) | \mathcal{F}_t] \\
&= \mathbf{1}_{\{T_{n-1}<t<T_n\}} f(t) + \int_t^\infty (p_{n-2-N_t}(x-t) - p_{n-1-N_t}(x-t)) f(x) dx \\
&= \mathbf{1}_{\{T_{n-1}<t<T_n\}} f(t) + \int_t^\infty f(x) p'_{n-1-N_t}(x-t) dx.
\end{aligned}
$$

As a consequence of Proposition 7.2.7 and (7.2.5) we find that $\tilde{D}$ satisfies the Clark formula, hence the Clark formula Assumption 3.2.1 is satisfied by $\tilde{D}$.

Proposition 7.2.8. *For any $F \in L^2(\Omega)$ we have*

$$
F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] d(N_t - t).
$$

In other words we have

$$
\begin{aligned}
F &= \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] d(N_t - t) \\
&= \mathbb{E}[F] + \int_0^\infty \mathbb{E}[\tilde{D}_t F | \mathcal{F}_t] d(N_t - t),
\end{aligned}
$$

$F \in L^2(\Omega)$.

Since the duality Assumption 3.1.1 and the Clark formula Assumption 3.2.1 are satisfied by $\tilde{D}$, it follows from Proposition 3.3.1 that the operator $\tilde{\delta}$ coincides with the compensated Poisson stochastic integral with respect to $(N_t - t)_{t \in \mathbb{R}_+}$ on the adapted square-integrable processes. This fact is stated in the next proposition with an independent proof.

Proposition 7.2.9. *The adjoint of $\tilde{D}$ extends the compensated Poisson stochastic integral, i.e. for all adapted square-integrable process $u \in L^2(\Omega \times \mathbb{R}_+)$ we have*

$$
\tilde{\delta}(u) = \int_0^\infty u_t d(N_t - t).
$$

Proof. We consider first the case where v is a cylindrical elementary predictable process $v = F \mathbf{1}_{(s,T]}(\cdot)$ with $F = f(T_1, \ldots, T_n)$, $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Since v is predictable, F is $\mathcal{F}_s$-measurable hence from Lemma 7.2.3 we have $\tilde{D}_t F = 0$, $s \geq t$, and

$$
\tilde{D}_t v_u = 0, \quad t \geq u.
$$

Hence from Definition 7.2.5 we get

$$\begin{aligned}\tilde{\delta}(v) &= F(\tilde{N}_T - \tilde{N}_t) \\ &= \int_0^\infty F \mathbf{1}_{(t,T]}(s) d\tilde{N}_s \\ &= \int_0^\infty v_s d\tilde{N}_s.\end{aligned}$$

We then use the fact that $\tilde{D}$ is linear to extend the property to the linear combinations of elementary predictable processes. The compensated Poisson stochastic integral coincides with $\tilde{\delta}$ on the predictable square-integrable processes from a density argument using the Itô isometry. □

Since the adjoint $\tilde{\delta}$ of $\tilde{D}$ extends the compensated Poisson stochastic integral, we may also use Proposition 3.3.2 to show that the Clark formula Assumption 3.2.1 is satisfied by $\tilde{D}$, and in this way we recover the fact that the adapted projections of $\tilde{D}$ and D coincide:

$$\mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] = \mathbb{E}[D_t F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+,$$

for $F \in L^2(\Omega)$.

7.3 Damped Gradient on a Compact Interval

In this section we work under the Poisson measure on the compact interval $[0, T]$, $T > 0$.

Definition 7.3.1. *We denote by $\mathcal{S}_c$ the space of Poisson functionals of the form*

$$F = h_n(T_1, \ldots, T_n), \qquad h_n \in \mathcal{C}_c((0,\infty)^n), \quad n \geq 1, \tag{7.3.1}$$

and by $\mathcal{S}_f$ the space of Poisson functionals of the form

$$F = f_0 \mathbf{1}_{\{N_T=0\}} + \sum_{n=1}^m \mathbf{1}_{\{N_T=n\}} f_n(T_1, \ldots, T_n), \tag{7.3.2}$$

where $f_0 \in \mathbb{R}$ and $f_n \in \mathcal{C}^1([0,T]^n)$, $1 \leq n \leq m$, is symmetric in n variables, $m \geq 1$.

The elements of $\mathcal{S}_c$ can be written as

$$F = f_0 \mathbf{1}_{\{N_T=0\}} + \sum_{n=1}^\infty \mathbf{1}_{\{N_T=n\}} f_n(T_1, \ldots, T_n),$$

where $f_0 \in \mathbb{R}$ and $f_n \in \mathcal{C}^1([0,T]^n)$, $1 \leq n \leq m$, is symmetric in n variables, $m \geq 1$, with the continuity condition

$$f_n(T_1, \ldots, T_n) = f_{n+1}(T_1, \ldots, T_n, T).$$

We also let

$$\mathcal{U}_c = \left\{ \sum_{i=1}^{n} F_i u_i \ : \ u_1, \ldots, u_n \in \mathcal{C}([0,T]), \ F_1, \ldots, F_n \in \mathcal{S}_c, \ n \geq 1 \right\},$$

and

$$\mathcal{U}_f = \left\{ \sum_{i=1}^{n} F_i u_i \ : \ u_1, \ldots, u_n \in \mathcal{C}([0,T]), \ F_1, \ldots, F_n \in \mathcal{S}_f, \ n \geq 1 \right\}.$$

Recall that under $\mathbb{P}$ we have, for all $F \in \mathcal{S}_f$ of the form (7.3.2):

$$\mathbb{E}[F] = \mathrm{e}^{-\lambda T} f_0 + \mathrm{e}^{-\lambda T} \sum_{n=1}^{m} \lambda^n \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n.$$

Definition 7.3.2. *Let $\bar{D}$ be defined on $F \in \mathcal{S}_f$ of the form (7.3.2) by*

$$\bar{D}_t F = -\sum_{n=1}^{m} \mathbf{1}_{\{N_T = n\}} \sum_{k=1}^{n} \mathbf{1}_{[0,T_k]}(t) \partial_k f_n(T_1, \ldots, T_n).$$

If F has the form (7.3.1) we have

$$\bar{D}_t F = -\sum_{k=1}^{n} \mathbf{1}_{[0,T_k]}(t) \partial_k f_n(T_1, \ldots, T_n),$$

where $\partial_k f_n$ denotes the partial derivative of f_n with respect to its k-th variable as in Definition 7.2.1.
We define $\bar{\delta}$ on $u \in \mathcal{U}_f$ by

$$\bar{\delta}(Fu) = F \int_0^T u_t dN_t - \int_0^\infty u_t \bar{D}_t F dt, \tag{7.3.3}$$

$F \in \mathcal{S}_f$, $u \in \mathcal{C}([0,T])$.

The following result shows that $\bar{D}$ and $\bar{\delta}$ also satisfy the duality Assumption 3.1.1.

Proposition 7.3.3. *The operators $\bar{D}$ and $\bar{\delta}$ satisfy the duality relation*

$$\mathbb{E}[\langle \bar{D}F, u \rangle] = \mathbb{E}[F \bar{\delta}(u)], \tag{7.3.4}$$

$F \in \mathcal{S}_f$, $u \in \mathcal{U}_f$.

Proof. By standard integration by parts we first prove (7.3.4) when $u \in \mathcal{C}([0,T])$ and F has the form (7.3.2). We have

$$\begin{aligned}
&\mathbb{E}[\langle \bar{D}F, u\rangle] \\
&= -\mathrm{e}^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \sum_{k=1}^{n} \int_0^T \cdots \int_0^T \int_0^{t_k} u(s) ds \partial_k f_n(t_1,\ldots,t_n) dt_1 \cdots dt_n \\
&= \mathrm{e}^{-\lambda T} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{k=1}^{n} \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n) u(t_k) dt_1 \cdots dt_n \\
&\quad -\mathrm{e}^{-\lambda T} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \int_0^T u(s) ds \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_{n-1},T) dt_1 \cdots dt_{n-1}.
\end{aligned}$$

The continuity condition

$$f_n(t_1,\ldots,t_{n-1},T) = f_{n-1}(t_1,\ldots,t_{n-1}) \tag{7.3.5}$$

yields

$$\begin{aligned}
&\mathbb{E}\left[\langle \bar{D}F, u\rangle\right] \\
&= \mathrm{e}^{-\lambda T} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n) \sum_{k=1}^{n} u(t_k) dt_1 \cdots dt_n \\
&\quad -\lambda \mathrm{e}^{-\lambda T} \int_0^T u(s) ds \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n) dt_1 \cdots dt_n \\
&= \mathbb{E}\left[F\left(\sum_{k=1}^{N_T} u(T_k) - \lambda \int_0^T u(s) ds\right)\right] \\
&= \mathbb{E}\left[F \int_0^T u(t) d\tilde{N}(t)\right].
\end{aligned}$$

Next we define $\bar{\delta}(uG)$, $G \in \mathcal{S}_f$, by (7.3.3), with for all $F \in \mathcal{S}_f$:

$$\begin{aligned}
\mathbb{E}\left[G\langle \bar{D}F, u\rangle\right] &= \mathbb{E}\left[\langle \bar{D}(FG), u\rangle - F\langle \bar{D}G, u\rangle\right] \\
&= \mathbb{E}\left[F\left(G\int_0^T u(t) dN_t - \langle \bar{D}G, u\rangle\right)\right] \\
&= \mathbb{E}\left[F\bar{\delta}(uG)\right],
\end{aligned}$$

which proves (7.3.4). □

Hence, the duality Assumption 3.1.1 of Section 3 is also satisfied by $\bar{D}$ and $\bar{\delta}$, which are closable from Proposition 3.1.2, with domains $\mathrm{Dom}\,(\bar{D})$ and $\mathrm{Dom}\,(\bar{\delta})$.
The stability Assumption 3.2.10 is also satisfied by $\bar{D}$ and Lemma 7.2.3 holds as well as a consequence of Proposition 3.2.11, i.e. for any $\mathcal{F}_T$-measurable random variable $F \in L^2(\Omega)$ we have

$$\tilde{D}_t F = 0, \qquad t \geq T.$$

Similarly, $\bar{\delta}$ coincides with the stochastic integral with respect to the compensated Poisson process, i.e.

$$\tilde{\delta}(u) = \int_0^\infty u_t d(N_t - t),$$

for all adapted square-integrable process $u \in L^2(\Omega \times \mathbb{R}_+)$, with the same proof as in Proposition 7.2.9.
Consequently, from Proposition 3.3.2 it follows that the Clark formula Assumption 3.2.1 is satisfied by $\bar{D}$, and the adapted projections of $\bar{D}$, $\tilde{D}$, and D coincide:

$$\begin{aligned}\mathbb{E}[\bar{D}_t F \mid \mathcal{F}_t] &= \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] \\ &= \mathbb{E}[D_t F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+,\end{aligned}$$

for $F \in L^2(\Omega)$.

Note that the gradients $\tilde{D}$ and $\bar{D}$ coincide on a common domain under the continuity condition (7.3.5). In case (7.3.5) is not satisfied by F the gradient $\bar{D}F$ can still be defined in $L^2(\Omega \times [0,T])$ on $F \in \mathcal{S}_f$ while $\tilde{D}F$ exists only in distribution sense due to the presence of the indicator function $\mathbf{1}_{\{N_T = k\}} = \mathbf{1}_{\{[T_k, T_{k+1})\}}(T)$ in (7.3.2).
Yet when (7.3.5) does not hold, we still get the integration by parts

$$\begin{aligned}\mathbb{E}\left[\langle \bar{D}F, u\rangle\right] &= \mathbb{E}\left[F \sum_{k=1}^{N_T} u(T_k)\right] \qquad (7.3.6)\\ &= \mathbb{E}\left[F \int_0^T u(t)dN(t)\right], \qquad F \in \mathcal{S}_f, \quad u \in \mathcal{U}_f,\end{aligned}$$

under the additional condition

$$\int_0^T u(s)ds = 0. \qquad (7.3.7)$$

However, in this case Proposition 3.1.2 does not apply to extend $\bar{D}$ by closability from its definition on $\mathcal{S}_f$ since the condition (7.3.7) is required in the integration by parts (7.3.6).

7.4 Chaos Expansions

In this section we review the application of $\tilde{D}$ to the computation of chaos expansions when $X = \mathbb{R}_+$. As noted above the gradient $\tilde{D}$ has some properties in common with D, namely its adapted projection coincides with that of D, and in particular from Proposition 7.2.7 we have

$$\mathbb{E}[D_t F] = \mathbb{E}[\tilde{D}_t F], \qquad t \in \mathbb{R}_+.$$

In addition, since the operator $\tilde{D}$ has the derivation property it is easier to manipulate than the finite difference operator D in recursive computations. We aim at applying Proposition 4.2.5 in order to compute the chaos expansions

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

with

$$f_n(t_1, \ldots, t_n) = \frac{1}{n!} \mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F],$$

$dt_1 \cdots dt_n d\mathbb{P}$-a.e., $n \geq 1$.

However, Proposition 4.2.5 cannot be applied since the gradient $\tilde{D}$ cannot be iterated in L^2 due to the non-differentiability of $\mathbf{1}_{[0,T_k]}(t)$ in T_k. In particular, an expression such as

$$\mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F] \tag{7.4.1}$$

makes a priori no sense and may differ from $\mathbb{E}[D_{t_1} \cdots D_{t_n} F]$ for $n \geq 2$.
Note that we have

$$\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} f(T_k) = (-1)^n \mathbf{1}_{[0,T_k]}(t_n) f^{(n)}(T_k), \qquad 0 < t_1 < \cdots < t_n,$$

and

$$\begin{aligned} \mathbb{E}[\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} f(T_k)] &= (-1)^n \, \mathbb{E}[\mathbf{1}_{[0,T_k]}(t_n) f^{(n)}(T_k)] \\ &= (-1)^n \int_{t_n}^{\infty} f^{(n)}(t) p_{k-1}(t) dt, \end{aligned}$$

$0 < t_1 < \cdots < t_n$, which differs from

$$\mathbb{E}[D_{t_n} \cdots D_{t_1} f(T_k)] = -\int_{t_n}^{\infty} f(t) P_k^{(n)}(t) dt,$$

computed in Theorem 1 of [110], where

$$P_k(t) = \int_0^t p_{k-1}(s)ds, \qquad t \in \mathbb{R}_+,$$

is the distribution function of T_k, cf. (6.3.5).
Hence on the Poisson space $\tilde{D}_{t_n} \cdots \tilde{D}_{t_1}$, $0 < t_1 < \cdots < t_n$, cannot be used in the L^2 sense as $D_{t_n} \cdots D_{t_1}$ to give the chaos decomposition of a random variable. Nevertheless we have the following proposition, see [112] for an approach to this problem gradient $\tilde{D}$ in distribution sense.

Proposition 7.4.1. *For any* $F \in \bigcap_{n=0}^{\infty} \mathrm{Dom}\,(D^n \tilde{D})$ *we have the chaos expansion*

$$F = \mathbb{E}[F] + \sum_{n\geq 1}^{\infty} \tilde{I}_n(1_{\Delta_n} f_n),$$

where

$$f_n(t_1, \ldots, t_n) = \mathbb{E}[D_{t_1} \cdots D_{t_{n-1}} \tilde{D}_{t_n} F],$$

$0 < t_1 < \cdots < t_n$, $n \geq 1$.

Proof. We apply Proposition 4.2.5 to $\tilde{D}_t F$, $t \in \mathbb{R}_+$:

$$\tilde{D}_t F = \mathbb{E}[\tilde{D}_t F] + \sum_{n=1}^{\infty} \tilde{I}_n(1_{\tilde{\Delta}_n} \mathbb{E}[D^n \tilde{D}_t F]),$$

which yields

$$\mathbb{E}[\tilde{D}_t F | \mathcal{F}_t] = \mathbb{E}[\tilde{D}_t F] + \sum_{n=1}^{\infty} \tilde{I}_n(1_{\tilde{\Delta}_{n+1}(*,t)} \mathbb{E}[D^n \tilde{D}_t F]).$$

Finally, integrating both sides with respect to $d(N_t - t)$ and using of the Clark formula Proposition 7.2.8 and the inductive definition (2.7.1) we get

$$F - \mathbb{E}[F] = \sum_{n=0}^{\infty} \tilde{I}_{n+1}(1_{\tilde{\Delta}_{n+1}} \mathbb{E}[D^n \tilde{D} F]).$$

□

The next lemma provides a way to compute the functions appearing in Proposition 7.4.1.

Lemma 7.4.2. *We have for* $f \in \mathcal{C}_c^1(\mathbb{R})$ *and* $n \geq 1$

$$D_t \tilde{D}_s f(T_n) = \tilde{D}_{s\vee t} f(T_{n-1}) - \tilde{D}_{s\vee t} f(T_n) - \mathbf{1}_{\{s<t\}} \mathbf{1}_{[T_{n-1}, T_n]}(s \vee t) f'(s \vee t),$$

$s, t \in \mathbb{R}_+$.

Proof. From Relation (6.4.15) we have

$$\begin{aligned}
D_t\tilde{D}_s f(T_n) &= -\mathbf{1}_{[0,T_{n-1}]}(t)\left(\mathbf{1}_{[0,T_{n-1}]}(s)f'(T_{n-1}) - \mathbf{1}_{[0,T_n]}(s)f'(T_n)\right)\\
&\quad -\mathbf{1}_{[T_{n-1},T_n]}(t)\left(\mathbf{1}_{[0,t]}(s)f'(t) - \mathbf{1}_{[0,T_n]}(s)f'(T_n)\right)\\
&= \mathbf{1}_{\{t<s\}}\left(\mathbf{1}_{[0,T_n]}(s)f'(T_n) - \mathbf{1}_{[0,T_{n-1}]}(s)f'(T_{n-1}),\right)\\
&\quad +\mathbf{1}_{\{s<t\}}\left(\mathbf{1}_{[0,T_n]}(t)f'(T_n) - \mathbf{1}_{[0,T_{n-1}]}(t)f'(T_{n-1}) - \mathbf{1}_{[T_{n-1},T_n]}(t)f'(t)\right),
\end{aligned}$$

$\mathbb{P}$-a.s. □

In the next proposition we apply Lemma 7.4.2 to the computation of the chaos expansion of $f(T_k)$.

Proposition 7.4.3. *For $k \geq 1$, the chaos expansion of $f(T_k)$ is given as*

$$f(T_k) = \mathbb{E}[f(T_k)] + \sum_{n\geq 1}\frac{1}{n!}I_n(f_n^k),$$

where $f_n^k(t_1,\ldots,t_n) = \alpha_n^k(f)(t_1\vee\cdots\vee t_n)$, $t_1,\ldots,t_n \in \mathbb{R}_+$, and

$$\begin{aligned}
\alpha_n^k(f)(t) &= -\int_t^\infty f'(s)\partial^{n-1}p_k(s)ds, &(7.4.2)\\
&= f(t)\partial^{n-1}p_k(t) + \langle f, \mathbf{1}_{[t,\infty[}\partial^n p_k\rangle_{L^2(\mathbb{R}_+)}, \qquad t\in\mathbb{R}_+, \qquad n\geq 1,
\end{aligned}$$

where the derivative f' in (7.4.2) is taken in the distribution sense.

We note the relation

$$\frac{d\alpha_n^k(f)}{dt}(t) = \alpha_n^k(f')(t) - \alpha_{n+1}^k(f)(t), \quad t\in\mathbb{R}_+.$$

From this proposition it is clearly seen that $f(T_n)\mathbf{1}_{[0,t]}(T_n)$ is $\mathcal{F}_{[0,t]}$-measurable, and that $f(T_n)\mathbf{1}_{[t,\infty[}(T_n)$ is not $\mathcal{F}_{[t,\infty[}$-measurable.

Proof. of Proposition 7.4.3. Let us first assume that $f\in\mathcal{C}_c^1(\mathbb{R}_+)$. We have

$$\begin{aligned}
f_1^k(t) &= \mathbb{E}[\tilde{D}_t f(T_k)]\\
&= -\mathbb{E}[\mathbf{1}_{[0,T_k]}(t)f'(T_k)]\\
&= -\int_t^\infty p_k(s)f'(s)ds.
\end{aligned}$$

Now, from Lemma 7.4.2, for $n\geq 2$ and $0\leq t_1<\cdots<t_n$,

$$D_{t_1}\cdots D_{t_{n-1}}\tilde{D}_{t_n}f(T_k) = D_{t_1}\cdots D_{t_{n-2}}(\tilde{D}_{t_n}f(T_{k-1}) - \tilde{D}_{t_n}f(T_k)),$$

hence taking expectations on both sides and using Proposition 7.4.1 we have

$$f_n^k(t_1,\ldots,t_n) = f_{n-1}^{k-1}(t_1,\ldots,t_{n-2},t_n) - f_{n-1}^k(t_1,\ldots,t_{n-2},t_n),$$

and we can show (4.3.3) by induction, for $n \geq 2$:

$$\begin{aligned} f_n^k(t_1,\ldots,t_n) &= f_{n-1}^{k-1}(t_1,\ldots,t_{n-2},t_n) - f_{n-1}^k(t_1,\ldots,t_{n-2},t_n), \\ &= -\int_{t_n}^\infty f'(s)\frac{\partial^{n-2}p_{k-1}}{\partial s^{n-2}}(s)ds + \int_{t_n}^\infty f'(s)\frac{\partial^{n-2}p_{k-1}}{\partial s^{n-2}}(s)ds \\ &= -\int_0^\infty f'(s)\frac{\partial^{n-1}p_{k-1}}{\partial s^{n-1}}p_k(s)ds. \end{aligned}$$

The conclusion is obtained by density of the $\mathcal{C}_c^1$ functions in $L^2(\mathbb{R}_+, p_k(t)dt)$, $k \geq 1$. □

7.5 Covariance Identities and Deviation Inequalities

Next we present a covariance identity for the gradient $\tilde{D}$, as an application of Theorem 3.4.4.

Corollary 7.5.1. *Let $n \in \mathbb{N}$ and $F, G \in \bigcap_{k=1}^{n+1} \mathbb{D}(\Delta_k)$. We have*

$$\begin{aligned} \mathrm{Cov}\,(F,G) = {}& \sum_{k=1}^n (-1)^{k+1}\,\mathbb{E}\left[\int_{\Delta_k} (\tilde{D}_{t_k}\cdots\tilde{D}_{t_1}F)(\tilde{D}_{t_k}\cdots\tilde{D}_{t_1}G)dt_1\cdots dt_k\right] \\ & +(-1)^n\,\mathbb{E}\left[\int_{\Delta_{n+1}} \mathbb{E}\left[\tilde{D}_{t_{n+1}}\cdots\tilde{D}_{t_1}F \mid \mathcal{F}_{t_{n+1}}\right]\right. \\ & \left.\times\,\mathbb{E}\left[\tilde{D}_{t_{n+1}}\cdots\tilde{D}_{t_1}G \mid \mathcal{F}_{t_{n+1}}\right] dt_1\cdots dt_{n+1}\right]. \end{aligned} \quad (7.5.1)$$

In particular,

$$\mathrm{Cov}\,(T_m, f(T_1,\ldots,T_m)) = \sum_{i=1}^m \mathbb{E}[T_i\partial_i f(T_1,\ldots,T_m)].$$

From the well-known fact that exponential random variables

$$(\tau_k)_{k\geq 1} := (T_k - T_{k-1})_{k\geq 1}$$

can be constructed as the half sums of squared independent Gaussian random variables we define a mapping Θ which sends Poisson functionals to Wiener functionals, cf. [103]. Given $F = f(\tau_1, \ldots, \tau_n)$ a Poisson functional, let ΘF denote the Gaussian functional defined by

$$\Theta F = f\left(\frac{X_1^2 + Y_1^2}{2}, \ldots, \frac{X_n^2 + Y_n^2}{2}\right),$$

where $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, denote two independent collections of standard Gaussian random variables. The random variables $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, may be constructed as Brownian single stochastic integrals on the Wiener space W. In the next proposition we let D denote the gradient operator of Chapter 5 on the Wiener space.

Proposition 7.5.2. *The mapping $\Theta : L^p(\Omega) \to L^p(W)$ is an isometry. Further, it satisfies the intertwining relation*

$$2\Theta|\tilde{D}F|^2_{L^2(\mathbb{R}_+)} = |D\Theta F|^2_{L^2(\mathbb{R}_+)}, \tag{7.5.2}$$

Proof. The proposition follows from the fact that F and ΘF have same distribution since the half sum of two independent Gaussian squares has an exponential distribution. Relation (7.5.2) follows by a direct calculation. □

Proposition 3.6.2 applies in particular to the damped gradient operator $\tilde{D}$:

Corollary 7.5.3. *Let $F \in \mathrm{Dom}\,(\tilde{D})$. We have*

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2\|\tilde{D}F\|^2_{L^2(\mathbb{R}_+, L^\infty(\Omega))}}\right), \qquad x > 0.$$

In particular if F is $\mathcal{F}_T$ measurable and $\|\tilde{D}F\|_\infty \leq K$ then

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2K^2T}\right), \qquad x \geq 0.$$

As an example we may consider $F = f(\tau_1, \ldots, \tau_n)$ with

$$\sum_{k=1}^n \tau_k |\partial_k f(\tau_1, \ldots, \tau_n)|^2 \leq K^2, \qquad a.s.$$

Applying Corollary 4.7.4 to ΘF, where Θ is the mapping defined in Definition 7.5.2 and using Relation (7.5.2) yields the following deviation result for the damped gradient $\tilde{D}$ on Poisson space.

Corollary 7.5.4. *Let $F \in \mathrm{Dom}\,(\tilde{D})$. Then*

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{4\|\tilde{D}F\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}}\right).$$

The above result can also be obtained via logarithmic Sobolev inequalities, i.e. by application of Corollary 2.5 of [76] to Theorem 0.7 in [4] (or Relation (4.4) in [76] for a formulation in terms of exponential random variables). A sufficient condition for the exponential integrability of F is $\| |\tilde{D}F|_{L^2(\mathbb{R}_+)} \|_\infty < \infty$, cf. Theorem 4 of [103].

7.6 Some Geometric Aspects of Poisson Analysis

In this section we use the operator $\tilde{D}$ to endow the configuration space on $\mathbb{R}_+$ with a (flat) differential structure.

We start by recalling some elements of differential geometry. Let M be a Riemannian manifold with volume measure dx, covariant derivative ∇, and exterior derivative d. Let ∇^*_μ and d^*_μ denote the adjoints of ∇ and d under a measure μ on M of the form $\mu(dx) = \mathrm{e}^{\phi(x)}dx$. The Weitzenböck formula under the measure μ states that

$$\mathrm{d}^*_\mu \mathrm{d} + \mathrm{d}\mathrm{d}^*_\mu = \nabla^*_\mu \nabla + R - \mathrm{Hess}\ \phi,$$

where R denotes the Ricci tensor on M. In terms of the de Rham Laplacian $H_R = \mathrm{d}^*_\mu \mathrm{d} + \mathrm{d}\mathrm{d}^*_\mu$ and of the Bochner Laplacian $H_B = \nabla^*_\mu \nabla$ we have

$$H_R = H_B + R - \mathrm{Hess}\ \phi. \tag{7.6.1}$$

In particular the term Hess ϕ plays the role of a curvature under the measure μ. The differential structure on $\mathbb{R}$ can be lifted to the space of configurations on $\mathbb{R}_+$. Here, $\mathcal{S}$ is defined as in Definition 7.1.8, and $\mathcal{U}$ denotes the space of smooth processes of the form

$$u(\omega, x) = \sum_{i=1}^{n} F_i(\omega) h_i(x), \qquad (\omega, x) \in \Omega \times \mathbb{R}_+, \tag{7.6.2}$$

$h_i \in \mathcal{C}^\infty_c(\mathbb{R}_+)$, $F_i \in \mathcal{S}$, $i = 1, \ldots, n$. The differential geometric objects to be introduced below have finite dimensional counterparts, and each of them has a stochastic interpretation. The following table describes the correspondence between geometry and probability.

Notation	Geometry	Probability
Ω	manifold	probability space
ω	element of Ω	point measure on $\mathbb{R}_+$
$\mathcal{C}^\infty_c(\mathbb{R}_+)$	tangent vectors to Ω	test functions on $\mathbb{R}_+$
σ	Riemannian metric on Ω	Lebesgue measure

d	gradient on Ω	stochastic gradient $\tilde{D}$
$\mathcal{U}$	vector field on Ω	stochastic process
du	exterior derivative of $u \in \mathcal{U}$	two-parameter process
$\{\cdot,\cdot\}$	bracket of vector fields on Ω	bracket on $\mathcal{U} \times \mathcal{U}$
R	curvature tensor on Ω	trilinear mapping on $\mathcal{U}$
d*	divergence on Ω	stochastic integral operator

We turn to the definition of a covariant derivative ∇_u in the direction $u \in L^2(\mathbb{R}_+)$, first for a vector field $v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ as

$$\nabla_u v(t) = -\dot{v}(t) \int_0^t u_s ds, \qquad t \in \mathbb{R}_+,$$

where $\dot{v}(t)$ denotes the derivative of $v(t)$, and then for a vector field

$$v = \sum_{i=1}^n F_i h_i \in \mathcal{U}$$

in the next definition.

Definition 7.6.1. *Given $u \in \mathcal{U}$ and $v = \sum_{i=1}^n F_i h_i \in \mathcal{U}$, let $\nabla_u v$ be defined as*

$$\nabla_u v(t) = \sum_{i=1}^n h_i(t) \tilde{D}_u F_i - F_i \dot{h}_i(t) \int_0^t u_s ds, \qquad t \in \mathbb{R}_+, \tag{7.6.3}$$

where

$$\tilde{D}_u F = \langle \tilde{D} F, u \rangle_{L^2(\mathbb{R}_+)}, \qquad F \in \mathcal{S}.$$

We have

$$\nabla_{uF}(vG) = Fv\tilde{D}_u G + FG\nabla_u v, \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+), \quad F, G \in \mathcal{S}. \tag{7.6.4}$$

We also let, by abuse of notation,

$$(\nabla_s v)(t) := \sum_{i=1}^n h_i(t) \tilde{D}_s F_i - F_i \dot{h}_i(t) \mathbf{1}_{[0,t]}(s),$$

for $s, t \in \mathbb{R}_+$, in order to write

$$\nabla_u v(t) = \int_0^\infty u_s \nabla_s v_t ds, \qquad t \in \mathbb{R}_+, \quad u, v \in \mathcal{U}.$$

The following is the definition of the Lie-Poisson bracket.

Definition 7.6.2. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ *is defined as the unique element of* $\mathcal{C}_c^\infty(\mathbb{R}_+)$ *satisfying*

$$(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u) F = \tilde{D}_w F, \qquad F \in \mathcal{S}.$$

The bracket $\{\cdot, \cdot\}$ is extended to $u, v \in \mathcal{U}$ via

$$\{Ff, Gg\}(t) = FG\{f, g\}(t) + g(t) F \tilde{D}_f G - f(t) G \tilde{D}_g F, \qquad t \in \mathbb{R}_+, \quad (7.6.5)$$

$f, g \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $F, G \in \mathcal{S}$. Given this definition we are able to prove the vanishing of the associated torsion term.

Proposition 7.6.3. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{U}$ *satisfies*

$$\{u, v\} = \nabla_u v - \nabla_v u, \quad (7.6.6)$$

i.e. the connection defined by ∇ *has a vanishing torsion*

$$T(u, v) = \nabla_u v - \nabla_v u - \{u, v\} = 0, \qquad u, v \in \mathcal{U}.$$

Proof. For all $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ we have

$$\begin{aligned}
(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u) T_n &= -\tilde{D}_u \int_0^{T_n} v_s ds + \tilde{D}_v \int_0^{T_n} u_s ds \\
&= v_{T_n} \int_0^{T_n} u_s ds - u_{T_n} \int_0^{T_n} v_s ds \\
&= \int_0^{T_n} \left(\dot{v}(t) \int_0^t u_s ds - \dot{u}(t) \int_0^t v_s ds \right) dt \\
&= \tilde{D}_{\nabla_u v - \nabla_v u} T_n.
\end{aligned}$$

Since $\tilde{D}$ is a derivation, this shows that

$$(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u) F = \tilde{D}_{\nabla_u v - \nabla_v u} F$$

for all $F \in \mathcal{S}$, hence

$$\tilde{D}_{\{u,v\}} = \tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u = \tilde{D}_{\nabla_u v - \nabla_v u}, \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

which shows that (7.6.6) holds for $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The extension to $u, v \in \mathcal{U}$ follows from (7.6.4) and (7.6.5). □

Similarly we show the vanishing of the associated curvature.

Proposition 7.6.4. *The Riemannian curvature tensor* R *of* ∇ *vanishes on* $\mathcal{U}$, *i.e.*

$$R(u, v)h := [\nabla_u, \nabla_v]h - \nabla_{\{u,v\}} h = 0, \qquad u, v, h \in \mathcal{U}.$$

Proof. We have, letting $\tilde{u}(t) = -\int_0^t u_s ds$, $t \in \mathbb{R}_+$:

$$[\nabla_u, \nabla_v]h = \tilde{u}\dot{\overbrace{\nabla_v h}} - \tilde{v}\dot{\overbrace{\nabla_u h}} = \tilde{u}\ \dot{\overbrace{\tilde{v}\dot{h}}}\ - \tilde{v}\ \dot{\overbrace{\tilde{u}\dot{h}}} = -\tilde{u}v\dot{h} + \tilde{v}u\dot{h},$$

and

$$\nabla_{\{u,v\}}h = \nabla_{\tilde{u}\dot{v}-\tilde{v}\dot{u}}h = (\widetilde{\tilde{u}\dot{v} - \tilde{v}\dot{u}})\dot{h} = (u\tilde{v} - v\tilde{u})\dot{h},$$

hence $R(u,v)h = 0$, $h, u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The extension of the result to $\mathcal{U}$ follows again from (7.6.4) and (7.6.5). □

Clearly, the bracket $\{\cdot,\cdot\}$ is antisymmetric, i.e.:

$$\{u, v\} = -\{v, u\}, \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+).$$

Proposition 7.6.5. *The bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity*

$$\{\{u, v\}, w\} + \{w, \{u, v\}\} + \{v, \{u, w\}\} = 0, \qquad u, v, w \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

hence $\mathcal{U}$ is a Lie algebra under $\{\cdot,\cdot\}$.

Proof. The vanishing of $R(u,v)$ in Proposition 7.6.4 shows that

$$[\nabla_u, \nabla_v] = \nabla_{\{u,v\}}h, \qquad u, v \in \mathcal{U},$$

hence

$$\begin{aligned}&\nabla_{\{\{u,v\},w\}} + \nabla_{\{w,\{u,v\}\}} + \nabla_{\{v,\{u,w\}\}}\\ &= [\nabla_{\{u,v\}}, \nabla_w] + [\nabla_w, \nabla_{\{u,v\}}] + [\nabla_v, \nabla_{\{u,w\}}]\\ &= 0, \qquad u, v, h \in \mathcal{U}.\end{aligned}$$

□

However, $\{\cdot,\cdot\}$ does not satisfy the Leibniz identity, thus it can not be considered as a Poisson bracket.

The exterior derivative $\tilde{D}u$ of a smooth vector field $u \in \mathcal{U}$ is defined from

$$\langle \tilde{D}u, h_1 \wedge h_2 \rangle_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} = \langle \nabla_{h_1} u, h_2 \rangle_{L^2(\mathbb{R}_+)} - \langle \nabla_{h_2} u, h_1 \rangle_{L^2(\mathbb{R}_+)},$$

$h_1, h_2 \in \mathcal{U}$, with the norm

$$\|\tilde{D}u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} := 2\int_0^\infty \int_0^\infty (\tilde{D}u(s,t))^2 ds dt, \tag{7.6.7}$$

where

$$\tilde{D}u(s,t) = \frac{1}{2}(\nabla_s u_t - \nabla_t u_s), \qquad s, t \in \mathbb{R}_+, \quad u \in \mathcal{U}.$$

The next result is analog to Proposition 4.1.4.

Lemma 7.6.6. *We have the commutation relation*

$$\tilde{D}_u \tilde{\delta}(v) = \tilde{\delta}(\nabla_u v) + \langle u, v\rangle_{L^2(\mathbb{R}_+)}, \tag{7.6.8}$$

$u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, *between* $\tilde{D}$ *and* δ.

Proof. We have

$$\begin{aligned}\tilde{D}_u \tilde{\delta}(v) &= -\sum_{k=1}^\infty \dot{v}(T_k) \int_0^{T_k} u_s ds \\ &= -\tilde{\delta}\left(\dot{v}_\cdot \int_0^\cdot u_s ds\right) - \int_0^\infty \dot{v}(t) \int_0^t u_s ds dt \\ &= \tilde{\delta}(\nabla_u v) + \langle u, v\rangle_{L^2(\mathbb{R}_+)},\end{aligned}$$

by (7.6.3). □

As an application we obtain a Skorohod type isometry for the operator $\tilde{\delta}$.

Proposition 7.6.7. *We have for* $u \in \mathcal{U}$:

$$\mathbb{E}\left[|\tilde{\delta}(u)|^2\right] = \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right] + \mathbb{E}\left[\int_0^\infty \int_0^\infty \nabla_s u_t \nabla_t u_s ds dt\right]. \tag{7.6.9}$$

Proof. Given $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ we have

$$\begin{aligned}&\mathbb{E}\left[\tilde{\delta}(h_i F_i)\tilde{\delta}(h_j F_j)\right] = \mathbb{E}\left[F_i \tilde{D}_{h_i} \tilde{\delta}(h_j F_j)\right] \\ &= \mathbb{E}\left[F_i \tilde{D}_{h_i}(F_j \tilde{\delta}(h_j) - \tilde{D}_{h_j} F_j)\right] \\ &= \mathbb{E}\left[(F_i F_j \tilde{D}_{h_i} \tilde{\delta} h_j + F_i \tilde{\delta}(h_j) \tilde{D}_{h_i} F_j - F_i \tilde{D}_{h_i} \tilde{D}_{h_j} F_j)\right] \\ &= \mathbb{E}\left[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + F_i F_j \tilde{\delta}(\nabla_{h_i} h_j) + F_i \tilde{\delta}(h_j) \tilde{D}_{h_i} F_j - F_i \tilde{D}_{h_i} \tilde{D}_{h_j} F_j)\right] \\ &= \mathbb{E}\left[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{h_i} h_j}(F_i F_j) + \tilde{D}_{h_j}(F_i \tilde{D}_{h_i} F_j) - F_i \tilde{D}_{h_i} \tilde{D}_{h_j} F_j)\right] \\ &= \mathbb{E}\left[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{h_i} h_j}(F_i F_j) + \tilde{D}_{h_j} F_i \tilde{D}_{h_i} F_j \right. \\ &\quad \left. + F_i(\tilde{D}_{h_j} \tilde{D}_{h_i} F_j - \tilde{D}_{h_i} \tilde{D}_{h_j} F_j))\right] \\ &= \mathbb{E}\left[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{h_i} h_j}(F_i F_j) + \tilde{D}_{h_j} F_i \tilde{D}_{h_i} F_j \right. \\ &\quad \left. + F_i \tilde{D}_{\nabla_{h_j} h_i - \nabla_{h_i} h_j} F_j)\right] \\ &= \mathbb{E}\left[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + F_j \tilde{D}_{\nabla_{h_i} h_j} F_i + F_i \tilde{D}_{\nabla_{h_j} h_i} F_j + \tilde{D}_{h_j} F_i \tilde{D}_{h_i} F_j)\right]\end{aligned}$$

$$= \mathbb{E}\left[F_iF_j\langle h_i,h_j\rangle_{L^2(\mathbb{R}_+)} + F_j\int_0^\infty \tilde{D}_sF_i\int_0^\infty \nabla_t h_j(s)h_i(t)dtds\right.$$
$$+F_i\int_0^\infty \tilde{D}_tF_j\int_0^\infty \nabla_s h_i(t)h_j(s)dsdt$$
$$\left.+\int_0^\infty h_i(t)\tilde{D}_tF_jdt\int_0^\infty h_j(s)\tilde{D}_sF_ids\right],$$

where we used the commutation relation (7.6.8). □

Proposition (7.6.7) is a version of the Skorohod isometry for the operator $\tilde{\delta}$ and it differs from from Propositions 4.3.1 and 6.5.4 which apply to finite difference operators on the Poisson space.
Finally we state a Weitzenböck type identity on configuration space under the form of the commutation relation

$$\tilde{D}\tilde{\delta} + \tilde{\delta}\tilde{D} = \nabla^*\nabla + \mathrm{Id}_{L^2(\mathbb{R}_+)},$$

i.e. the Ricci tensor under the Poisson measure is the identity $\mathrm{Id}_{L^2(\mathbb{R}_+)}$ on $L^2(\mathbb{R}_+)$ by comparison with (7.6.1).

Theorem 7.6.8. *We have for $u \in \mathcal{U}$:*

$$\mathbb{E}\left[|\tilde{\delta}(u)|^2\right] + \mathbb{E}\left[\|\tilde{D}u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)}\right] \qquad (7.6.10)$$
$$= \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right] + \mathbb{E}\left[\|\nabla u\|^2_{L^2(\mathbb{R}_+)\otimes L^2(\mathbb{R}_+)}\right].$$

Proof. Relation (7.6.10) for $u = \sum_{i=1}^n h_iF_i \in \mathcal{U}$ follows from Relation (7.6.7) and Proposition 7.6.7. □

7.7 Chaos Interpretation of Time Changes

In this section we study the Poisson probabilistic interpretation of the operators introduced in Section 4.8. We refer to Section 5.8 for their interpretation on the Wiener space. We now prove that $\nabla^\ominus + D$ is identified to the operator $\tilde{D}$ under the Poisson identification of Φ and $L^2(B)$.

Lemma 7.7.1. *On the Poisson space, $\nabla^\ominus$ satisfies the relation*

$$\nabla_t^\ominus(FG) = F\nabla_t^\ominus G + G\nabla_t^\ominus F - D_tFD_tG, \qquad t\in\mathbb{R}_+,\ F,G\in\mathcal{S}. \qquad (7.7.1)$$

Proof. We will use the multiplication formula for multiple Poisson stochastic integrals of Proposition 6.2.5:

$$I_n(f^{\circ n})I_1(g) = I_{n+1}(f^{\circ n} \circ g) + n\langle f, g\rangle I_{n-1}(f^{\circ n-1}) + nI_n((fg) \circ f^{\circ n-1}),$$

$f, g \in L^4(\mathbb{R}_+)$. We first show that

$$\nabla_t^{\ominus}(I_n(f^{\circ n})I_1(g)) = I_n(f^{\circ n})\nabla_t^{\ominus} I_1(g) + I_1(g)\nabla_t^{\ominus} I_n(f^{\circ n}) - D_t I_1(g) D_t I_n(f^{\circ n}),$$

$t \in \mathbb{R}_+$, when $f, g \in \mathcal{C}_c^1(\mathbb{R}_+)$ and $\langle f, f\rangle_{L^2(\mathbb{R}_+)} = 1$. Indeed, we have

$$\begin{aligned}
&I_n(f^{\circ n})\nabla_t^{\ominus} I_1(g) + I_1(g)\nabla_t^{\ominus} I_n(f^{\circ n})\\
&= -I_n(f^{\circ n})I_1(g'1_{[t,\infty)}) - nI_1(g)I_n((f'1_{[t,\infty)}) \circ f^{\circ(n-1)})\\
&= -n\Big(I_{n+1}((f'1_{[t,\infty)}) \circ f^{\circ(n-1)} \circ g) + (n-1)I_n((fg) \circ (f'1_{[t,\infty)}) \circ f^{\circ(n-2)})\\
&\quad + I_n((gf'1_{[t,\infty)}) \circ f^{\circ(n-1)}) + \langle f'1_{[t,\infty)}, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\circ(n-1)})\\
&\quad + (n-1)\langle f, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}((f'1_{[t,\infty)}) \circ f^{\circ(n-2)})\Big)\\
&\quad - I_{n+1}((g'1_{[t,\infty)}) \circ f^{\circ n}) - nI_n((g'1_{[t,\infty)}f) \circ f^{\circ(n-1)})\\
&\quad - n\langle g'1_{[t,\infty)}, f\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\circ(n-1)})\\
&= -nI_{n+1}((f'1_{[t,\infty)}) \circ f^{\circ(n-1)} \circ g) - I_{n+1}((g'1_{[t,\infty)}) \circ f^{\circ n})\\
&\quad - n(n-1)I_n((f'1_{[t,\infty)}) \circ (fg) \circ f^{\circ(n-2)})\\
&\quad - nI_n((gf'1_{[t,\infty)}) \circ f^{\circ(n-1)}) - nI_n((fg'1_{[t,\infty)}) \circ f^{\circ(n-1)})\\
&\quad + nf(t)g(t)I_{n-1}(f^{\circ(n-1)}) - n(n-1)\langle f, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}((f'1_{[t,\infty)}) \circ f^{\circ(n-2)})\\
&= \nabla_t^{\ominus}\Big(I_{n+1}(f^{\circ n} \circ g) + nI_n(f^{\circ(n-1)} \circ (fg)) + n\langle f, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\circ(n-1)})\Big)\\
&\quad + nf(t)g(t)I_{n-1}(f^{\circ(n-1)})\\
&= \nabla_t^{\ominus}(I_n(f^{\circ n})I_1(g)) + D_t I_1(g) D_t I_n(f^{\circ n}), \qquad f, g \in \mathcal{C}_c^1(\mathbb{R}_+).
\end{aligned}$$

We now make use of the multiplication formula for Poisson stochastic integrals to prove the result on $\mathcal{S}$ by induction. Assume that (7.7.1) holds for $F = I_n(f^{\circ n})$ and $G = I_1(g)^k$ for some $k \geq 1$. Then, using the product rule Proposition 4.5.2 or Proposition 6.4.8 for the operator D_t we have

$$\begin{aligned}
&\nabla_t^{\ominus}(I_n(f^{\circ n})I_1(g)^{k+1})\\
&= I_1(g)\nabla_t^{\ominus}(I_n(f^{\circ n})I_1(g)^k) + I_n(f^{\circ n})I_1(g)^k\nabla_t^{\ominus} I_1(g)\\
&\quad - D_t I_1(g) D_t(I_1(g)^k I_n(f^{\circ n}))\\
&= I_1(g)\left(I_1(g)^k\nabla_t^{\ominus} I_n(f^{\circ n}) + I_n(f^{\circ n})\nabla_t^{\ominus}\left(I_1(g)^k\right) - D_t\left(I_1(g)^k\right) D_t I_n(f^{\circ n})\right)\\
&\quad + I_n(f^{\circ n})I_1(g)^k\nabla_t^{\ominus} I_1(g) - D_t I_1(g)\left(I_1(g)^k D_t I_n(f^{\circ n})\right.\\
&\quad \left. + I_n(f^{\circ n}) D_t\left(I_1(g)^k\right)\right) - D_t I_1(g) D_t I_1(g)^k D_t I_n(f^{\circ n})\\
&= I_1(g)^{k+1}\nabla_t^{\ominus} I_n(f^{\circ n}) + I_n(f^{\circ n})\nabla_t^{\ominus}\left(I_1(g)^{k+1}\right) - D_t\left(I_1(g)^{k+1}\right) D_t I_n(f^{\circ n}),
\end{aligned}$$

$t \in \mathbb{R}_+$. □

Proposition 7.7.2. *We have the identity*

$$\tilde{D} = D + \nabla^{\ominus}$$

on the space $\mathcal{S}$.

Proof. Lemma 7.7.1 shows that $(\nabla^{\ominus} + D)$ is a derivation operator since

$$\begin{aligned}(\nabla_t^{\ominus} + D_t)(FG) &= \nabla_t^{\ominus}(FG) + D_t(FG)\\ &= F\nabla_t^{\ominus}G + G\nabla_t^{\ominus}F - D_tFD_tG + D_t(FG)\\ &= F(\nabla_t^{\ominus} + D_t)G + G(\nabla_t^{\ominus} + D_t)F, \qquad F, G \in \mathcal{S}.\end{aligned}$$

Thus it is sufficient to show that

$$(D_t + \nabla_t^{\ominus})f(T_k) = \tilde{D}f(T_k), \qquad k \geq 1, \quad f \in \mathcal{C}_b^1(\mathbb{R}). \tag{7.7.2}$$

Letting $\pi_{[t}$ denote the projection

$$\pi_{[t}f = f1_{[t,\infty)}, \qquad f \in L^2(\mathbb{R}_+),$$

we have

$$\begin{aligned}&(D_t + \nabla_t^{\ominus})f(T_k) = (D_t + \nabla_t^{\ominus})\sum_{n\in\mathbb{N}}\frac{1}{n!}I_n(f_n^k)\\ &= \sum_{n\geq 1}\frac{1}{(n-1)!}I_{n-1}(f_n^k(\cdot,t)) - \sum_{n\geq 1}\frac{1}{(n-1)!}I_n(\pi_{[t}\otimes \mathrm{Id}^{\otimes(n-1)}\partial_1 f_n^k)\\ &= \sum_{n\in\mathbb{N}}\frac{1}{n!}I_n\left(f_{n+1}^k(\cdot,t) - n\pi_{[t}\otimes \mathrm{Id}^{\otimes(n-1)}\partial_1 f_n^k\right),\end{aligned}$$

where $\mathrm{Id} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is the identity operator. Now,

$$\begin{aligned}&f_{n+1}^k(t, t_1, \ldots, t_n) - n\pi_{[t}\otimes \mathrm{Id}^{\otimes(n-1)}\partial_1 f_n^k(t_1, \ldots, t_n)\\ &\quad = \alpha_{n+1}^k(f)(t_1 \vee \cdots \vee t_n \vee t)\\ &\qquad -\mathbf{1}_{\{t<t_1\vee\cdots\vee t_n\}}\left(\alpha_n^k(f') + \alpha_{n+1}^k(f)\right)(t_1 \vee \cdots \vee t_n)\\ &\quad = \alpha_{n+1}^k(f)\mathbf{1}_{\{t_1\vee\cdots\vee t_n<t\}} - \alpha_n^k(f')(t_1 \vee \cdots \vee t_n)\mathbf{1}_{\{t_1\vee\cdots\vee t_n>t\}}\\ &\quad = \alpha_n^k(-f'_{[t})(t_1 \vee \cdots \vee t_n),\end{aligned}$$

which coincides with n-th term, in the chaos expansion of $-\mathbf{1}_{[0,T_k]}f'(T_k)$ by Proposition 7.4.3, $k \in \mathbb{N}$, $n \geq 1$. Hence Relation (7.7.2) holds and we have $D + \nabla^{\ominus} = \tilde{D}$. □

Since both δ and $\tilde{\delta} = \delta + \nabla^{\oplus}$ coincide with the Itô integral on adapted processes, it follows that $\nabla^{\oplus}$ vanishes on adapted processes. By duality this implies that the adapted projection of $\nabla^{\ominus}$ is zero, hence by Proposition 7.7.2, $\tilde{D}$ is written as a perturbation of D by a gradient process with vanishing adapted projection.

7.8 Notes and References

The notion of lifting of the differential geometry on a Riemannian manifold X to a differential geometry on Ω^X has been introduced in [3], and the integration by parts formula (7.1.2) can be found been obtained therein, cf. also [16]. In Corollary 7.1.7, our pointwise lifting of gradients allows us to recover Theorem 5-2 of [3], page 489, as a particular case by taking expectations in Relation (7.1.5). See [20], [93], [106], for the locality of $\tilde{D}$ and $\tilde{\delta}$. See [2] and [30] for another approaches to the Weitzenböck formula on configuration spaces under Poisson measures. The proof of Proposition 7.6.7 is based on an argument of [43] for path spaces over Lie groups. The gradient $\tilde{D}$ is called "damped" in reference to [44], cf. Section 5.7. The gradient $\tilde{D}$ of Definition 7.2.1 is a modification of the gradient introduced in [23], see also [36]. However, the integration by parts formula of [23] deals with processes of zero integral only, as in (7.3.6). A different version of the gradient $\bar{D}$, which solves the closability issue mentioned at the end of Section 7.3, has been used for sensitivity analysis in [71], [117], [118]. The combined use of D^n and $\tilde{D}$ for the computation of the chaos expansion of the jump time T_d, $d \geq 1$, and the Clark representation formula for $\tilde{D}$ can be found in [102]. The construction of $\tilde{D}$ and D can also be extended to arbitrary Poisson processes with adapted intensities, cf. [32], [104], [105].