Chapter 5
Analysis on the Wiener Space

In this chapter we consider the particular case where the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. The general results stated in Chapters 3 and 4 are developed in this particular setting of a continuous martingale. Here, the gradient operator has the derivation property and can be interpreted as a derivative in the directions of Brownian paths, while the multiple stochastic integrals are connected to the Hermite polynomials. The connection is also made between the gradient and divergence operators and other transformations of Brownian motion, e.g. by time changes. We also describe in more detail the specific forms of covariance identities and deviation inequalities that can be obtained on the Wiener space and on Riemannian path space.

5.1 Multiple Wiener Integrals

In this chapter we consider in detail the particular case where $(M_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, i.e. $(M_t)_{t\in\mathbb{R}_+}$ solves the structure equation (2.10.1) with $\phi_t = 0$, $t \in \mathbb{R}_+$, i.e.

$$[M, M]_t = t, \qquad t \in \mathbb{R}_+.$$

The Hermite polynomials will be used to represent the multiple Wiener integrals.

Definition 5.1.1. *The Hermite polynomial $H_n(x;\sigma^2)$ of degree $n \in \mathbb{N}$ and parameter $\sigma^2 > 0$ is defined with*

$$H_0(x;\sigma^2) = 1, \qquad H_1(x;\sigma^2) = x, \qquad H_2(x;\sigma^2) = x^2 - \sigma^2,$$

and more generally from the recurrence relation

$$H_{n+1}(x;\sigma^2) = xH_n(x;\sigma^2) - n\sigma^2 H_{n-1}(x;\sigma^2), \qquad n \geq 1. \tag{5.1.1}$$

N. Privault, *Stochastic Analysis in Discrete and Continuous Settings*,
Lecture Notes in Mathematics 1982, DOI 10.1007/978-3-642-02380-4_5,

In particular we have

$$H_n(x;0) = x^n, \qquad n \in \mathbb{N}.$$

The generating function of Hermite polynomials is defined as

$$\psi_\lambda(x,\sigma^2) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(x;\sigma^2), \quad \lambda \in (-1,1).$$

Proposition 5.1.2. *The following statements hold on the Hermite polynomials:*
i) Generating function:

$$\psi_\lambda(x,\sigma) = \mathrm{e}^{\lambda x - \frac{1}{2}\lambda^2\sigma^2}, \quad x, \lambda \in \mathbb{R}.$$

ii) Derivation rule:

$$\frac{\partial H_n}{\partial x}(x;\sigma^2) = n H_{n-1}(x;\sigma^2), \tag{5.1.2}$$

iii) Creation rule:

$$H_{n+1}(x;\sigma^2) = \left(x - \sigma^2 \frac{\partial}{\partial x} \right) H_n(x;\sigma^2).$$

Proof. The recurrence relation (5.1.1) shows that the generating function ψ_λ satisfies the differential equation

$$\begin{cases} \dfrac{\partial \psi_\lambda}{\partial \lambda}(x,\sigma) = (x - \lambda\sigma^2)\psi_\lambda(x,\sigma), \\ \\ \psi_0(x,\sigma) = 1, \end{cases}$$

which proves (i). From the expression of the generating function we deduce (ii), and by rewriting (5.1.1) we obtain (iii). □

Let

$$\phi_d^\sigma(s_1,\ldots,s_d) = \frac{1}{(2\pi)^{d/2}} \mathrm{e}^{-(s_1^2+\cdots+s_d^2)/2}, \qquad (s_1,\ldots,s_d) \in \mathbb{R}^d,$$

denote the standard Gaussian density function with covariance $\sigma^2\mathrm{Id}$ on $\mathbb{R}^n$. From Relation (5.1.2) we have

$$\begin{aligned} \frac{\partial}{\partial x}(\phi_1^\sigma(x) H_n(x;\sigma^2)) &= \phi_1^\sigma(x) \left(\frac{\partial H_n}{\partial x}(x;\sigma^2) - \frac{x}{\sigma^2} H_n(x;\sigma^2) \right) \\ &= -\frac{\phi_1^\sigma(x)}{\sigma^2} H_{n+1}(x;\sigma^2), \end{aligned}$$

hence by induction, Proposition 5.1.2-(iii) implies

$$\sigma^{2(k_1+\cdots+k_d)}\frac{(-1)^{k_1+\cdots+k_d}}{\phi_d^\sigma(x_1,\ldots x_d)}\partial_{x_1}^{k_1}\cdots\partial_{x_d}^{k_d}\phi_d^\sigma(x_1,\ldots,x_d)=\prod_{i=1}^d H_{k_i}(x_i;\sigma^2). \tag{5.1.3}$$

Let $I_n(f_n)$ denote the multiple stochastic integral of $f_n\in L^2(\mathbb{R}_+)^{\circ n}$ with respect to $(B_t)_{t\in\mathbb{R}_+}$, as defined in Section 2.7. Note that here $I_1(u)$ coincides with $J_1(u)$ defined in (2.2.2), and in particular it has a centered Gaussian distribution with variance $\|u\|_2^2:=\|u\|_{L^2(\mathbb{R}_+)}^2$, $u\in L^2(\mathbb{R}_+)$.
In addition, the multiplication formula (4.5.1) of Proposition 4.5.1 reads

$$I_1(u)I_n(v^{\otimes n})=I_{n+1}(v^{\otimes n}\circ u)+n\langle u,v\rangle_{L^2(\mathbb{R}_+)}I_{n-1}(v^{\otimes(n-1)}) \tag{5.1.4}$$

for $n\geq 1$, since with $\phi_t=0$, $t\in\mathbb{R}_+$, and we have in particular

$$I_1(u)I_1(v)=I_2(v\circ u)+\langle u,v\rangle_{L^2(\mathbb{R}_+)}$$

for $n=1$. More generally, Relation (4.5.7) of Proposition 4.5.6 reads

$$I_n(f_n)I_m(g_m)=\sum_{s=0}^{n\wedge m}\binom{n}{s}\binom{m}{s}I_{n+m-2s}(h_{n,m,2s}),$$

where $h_{n,m,2s}$ is the symmetrization in $n+m-2s$ variables of

$$(x_{s+1},\ldots,x_n,y_{s+1},\ldots,y_m)\mapsto \int_{\mathbb{R}_+^s} f_n(x_1,\ldots,x_n)g_m(x_1,\ldots,x_i,y_{s+1},\ldots,y_m)dx_1\cdots dx_s.$$

Proposition 5.1.3. *For any orthogonal family $\{u_1,\ldots,u_d\}$ in $L^2(\mathbb{R}_+)$ we have*

$$I_n(u_1^{\otimes n_1}\circ\cdots\circ u_d^{\otimes n_d})=\prod_{k=1}^d H_{n_k}(I_1(u_k);\|u_k\|_2^2),$$

where $n=n_1+\cdots+n_d$.

Proof. We have

$$H_0(I_1(u);\|u\|_2^2)=I_0(u^{\otimes 0})=1 \quad\text{and}\quad H_1(I_1(u);\|u\|_2^2)=I_1(u),$$

hence the proof follows by induction on $n\geq 1$, by comparison of the recurrence formula (5.1.1) with the multiplication formula (5.1.4). □

In particular,

$$\begin{aligned}I_n\left(\mathbf{1}_{[0,t]}^{\otimes n}\right)&=n!\int_0^t\int_0^{s_n}\cdots\int_0^{s_2}dB_{s_1}\cdots dB_{s_n}\\&=H_n(B_t;t),\end{aligned} \tag{5.1.5}$$

and from (4.5.3) we have

$$I_n\left(\mathbf{1}_{[t_0,t_1]}^{\otimes n_1}\circ\cdots\circ\mathbf{1}_{[t_{d-1},t_d]}^{\otimes n_d}\right)=\prod_{k=1}^{d} I_{n_k}\left(\mathbf{1}_{[t_{k-1},t_k]}^{\otimes n_k}\right)$$
$$=\prod_{k=1}^{d} H_{n_k}(B_{t_k}-B_{t_{k-1}};t_k-t_{k-1}).$$

From this we recover the orthonormality properties of the Hermite polynomials with respect to the Gaussian density:

$$\begin{aligned}\int_{-\infty}^{\infty} H_n(x;t)H_m(x;t)\mathrm{e}^{-\frac{x^2}{2t}}\frac{dx}{\sqrt{2\pi t}} &= \mathbb{E}[H_n(B_t;t)H_m(B_t;t)]\\ &= \mathbb{E}[I_n(\mathbf{1}_{[0,t]}^{\otimes n})I_m(\mathbf{1}_{[0,t]}^{\otimes m})]\\ &= \mathbf{1}_{\{n=m\}}n!t^n.\end{aligned}$$

In addition, by Lemma 2.7.2 we have

$$\begin{aligned}H_n(B_t;t) &= I_n\left(\mathbf{1}_{[0,t]}^{\otimes n}\right)\\ &= \mathbb{E}\left[I_n(\mathbf{1}_{[0,T]}^{\otimes n})\Big|\mathcal{F}_t\right], \qquad t\in\mathbb{R}_+,\end{aligned}$$

is a martingale which, from Itô's formula, can be written as

$$\begin{aligned}&H_n(B_t;t)=I_n(\mathbf{1}_{[0,t]}^{\otimes n})\\ &=H_n(0;0)+\int_0^t\frac{\partial H_n}{\partial x}(B_s;s)dB_s+\frac{1}{2}\int_0^t\frac{\partial^2 H_n}{\partial x^2}(B_s;s)ds+\int_0^t\frac{\partial H_n}{\partial s}(B_s;s)ds\\ &=n\int_0^t I_{n-1}(\mathbf{1}_{[0,s]}^{\otimes(n-1)})dB_s\\ &=n\int_0^t H_{n-1}(B_s;s)dB_s\end{aligned}$$

from Proposition 2.12.1. By identification we recover Proposition 5.1.2-(ii), i.e.

$$\frac{\partial H_n}{\partial x}(x;s)=nH_{n-1}(x;s), \tag{5.1.6}$$

and the partial differential equation

$$\frac{\partial H_n}{\partial s}(x;s)=-\frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(x;s),$$

i.e the heat equation with initial condition

$$H_n(x;0)=x^n, \qquad x\in\mathbb{R}, \quad n\in\mathbb{N}.$$

Given $f_n \in L^2(\mathbb{R}_+)^{\otimes n}$ with orthogonal expansion

$$f_n = \sum_{\substack{n_1+\cdots+n_d=n \\ k_1,\ldots,k_d \geq 0}} a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d} e^{\otimes n_1}_{k_1} \circ \cdots \circ e^{\otimes n_d}_{k_d},$$

in an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}_+)$, we have

$$I_n(f_n) = \sum_{\substack{n_1+\cdots+n_d=n \\ k_1,\ldots,k_d \geq 0}} a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d} H_{n_1}(I_1(e_{k_1});1)\cdots H_{n_d}(I_1(e_{k_d});1),$$

where the coefficients $a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d}$ are given by

$$\begin{aligned} a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d} &= \frac{1}{n_1!\cdots n_d!}\langle I_n(f_n), I_k(e^{\otimes n_1}_{k_1} \circ \cdots \circ u^{\otimes n_d}_{k_d})\rangle_{L^2(\Omega)} \\ &= \langle f_n, e^{\otimes n_1}_{k_1} \circ \cdots \circ e^{\otimes n_d}_{k_d}\rangle_{L^2(\mathbb{R}^n_+)}. \end{aligned}$$

Proposition 2.13.1 implies the following relation for exponential vectors, that can be recovered independently using the Hermite polynomials.

Proposition 5.1.4. *We have*

$$\xi(u) = \sum_{k=0}^{\infty} \frac{1}{n!} I_n(u^{\otimes n}) = \exp\left(I_1(u) - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)}\right). \tag{5.1.7}$$

Proof. Relation (5.1.7) follows from Proposition 5.1.2-i) and Proposition 5.1.3 which reads $I_n(u^{\otimes n}) = H_n(I_1(u); \|u\|^2_{L^2(\mathbb{R}_+)})$, $n \geq 1$. □

Proposition 5.1.5. *The Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ has the chaos representation property.*

Proof. Theorem 4.1, p. 134 of [50], shows by a Fourier transform argument that the linear space spanned by the exponential vectors

$$\left\{\exp\left(I_1(u) - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)}\right) \quad : \quad u \in L^2(\mathbb{R}_+)\right\}$$

is dense in $L^2(\Omega)$. To conclude we note that the exponential vectors belong to the closure of $\mathcal{S}$ in $L^2(\Omega)$. □

From Proposition 5.1.5, any $F \in L^2(\Omega)$ has a chaos decomposition

$$F = \sum_{k=0}^{\infty} I_k(g_k),$$

where

$$I_k(g_k) \tag{5.1.8}$$
$$= \sum_{d=1}^{k} \sum_{k_1+\cdots+k_d=k} \frac{1}{k_1!\cdots k_d!} \mathbb{E}[F I_k(u_1^{\otimes k_1} \circ \cdots \circ u_d^{\otimes k_d})] I_k(u_1^{\otimes k_1} \circ \cdots \circ u_d^{\otimes k_d})$$
$$= \sum_{d=1}^{k} \sum_{k_1+\cdots+k_d=k} \frac{1}{k_1!\cdots k_d!} \mathbb{E}[F I_k(u_1^{\otimes k_1} \circ \cdots \circ u_d^{\otimes k_d})] \prod_{i=1}^{n} H_{k_i}(I_1(u_i); \|u_i\|_2^2),$$

is a finite sum since for all $m \geq 1$ and $l > k$,

$$\mathbb{E}[I_m(e_l^{\otimes m}) g(I_1(e_1), \ldots, I_1(e_k))] = 0.$$

Lemma 5.1.6. *Assume that F has the form $F = g(I_1(e_1), \ldots, I_1(e_k))$ for some*

$$g \in L^2(\mathbb{R}^k, (2\pi)^{-k/2} \mathrm{e}^{-|x|^2/2} dx),$$

and admits the chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

Then for all $n \geq 1$ there exists a (multivariate) Hermite polynomial P_n of degree n such that

$$I_n(f_n) = P_n(I_1(e_1), \ldots, I_1(e_k)).$$

Proof. The polynomial P_n is given by (5.1.8) above, which is a finite sum. □

Lemma 5.1.6 can also be recovered from the relation

$$f(I_1(e_1), \ldots, I_1(e_d)) \tag{5.1.9}$$
$$= \sum_{n=0}^{\infty} \sum_{\substack{k_1+\cdots+k_d=n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{(-1)^n}{k_1!\cdots k_d!} \langle f, \partial_1^{k_1} \cdots \partial_d^{k_d} \phi_d^1 \rangle_{L^2(\mathbb{R}^d)} I_n(e_1^{\otimes k_1} \circ \cdots \circ e_d^{\otimes k_d}),$$

which follows from (5.1.6) and (5.1.3).

5.2 Gradient and Divergence Operators

In the Brownian case D has the derivation property, as an application of Proposition 4.5.2 with $\phi_t = 0$, $t \in \mathbb{R}_+$, i.e.

$$D_t(FG) = F D_t G + G D_t F, \qquad F, G \in \mathcal{S}. \tag{5.2.1}$$

More precisely we have the following result.

Proposition 5.2.1. *Let $u_1, \ldots, u_n \in L^2(\mathbb{R}_+)$ and*

$$F = f(I_1(u_1), \ldots, I_1(u_n)),$$

where f is a polynomial or $f \in \mathcal{C}_b^1(\mathbb{R}^n)$. We have

$$D_t F = \sum_{i=1}^{n} u_i(t) \frac{\partial f}{\partial x_i}(I_1(u_1), \ldots, I_1(u_n)), \qquad t \in \mathbb{R}_+. \tag{5.2.2}$$

Proof. Using the derivation rule (5.1.2), Definition 4.1.1 and Proposition 5.1.3, this statement is obvious when

$$\begin{aligned} F &= I_n(u^{\otimes n}) \\ &= H_n(I_1(u); \|u\|_2^2), \qquad u \in L^2(\mathbb{R}_+). \end{aligned}$$

Using the product rule (5.2.1) it extends to polynomial f (precisely, to products of Hermite polynomials) and to $F \in \mathcal{S}$. In the general case we may assume that $u_1, \ldots, u_n \in L^2(\mathbb{R}_+)$ are orthonormal, and that $f \in \mathcal{C}_c^1(\mathbb{R}^n)$. Then from Lemma 5.1.6, we have the chaotic decomposition

$$\begin{aligned} F &= f(I_1(u_1), \ldots, I_1(u_n)) \\ &= \sum_{k=0}^{\infty} I_k(g_k), \end{aligned}$$

where $I_k(g_k)$ is a polynomial in $I_1(u_1), \ldots, I_1(u_n)$. The sequence

$$F_k = \sum_{l=0}^{k} I_l(g_l), \quad k \in \mathbb{N}, \qquad k \in \mathbb{N},$$

is a sequence of polynomial functionals contained in $\mathcal{S}$ that converges to F in $L^2(\Omega)$. By tensorization of the finite dimensional integration by parts

$$\begin{aligned} &\int_{-\infty}^{\infty} f'(x) H_n(x;1) \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} f(x)(x H_n(x;1) - H_n'(x;1)) \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} f(x)(x H_n(x;1) - n H_{n-1}(x;1)) \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} f(x) H_{n+1}(x;1) \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

we get

$$
\begin{aligned}
&\mathbb{E}\left[I_k(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n})\frac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n))\right]\\
&= \mathbb{E}[f(I_1(u_1),\ldots,I_1(u_n))I_{k+1}(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n}\circ u_i)]\\
&= \mathbb{E}[I_{k+1}(g_{k+1})I_{k+1}(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n}\circ u_i)]\\
&= \mathbb{E}[\langle DI_{k+1}(g_{k+1}),u_i\rangle_{L^2(\mathbb{R}_+)}I_k(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n})]
\end{aligned}
$$

hence $\displaystyle\frac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n))$ has the chaotic decomposition

$$
\frac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n)) = \sum_{k=1}^{\infty}\langle DI_k(g_k),u_i\rangle_{L^2(\mathbb{R}_+)},
$$

that converges in $L^2(\Omega)$, hence $(DF_k)_{k\in\mathbb{N}}$ converges in $L^2(\Omega\times\mathbb{R}_+)$ to

$$
\begin{aligned}
\sum_{i=1}^n u_i\frac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n)) &= \sum_{i=1}^n u_i\sum_{k=1}^{\infty}\langle DI_k(g_k),u_i\rangle_{L^2(\mathbb{R}_+)}\\
&= \sum_{k=1}^{\infty} DI_k(g_k).
\end{aligned}
$$

$\square$

In particular for f polynomial and for $f\in\mathcal{C}_b^1(\mathbb{R}^n)$ we have

$$
D_t f(B_{t_1},\ldots B_{t_n}) = \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)\frac{\partial f}{\partial x_i}(B_{t_1},\ldots B_{t_n}), \qquad 0\le t_1<\cdots<t_n, \tag{5.2.3}
$$

and (5.2.2) can also be written as

$$
\begin{aligned}
&\langle DF,h\rangle_{L^2(\mathbb{R}_+)} \qquad (5.2.4)\\
&= \frac{d}{d\varepsilon}f\left(\int_0^\infty u_1(t)(dB(t)+\varepsilon h(t)dt),\ldots,\int_0^\infty u_n(t)(dB(t)+\varepsilon h(t)dt)\right)_{|\varepsilon=0},\\
&= \frac{d}{d\varepsilon}F(\omega+\epsilon h)_{|\varepsilon=0},
\end{aligned}
$$

$h\in L^2(\mathbb{R}_+)$, where the limit exists in $L^2(\Omega)$. We refer to the above identity as the *probabilistic interpretation* of the gradient operator D on the Wiener space.

From Proposition 4.2.3, the operator D satisfies the Clark formula Assumption 3.2.1.

Corollary 5.2.2. *For all $F \in L^2(\Omega)$ we have*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dB_t.$$

Moreover, since $\phi_t = 0$, $t \in \mathbb{R}_+$, Proposition 4.5.4 becomes a divergence formula as in the next proposition.

Proposition 5.2.3. *For all $u \in \mathcal{U}$ and $F \in \mathcal{S}$ we have*

$$\delta(u)F = \delta(uF) + \langle DF, u \rangle_{L^2(\mathbb{R}_+)}.$$

On the other hand, applying Proposition 4.5.6 yields the following multiplication formula for Wiener multiple stochastic integrals:

$$I_n(f_n) I_m(g_m) = \sum_{k=0}^{n \wedge m} \frac{1}{k!} \binom{n}{k} \binom{m}{k} I_{n+m-2k}(f_n \otimes_k g_m),$$

where $f_n \otimes_k g_m$ is the contraction

$$(t_{k+1}, \ldots, t_n, s_{k+1}, \ldots, s_m) \mapsto \int_0^\infty \cdots \int_0^\infty f_n(t_1, \ldots t_n) g_m(t_1, \ldots, t_k, s_{k+1}, \ldots, s_m) dt_1 \ldots dt_k,$$

$t_{k+1}, \ldots, t_n, s_{k+1}, \ldots, s_m \in \mathbb{R}_+$.
From Proposition 4.3.4, the Skorohod integral $\delta(u)$ coincides with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t,$$

when u is square-integrable and adapted with respect to the Brownian filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$.
We have the following corollary, that completes Proposition 4.2.2 and can be proved using the density property of smooth functions in finite-dimensional Sobolev spaces, cf. e.g. Lemma 1.2 of [91] or [96].
For simplicity we work with a Brownian motion $(B_t)_{t \in [0,1]}$ on $[0, 1]$ and we assume that $(e_n)_{n \in \mathbb{N}}$ is the dyadic basis of $L^2([0, 1])$ given by

$$e_k = 2^{n/2} \mathbf{1}_{\left[\frac{k-2^n}{2^n}, \frac{k+1-2^n}{2^n}\right]}, \qquad 2^n \leq k \leq 2^{n+1} - 1, \quad n \in \mathbb{N}. \tag{5.2.5}$$

Corollary 5.2.4. *Given $F \in L^2(\Omega)$, let for all $n \in \mathbb{N}$:*

$$\mathcal{G}_n = \sigma(I_1(e_{2^n}), \ldots, I_1(e_{2^{n+1}-1})),$$

and $F_n = \mathbb{E}[F|\mathcal{G}_n]$, *and consider* f_n *a square-integrable function with respect to the standard Gaussian measure on* $\mathbb{R}^{2^n}$, *such that*

$$F_n = f_n(I_1(e_{2^n}), \ldots, I_1(e_{2^{n+1}-1})).$$

Then $F \in \mathrm{Dom}\,(D)$ *if and only if* f_n *belongs for all* $n \geq 1$ *to the Sobolev space* $W^{2,1}(\mathbb{R}^{2^n})$ *with respect to the standard Gaussian measure on* $\mathbb{R}^{2^n}$, *and the sequence*

$$D_t F_n := \sum_{i=1}^{2^n} e_{2^n+i-1}(t) \frac{\partial f_n}{\partial x_i}(I_1(e_{2^n}), \ldots, I_1(e_{2^{n+1}-1})), \qquad t \in [0,1],$$

converges in $L^2(\Omega \times [0,1])$. *In this case we have*

$$DF = \lim_{n\to\infty} DF_n.$$

We close this section by considering the case of a d-dimensional Brownian motion $(B_t)_{0\leq t\leq T}=(B_t^{(1)}, \ldots, B_t^{(d)})_{0\leq t\leq T}$, where $(B_t^{(1)})_{t\in\mathbb{R}_+}, \ldots, (B_t^{(d)})_{t\in\mathbb{R}_+}$, are independent copies of Brownian motion. In this case the gradient D can be defined with values in $H = L^2(\mathbb{R}_+, X \otimes \mathbb{R}^d)$, where X is a Hilbert space, by

$$D_t F = \sum_{i=1}^{n} \mathbf{1}_{[0,t_i]}(t) \nabla_i f(B_{t_1}, \ldots, B_{t_n}), \qquad t \in \mathbb{R}_+,$$

for F of the form

$$F = f(B_{t_1}, \ldots, B_{t_n}), \tag{5.2.6}$$

$f \in \mathcal{C}_b^\infty(\mathbb{R}^n, X)$, $t_1, \ldots, t_n \in \mathbb{R}_+$, $n \geq 1$.
We let $I\!D_{p,k}(X)$ denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{I\!D_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p(W, X\otimes H^{\otimes l})}, \qquad p > 1,$$

where $X \otimes H$ denotes the completed symmetric tensor product of X and H.
For all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $k \geq 1$, the Skorohod integral operator

$$\delta : I\!D_{p,k}(X \otimes H) \to I\!D_{q,k-1}(X)$$

adjoint of

$$D : I\!D_{p,k}(X) \to I\!D_{q,k-1}(X \otimes H),$$

satisfies

$$E[\langle F, \delta(u)\rangle_X] = E[\langle DF, u\rangle_{X\otimes H}],$$

$F \in I\!D_{p,k}(X)$, $u \in I\!D_{q,k}(X \otimes H)$.
Finally we note that the chaos representation property extends to d-dimensional Brownian motion.

Theorem 5.2.5. *For any $F \in L^2(\Omega)$ there exists a unique sequence $(f_n)_{n\in\mathbb{N}}$ of deterministic symmetric functions*

$$f_n = (f_n^{(i_1,\ldots,i_n)})_{i_1,\ldots,i_n\in\{1,\ldots,d\}} \in L^2([0,T],\mathbb{R}^d)^{\circ n}$$

such that

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n).$$

Moreover we have

$$\|F\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} n!\|f_n^{(i_1,\ldots,i_n)}\|^2_{L^2([0,T]^n)}.$$

Given $F = f(B_{t_1},\ldots,B_{t_n}) \in L^2(\Omega)$ where $(t_1,\ldots,t_n) \in [0,T]^n$ and

$$f(x^{1,1},\ldots,x^{d,1},\ldots,x^{1,n},\ldots,x^{d,n})$$

is in $\mathcal{C}_b^\infty(\mathbb{R}^{dn})$, for $l = 1,\ldots,d$ we have:

$$D_t^{(l)} F = \sum_{k=1}^{n} \frac{\partial f}{\partial x^{l,k}}(B_{t_1},\ldots,B_{t_n}) \mathbf{1}_{[0,t_k]}(t).$$

Similarly the Clark formula of Corollary 5.2.2 extends to the d-dimensional case as

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F|\mathcal{F}_t] \cdot dB_t, \tag{5.2.7}$$

$F \in L^2(\Omega)$.

5.3 Ornstein-Uhlenbeck Semi-Group

Recall the Definition 4.4.1 of the Ornstein-Uhlenbeck semi-group $(P_t)_{t\in\mathbb{R}_+}$ as

$$P_t F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \mathrm{e}^{-nt} I_n(f_n), \qquad t \in \mathbb{R}_+, \tag{5.3.1}$$

for any $F \in L^2(\Omega)$ with the chaos representation

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n).$$

In this section we show that on the Wiener space, P_t admits the integral representation, known as the Mehler formula,

$$P_t F(\omega) = \int_\Omega F(\mathrm{e}^{-t}\omega + \sqrt{1-\mathrm{e}^{-2t}}\tilde{\omega})d\mathbb{P}(\tilde{\omega}), \qquad \mathbb{P}(d\omega)-a.s., \qquad (5.3.2)$$

$F \in L^2(\Omega)$, $t \in \mathbb{R}_+$, cf. e.g. [92], [143], [147]. Precisely we have the following.

Lemma 5.3.1. *Let F of the form*

$$F = f(I_1(u_1), \ldots, I_1(u_n)),$$

where $f \in \mathcal{C}_b(\mathbb{R}^n)$ and $u_1, \ldots, u_n \in L^2(\mathbb{R}_+)$ are mutually orthogonal. For all $t \in \mathbb{R}_+$ we have:

$$P_t F(\omega) = \int_\Omega f(\mathrm{e}^{-t} I_1(u_1)(\omega) + \sqrt{1-\mathrm{e}^{-2t}} I_1(u_1)(\tilde{\omega}), \ldots$$
$$\ldots, \mathrm{e}^{-t} I_1(u_n)(\omega) + \sqrt{1-\mathrm{e}^{-2t}} I_1(u_n)(\tilde{\omega}))\mathbb{P}(d\tilde{\omega}).$$

Proof. Since, by Proposition 5.1.5, the exponential vectors are total in $L^2(\Omega)$ and P_t is continuous on $L^2(\Omega)$, it suffices to consider

$$f_u(x) = \exp\left(x - \frac{1}{2}\|u\|_2^2\right),$$

and to note that by Proposition 5.1.4 we have

$$\begin{aligned}
\xi(f_u) &= \exp\left(I_1(u) - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)}\right) \\
&= \sum_{k=0}^\infty \frac{1}{n!} H_n(I_1(u); \|u\|^2_{L^2(\mathbb{R}_+)}) \\
&= \sum_{k=0}^\infty \frac{1}{n!} I_n(u^{\otimes n}).
\end{aligned}$$

Hence

$$\begin{aligned}
P_t \xi(f_u) &= \sum_{k=0}^\infty \frac{\mathrm{e}^{-nt}}{n!} I_n(u^{\otimes n}) \\
&= \exp\left(\mathrm{e}^{-t} I_1(u) - \frac{1}{2}\mathrm{e}^{-2t}\|u\|^2_{L^2(\mathbb{R}_+)}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega f_u(\mathrm{e}^{-t} I_1(u)(\omega) + \sqrt{1-\mathrm{e}^{-2t}} I_1(u)(\tilde{\omega}))\mathbb{P}(d\tilde{\omega}) \\
&= \int_{-\infty}^\infty \exp\left(\mathrm{e}^{-t} I_1(u)(\omega) + \sqrt{1-\mathrm{e}^{-2t}} y - \frac{1}{2}\|u\|_2^2 - \frac{y^2}{2\|u\|_2^2}\right) \frac{dy}{\sqrt{2\pi}\|u\|_2}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \exp\left(\mathrm{e}^{-t} I_1(u)(\omega) - \frac{1}{2}\|\mathrm{e}^{-t}u\|_2^2 - \frac{(y - \sqrt{1-\mathrm{e}^{-2t}}\|u\|_2^2)^2}{2\|u\|_2^2} \right) \frac{dy}{\sqrt{2\pi}\|u\|_2} \\
&= \exp\left(\mathrm{e}^{-t} I_1(u)(\omega) - \frac{1}{2}\|\mathrm{e}^{-t}u\|_2^2 \right) \\
&= P_t f_u(I_1(u))(\omega).
\end{aligned}
$$

The result is extended by density of the exponential vectors in $L^2(\Omega)$ since Brownian motion has the chaos representation property from Proposition 5.2.5. $\square$

The integral representation of Lemma 5.3.1 together with Jensen's inequality (9.3.1) imply the following bound which is used in the proof of deviation inequalities in Section 4.7, cf. Lemma 4.7.1.

Lemma 5.3.2. *We have for $u \in L^2(\Omega \times \mathbb{R}_+)$:*

$$
\|P_t u\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leq \|u\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad t \in \mathbb{R}_+.
$$

Proof. Due to Lemma 5.3.1 we have

$$
\begin{aligned}
\|P_s u(\omega)\|_{L^2(\mathbb{R}_+)}^2 &= \int_0^\infty |P_s u_t(\omega)|^2 dt \\
&\leq \int_0^\infty P_s |u_t(\omega)|^2 dt \\
&\leq \|u\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}^2.
\end{aligned}
$$

$\square$

5.4 Covariance Identities and Inequalities

In this section we present some covariance identities and inequalities that can be obtained in the particular setting of the Wiener space, in addition to the general results of Section 3.4 and 4.4.

We consider the order relation introduced in [11] when $\Omega = \mathcal{C}_0(\mathbb{R}_+)$ is the space of continuous functions on $\mathbb{R}_+$ starting at 0.

Definition 5.4.1. *Given $\omega_1, \omega_2 \in \Omega$, we say that $\omega_1 \preceq \omega_2$ if and only if we have*

$$
\omega_1(t_2) - \omega_1(t_1) \leq \omega_2(t_2) - \omega_2(t_1), \qquad 0 \leq t_1 \leq t_2.
$$

The class of non-decreasing functionals with respect to $\preceq$ is larger than that of non-decreasing functionals with respect to the pointwise order on Ω defined by

$$
\omega_1(t) \leq \omega_2(t), \qquad t \in \mathbb{R}_+, \quad \omega_1, \omega_2 \in \Omega.
$$

Definition 5.4.2. *A random variable $F : \Omega \to \mathbb{R}$ is said to be non-decreasing if*

$$\omega_1 \preceq \omega_2 \;\Rightarrow\; F(\omega_1) \le F(\omega_2), \qquad \mathbb{P}(d\omega_1) \otimes \mathbb{P}(d\omega_2) - a.s.$$

The next result is the FKG inequality on the Wiener space. It recovers Theorem 4 of [11] under weaker (i.e. almost-sure) hypotheses.

Theorem 5.4.3. *For any non-decreasing functionals $F, G \in L^2(\Omega)$ we have*

$$\mathrm{Cov}\,(F, G) \ge 0.$$

The proof of this result is a direct consequence of Lemma 3.4.2 and the next lemma.

Lemma 5.4.4. *For every non-decreasing $F \in \mathrm{Dom}\,(D)$ we have*

$$D_t F \ge 0, \qquad dt \times d\mathbb{P} - a.e.$$

Proof. Without loss of generality we state the proof for a Brownian motion on the interval $[0,1]$. For $n \in \mathbb{N}$, let π_n denotes the orthogonal projection from $L^2([0,1])$ onto the linear space generated by the sequence $(e_k)_{2^n \le k < 2^{n+1}}$ introduced in Definition 5.2.5. Let

$$H = \left\{ h : [0,1] \to \mathbb{R} \;:\; \int_0^1 |\dot{h}(s)|^2 ds < \infty \right\}$$

denote the Cameron-Martin space, i.e. the space of absolutely continuous functions with square-integrable derivative.
Given $h \in H$, let

$$h_n(t) = \int_0^t [\pi_n \dot{h}](s) ds, \qquad t \in [0,1], \quad n \in \mathbb{N}.$$

Let Λ_n denote the square-integrable and $\mathcal{G}_n$-measurable random variable

$$\Lambda_n = \exp\left(\int_0^1 [\pi_n \dot{h}](s) dB_s - \frac{1}{2} \int_0^1 |[\pi_n \dot{h}](s)|^2 ds \right).$$

Letting $F_n = \mathbb{E}[F \mid \mathcal{G}_n]$, $n \in \mathbb{N}$, a suitable change of variable on $\mathbb{R}^n$ with respect to the standard Gaussian density (or an application of the Cameron-Martin theorem cf. e.g. [146]) shows that for all $n \in \mathbb{N}$ and $\mathcal{G}_n$-measurable bounded random variable G_n shows that

$$\begin{aligned}
\mathbb{E}[F_n(\cdot + h_n) G_n] &= \mathbb{E}[\Lambda_n F_n G_n(\cdot - h_n)] \\
&= \mathbb{E}[\Lambda_n \mathbb{E}[F|\mathcal{G}_n] G_n(\cdot - h_n)] \\
&= \mathbb{E}[\mathbb{E}[\Lambda_n F G_n(\cdot - h_n)|\mathcal{G}_n]] \\
&= \mathbb{E}[\Lambda_n F G_n(\cdot - h_n)] \\
&= \mathbb{E}[F(\cdot + h_n) G_n],
\end{aligned}$$

hence

$$F_n(\omega + h_n) = \mathbb{E}[F(\cdot + h_n)|\mathcal{G}_n](\omega), \qquad \mathbb{P}(d\omega) - a.s.$$

If $\dot{h}$ is non-negative, then $\pi_n \dot{h}$ is non-negative by construction hence $\omega \preceq \omega + h_n$, $\omega \in \Omega$, and we have

$$F(\omega) \leq F(\omega + h_n), \qquad \mathbb{P}(d\omega) - a.s.,$$

since from the Cameron-Martin theorem, $\mathbb{P}(\{\omega + h_n \ : \ \omega \in \Omega\}) = 1$. Hence we have

$$\begin{aligned}
&F_n(\omega + h) \\
&= f_n(I_1(e_{2^n}) + \langle e_{2^n}, \dot{h}\rangle_{L^2([0,1])}, \ldots, I_1(e_{2^{n+1}-1}) + \langle e_{2^{n+1}-1}, \dot{h}\rangle_{L^2([0,1])}) \\
&= f_n(I_1(e_{2^n}) + \langle e_{2^n}, \pi_n\dot{h}\rangle_{L^2([0,1])}, \ldots, I_1(e_{2^{n+1}-1}) + \langle e_{2^{n+1}-1}, \pi_n\dot{h}\rangle_{L^2([0,1])}) \\
&= F_n(\omega + h_n) \\
&= \mathbb{E}[F(\cdot + h_n)|\mathcal{G}_n](\omega) \\
&\geq \mathbb{E}[F|\mathcal{G}_n](\omega) \\
&= F_n(\omega), \qquad \mathbb{P}(d\omega) - a.s.,
\end{aligned}$$

where $(e_k)_{k\in\mathbb{N}}$ is the dyadic basis defined in (5.2.5). Consequently, for any $\varepsilon_1 \leq \varepsilon_2$ and $h \in H$ such that $\dot{h}$ is non-negative we have

$$F_n(\omega + \varepsilon_1 h) \leq F_n(\omega + \varepsilon_2 h),$$

i.e. the smooth function $\varepsilon \mapsto F_n(\omega + \varepsilon h)$ is non-decreasing in ε on $[-1, 1]$, $\mathbb{P}(d\omega)$-a.s. As a consequence,

$$\langle DF_n, \dot{h}\rangle_{L^2([0,1])} = \frac{d}{d\varepsilon} F_n(\omega + \epsilon h)_{|\varepsilon=0} \geq 0,$$

for all $h \in H$ such that $\dot{h} \geq 0$, hence $DF_n \geq 0$. Taking the limit of $(DF_n)_{n\in\mathbb{N}}$ as n goes to infinity shows that $DF \geq 0$. □

Next, we extend Lemma 5.4.4 to $F \in L^2(\Omega)$.

Proposition 5.4.5. *For any non-decreasing functional $F \in L^2(\Omega)$ we have*

$$\mathbb{E}[D_t F|\mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Proof. Assume that $F \in L^2(\Omega)$ is non-decreasing. Then $P_{1/n}F$, $n \geq 1$, is non-decreasing from (5.3.2), and belongs to Dom (D) from Relation (5.3.1). From Lemma 5.4.4 we have

$$D_t P_{1/n} F \geq 0, \qquad dt \times d\mathbb{P} - a.e.,$$

hence

$$\mathbb{E}[D_t P_{1/n} F|\mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Taking the limit as n goes to infinity yields $\mathbb{E}[D_tF|\mathcal{F}_t] \geq 0$, $dt \times d\mathbb{P}$-a.e. from Proposition 3.2.6 and the fact that $P_{1/n}F$ converges to F in $L^2(\Omega)$ as n goes to infinity. $\square$

Finally, using the change of variable $\alpha = \mathrm{e}^{-t}$, the covariance identity (4.4.2) can be rewritten with help of Lemma 5.3.1 as

$$\begin{aligned}
&\mathrm{Cov}\,(F,G)\\
&= \int_0^1 \int_\Omega \int_\Omega \langle \nabla f(I_1(u_1),\ldots,I_1(u_n))(\omega), \nabla g(\alpha I_1(u_1)(\omega) + \sqrt{1-\alpha^2} I_1(u_1)(\tilde{\omega}),\\
&\quad \ldots, \alpha I_1(u_n)(\omega) + \sqrt{1-\alpha^2} I_1(u_n)(\tilde{\omega}))\rangle_{\mathbb{R}^n} \mathbb{P}(d\omega)\mathbb{P}(d\tilde{\omega})d\alpha.
\end{aligned}$$

This identity can be recovered using characteristic function: letting

$$\varphi(t) = \mathbb{E}[\mathrm{e}^{itI_1(u)}] = \mathrm{e}^{-t^2\|u\|_2^2/2}$$

and

$$\varphi_\alpha(s,t) := \mathbb{E}[\mathrm{e}^{is\alpha I_1(u)(\omega) + it\sqrt{1-\alpha^2} I_1(u)(\tilde{\omega})}] = (\varphi(s+t))^\alpha (\varphi(s))^{1-\alpha} (\varphi(t))^{1-\alpha},$$

we have

$$\begin{aligned}
&\mathrm{Var}\,[\mathrm{e}^{isI_1(u)}] = \varphi_1(s,t) - \varphi_0(s,t)\\
&= \int_0^1 \frac{\partial \varphi_\alpha}{\partial \alpha}(s,t) d\alpha\\
&= \int_0^1 \frac{\partial}{\partial \alpha}((\varphi(t))^{1-\alpha}(\varphi(t+s))^\alpha (\varphi(s))^{1-\alpha}) d\alpha\\
&= \int_0^1 \log\left(\frac{\varphi(s+t)}{\varphi(s)\varphi(t)}\right) \varphi_\alpha(s,t) d\alpha\\
&= -st\|u\|^2_{L^2(\mathbb{R}_+)} \int_0^1 \varphi_\alpha(s,t) d\alpha\\
&= \int_0^1 \int_\Omega \int_\Omega \langle D\mathrm{e}^{isI_1(u)}(\omega), D\mathrm{e}^{itI_1(u)}(\alpha\omega + \sqrt{1-\alpha^2}\tilde{\omega}))\rangle_{L^2(\mathbb{R}_+)} \mathbb{P}(d\omega)\mathbb{P}(d\tilde{\omega})d\alpha,
\end{aligned}$$

hence

$$\begin{aligned}
&\mathrm{Cov}\,(\mathrm{e}^{isI_1(u)}, \mathrm{e}^{isI_1(v)})\\
&= \int_0^1 \int_\Omega \int_\Omega \langle D\mathrm{e}^{isI_1(u)}(\omega), D\mathrm{e}^{itI_1(v)}(\alpha\omega + \sqrt{1-\alpha^2}\tilde{\omega}))\rangle_{L^2(\mathbb{R}_+)} \mathbb{P}(d\omega)\mathbb{P}(d\tilde{\omega})d\alpha.
\end{aligned}$$

Since D is a derivation operator from Proposition 5.2.1, the deviation results of Proposition 3.6.2 hold, i.e. for any $F \in \mathrm{Dom}\,(D)$ such that $\|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))} \leq C$ for some $C > 0$ we have

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2C\|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega))}}\right), \qquad x \geq 0.$$

5.5 Moment Identities for Skorohod Integrals

In this section we prove a moment identity that extends the Skorohod isometry to arbitrary powers of the Skorohod integral on the Wiener space. As simple consequences of this identity we obtain sufficient conditions for the Gaussianity of the law of the Skorohod integral and a recurrence relation for the moments of second order Wiener integrals.

Here, $(B_t)_{t\in\mathbb{R}_+}$ is a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space (W,μ) with $W = \mathcal{C}_0(\mathbb{R}_+,\mathbb{R}^d)$.

Each element of $X \otimes H$ is naturally identified to a linear operator from H to X via

$$(a \otimes b)c = a\langle b, c\rangle, \qquad a \otimes b \in X \otimes H, \quad c \in H.$$

For $u \in \mathbb{D}_{2,1}(H)$ we identify $Du = (D_t u_s)_{s,t\in\mathbb{R}_+}$ to the random operator $Du : H \to H$ almost surely defined by

$$(Du)v(s) = \int_0^\infty (D_t u_s) v_t dt, \quad s \in \mathbb{R}_+, \quad v \in L^2(W;H),$$

and define its adjoint D^*u on $H \otimes H$ as

$$(D^*u)v(s) = \int_0^\infty (D_s^\dagger u_t) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W;H),$$

where $D_s^\dagger u_t$ denotes the transpose matrix of $D_s u_t$ in $\mathbb{R}^d \otimes \mathbb{R}^d$.

The Skorohod isometry of Proposition 4.3.1 reads

$$\mathbb{E}[|\delta(u)|^2] = \mathbb{E}[\langle u, u\rangle_H] + \mathbb{E}\left[\mathrm{trace}\,(Du)^2\right], \qquad u \in \mathbb{D}_{2,1}(H), \tag{5.5.1}$$

with

$$\begin{aligned}\mathrm{trace}\,(Du)^2 &= \langle Du, D^*u\rangle_{H\otimes H}\\ &= \int_0^\infty \int_0^\infty \langle D_s u_t, D_t^\dagger u_s\rangle_{\mathbb{R}^d\otimes\mathbb{R}^d} ds dt,\end{aligned}$$

and the commutation relation

$$D\delta(u) = u + \delta(D^*u), \qquad u \in \mathbb{D}_{2,2}(H). \tag{5.5.2}$$

Next we state a moment identity for Skorohod integrals.

Theorem 5.5.1. *For any $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have*

$$\mathbb{E}[(\delta(u))^{n+1}] = \sum_{k=1}^{n} \frac{n!}{(n-k)!} \mathbb{E}\Big[(\delta(u))^{n-k} \tag{5.5.3}$$

$$\left(\langle (Du)^{k-1}u, u\rangle_H + \mathrm{trace}\,(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i}u, D\mathrm{trace}\,(Du)^i\rangle_H \right)\Big],$$

where

$$\begin{aligned} &\mathrm{trace}\,(Du)^{k+1} \\ &= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} D_{t_k} u_{t_0} \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} dt_0 \cdots dt_k. \end{aligned}$$

For $n = 1$ the above identity coincides with the Skorohod isometry (5.5.1). The proof of Theorem 5.5.1 will be given at the end of this section.
In particular we obtain the following immediate consequence of Theorem 5.5.1. Recall that $\mathrm{trace}\,(Du)^k = 0$, $k \geq 2$, when the process u is adapted with respect to the Brownian filtration.

Corollary 5.5.2. *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u\rangle_H$ is deterministic and*

$$\mathrm{trace}\,(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i}u, D\mathrm{trace}\,(Du)^i\rangle_H = 0, \quad a.s., \qquad 1 \leq k \leq n. \tag{5.5.4}$$

Then $\delta(u)$ has the same first $n+1$ moments as the centered Gaussian distribution with variance $\langle u, u\rangle_H$.

Proof. We have

$$\begin{aligned} D_t\langle u, u\rangle &= D_t \int_0^\infty \langle u_s, u_s\rangle ds \\ &= \int_0^\infty \langle u_s, D_t u_s\rangle ds + \int_0^\infty \langle D_t u_s, u_s\rangle ds \\ &= 2\int_0^\infty \langle D^\dagger_t u_s, u_s\rangle ds \\ &= 2(D^* u)u, \end{aligned}$$

shows that

$$\begin{aligned} \langle (D^{k-1}u)u, u\rangle &= \langle (D^*u)^{k-1}u, u\rangle \\ &= \frac{1}{2}\langle u, (D^*)^{k-2} D\langle u, u\rangle\rangle \\ &= 0, \end{aligned} \tag{5.5.5}$$

$k \geq 2$, when $\langle u, u\rangle$ is deterministic, $u \in \mathbb{D}_{2,1}(H)$. Hence under Condition (5.5.4), Theorem 5.5.1 yields

$$\mathbb{E}[(\delta(u))^{n+1}] = n\langle u, u\rangle_H \, \mathbb{E}\left[(\delta(u))^{n-1}\right],$$

and by induction

$$\mathbb{E}[(\delta(u))^{2m}] = \frac{(2m)!}{2^m m!}\langle u, u\rangle_H^m, \qquad 0 \leq 2m \leq n+1,$$

and $\mathbb{E}[(\delta(u))^{2m+1}] = 0$, $0 \leq 2m \leq n$, while $\mathbb{E}[\delta(u)] = 0$ for all $u \in \mathbb{D}_{2,1}(H)$. □

As a consequence of Corollary 5.5.2 we recover Theorem 2.1-b) of [145], i.e. $\delta(Rh)$ has a centered Gaussian distribution with variance $\langle h, h\rangle_H$ when $u = Rh$, $h \in H$, and R is a random mapping with values in the isometries of H, such that $Rh \in \cap_{p>1}\mathbb{D}_{p,2}(H)$ and $\operatorname{trace}(DRh)^{k+1} = 0$, $k \geq 1$. Note that in [145] the condition $Rh \in \cap_{p>1,k\geq 2}\mathbb{D}_{p,k}(H)$ is assumed instead of $Rh \in \cap_{p>1}\mathbb{D}_{p,2}(H)$.
In the sequel, all scalar products will be simply denoted by $\langle \cdot, \cdot\rangle$.
We will need the following lemma.

Lemma 5.5.3. *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$. Then for all $1 \leq k \leq n$ we have*

$$\begin{aligned}
&\mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] - (n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle (Du)^k u, D\delta(u)\rangle\right] \\
&= \mathbb{E}\Bigg[(\delta(u))^{n-k} \\
&\qquad \left(\langle (Du)^{k-1}u, u\rangle + \operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i}\langle (Du)^{k-i}u, D\operatorname{trace}(Du)^i\rangle\right)\Bigg].
\end{aligned}$$

Proof. We have $(Du)^{k-1}u \in \mathbb{D}_{(n+1)/k,1}(H)$, $\delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}(\mathbb{R})$, and using Relation (5.5.2) we obtain

$$\begin{aligned}
&\mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u + \delta(D^*u)\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, \delta(Du)\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + \mathbb{E}\left[\langle D^*u, D((\delta(u))^{n-k}(Du)^{k-1}u)\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + \mathbb{E}\left[(\delta(u))^{n-k}\langle D^*u, D((Du)^{k-1}u)\rangle\right] \\
&\quad + \mathbb{E}\left[\langle D^*u, ((Du)^{k-1}u) \otimes D(\delta(u))^{n-k}\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\left(\langle (Du)^{k-1}u, u\rangle + \langle D^*u, D((Du)^{k-1}u)\rangle\right)\right] \\
&\quad + (n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle D^*u, ((Du)^{k-1}u) \otimes D\delta(u)\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\left(\langle (Du)^{k-1}u, u\rangle + \langle D^*u, D((Du)^{k-1}u)\rangle\right)\right] \\
&\quad + (n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle (Du)^k u, D\delta(u)\rangle\right].
\end{aligned}$$

Next,

$$
\begin{aligned}
&\langle D^* u, D((Du)^{k-1}u)\rangle \\
&= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k}(D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} u_{t_0})\rangle dt_0 \cdots dt_k \\
&= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} D_{t_k} u_{t_0}\rangle dt_0 \cdots dt_k \\
&\quad + \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k}(D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1}) u_{t_0}\rangle dt_0 \cdots dt_k \\
&= \mathrm{trace}\,(Du)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \\
&\quad \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k+1}} \cdots D_{t_{i+1}} u_{t_{i+2}} (D_{t_i} D_{t_k} u_{t_{i+1}}) D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} u_{t_0}\rangle \\
&\quad dt_0 \cdots dt_k \\
&= \mathrm{trace}\,(Du)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \int_0^\infty \cdots \int_0^\infty \\
&\quad \langle D_{t_i} \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k+1}} \cdots D_{t_{i+1}} u_{t_{i+2}} D_{t_k} u_{t_{i+1}}\rangle, D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} u_{t_0}\rangle \\
&\quad dt_0 \cdots dt_k \\
&= \mathrm{trace}\,(Du)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \langle (Du)^i u, D\mathrm{trace}\,(Du)^{k-i}\rangle .
\end{aligned}
$$

□

Proof of Theorem 5.5.1. We decompose

$$
\begin{aligned}
&\mathbb{E}[(\delta(u))^{n+1}] = \mathbb{E}[\langle u, D(\delta(u))^n\rangle] \\
&= n\,\mathbb{E}[(\delta(u))^{n-1}\langle u, D\delta(u)\rangle] \\
&= \sum_{k=1}^{n} \frac{n!}{(n-k)!} \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] \\
&\quad - \sum_{k=1}^{n} \frac{n!}{(n-k)!} (n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle (Du)^{k}u, D\delta(u)\rangle\right],
\end{aligned}
$$

as a telescoping sum and then apply Lemma 5.5.3, which yields (5.5.3). □

5.6 Differential Calculus on Random Morphisms

In this section, in addition to the shift of Brownian paths by absolutely continuous functions as in (5.2.4), we consider a general class of transformations of Brownian motion and its associated differential calculus. The main result of this section Corollary 5.6.5 will be applied in Section 5.7 to construct another example of a gradient operator satisfying the assumptions of Chapter 3. Here we work with a d-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ as in Section 2.14. Let

$$U : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$$

be a random linear operator such that $Uf \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ is adapted for all f in a space $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$ of functions dense in $L^2(\mathbb{R}_+;\mathbb{R}^d)$.
The operator U is extended by linearity to the algebraic tensor product $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\otimes\mathcal{S}$, in this case Uf is not necessarily adapted if $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\otimes\mathcal{S}$.

Definition 5.6.1. *Let $(h(t))_{t\in\mathbb{R}_+} \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ be a square-integrable process, and let the transformation*

$$\Lambda(U,h) : \mathcal{S} \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$$

be defined as

$$\begin{aligned}&\Lambda(U,h)F\\&= f\left(I_1(Uu_1) + \int_0^\infty \langle u_1(t), h(t)\rangle dt, \ldots, I_1(Uu_n) + \int_0^\infty \langle u_n(t), h(t)\rangle dt\right),\end{aligned}$$

for $F \in \mathcal{S}$ of the form

$$F = f(I_1(u_1), \ldots, I_1(u_n)),$$

$u_1, \ldots, u_n \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R})$.

In the particular case where

$$U : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$$

is given as

$$[Uf](t) = V(t)f(t), \qquad t \in \mathbb{R}_+,$$

by an adapted family of random endomorphisms

$$V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad t \in \mathbb{R}_+,$$

this definition states that $\Lambda(U,h)F$ is the evaluation of F on the perturbed process of differential $V^*(t)dB(t) + h(t)dt$ instead of $dB(t)$, where

$$V^*(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

denotes the dual of $V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $t \in \mathbb{R}_+$.

We are going to define $\Lambda(U,h)$ on the space $\mathcal{S}$ of smooth functionals. For this we need to show that the definition of $\Lambda(U,h)F$ is independent of the particular representation

$$F = f(I_1(u_1), \ldots, I_1(u_n)), \qquad u_1, \ldots, u_n \in H,$$

chosen for $F \in \mathcal{S}$.

Lemma 5.6.2. *Let $F, G \in \mathcal{S}$ be written as*

$$F = f\left(I_1(u_1), \ldots, I_1(u_n)\right), \qquad u_1, \ldots, u_n \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d), \quad f \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}),$$

and

$$G = g\left(I_1(v_1), \ldots, I_1(v_m)\right), \qquad v_1, \ldots, v_m \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d), \quad g \in \mathcal{C}^1(\mathbb{R}^m; \mathbb{R}).$$

If $F = G$ $\mathbb{P}$-a.s. then $\Lambda(U,h)F = \Lambda(U,h)G$, $\mathbb{P}$-a.s.

Proof. Let $e_1, \ldots, e_k \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d)$ be orthonormal vectors that generate $u_1, \ldots, u_n$, $v_1, \ldots, v_m$. Assume that u_i and v_i are written as

$$u_i = \sum_{j=1}^k \alpha_i^j e_j \quad \text{and} \quad v_i = \sum_{j=1}^k \beta_i^j e_j, \qquad i = 1, \ldots, n,$$

in the basis $e_1, \ldots, e_k$. Then F and G are also represented as

$$F = \tilde{f}\left(I_1(e_1), \ldots, I_1(e_k)\right),$$

and $G = \tilde{g}\left(I_1(e_1), \ldots, I_1(e_k)\right)$, where the functions $\tilde{f}$ and $\tilde{g}$ are defined by

$$\tilde{f}(x_1, \ldots, x_k) = f\left(\sum_{j=1}^k \alpha_1^j x_j, \ldots, \sum_{j=1}^k \alpha_n^j x_j\right), \qquad x_1, \ldots, x_k \in \mathbb{R},$$

and

$$\tilde{g}(x_1, \ldots, x_k) = f\left(\sum_{j=1}^k \beta_1^j x_j, \ldots, \sum_{j=1}^k \beta_n^j x_j\right), \qquad x_1, \ldots, x_k \in \mathbb{R}.$$

Since $F = G$ and $I_1(e_1), \ldots, I_1(e_k)$ are independent, we have $\tilde{f} = \tilde{g}$ *a.e.*, hence everywhere, and by linearity,

$$\Lambda(U,h)F = \tilde{f}\left(I_1(Ue_1) + \int_0^\infty \langle e_1(t), h(t)\rangle dt, \ldots, I_1(Ue_k) + \int_0^\infty \langle e_k(t), h(t)\rangle dt\right),$$

and

$$\begin{aligned}
&\Lambda(U,h)G \\
&= \tilde{g}\left(I_1(Ue_1) + \int_0^\infty \langle e_1(t), h(t)\rangle dt, \ldots, I_1(Ue_k) + \int_0^\infty \langle e_k(t), h(t)\rangle dt\right),
\end{aligned}$$

hence $\Lambda(U,h)F = \Lambda(U,h)G$. □

Moreover, $\Lambda(U,h)$ is linear and multiplicative:

$$\Lambda(U,h)f(F_1,\ldots,F_n) = f(\Lambda(U,h)F_1,\ldots,\Lambda(U,h)F_n),$$

$F_1,\ldots,F_n \in \mathcal{S}$, $f \in \mathcal{C}_b^1(\mathbb{R}^n;\mathbb{R})$.

Definition 5.6.3. *Let $(U_\varepsilon)_{\varepsilon\in[0,1]}$ be a family of linear operators*

$$U_\varepsilon : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d),$$

such that

i) U_0 is the identity of $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, i.e. we have $U_0 f = f$, $\mathbb{P}$-a.s., $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$.
ii) for any $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, $U_\varepsilon f \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ and is adapted, $\varepsilon\in[0,1]$,
iii) the family $(U_\varepsilon)_{\varepsilon\in[0,1]}$ admits a derivative at $\varepsilon = 0$ in the form of an operator

$$\mathcal{L} : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d),$$

such that

$$((U_\varepsilon f - f)/\varepsilon)_{\varepsilon\in[0,1]}$$

converges in $L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ to $\mathcal{L}f = (\mathcal{L}_t f)_{t\in\mathbb{R}_+}$ as ε goes to zero, $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$.

Let $h \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ be a square-integrable adapted process.

The operator $\mathcal{L}$ is extended by linearity to $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\otimes\mathcal{S}$. The family $(U_\varepsilon)_{\varepsilon\in[0,1]}$ needs not have the semigroup property. The above assumptions imply that $\mathcal{L}DF \in \mathrm{Dom}\,(\delta)$, $F\in\mathcal{S}$, with

$$\begin{aligned}
\delta(\mathcal{L}DF) = {} & \sum_{i=1}^n \partial_i f(I_1(u_1),\ldots,I_1(u_n))\delta(\mathcal{L}u_i) \qquad (5.6.1)\\
& - \sum_{i,j=1}^n \langle u_i, \mathcal{L}u_j\rangle_{L^2(\mathbb{R}_+;\mathbb{R}^d)} \partial_i\partial_j f(I_1(u_1),\ldots,I_1(u_n)),
\end{aligned}$$

for $F = f(I_1(u_1), \ldots, I_1(u_n))$, where we used Relation 5.2.3. We now compute on $\mathcal{S}$ the derivative at $\varepsilon = 0$ of one-parameter families

$$\Lambda(U_\varepsilon, \varepsilon h) : \mathcal{S} \longrightarrow L^2(\Omega), \qquad \varepsilon \in \mathbb{R},$$

of transformations of Brownian functionals. Let the linear operator trace be defined on the algebraic tensor product $H \otimes H$ as

$$\operatorname{trace} u \otimes v = (u, v)_H, \qquad u, v \in H.$$

Proposition 5.6.4. *For $F \in \mathcal{S}$, we have in $L^2(\Omega)$:*

$$\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h)F_{|\varepsilon=0} = \int_0^\infty \langle h_0(t), D_t F\rangle dt + \delta(\mathcal{L}DF) + \operatorname{trace}(\mathrm{Id}_H \otimes \mathcal{L})DDF. \tag{5.6.2}$$

Proof. Let $A : \mathcal{S} \longrightarrow \mathcal{S}$ be defined by

$$AF = \delta(\mathcal{L}DF) + \operatorname{trace}(\mathrm{Id}_H \otimes \mathcal{L})DDF + \int_0^\infty \langle h_0(t), D_t F\rangle dt, \qquad F \in \mathcal{S}.$$

For $F = I_1(u)$, $u \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d)$, we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h)F_{|\varepsilon=0} &= \int_0^\infty \langle h_0(t), u(t)\rangle dt + I_1(\mathcal{L}u) \\
&= \int_0^\infty \langle h_0(t), D_t F\rangle dt + \delta(\mathcal{L}DF) + \operatorname{trace}(\mathrm{Id}_H \otimes \mathcal{L})DDF \\
&= AF
\end{aligned}$$

since $DDF = 0$. From (5.6.1), for $F_1, \ldots, F_n \in \mathcal{S}$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n; \mathbb{R})$ we have

$$\begin{aligned}
&Af(F_1, \ldots, F_n) = \\
&\quad \delta(\mathcal{L}Df(F_1, \ldots, F_n)) + \operatorname{trace}(\mathrm{Id}_H \otimes \mathcal{L})DDf(F_1, \ldots, F_n) \\
&\quad + \int_0^\infty \langle h_0(t), D_t f(F_1, \ldots, F_n)\rangle dt \\
&= \sum_{i=1}^n \delta\left(\partial_i f(F_1, \ldots, F_n)\mathcal{L}DF_i\right) \\
&\quad + \sum_{i=1}^n \partial_i f(F_1, \ldots, F_n) \operatorname{trace}(\mathrm{Id}_H \otimes \mathcal{L})DDF_i
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{n} \partial_i \partial_j f(F_1, , \ldots, F_n) \int_0^\infty \langle \mathcal{L}_s DF_i, D_s F_j \rangle ds \\
&+\sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) \int_0^\infty \langle h_0(t), D_t F_i \rangle dt \\
&= \sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) \delta(\mathcal{L} DF_i) + \sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) \mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L}) DDF_i \\
&+\sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) \int_0^\infty \langle h_0(t), D_t F_i \rangle dt \\
&= \sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) \left(\delta(\mathcal{L} DF_i) + \mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L}) DDF_i \right. \\
&\left. + \int_0^\infty \langle h_0(t), D_t F_i \rangle dt \right) \\
&= \sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) A F_i .
\end{aligned}
$$

Hence for $F_1 = I_1(u_1), \ldots, F_n = I_1(u_n) \in \mathcal{S}$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n; \mathbb{R})$:

$$
\begin{aligned}
Af(F_1, \ldots, F_n) &= \sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) A F_i \\
&= \sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n) \left(\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, \varepsilon h) F_i \right)_{|\varepsilon=0} \\
&= \left(\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, \varepsilon h) f(F_1, \ldots, F_n) \right)_{|\varepsilon=0} .
\end{aligned}
$$

Consequently, Relation (5.6.2) holds on $\mathcal{S}$. □

Corollary 5.6.5. *Assume that* $\mathcal{L} : L^2(\mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^d)$ *is antisymmetric as an endomorphism of* $L^2(\mathbb{R}_+; \mathbb{R}^d)$*,* $\mathbb{P}$*-a.s., we have in* $L^2(\Omega)$*:*

$$
\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, \varepsilon h) F_{|\varepsilon=0} = \int_0^\infty \langle h_0(t), D_t F \rangle dt + \delta(\mathcal{L} DF), \qquad F \in \mathcal{S}.
$$

Proof. Since $\mathcal{L}$ is antisymmetric, we have for any symmetric tensor $u \otimes u \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d) \otimes \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d)$:

$$
\mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L}) u \otimes u = \mathrm{trace}\, u \otimes \mathcal{L} u = \langle u, \mathcal{L} u \rangle_H = -\langle \mathcal{L} u, u \rangle_H = 0.
$$

Hence the term $\mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L}) DDF$ of Proposition 5.6.4 vanishes $\mathbb{P}$-a.s. since DDF is a symmetric tensor. □

5.7 Riemannian Brownian Motion

In this section we mention another example of a gradient operator satisfying the Clark formula Assumption 3.2.1 of Chapter 3. As an application we derive concentration inequalities on path space using the method of covariance representation. This section is not self-contained and we refer to [40], [41], [44], [83] for details on Riemannian Brownian motion.

Let $(B(t))_{t\in\mathbb{R}_+}$ denote a $\mathbb{R}^d$-valued Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$, generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Let M be a Riemannian manifold of dimension d whose Ricci curvature is uniformly bounded from below, and let $O(M)$ denote the bundle of orthonormal frames over M. The Levi-Civita parallel transport defines d canonical horizontal vector fields $A_1,\ldots,A_d$ on $O(M)$, and the Stratonovich stochastic differential equation

$$\begin{cases} dr(t) = \displaystyle\sum_{i=1}^{i=d} A_i(r(t))\circ dx^i(t), \qquad t\in\mathbb{R}_+, \\ \\ r(0)=(m_0,r_0)\in O(M), \end{cases}$$

defines an $O(M)$-valued process $(r(t))_{t\in\mathbb{R}_+}$. Let $\pi : O(M) \longrightarrow M$ be the canonical projection, let

$$\gamma(t) = \pi(r(t)), \qquad t\in\mathbb{R}_+,$$

be the Brownian motion on M and let the Itô parallel transport along $(\gamma(t))_{t\in\mathbb{R}_+}$ is defined as

$$t_{t\leftarrow 0} = r(t)r_0^{-1} : T_{m_0}M \simeq \mathbb{R}^d \longrightarrow T_{\gamma(t)}M, \qquad t\in[0,T].$$

Let $\mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$ vanishing at the origin. Let also $\mathbf{P}(M)$ denote the set of continuous paths on M starting at m_0, let

$$\begin{aligned} I : \mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d) &\longrightarrow \mathbf{P}(M) \\ (\omega(t))_{t\in\mathbb{R}_+} &\mapsto I(\omega) = (\gamma(t))_{t\in\mathbb{R}_+} \end{aligned}$$

be the Itô map, and let ν denote the image measure on $\mathbf{P}(M)$ of the Wiener measure $\mathbb{P}$ by I. In order to simplify the notation we write F instead of $F\circ I$, for random variables and stochastic processes. Let Ω_r denote the curvature tensor and $\mathrm{ric}_r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ the Ricci tensor of M at the frame $r\in O(M)$. Given an adapted process $(z_t)_{t\in\mathbb{R}_+}$ with absolutely continuous trajectories, we let $(\hat{z}(t))_{t\in\mathbb{R}_+}$ be defined by

$$\dot{\hat{z}}(t) = \dot{z}(t) + \frac{1}{2}\mathrm{ric}_{r(t)}z(t), \qquad t \in \mathbb{R}_+, \qquad \hat{z}(0) = 0. \tag{5.7.1}$$

We recall that $z \mapsto \hat{z}$ can be inverted, i.e. there exists a process $(\tilde{z}_t)_{t\in\mathbb{R}_+}$ such that $\hat{\tilde{z}} = z$, cf. Section 3.7 of [44]. Finally, let $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, be defined as

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}\mathrm{ric}_{r(t)}Q_{t,s}, \qquad Q_{s,s} = \mathrm{Id}_{T_{m_0}}, \qquad 0 \le s \le t,$$

and let

$$q(t,z) = -\int_0^t \Omega_{r(s)}(\circ dB(s), z(s)), \qquad t \in \mathbb{R}_+,$$

where $\circ dB(s)$ denotes the Stratonovich differential. Let $Q^*_{t,s}$ be the adjoint of $Q_{t,s}$, let $\mathbb{H} = L^2(\mathbb{R}_+, \mathbb{R}^d)$, and let $\mathbf{H} = L^\infty(\mathbf{P}(M), \mathbb{H}; d\nu)$. Let finally $\mathcal{C}^\infty_c(M^n)$ denote the space of infinitely differentiable functions with compact support in M^n.

In the sequel we endow $\mathbf{P}(M)$ with the σ-algebra $\mathcal{F}^P$ on $\mathbf{P}(M)$ generated by subsets of the form

$$\{\gamma \in \mathbf{P}(M) \ : \ (\gamma(t_1), \ldots, \gamma(t_n)) \in B_1 \times \cdots \times B_n\},$$

where $0 \le t_1 < \cdots < t_n$, $B_1, \ldots, B_n \in \mathcal{B}(M)$, $n \ge 1$.

Let

$$\mathcal{S}(\mathbf{P}(M); \mathbb{R}) = \{F = f(\gamma(t_1), \ldots, \gamma(t_n)) \ : \ f \in \mathcal{C}^\infty_b(M^n; \mathbb{R}), \\ 0 \le t_1 \le \cdots \le t_n \le 1, \ n \ge 1\},$$

and

$$\mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) = \left\{ \sum_{k=1}^{k=n} F_k \int_0^{\cdot} u_k(s)ds \ : \ F_1, \ldots, F_n \in \mathcal{S}(\mathbf{P}(M); \mathbb{R}), \right. \\ \left. u_1, \ldots, u_n \in L^2(\mathbb{R}_+; \mathbb{R}^d), \ n \ge 1 \right\}$$

In the following, the space $L^2(\mathbf{P}(M), \mathcal{F}^P, \nu)$ will be simply denoted by $L^2(\mathbf{P}(M))$. Note that the spaces $\mathcal{S}(\mathbf{P}(M); \mathbb{R})$ and $\mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ are dense in $L^2(\mathbf{P}(M); \mathbb{R})$ and in $L^2(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ respectively. The following definition of the intrisic gradient on $\mathbf{P}(M)$ can be found in [44].

Definition 5.7.1. *Let $\hat{D} : L^2(\mathbf{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ be the gradient operator defined as*

$$\hat{D}_t F = \sum_{i=1}^{i=n} t_{0 \leftarrow t_i} \nabla^M_i f(\gamma(t_1), \ldots, \gamma(t_n)) \mathbf{1}_{[0,t_i]}(t), \qquad t \in \mathbb{R}_+,$$

for $F \in \mathcal{S}(\mathbf{P}(M); \mathbb{R})$ of the form $F = f(\gamma(t_1), \ldots, \gamma(t_n))$, where ∇^M_i denotes the gradient on M applied to the i-th variable of f.

Given an adapted vector field $(Z(t))_{t\in\mathbb{R}_+}$ on M with $Z(t)\in T_{\gamma(t)}M$, $t\in\mathbb{R}_+$, we let $z(t)=t_{0\leftarrow t}Z(t)$, $t\in\mathbb{R}_+$, and assume that $\dot z(t)$ exists, $\forall t\in\mathbb{R}_+$. Let

$$\nabla Z(t)=\lim_{\varepsilon\to 0}\frac{t_{t\leftarrow t+\varepsilon}Z(t+\varepsilon)-Z(t)}{\varepsilon}.$$

Then

$$\dot z(t)=t_{0\leftarrow t}\nabla Z(t),\qquad t\in\mathbb{R}_+.$$

let Ω_r denote the curvature tensor of M and let $\mathrm{ric}_r:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ denote the Ricci tensor at the frame $r\in O(M)$, and let the process $(\hat z(t))_{t\in\mathbb{R}_+}$ be defined by

$$\begin{cases}\dot{\hat z}(t)=\dot z(t)+\dfrac{1}{2}\mathrm{ric}_{r(t)}z(t), & t\in\mathbb{R}_+,\\[2ex] \hat z(0)=0.\end{cases}\tag{5.7.2}$$

As a consequence of Corollary 5.6.5 we obtain the following relation between the gradient $\hat D$ and the operators D and δ, cf. Theorem 2.3.8 and Theorem 2.6 of [27].

Corollary 5.7.2. *Assume that the Ricci curvature of M is uniformly bounded, and let $z\in\mathcal{U}(\mathbf{P}(M)\times\mathbb{R}_+;\mathbb{R}^d)$ be adapted. We have*

$$\int_0^\infty\langle\hat D_tF,\dot z(t)\rangle dt=\int_0^\infty\langle D_tF,\dot{\hat z}(t)\rangle dt+\delta(q(\cdot,z)D.F),\tag{5.7.3}$$

$F\in\mathcal{S}(\mathbf{P}(M);\mathbb{R})$, where $q(t,z):\mathbb{R}^d\longrightarrow\mathbb{R}^d$ is defined as

$$q(t,z)=-\int_0^t\Omega_{r(s)}(\circ dB(s),z(s)),\qquad t\in\mathbb{R}_+.$$

Proof. We let $V_\varepsilon(t)=\exp(\varepsilon q(t,z))$, $t\in\mathbb{R}_+$, $\varepsilon\in\mathbb{R}$. Then from Proposition 3.5.3 of [44] we have

$$\int_0^\infty\langle\hat DF,\dot z(t)\rangle dt=\frac{d}{d\varepsilon}\Lambda(U_\varepsilon,\varepsilon\dot{\hat z})F_{|\varepsilon=0}.$$

Since the Ricci curvature of M is bounded, we have $\dot{\hat z}\in L^2(\mathbb{R}_+;L^\infty(W;\mathbb{R}))$ from (5.7.2). Moreover, from Theorem 2.2.1 of [44], $\varepsilon\mapsto\Lambda(U_\varepsilon,0)r(t)$ is differentiable in $L^2(W;\mathbb{R})$, hence continuous, $\forall t\in\mathbb{R}_+$. Consequently, from (5.7.2) and by construction of $\mathcal{U}(\mathbf{P}(M)\times\mathbb{R}_+;\mathbb{R}^d)$, $\varepsilon\mapsto\Lambda(U_\varepsilon,0)\dot{\hat z}$ is continuous in $L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ and we can apply Corollary 5.6.5 with $\mathcal{L}_t=q(t,z)$ to obtain (5.7.3). □

If $u\in\mathcal{U}(\mathbf{P}(M)\times\mathbb{R}_+;\mathbb{R}^d)$ is written as $u=\sum_{i=1}^{i=n}G_iz_i$, z_i deterministic, $G_i\in\mathcal{S}(\mathbf{P}(M);\mathbb{R})$, $i=1,\ldots,n$, we let

$$\mathrm{trace}\, q(t, D_t u) = \sum_{i=1}^{i=n} q(t, z_i) D_t G_i.$$

Given $u \in \mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ written as $u = \sum_{i=1}^{i=n} G_i z_i$, z_i deterministic, $G_i \in \mathcal{S}(\mathbf{P}(M); \mathbb{R})$, $i = 1, \ldots, n$, we let

$$\hat{u} = \sum_{i=1}^{i=n} G_i \hat{z}_i.$$

We now recall the inversion of $z \mapsto \hat{z}$ by the method of variation of constants described in Section 3.7 of [44]. Let $\mathrm{Id}_{\gamma(t)}$ denote the identity of $T_{\gamma(t)}M$. We have

$$\dot{z}(t) = \dot{\tilde{z}}(t) + \frac{1}{2}\mathrm{ric}_{r(t)} \tilde{z}(t), \qquad t \in \mathbb{R}_+,$$

where $(\tilde{z}(t))_{t \in \mathbb{R}_+}$ is defined as

$$\tilde{z}(t) = \int_0^t Q_{t,s} \dot{z}(s) ds, \qquad t \in \mathbb{R}_+,$$

and $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}\mathrm{ric}_{r(t)} Q_{t,s}, \qquad Q_{s,s} = \mathrm{Id}_{\gamma(0)}, \qquad 0 \leq s \leq t.$$

Let also the process $(\hat{Z}(t))_{t \in \mathbb{R}_+}$ be defined by

$$\begin{cases} \nabla \hat{Z}(t) = \nabla Z(t) + \dfrac{1}{2}\mathrm{Ric}_{\gamma(t)} Z(t), \qquad t \in \mathbb{R}_+, \\ \\ \hat{Z}(0) = 0, \end{cases}$$

with $\hat{z}(t) = \tau_{0 \leftarrow t} \hat{Z}(t)$, $t \in \mathbb{R}_+$. In order to invert the mapping $Z \mapsto \hat{Z}$, let

$$\tilde{Z}(t) = \int_0^t R_{t,s} \nabla Z(s) ds, \qquad t \in \mathbb{R}_+,$$

where $R_{t,s} : T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M$ is defined by the equation

$$\nabla_t R_{t,s} = -\frac{1}{2}\mathrm{Ric}_{\gamma(t)} R_{t,s}, \qquad R_{s,s} = \mathrm{Id}_{\gamma(s)}, \qquad 0 \leq s \leq t,$$

∇_t denotes the covariant derivative along $(\gamma(t))_{t \in \mathbb{R}_+}$, and

$$\mathrm{Ric}_m : T_m M \longrightarrow T_m M$$

denotes the Ricci tensor at $m \in M$, with the relation

$$\mathrm{ric}_{r(t)} = t_{0\leftarrow t} \circ \mathrm{Ric}_{\gamma(t)} \circ t_{t\leftarrow 0}.$$

Then we have

$$\begin{cases} \nabla Z(t) = \nabla \tilde{Z}(t) + \frac{1}{2}\mathrm{Ric}_{\gamma(t)}\tilde{Z}(t), & t \in \mathbb{R}_+, \\ \\ Z(0) = 0. \end{cases}$$

We refer to [44] for the next definition.

Definition 5.7.3. *The damped gradient*

$$\tilde{D} : L^2(\mathbf{P}(M);\mathbb{R}) \longrightarrow L^2(\mathbf{P}(M) \times \mathbb{R}_+;\mathbb{R}^d)$$

is defined as

$$\tilde{D}_t F = \sum_{i=1}^{i=n} \mathbf{1}_{[0,t_i]}(t) Q^*_{t_i,t} t_{0\leftarrow t_i} \nabla_i^M f(\gamma(t_1),\ldots,\gamma(t_n)), \qquad t \in \mathbb{R}_+,$$

for $F \in \mathcal{S}(\mathbf{P}(M);\mathbb{R})$ of the form $F = f(\gamma(t_1),\ldots,\gamma(t_n))$, where

$$Q^*_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

denotes the adjoint of $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $0 \leq s < t$.

Given $f \in \mathcal{C}_c^\infty(M^n)$ we also have

$$\tilde{D}_t F = \sum_{i=1}^{i=n} \mathbf{1}_{[0,t_i]}(t) t_{0\leftarrow t} R^*_{t_i,t} \nabla_i^M f(\gamma(t_1),\ldots,\gamma(t_n)), \qquad t \in \mathbb{R}_+,$$

where $R^*_{t_i,t} : T_{\gamma(t_i)} \longrightarrow T_{\gamma(t)}$ is the adjoint of $R_{t_i,t} : T_{\gamma(t)} \longrightarrow T_{\gamma(t_i)}$.

Proposition 5.7.4. *We have for $z \in \mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+;\mathbb{R}^d)$:*

$$\int_0^\infty \langle \tilde{D}_t F, \dot{z}(t)\rangle dt = \int_0^\infty \langle \hat{D}_t F, \dot{\tilde{z}}(t)\rangle dt, \qquad F \in \mathcal{S}(\mathbf{P}(M);\mathbb{R}). \tag{5.7.4}$$

Proof. We compute

$$\int_0^\infty \langle \tilde{D}_t F, \dot{z}(t)\rangle dt = \sum_{i=1}^{i=n} \int_0^{t_i} \langle Q^*_{t_i,s} t_{0\leftarrow t_i} \nabla_i^M f(\gamma(t_1),\ldots,\gamma(t_n)), \dot{z}(s)\rangle ds$$

$$= \sum_{i=1}^{i=n} \int_0^{t_i} \langle t_{0 \leftarrow t_i} \nabla_i^M f(\gamma(t_1), \ldots, \gamma(t_n)), Q_{t_i,s} \dot{z}(s) \rangle dt$$
$$= \int_0^\infty \langle \hat{D}_t F, \dot{\hat{z}}(t) \rangle dt, \qquad F \in \mathcal{S}(\mathbf{P}(M); \mathbb{R}).$$

□

We also have

$$\int_0^\infty \langle \tilde{D}_t F, \dot{\hat{z}}(t) \rangle dt = \int_0^\infty \langle \hat{D}_t F, \dot{z}(t) \rangle dt, \qquad F \in \mathcal{S}(\mathbf{P}(M); \mathbb{R}).$$

Taking expectation on both sides of (5.7.3) and (5.7.4) it follows that the processes DF and $\tilde{D}F$ have the same adapted projections:

$$\mathbb{E}[D_t F \mid \mathcal{F}_t] = \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+, \tag{5.7.5}$$

$F = f(\gamma(t_1), \ldots, \gamma(t_n))$. Using this relation and the Clark formula

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F \mid \mathcal{F}_t] \cdot dB(t),$$

on the Wiener space, cf. Proposition 5.2.7 we obtain the expression of the Clark formula on path space, i.e. Assumption 3.2.1 is satisfied by $\tilde{D}$.

Proposition 5.7.5. *Let* $F \in \mathrm{Dom}\,(\tilde{D})$, *then*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] \cdot dB(t).$$

The following covariance identity is then a consequence of Proposition 3.4.1.

Proposition 5.7.6. *Let* $F, G \in \mathrm{Dom}\,(\tilde{D})$, *then*

$$\mathrm{Cov}\,(F, G) = \mathbb{E}\left[\int_0^\infty \tilde{D}_t F \cdot \mathbb{E}[\tilde{D}_t G \mid \mathcal{F}_t]\ dt\right]. \tag{5.7.6}$$

From Proposition 3.6.2 we obtain a concentration inequality on path space.

Lemma 5.7.7. *Let* $F \in \mathrm{Dom}\,(\tilde{D})$. *If* $\|\tilde{D}F\|_{L^2(\mathbb{R}_+, L^\infty(\mathbf{P}(M)))} \leq C$, *for some* $C > 0$, *then*

$$\nu(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2C\|\tilde{D}F\|_{\mathbf{H}}}\right), \qquad x \geq 0. \tag{5.7.7}$$

In particular, $\mathbb{E}[e^{\lambda F^2}] < \infty$, *for* $\lambda < (2C\|\tilde{D}F\|_{\mathbf{H}})^{-1}$.

5.8 Time Changes on Brownian Motion

In this section we study the transformations given by time changes on Brownian motion, in connection with the operator $\nabla^{\ominus}$ of Definition 4.8.1.

Proposition 5.8.1. *On the Wiener space, $\nabla^{\ominus}$ satisfies the relation*

$$\nabla_t^{\ominus}(FG) = F\nabla_t^{\ominus}G + G\nabla_t^{\ominus}F - D_tFD_tG, \qquad t \in \mathbb{R}_+. \tag{5.8.1}$$

Proof. We will show by induction using the derivation property (5.2.3) of D that for all $k \geq 1$,

$$\begin{aligned}\nabla_t^{\ominus}(I_n(f^{\otimes n})I_1(g)^k) = I_n(f^{\otimes n})\nabla_t^{\ominus}\left(I_1(g)^k\right) + I_1(g)^k\nabla_t^{\ominus}I_n(f^{\otimes n})\\ -D_tI_n(f^{\otimes n})D_t\left(I_1(g)^k\right),\end{aligned} \tag{5.8.2}$$

$t \in \mathbb{R}_+$. We have

$$\begin{aligned}
&\nabla^{\ominus}\left(I_n(f^{\otimes n})I_1(g)\right) = \nabla^{\ominus}\left(I_{n+1}(f^{\otimes n}\circ g) + n\langle f,g\rangle_{L^2(\mathbb{R}_+)}I_{n-1}(f^{\otimes(n-1)})\right)\\
&= -I_{n+1}((g'\mathbf{1}_{[t,\infty)})\circ f^{\otimes n}) - nI_{n+1}((f'\mathbf{1}_{[t,\infty)})\circ g\circ f^{\otimes(n-1)})\\
&\quad -n(n-1)\langle f,g\rangle_{L^2(\mathbb{R}_+)}I_{n-1}((f'\mathbf{1}_{[t,\infty)})\circ f^{\otimes(n-2)})\\
&= -nI_{n+1}((f'\mathbf{1}_{[t,\infty)})\circ f^{\otimes(n-1)}\circ g) - n\langle g,(f'\mathbf{1}_{[t,\infty)})\rangle_{L^2(\mathbb{R}_+)}I_{n-1}(f^{\otimes(n-1)})\\
&\quad -I_{n+1}(f^{\otimes n}\circ(g'\mathbf{1}_{[t,\infty)})) - n(n-1)\langle f,g\rangle_{L^2(\mathbb{R}_+)}I_{n-1}(f^{\otimes(n-2)}\circ(f'\mathbf{1}_{[t,\infty)}))\\
&\quad -n\langle f,g'\mathbf{1}_{[t,\infty)}\rangle_{L^2(\mathbb{R}_+)}I_{n-1}(f^{\otimes(n-1)})\\
&\quad +n\langle f'\mathbf{1}_{[t,\infty)},g\rangle_{L^2(\mathbb{R}_+)}I_{n-1}(f^{\otimes(n-1)}) + n\langle g'\mathbf{1}_{[t,\infty)},f\rangle_{L^2(\mathbb{R}_+)}I_{n-1}(f^{\otimes(n-1)})\\
&= -nI_n((f'\mathbf{1}_{[t,\infty)})\circ f^{\otimes(n-1)})I_1(g) - I_n(f^{\otimes n})I_1(g'\mathbf{1}_{[t,\infty)})\\
&\quad -nf(t)g(t)I_{n-1}(f^{\otimes(n-1)})\\
&= I_1(g)\nabla_t^{\ominus}I_n(f^{\otimes n}) + I_n(f^{\otimes n})\nabla_t^{\ominus}I_1(g) - D_tI_1(g)D_tI_n(f^{\otimes n}), \qquad t\in\mathbb{R}_+,
\end{aligned}$$

which shows (5.8.2) for $k = 1$. Next, assuming that (5.8.2) holds for some $k \geq 1$, we have

$$\begin{aligned}
\nabla_t^{\ominus}(I_n(f^{\otimes n})I_1(g)^{k+1}) &= I_1(g)\nabla_t^{\ominus}(I_n(f^{\otimes n})I_1(g)^k) + I_n(f^{\otimes n})I_1(g)^k\nabla_t^{\ominus}I_1(g)\\
&\quad -D_tI_1(g)D_t(I_1(g)^kI_n(f^{\otimes n}))\\
&= I_1(g)\left(I_1(g)^k\nabla_t^{\ominus}I_n(f^{\otimes n}) + I_n(f^{\otimes n})\nabla_t^{\ominus}\left(I_1(g)^k\right)\right.\\
&\quad \left.-D_t\left(I_1(g)^k\right)D_tI_n(f^{\otimes n})\right)
\end{aligned}$$

$$
\begin{aligned}
&+I_n(f^{\otimes n})I_1(g)^k\nabla_t^{\ominus}I_1(g) - D_tI_1(g)\left(I_1(g)^kD_tI_n(f^{\otimes n})\right.\\
&\left.+I_n(f^{\otimes n})D_t\left(I_1(g)^k\right)\right)\\
&= I_1(g)^{k+1}\nabla_t^{\ominus}I_n(f^{\otimes n}) + I_n(f^{\otimes n})\nabla_t^{\ominus}\left(I_1(g)^{k+1}\right)\\
&-D_t\left(I_1(g)^{k+1}\right)D_tI_n(f^{\otimes n}),
\end{aligned}
$$

$t \in \mathbb{R}_+$, which shows that (5.8.2) holds at the rank $k+1$. □

Definition 5.8.2. *Let $h \in L^2(\mathbb{R}_+)$, with $\|h\|_{L^\infty(\mathbb{R}_+)} < 1$, and*

$$
\nu_h(t) = t + \int_0^t h(s)ds, \qquad t \in \mathbb{R}_+.
$$

We define a mapping $\mathcal{T}_h : \Omega \to \Omega$, $t, \varepsilon \in \mathbb{R}_+$, as

$$
\mathcal{T}_h(\omega) = \omega \circ \nu_h^{-1}, \qquad h \in L^2(\mathbb{R}_+), \ \sup_{x\in\mathbb{R}_+} \mid h(x) \mid < 1.
$$

The transformation $\mathcal{T}_h$ acts on the trajectory of $(B_s)_{s\in\mathbb{R}_+}$ by change of time, or by perturbation of its predictable quadratic variation. Although $\mathcal{T}_h$ is not absolutely continuous, the functional $F \circ \mathcal{T}_h$ is well-defined for $F \in \mathcal{S}$, since elements of $\mathcal{S}$ can be defined trajectory by trajectory.

Proposition 5.8.3. *We have for $F \in \mathcal{S}$*

$$
\int_0^\infty h(t)\left(\nabla_t^{\ominus} + \frac{1}{2}D_tD_t\right)Fdt = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(F\circ\mathcal{T}_{\varepsilon h} - F).
$$

Proof. We first notice that as a consequence of Proposition 5.8.1, the operator

$$
\nabla_t^{\ominus} + \frac{1}{2}D_tD_t
$$

$t \in \mathbb{R}_+$, has the derivation property. Indeed, by Proposition 5.8.1 we have

$$
\begin{aligned}
\nabla_t^{\ominus}(FG) + \frac{1}{2}D_tD_t(FG) &= F\nabla_t^{\ominus}G + G\nabla_t^{\ominus}F - D_tFD_tG\\
&\quad+\frac{1}{2}\left(FD_tD_tG + GD_tD_tF + 2D_tFD_tG\right)\\
&= F\left(\nabla_t^{\ominus}G + \frac{1}{2}D_tD_tG\right) + G\left(\nabla_t^{\ominus}F + \frac{1}{2}D_tD_tF\right).
\end{aligned}
$$

Moreover, $\mathcal{T}_{\varepsilon h}$ is multiplicative, hence we only need to treat the particular case of $F = I_1(f)$. We have

$$\begin{aligned} I_1(f) \circ \mathcal{T}_{\varepsilon h} - I_1(f) &= \int_0^\infty f(s) dB(\nu_{\varepsilon h}^{-1}(s)) - I_1(f) \\ &= \int_0^\infty f(\nu_{\varepsilon h}(s)) dB_s - \int_0^\infty f(s) dB_s \\ &= \int_0^\infty \left(f\left(t + \varepsilon \int_0^t h(s) ds \right) - f(t) \right) dB_t. \end{aligned}$$

After division by $\varepsilon > 0$, this converges in $L^2(\Omega)$ as $\varepsilon \to 0$ to

$$\begin{aligned} \int_0^\infty f'(t) \int_0^t h(s) ds dB_t &= \int_0^\infty h(t) \int_t^\infty f'(s) dB_s dt \\ &= -\int_0^\infty h(t) \nabla_t^\ominus I_1(f) dt \\ &= -\int_0^\infty h(t) \left(\nabla_t^\ominus + \frac{1}{2} D_t D_t \right) I_1(f) dt. \end{aligned}$$

□

5.9 Notes and References

Proposition 5.2.1 is usually taken as a definition of the Malliavin derivative D, see for example [92]. The relation between multiple Wiener integrals and Hermite polynomials originates in [132]. Corollary 5.2.4 can be found in Lemma 1.2 of [91] and in [96]. Finding the probabilistic interpretation of D for normal martingales other than the Brownian motion or the Poisson process, e.g. for the Azéma martingales, is still an open problem. In relation to Proposition 5.6.1, see [27] for a treatment of transformations called Euclidean motions, in which case the operator $V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is chosen to be an isometry and h is adapted, so that $\Lambda(U, h)$ is extended by quasi-invariance of the Wiener measure, see also [62]. Corollary 5.5.2 recovers and extend the sufficient conditions for the invariance of the Wiener measure under random rotations given in [145], i.e. the Skorohod integral $\delta(Rh)$ to has a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and R is a random isometry of H. We refer to [42], [44] for the Clark formula and the construction of gradient and divergence operators on Riemannian path space, and to [60] for the corresponding deviation results stated in Section 5.7.