Chapter 1
The Discrete Time Case

In this chapter we introduce the tools of stochastic analysis in the simple framework of discrete time random walks. Our presentation relies on the use of finite difference gradient and divergence operators which are defined along with single and multiple stochastic integrals. The main applications of stochastic analysis to be considered in the following chapters, including functional inequalities and mathematical finance, are discussed in this elementary setting. Some technical difficulties involving measurability and integrability conditions, that are typical of the continuous-time case, are absent in the discrete time case.

1.1 Normal Martingales

Consider a sequence $(Y_k)_{k\in\mathbb{N}}$ of (not necessarily independent) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_n)_{n\geq -1}$ denote the filtration generated by $(Y_n)_{n\in\mathbb{N}}$, i.e.

$$\mathcal{F}_{-1} = \{\emptyset, \Omega\},$$

and

$$\mathcal{F}_n = \sigma(Y_0, \ldots, Y_n), \qquad n \geq 0.$$

Recall that a random variable F is said to be $\mathcal{F}_n$-measurable if it can be written as a function

$$F = f_n(Y_0, \ldots, Y_n)$$

of $Y_0, \ldots, Y_n$, where $f_n : \mathbb{R}^{n+1} \to \mathbb{R}$.

Assumption 1.1.1. We make the following assumptions on the sequence $(Y_n)_{n\in\mathbb{N}}$:

a) it is conditionally centered:

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = 0, \qquad n \geq 0, \tag{1.1.1}$$

N. Privault, *Stochastic Analysis in Discrete and Continuous Settings*,
Lecture Notes in Mathematics 1982, DOI 10.1007/978-3-642-02380-4_1,

b) its conditional quadratic variation satisfies:

$$\mathbb{E}[Y_n^2 \mid \mathcal{F}_{n-1}] = 1, \qquad n \geq 0.$$

Condition (1.1.1) implies that the process $(Y_0 + \cdots + Y_n)_{n\geq 0}$ is an $\mathcal{F}_n$-martingale, cf. Section 9.4 in the Appendix. More precisely, the sequence $(Y_n)_{n\in\mathbb{N}}$ and the process $(Y_0 + \cdots + Y_n)_{n\geq 0}$ can be viewed respectively as a (correlated) noise and as a normal martingale in discrete time.

1.2 Stochastic Integrals

In this section we construct the discrete stochastic integral of predictable square-summable processes with respect to a discrete-time normal martingale.

Definition 1.2.1. *Let $(u_k)_{k\in\mathbb{N}}$ be a uniformly bounded sequence of random variables with finite support in $\mathbb{N}$, i.e. there exists $N \geq 0$ such that $u_k = 0$ for all $k \geq N$. The stochastic integral $J(u)$ of $(u_n)_{n\in\mathbb{N}}$ is defined as*

$$J(u) = \sum_{k=0}^{\infty} u_k Y_k.$$

The next proposition states a version of the Itô isometry in discrete time. A sequence $(u_n)_{n\in\mathbb{N}}$ of random variables is said to be $\mathcal{F}_n$-predictable if u_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{N}$, in particular u_0 is constant in this case.

Proposition 1.2.2. *The stochastic integral operator $J(u)$ extends to square-integrable predictable processes $(u_n)_{n\in\mathbb{N}} \in L^2(\Omega \times \mathbb{N})$ via the (conditional) isometry formula*

$$\mathbb{E}[|J(\mathbf{1}_{[n,\infty)}u)|^2| \mid \mathcal{F}_{n-1}] = \mathbb{E}[\|\mathbf{1}_{[n,\infty)}u\|^2_{\ell^2(\mathbb{N})} \mid \mathcal{F}_{n-1}], \qquad n \in \mathbb{N}. \tag{1.2.1}$$

Proof. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be bounded predictable processes with finite support in $\mathbb{N}$. The product $u_k Y_k v_l$, $0 \leq k < l$, is $\mathcal{F}_{l-1}$-measurable, and $u_k Y_l v_l$ is $\mathcal{F}_{k-1}$-measurable, $0 \leq l < k$. Hence

$$\mathbb{E}\left[\sum_{k=n}^{\infty} u_k Y_k \sum_{l=n}^{\infty} v_l Y_l \Big| \mathcal{F}_{n-1}\right] = \mathbb{E}\left[\sum_{k,l=n}^{\infty} u_k Y_k v_l Y_l \Big| \mathcal{F}_{n-1}\right]$$

$$= \mathbb{E}\left[\sum_{k=n}^{\infty} u_k v_k Y_k^2 + \sum_{n\leq k<l} u_k Y_k v_l Y_l + \sum_{n\leq l<k} u_k Y_k v_l Y_l \Big| \mathcal{F}_{n-1}\right]$$

$$
\begin{aligned}
&= \sum_{k=n}^{\infty} \mathbb{E}[\mathbb{E}[u_k v_k Y_k^2 \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_{n-1}] + \sum_{n \le k < l} \mathbb{E}[\mathbb{E}[u_k Y_k v_l Y_l \mid \mathcal{F}_{l-1}] \mid \mathcal{F}_{n-1}] \\
&\quad + \sum_{n \le l < k} \mathbb{E}[\mathbb{E}[u_k Y_k v_l Y_l \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_{n-1}] \\
&= \sum_{k=0}^{\infty} \mathbb{E}[u_k v_k \mathbb{E}[Y_k^2 \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_{n-1}] + 2 \sum_{n \le k < l} \mathbb{E}[u_k Y_k v_l \mathbb{E}[Y_l \mid \mathcal{F}_{l-1}] \mid \mathcal{F}_{n-1}] \\
&= \sum_{k=n}^{\infty} \mathbb{E}[u_k v_k \mid \mathcal{F}_{n-1}] \\
&= \mathbb{E}\left[\sum_{k=n}^{\infty} u_k v_k \middle| \mathcal{F}_{n-1} \right].
\end{aligned}
$$

This proves the isometry property (1.2.1) for J. The extension to $L^2(\Omega \times \mathbb{N})$ is proved using the following Cauchy sequence argument. Consider a sequence of bounded predictable processes with finite support converging to u in $L^2(\Omega \times \mathbb{N})$, for example the sequence $(u^n)_{n\in\mathbb{N}}$ defined as

$$
u^n = (u_k^n)_{k\in\mathbb{N}} = (u_k \mathbf{1}_{\{0\le k\le n\}} \mathbf{1}_{\{|u_k|\le n\}})_{k\in\mathbb{N}}, \qquad n \in \mathbb{N}.
$$

Then the sequence $(J(u^n))_{n\in\mathbb{N}}$ is Cauchy and converges in $L^2(\Omega)$, hence we may define

$$
J(u) := \lim_{k\to\infty} J(u^k).
$$

From the isometry property (1.2.1) applied with $n = 0$, the limit is clearly independent of the choice of the approximating sequence $(u^k)_{k\in\mathbb{N}}$. □

Note that by polarization, (1.2.1) can also be written as

$$
\mathbb{E}[J(\mathbf{1}_{[n,\infty)}u)J(\mathbf{1}_{[n,\infty)}v)|\mathcal{F}_{n-1}] = \mathbb{E}[\langle \mathbf{1}_{[n,\infty)}u, \mathbf{1}_{[n,\infty)}v\rangle_{\ell^2(\mathbb{N})} \mid \mathcal{F}_{n-1}], \qquad n \in \mathbb{N},
$$

and that for $n = 0$ we get

$$
\mathbb{E}\left[J(u)J(v)\right] = \mathbb{E}[\langle u, v\rangle_{\ell^2(\mathbb{N})}], \tag{1.2.2}
$$

and

$$
\mathbb{E}[|J(u)|^2] = \mathbb{E}\left[\|u\|^2_{\ell^2(\mathbb{N})}\right], \tag{1.2.3}
$$

for all square-integrable predictable processes $u = (u_k)_{k\in\mathbb{N}}$ and $v = (v_k)_{k\in\mathbb{N}}$.

Proposition 1.2.3. *Let $(u_k)_{k\in\mathbb{N}} \in L^2(\Omega \times \mathbb{N})$ be a predictable square-integrable process. We have*

$$
\mathbb{E}[J(u) \mid \mathcal{F}_k] = J(u\mathbf{1}_{[0,k]}), \quad k \in \mathbb{N}.
$$

Proof. In case $(u_k)_{k\in\mathbb{N}}$ has finite support in $\mathbb{N}$ it suffices to note that

$$\begin{aligned}\mathbb{E}[J(u) \mid \mathcal{F}_k] &= \mathbb{E}\left[\sum_{i=0}^{k} u_i Y_i \Big| \mathcal{F}_k\right] + \sum_{i=k+1}^{\infty} \mathbb{E}\left[u_i Y_i \mid \mathcal{F}_k\right] \\ &= \sum_{i=0}^{k} u_i Y_i + \sum_{i=k+1}^{\infty} \mathbb{E}\left[\mathbb{E}\left[u_i Y_i \mid \mathcal{F}_{i-1}\right] \mid \mathcal{F}_k\right] \\ &= \sum_{i=0}^{k} u_i Y_i + \sum_{i=k+1}^{\infty} \mathbb{E}\left[u_i \mathbb{E}\left[Y_i \mid \mathcal{F}_{i-1}\right] \mid \mathcal{F}_k\right] \\ &= \sum_{i=0}^{k} u_i Y_i \\ &= J(u\mathbf{1}_{[0,k]}).\end{aligned}$$

The formula extends to the general case by linearity and density, using the continuity of the conditional expectation on L^2 and the sequence $(u^n)_{n\in\mathbb{N}}$ defined as $u^n = (u^n_k)_{k\in\mathbb{N}} = (u_k \mathbf{1}_{\{0\le k\le n\}})_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, i.e.

$$\begin{aligned}\mathbb{E}\left[\left(J(u\mathbf{1}_{[0,k]}) - \mathbb{E}[J(u) \mid \mathcal{F}_k]\right)^2\right] &= \lim_{n\to\infty} \mathbb{E}\left[\left(J(u^n\mathbf{1}_{[0,k]}) - \mathbb{E}[J(u) \mid \mathcal{F}_k]\right)^2\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[J(u^n) - J(u) \mid \mathcal{F}_k\right]\right)^2\right] \\ &\le \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(J(u^n) - J(u)\right)^2 \Big| \mathcal{F}_k\right]\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left(J(u^n) - J(u)\right)^2\right] \\ &= 0,\end{aligned}$$

by (1.2.3). □

Corollary 1.2.4. *The indefinite stochastic integral* $(J(u\mathbf{1}_{[0,k]}))_{k\in\mathbb{N}}$ *is a discrete time martingale with respect to* $(\mathcal{F}_n)_{n\ge -1}$.

Proof. We have

$$\begin{aligned}\mathbb{E}[J(u\mathbf{1}_{[0,k+1]}) \mid \mathcal{F}_k] &= \mathbb{E}[\mathbb{E}[J(u\mathbf{1}_{[0,k+1]}) \mid \mathcal{F}_{k+1} \mid \mathcal{F}_k] \\ &= \mathbb{E}[\mathbb{E}[J(u) \mid \mathcal{F}_{k+1} \mid \mathcal{F}_k] \\ &= \mathbb{E}[J(u) \mid \mathcal{F}_k] \\ &= J(u\mathbf{1}_{[0,k]}).\end{aligned}$$

□

1.3 Multiple Stochastic Integrals

The role of multiple stochastic integrals in the orthogonal expansion of a random variable is similar to that of polynomials in the series expansion of a function of a real variable. In some situations, multiple stochastic integrals can be expressed using polynomials, such as in the symmetric case $p_n = q_n = 1/2$, $n \in \mathbb{N}$, in which the Krawtchouk polynomials are used, see Relation (1.5.2) below.

Definition 1.3.1. *Let $\ell^2(\mathbb{N})^{\circ n}$ denote the subspace of $\ell^2(\mathbb{N})^{\otimes n} = \ell^2(\mathbb{N}^n)$ made of functions f_n that are symmetric in n variables, i.e. such that for every permutation σ of $\{1, \ldots, n\}$,*

$$f_n(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = f_n(k_1, \ldots, k_n), \quad k_1, \ldots, k_n \in \mathbb{N}.$$

Given $f_1 \in \ell^2(\mathbb{N})$ we let

$$J_1(f_1) = J(f_1) = \sum_{k=0}^{\infty} f_1(k) Y_k.$$

As a convention we identify $\ell^2(\mathbb{N}^0)$ to $\mathbb{R}$ and let $J_0(f_0) = f_0$, $f_0 \in \mathbb{R}$. Let

$$\Delta_n = \{(k_1, \ldots, k_n) \in \mathbb{N}^n \ : \ k_i \neq k_j, \ 1 \leq i < j \leq n\}, \qquad n \geq 1.$$

The following proposition gives the definition of multiple stochastic integrals by iterated stochastic integration of predictable processes in the sense of Proposition 1.2.2.

Proposition 1.3.2. *The multiple stochastic integral $J_n(f_n)$ of $f_n \in \ell^2(\mathbb{N})^{\circ n}$, $n \geq 1$, is defined as*

$$J_n(f_n) = \sum_{(i_1, \ldots, i_n) \in \Delta_n} f_n(i_1, \ldots, i_n) Y_{i_1} \cdots Y_{i_n}.$$

It satisfies the recurrence relation

$$J_n(f_n) = n \sum_{k=1}^{\infty} Y_k J_{n-1}(f_n(*, k) \mathbf{1}_{[0,k-1]^{n-1}}(*)) \tag{1.3.1}$$

and the isometry formula

$$\mathbb{E}[J_n(f_n) J_m(g_m)] = \begin{cases} n! \langle \mathbf{1}_{\Delta_n} f_n, g_m \rangle_{\ell^2(\mathbb{N})^{\otimes n}} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{1.3.2}$$

Proof. Note that we have

$$J_n(f_n) = n! \sum_{0 \le i_1 < \cdots < i_n} f_n(i_1, \ldots, i_n) Y_{i_1} \cdots Y_{i_n}$$
$$= n! \sum_{i_n=0}^{\infty} \sum_{0 \le i_{n-1} < i_n} \cdots \sum_{0 \le i_1 < i_2} f_n(i_1, \ldots, i_n) Y_{i_1} \cdots Y_{i_n}. \quad (1.3.3)$$

Note that since $0 \le i_1 < i_2 < \cdots < i_n$ and $0 \le j_1 < j_2 < \cdots < j_n$ we have

$$\mathbb{E}[Y_{i_1} \cdots Y_{i_n} Y_{j_1} \cdots Y_{j_n}] = \mathbf{1}_{\{i_1 = j_1, \ldots, i_n = j_n\}}.$$

Hence

$$\mathbb{E}[J_n(f_n) J_n(g_n)]$$
$$= (n!)^2 \mathbb{E}\left[\sum_{0 \le i_1 < \cdots < i_n} f_n(i_1, \ldots, i_n) Y_{i_1} \cdots Y_{i_n} \sum_{0 \le j_1 < \cdots < j_n} g_n(j_1, \ldots, j_n) Y_{j_1} \cdots Y_{j_n} \right]$$
$$= (n!)^2 \sum_{0 \le i_1 < \cdots < i_n, \ 0 \le j_1 < \cdots < j_n} f_n(i_1, \ldots, i_n) g_n(j_1, \ldots, j_n) \mathbb{E}[Y_{i_1} \cdots Y_{i_n} Y_{j_1} \cdots Y_{j_n}]$$
$$= (n!)^2 \sum_{0 \le i_1 < \cdots < i_n} f_n(i_1, \ldots, i_n) g_n(i_1, \ldots, i_n)$$
$$= n! \sum_{(i_1, \ldots, i_n) \in \Delta_n} f_n(i_1, \ldots, i_n) g_n(i_1, \ldots, i_n)$$
$$= n! \langle \mathbf{1}_{\Delta_n} f_n, g_m \rangle_{\ell^2(\mathbb{N})^{\otimes n}}.$$

When $n < m$ and $(i_1, \ldots, i_n) \in \Delta_n$ and $(j_1, \ldots, j_m) \in \Delta_m$ are two sets of indices, there necessarily exists $k \in \{1, \ldots, m\}$ such that $j_k \notin \{i_1, \ldots, i_n\}$, hence

$$\mathbb{E}[Y_{i_1} \cdots Y_{i_n} Y_{j_1} \cdots Y_{j_m}] = 0,$$

and this implies the orthogonality of $J_n(f_n)$ and $J_m(g_m)$. The recurrence relation (1.3.1) is a direct consequence of (1.3.3). The isometry property (1.3.2) of J_n also follows by induction from (1.2.1) and the recurrence relation. □

If $f_n \in \ell^2(\mathbb{N}^n)$ is not symmetric we let $J_n(f_n) = J_n(\tilde{f}_n)$, where $\tilde{f}_n$ is the symmetrization of f_n, defined as

$$\tilde{f}_n(i_1, \ldots, i_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(i_{\sigma(1)}, \ldots, i_{\sigma_n}), \qquad i_1, \ldots, i_n \in \mathbb{N}^n,$$

and Σ_n is the set of all permutations of $\{1, \ldots, n\}$.

In particular, if $(k_1, \ldots, k_n) \in \Delta_n$, the symmetrization of $\mathbf{1}_{\{(k_1,\ldots,k_n)\}}$ in n variables is given by

$$\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}(i_1, \ldots, i_n) = \frac{1}{n!}\mathbf{1}_{\{\{i_1,\ldots,i_n\}=\{k_1,\ldots,k_n\}\}}, \quad i_1, \ldots, i_n \in \mathbb{N},$$

and

$$J_n(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}) = Y_{k_1} \cdots Y_{k_n}.$$

Lemma 1.3.3. *For all $n \geq 1$ we have*

$$\mathbb{E}[J_n(f_n) \mid \mathcal{F}_k] = J_n(f_n \mathbf{1}_{[0,k]^n}),$$

$k \in \mathbb{N}$, $f_n \in \ell^2(\mathbb{N})^{\circ n}$.

Proof. This lemma can be proved in two ways, either as a consequence of Proposition 1.2.3 and Proposition 1.3.2 or via the following direct argument, noting that for all $m = 0, \ldots, n$ and $g_m \in \ell^2(\mathbb{N})^{\circ m}$ we have:

$$\begin{aligned}
&\mathbb{E}[(J_n(f_n) - J_n(f_n \mathbf{1}_{[0,k]^n}))J_m(g_m \mathbf{1}_{[0,k]^m})] \\
&= \mathbf{1}_{\{n=m\}} n! \langle f_n(1 - \mathbf{1}_{[0,k]^n}), g_m \mathbf{1}_{[0,k]^m} \rangle_{\ell^2(\mathbb{N}^n)} \\
&= 0,
\end{aligned}$$

hence $J_n(f_n \mathbf{1}_{[0,k]^n}) \in L^2(\Omega, \mathcal{F}_k)$, and $J_n(f_n) - J_n(f_n \mathbf{1}_{[0,k]^n})$ is orthogonal to $L^2(\Omega, \mathcal{F}_k)$. □

In other terms we have

$$\mathbb{E}[J_n(f_n)] = 0, \qquad f_n \in \ell^2(\mathbb{N})^{\circ n}, \qquad n \geq 1,$$

the process $(J_n(f_n \mathbf{1}_{[0,k]^n}))_{k \in \mathbb{N}}$ is a discrete-time martingale, and $J_n(f_n)$ is $\mathcal{F}_k$-measurable if and only if

$$f_n \mathbf{1}_{[0,k]^n} = f_n, \qquad 0 \leq k \leq n.$$

1.4 Structure Equations

Assume now that the sequence $(Y_n)_{n \in \mathbb{N}}$ satisfies the discrete structure equation:

$$Y_n^2 = 1 + \varphi_n Y_n, \qquad n \in \mathbb{N}, \tag{1.4.1}$$

where $(\varphi_n)_{n \in \mathbb{N}}$ is an $\mathcal{F}_n$-predictable process. Condition (1.1.1) implies that

$$\mathbb{E}[Y_n^2 \mid \mathcal{F}_{n-1}] = 1, \qquad n \in \mathbb{N},$$

hence the hypotheses of the preceding sections are satisfied. Since (1.4.1) is a second order equation, there exists an $\mathcal{F}_n$-adapted process $(X_n)_{n\in\mathbb{N}}$ of Bernoulli $\{-1,1\}$-valued random variables such that

$$Y_n = \frac{\varphi_n}{2} + X_n\sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}, \qquad n \in \mathbb{N}. \tag{1.4.2}$$

Consider the conditional probabilities

$$p_n = \mathbb{P}(X_n = 1 \mid \mathcal{F}_{n-1}) \quad \text{and} \quad q_n = \mathbb{P}(X_n = -1 \mid \mathcal{F}_{n-1}), \qquad n \in \mathbb{N}. \tag{1.4.3}$$

From the relation $\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = 0$, rewritten as

$$p_n\left(\frac{\varphi_n}{2} + \sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}\right) + q_n\left(\frac{\varphi_n}{2} - \sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}\right) = 0, \qquad n \in \mathbb{N},$$

we get

$$p_n = \frac{1}{2}\left(1 - \frac{\varphi_n}{\sqrt{4+\varphi_n^2}}\right), \qquad q_n = \frac{1}{2}\left(1 + \frac{\varphi_n}{\sqrt{4+\varphi_n^2}}\right), \tag{1.4.4}$$

and

$$\varphi_n = \sqrt{\frac{q_n}{p_n}} - \sqrt{\frac{p_n}{q_n}} = \frac{q_n - p_n}{\sqrt{p_n q_n}}, \qquad n \in \mathbb{N},$$

hence

$$Y_n = \mathbf{1}_{\{X_n=1\}}\sqrt{\frac{q_n}{p_n}} - \mathbf{1}_{\{X_n=-1\}}\sqrt{\frac{p_n}{q_n}}, \qquad n \in \mathbb{N}. \tag{1.4.5}$$

Letting

$$Z_n = \frac{X_n + 1}{2} \in \{0,1\}, \qquad n \in \mathbb{N},$$

we also have the relations

$$Y_n = \frac{q_n - p_n + X_n}{2\sqrt{p_n q_n}} = \frac{Z_n - p_n}{\sqrt{p_n q_n}}, \quad n \in \mathbb{N}, \tag{1.4.6}$$

which yield

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n) = \sigma(Z_0, \ldots, Z_n), \qquad n \in \mathbb{N}.$$

Remark 1.4.1. In particular, one can take $\Omega = \{-1,1\}^{\mathbb{N}}$ and construct the Bernoulli process $(X_n)_{n\in\mathbb{N}}$ as the sequence of canonical projections on $\Omega = \{-1,1\}^{\mathbb{N}}$ under a countable product $\mathbb{P}$ of Bernoulli measures on $\{-1,1\}$. In this case the sequence $(X_n)_{n\in\mathbb{N}}$ can be viewed as the dyadic expansion of

$X(\omega) \in [0,1]$ defined as:

$$X(\omega) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} X_n(\omega).$$

In the symmetric case $p_k = q_k = 1/2$, $k \in \mathbb{N}$, the image measure of $\mathbb{P}$ by the mapping $\omega \mapsto X(\omega)$ is the Lebesgue measure on $[0,1]$, see [139] for the non-symmetric case.

1.5 Chaos Representation

From now on we assume that the sequence $(p_k)_{k\in\mathbb{N}}$ defined in (1.4.3) is deterministic, which implies that the random variables $(X_n)_{n\in\mathbb{N}}$ are independent. Precisely, X_n will be constructed as the canonical projection $X_n : \Omega \to \{-1,1\}$ on $\Omega = \{-1,1\}^{\mathbb{N}}$ under the measure $\mathbb{P}$ given on cylinder sets by

$$\mathbb{P}(\{\epsilon_0,\ldots,\epsilon_n\} \times \{-1,1\}^{\mathbb{N}}) = \prod_{k=0}^{n} p_k^{(1+\varepsilon_k)/2} q_k^{(1-\varepsilon_k)/2},$$

$\{\epsilon_0,\ldots,\epsilon_n\} \in \{-1,1\}^{n+1}$. The sequence $(Y_k)_{k\in\mathbb{N}}$ can be constructed as a family of independent random variables given by

$$Y_n = \frac{\varphi_n}{2} + X_n\sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}, \qquad n \in \mathbb{N},$$

where the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is deterministic. In this case, all spaces $L^r(\Omega,\mathcal{F}_n)$, $r \geq 1$, have finite dimension 2^{n+1}, with basis

$$\left\{\mathbf{1}_{\{Y_0=\epsilon_0,\ldots,Y_n=\epsilon_n\}} \ : \ (\epsilon_0,\ldots,\epsilon_n) \in \prod_{k=0}^{n}\left\{\sqrt{\frac{q_k}{p_k}}, -\sqrt{\frac{p_k}{q_k}}\right\}\right\}$$
$$= \left\{\mathbf{1}_{\{X_0=\epsilon_0,\ldots,X_n=\epsilon_n\}} \ : \ (\epsilon_0,\ldots,\epsilon_n) \in \prod_{k=0}^{n}\{-1,1\}\right\}.$$

An orthogonal basis of $L^r(\Omega,\mathcal{F}_n)$ is given by

$$\left\{Y_{k_1}\cdots Y_{k_l} = J_l(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_l)\}}) \ : \ 0 \leq k_1 < \cdots < k_l \leq n, \ l = 0,\ldots,n+1\right\}.$$

Let

$$S_n = \sum_{k=0}^{n} \frac{1+X_k}{2} \tag{1.5.1}$$
$$= \sum_{k=0}^{n} Z_k, \qquad n \in \mathbb{N},$$

denote the random walk associated to $(X_k)_{k\in\mathbb{N}}$. If $p_k = p$, $k \in \mathbb{N}$, then

$$J_n(\mathbf{1}_{[0,N]}^{\circ n}) = K_n(S_N; N+1, p) \tag{1.5.2}$$

coincides with the Krawtchouk polynomial $K_n(\cdot; N+1, p)$ of order n and parameter $(N+1, p)$, evaluated at S_N, cf. Proposition 4 of [115].
Let now $\mathcal{H}_0 = \mathbb{R}$ and let $\mathcal{H}_n$ denote the subspace of $L^2(\Omega)$ made of integrals of order $n \geq 1$, and called chaos of order n:

$$\mathcal{H}_n = \{J_n(f_n) \ : \ f_n \in \ell^2(\mathbb{N})^{\circ n}\}.$$

The space of $\mathcal{F}_n$-measurable random variables is denoted by $L^0(\Omega, \mathcal{F}_n)$.

Lemma 1.5.1. *For all $n \in \mathbb{N}$ we have*

$$L^0(\Omega, \mathcal{F}_n) = (\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}) \bigcap L^0(\Omega, \mathcal{F}_n). \tag{1.5.3}$$

Proof. It suffices to note that $\mathcal{H}_l \cap L^0(\Omega, \mathcal{F}_n)$ has dimension $\binom{n+1}{l}$, $1 \leq l \leq n+1$. More precisely it is generated by the orthonormal basis

$$\left\{Y_{k_1} \cdots Y_{k_l} = J_l(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_l)\}}) \ : \ 0 \leq k_1 < \cdots < k_l \leq n\right\},$$

since any element F of $\mathcal{H}_l \cap L^0(\Omega, \mathcal{F}_n)$ can be written as $F = J_l(f_l \mathbf{1}_{[0,n]^l})$. Hence $L^0(\Omega, \mathcal{F}_n)$ and $(\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}) \bigcap L^0(\Omega, \mathcal{F}_n)$ have same dimension $2^{n+1} = \displaystyle\sum_{k=0}^{n+1} \binom{n+1}{k}$, and this implies (1.5.3) since

$$L^0(\Omega, \mathcal{F}_n) \supset (\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}) \bigcap L^0(\Omega, \mathcal{F}_n).$$

□

As a consequence of Lemma 1.5.1 we have

$$L^0(\Omega, \mathcal{F}_n) \subset \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}.$$

Alternatively, Lemma 1.5.1 can be proved by noting that

$$J_n(f_n \mathbf{1}_{[0,N]^n}) = 0, \qquad n > N+1, \quad f_n \in \ell^2(\mathbb{N})^{\circ n},$$

and as a consequence, any $F \in L^0(\Omega, \mathcal{F}_N)$ can be expressed as

$$F = \mathbb{E}[F] + \sum_{n=1}^{N+1} J_n(f_n \mathbf{1}_{[0,N]^n}).$$

Definition 1.5.2. *Let $\mathcal{S}$ denote the linear space spanned by multiple stochastic integrals, i.e.*

$$\begin{aligned}\mathcal{S} &= \text{Vect}\left\{\bigcup_{n=0}^{\infty} \mathcal{H}_n\right\} \qquad (1.5.4)\\ &= \left\{\sum_{k=0}^{n} J_k(f_k) \ : \ f_k \in \ell^2(\mathbb{N})^{\circ k}, \ k = 0, \ldots, n, \ n \in \mathbb{N}\right\}.\end{aligned}$$

The completion of $\mathcal{S}$ in $L^2(\Omega)$ is denoted by the direct sum

$$\bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

The next result is the chaos representation property for Bernoulli processes, which is analogous to the Walsh decomposition, cf. [78]. Here this property is obtained under the assumption that the sequence $(X_n)_{n\in\mathbb{N}}$ is made of independent random variables since $(p_k)_{k\in\mathbb{N}}$ is deterministic, which corresponds to the setting of Proposition 4 in [38]. See [38] and Proposition 5 therein for other instances of the chaos representation property without this independence assumption.

Proposition 1.5.3. *We have the identity*

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Proof. It suffices to show that $\mathcal{S}$ is dense in $L^2(\Omega)$. Let F be a bounded random variable. Relation (1.5.3) of Lemma 1.5.1 shows that $\mathbb{E}[F \mid \mathcal{F}_n] \in \mathcal{S}$. The martingale convergence theorem, cf. e.g. Theorem 27.1 in [67], implies that $(\mathbb{E}[F \mid \mathcal{F}_n])_{n\in\mathbb{N}}$ converges to F a.s., hence every bounded F is the $L^2(\Omega)$-limit of a sequence in $\mathcal{S}$. If $F \in L^2(\Omega)$ is not bounded, F is the limit in $L^2(\Omega)$ of the sequence $(\mathbf{1}_{\{|F|\leq n\}} F)_{n\in\mathbb{N}}$ of bounded random variables. □

As a consequence of Proposition 1.5.3, any $F \in L^2(\Omega, \mathbb{P})$ has a unique decomposition

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n), \qquad f_n \in \ell^2(\mathbb{N})^{\circ n}, \ n \in \mathbb{N},$$

as a series of multiple stochastic integrals. Note also that the statement of Lemma 1.5.1 is sufficient for the chaos representation property to hold.

1.6 Gradient Operator

We start by defining the operator D on the space $\mathcal{S}$ of finite sums of multiple stochastic integrals, which is dense in in $L^2(\Omega)$ by Proposition 1.5.3.

Definition 1.6.1. *We densely define the linear gradient operator*

$$D : \mathcal{S} \longrightarrow L^2(\Omega \times \mathbb{N})$$

by

$$D_k J_n(f_n) = n J_{n-1}(f_n(*, k) \mathbf{1}_{\Delta_n}(*, k)),$$

$k \in \mathbb{N}$, $f_n \in \ell^2(\mathbb{N})^{\circ n}$, $n \in \mathbb{N}$.

Note that for all $k_1, \ldots, k_{n-1}, k \in \mathbb{N}$, we have

$$\mathbf{1}_{\Delta_n}(k_1, \ldots, k_{n-1}, k) = \mathbf{1}_{\{k \notin (k_1, \ldots, k_{n-1})\}} \mathbf{1}_{\Delta_{n-1}}(k_1, \ldots, k_{n-1}),$$

hence we can write

$$D_k J_n(f_n) = n J_{n-1}(f_n(*, k) \mathbf{1}_{\{k \notin *\}}), \quad k \in \mathbb{N},$$

where in the above relation, "$*$" denotes the first $k-1$ variables $(k_1, \ldots, k_{n-1})$ of $f_n(k_1, \ldots, k_{n-1}, k)$. We also have $D_k F = 0$ whenever $F \in \mathcal{S}$ is $\mathcal{F}_{k-1}$-measurable.
On the other hand, D_k is a continuous operator on the chaos $\mathcal{H}_n$ since

$$\begin{aligned} \|D_k J_n(f_n)\|^2_{L^2(\Omega)} &= n^2 \|J_{n-1}(f_n(*, k))\|^2_{L^2(\Omega)} \\ &= n n! \|f_n(*, k)\|^2_{\ell^2(\mathbb{N}^{\otimes(n-1)})}, \qquad f_n \in \ell^2(\mathbb{N}^{\otimes n}), \quad k \in \mathbb{N}. \end{aligned} \tag{1.6.1}$$

The following result gives the probabilistic interpretation of D_k as a finite difference operator. Given

$$\omega = (\omega_0, \omega_1, \ldots) \in \{-1, 1\}^{\mathbb{N}},$$

let

$$\omega_+^k = (\omega_0, \omega_1, \ldots, \omega_{k-1}, +1, \omega_{k+1}, \ldots)$$

and

$$\omega_-^k = (\omega_0, \omega_1, \ldots, \omega_{k-1}, -1, \omega_{k+1}, \ldots).$$

Proposition 1.6.2. *We have for any $F \in \mathcal{S}$:*

$$D_k F(\omega) = \sqrt{p_k q_k}(F(\omega_+^k) - F(\omega_-^k)), \quad k \in \mathbb{N}. \tag{1.6.2}$$

Proof. We start by proving the above statement for an $\mathcal{F}_n$-measurable $F \in \mathcal{S}$. Since $L^0(\Omega, \mathcal{F}_n)$ is finite dimensional it suffices to consider

$$F = Y_{k_1} \cdots Y_{k_l} = f(X_0, \ldots, X_{k_l}),$$

with from (1.4.6):

$$f(x_0, \ldots, x_{k_l}) = \frac{1}{2^l} \prod_{i=1}^{l} \frac{q_{k_i} - p_{k_i} + x_{k_i}}{\sqrt{p_{k_i} q_{k_i}}}.$$

First we note that from (1.5.3) we have for $(k_1, \ldots, k_n) \in \Delta_n$:

$$\begin{aligned}
&D_k (Y_{k_1} \cdots Y_{k_n}) = D_k J_n(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}) \\
&= n J_{n-1}(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}(*, k)) \\
&= \frac{1}{(n-1)!} \sum_{i=1}^{n} \mathbf{1}_{\{k_i\}}(k) \sum_{(i_1,\ldots,i_{n-1}) \in \Delta_{n-1}} \tilde{\mathbf{1}}_{\{\{i_1,\ldots,i_{n-1}\} = \{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n\}\}} \\
&= \sum_{i=1}^{n} \mathbf{1}_{\{k_i\}}(k) J_{n-1}(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n)\}}) \\
&= \mathbf{1}_{\{k_1,\ldots,k_n\}}(k) \prod_{\substack{i=1 \\ k_i \neq k}}^{n} Y_{k_i}.
\end{aligned} \tag{1.6.3}$$

If $k \notin \{k_1, \ldots, k_l\}$ we clearly have $F(\omega_+^k) = F(\omega_-^k) = F(\omega)$, hence

$$\sqrt{p_k q_k}(F(\omega_+^k) - F(\omega_-^k)) = 0 = D_k F(\omega).$$

On the other hand if $k \in \{k_1, \ldots, k_l\}$ we have

$$F(\omega_+^k) = \sqrt{\frac{q_k}{p_k}} \prod_{\substack{i=1 \\ k_i \neq k}}^{l} \frac{q_{k_i} - p_{k_i} + \omega_{k_i}}{2\sqrt{p_{k_i} q_{k_i}}},$$

$$F(\omega_-^k) = -\sqrt{\frac{p_k}{q_k}} \prod_{\substack{i=1 \\ k_i \neq k}}^{l} \frac{q_{k_i} - p_{k_i} + \omega_{k_i}}{2\sqrt{p_{k_i} q_{k_i}}},$$

hence from (1.6.3) we get

$$\begin{aligned}\sqrt{p_k q_k}(F(\omega_+^k) - F(\omega_-^k)) &= \frac{1}{2^{l-1}} \prod_{\substack{i=1\\k_i\neq k}}^{l} \frac{q_{k_i} - p_{k_i} + \omega_{k_i}}{\sqrt{p_{k_i} q_{k_i}}} \\ &= \prod_{\substack{i=1\\k_i\neq k}}^{l} Y_{k_i}(\omega) \\ &= D_k\left(Y_{k_1} \cdots Y_{k_l}\right)(\omega) \\ &= D_k F(\omega).\end{aligned}$$

In the general case, $J_l(f_l)$ is the L^2-limit of the sequence $\mathbb{E}[J_l(f_l) \mid \mathcal{F}_n] = J_l(f_l \mathbf{1}_{[0,n]^l})$ as n goes to infinity, and since from (1.6.1) the operator D_k is continuous on all chaoses $\mathcal{H}_n$, $n \geq 1$, we have

$$\begin{aligned}D_k F &= \lim_{n\to\infty} D_k \mathbb{E}[F \mid \mathcal{F}_n] \\ &= \sqrt{p_k q_k} \lim_{n\to\infty} (\mathbb{E}[F \mid \mathcal{F}_n](\omega_+^k) - \mathbb{E}[F \mid \mathcal{F}_n](\omega_-^k)) \\ &= \sqrt{p_k q_k}(F(\omega_+^k) - F(\omega_-^k)), \quad k \in \mathbb{N}.\end{aligned}$$

□

The next property follows immediately from Proposition 1.6.2.

Corollary 1.6.3. *A random variable $F : \Omega \to \mathbb{R}$ is $\mathcal{F}_n$-measurable if and only if*

$$D_k F = 0$$

for all $k > n$.

If F has the form $F = f(X_0, \ldots, X_n)$, we may also write

$$D_k F = \sqrt{p_k q_k}(F_k^+ - F_k^-), \qquad k \in \mathbb{N},$$

with

$$F_k^+ = f(X_0, \ldots, X_{k-1}, +1, X_{k+1}, \ldots, X_n),$$

and

$$F_k^- = f(X_0, \ldots, X_{k-1}, -1, X_{k+1}, \ldots, X_n).$$

The gradient D can also be expressed as

$$D_k F(S_\cdot) = \sqrt{p_k q_k}\left(F\left(S_\cdot + \mathbf{1}_{\{X_k=-1\}}\mathbf{1}_{\{k\leq\cdot\}}\right) - F\left(S_\cdot - \mathbf{1}_{\{X_k=1\}}\mathbf{1}_{\{k\leq\cdot\}}\right)\right),$$

where $F(S_\cdot)$ is an informal notation for the random variable F estimated on a given path of $(S_n)_{n\in\mathbb{N}}$ defined in (1.5.1) and $S_\cdot + \mathbf{1}_{\{X_k=\mp1\}}\mathbf{1}_{\{k\leq\cdot\}}$ denotes the path of $(S_n)_{n\in\mathbb{N}}$ perturbed by forcing X_k to be equal to ±1.

We will also use the gradient ∇_k defined as

$$\begin{aligned}\nabla_k F = X_k \big(& f(X_0, \ldots, X_{k-1}, -1, X_{k+1}, \ldots, X_n) \\ & - f(X_0, \ldots, X_{k-1}, 1, X_{k+1}, \ldots, X_n) \big),\end{aligned}$$

$k \in \mathbb{N}$, with the relation

$$D_k = -X_k \sqrt{p_k q_k} \nabla_k, \qquad k \in \mathbb{N},$$

hence $\nabla_k F$ coincides with $D_k F$ after squaring and multiplication by $p_k q_k$. From now on, D_k denotes the finite difference operator which is extended to any $F : \Omega \to \mathbb{R}$ using Relation (1.6.2).
The L^2 domain of D, denoted $\mathrm{Dom}\,(D)$, is naturally defined as the space of functionals $F \in L^2(\Omega)$ such that

$$\mathbb{E}\left[\|DF\|^2_{\ell^2(\mathbb{N})} \right] < \infty,$$

or equivalently by (1.6.1),

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{\ell^2(\mathbb{N}^n)} < \infty,$$

if $F = \sum_{n=0}^{\infty} J_n(f_n)$.
The following is the product rule for the operator D.

Proposition 1.6.4. *Let $F, G : \Omega \to \mathbb{R}$. We have*

$$D_k(FG) = F D_k G + G D_k F - \frac{X_k}{\sqrt{p_k q_k}} D_k F D_k G, \qquad k \in \mathbb{N}.$$

Proof. Let $F^k_+(\omega) = F(\omega^k_+)$, $F^k_-(\omega) = F(\omega^k_-)$, $k \geq 0$. We have

$$\begin{aligned} D_k(FG) &= \sqrt{p_k q_k} (F^k_+ G^k_+ - F^k_- G^k_-) \\ &= \mathbf{1}_{\{X_k=-1\}} \sqrt{p_k q_k} \left(F(G^k_+ - G) + G(F^k_+ - F) + (F^k_+ - F)(G^k_+ - G) \right) \\ &\quad + \mathbf{1}_{\{X_k=1\}} \sqrt{p_k q_k} \left(F(G - G^k_-) + G(F - F^k_-) - (F - F^k_-)(G - G^k_-) \right) \\ &= \mathbf{1}_{\{X_k=-1\}} \left(F D_k G + G D_k F + \frac{1}{\sqrt{p_k q_k}} D_k F D_k G \right) \\ &\quad + \mathbf{1}_{\{X_k=1\}} \left(F D_k G + G D_k F - \frac{1}{\sqrt{p_k q_k}} D_k F D_k G \right). \end{aligned}$$

□

1.7 Clark Formula and Predictable Representation

In this section we prove a predictable representation formula for the functionals of $(S_n)_{n\geq 0}$ defined in (1.5.1).

Proposition 1.7.1. *For all $F \in \mathcal{S}$ we have*

$$F = \mathbb{E}[F] + \sum_{k=0}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k \tag{1.7.1}$$
$$= \mathbb{E}[F] + \sum_{k=0}^{\infty} Y_k D_k \mathbb{E}[F \mid \mathcal{F}_k].$$

Proof. The formula is obviously true for $F = J_0(f_0)$. Given $n \geq 1$, as a consequence of Proposition 1.3.2 above and Lemma 1.3.3 we have:

$$\begin{aligned} J_n(f_n) &= n \sum_{k=0}^{\infty} J_{n-1}(f_n(*,k) \mathbf{1}_{[0,k-1]^{n-1}}(*)) Y_k \\ &= n \sum_{k=0}^{\infty} J_{n-1}(f_n(*,k) \mathbf{1}_{\Delta_n}(*,k) \mathbf{1}_{[0,k-1]^{n-1}}(*)) Y_k \\ &= n \sum_{k=0}^{\infty} \mathbb{E}[J_{n-1}(f_n(*,k) \mathbf{1}_{\Delta_n}(*,k)) \mid \mathcal{F}_{k-1}] Y_k \\ &= \sum_{k=0}^{\infty} \mathbb{E}[D_k J_n(f_n) \mid \mathcal{F}_{k-1}] Y_k, \end{aligned}$$

which yields (1.7.1) for $F = J_n(f_n)$, since $\mathbb{E}[J_n(f_n)] = 0$. By linearity the formula is established for $F \in \mathcal{S}$.
For the second identity we use the relation

$$\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] = D_k \mathbb{E}[F \mid \mathcal{F}_k]$$

which clearly holds since $D_k F$ is independent of X_k, $k \in \mathbb{N}$. □

Although the operator D is unbounded we have the following result, which states the boundedness of the operator that maps a random variable to the unique process involved in its predictable representation.

Lemma 1.7.2. *The operator*

$$\begin{aligned} L^2(\Omega) &\longrightarrow L^2(\Omega \times \mathbb{N}) \\ F &\mapsto (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])_{k\in\mathbb{N}} \end{aligned}$$

is bounded with norm equal to one.

Proof. Let $F \in \mathcal{S}$. From Relation (1.7.1) and the isometry formula (1.2.2) for the stochastic integral operator J we get

$$\begin{aligned} \|\mathbb{E}[D_{\cdot}F \mid \mathcal{F}_{\cdot-1}]\|^2_{L^2(\Omega\times\mathbb{N})} &= \|F - \mathbb{E}[F]\|^2_{L^2(\Omega)} \\ &\leq \|F - \mathbb{E}[F]\|^2_{L^2(\Omega)} + (\mathbb{E}[F])^2 \\ &= \|F\|^2_{L^2(\Omega)}, \end{aligned} \tag{1.7.2}$$

with equality in case $F = J_1(f_1)$. □

As a consequence of Lemma 1.7.2 we have the following corollary.

Corollary 1.7.3. *The Clark formula of Proposition 1.7.1 extends to any $F \in L^2(\Omega)$.*

Proof. Since $F \mapsto \mathbb{E}[D_{\cdot}F \mid \mathcal{F}_{\cdot-1}]$ is bounded from Lemma 1.7.2, the Clark formula extends to $F \in L^2(\Omega)$ by a standard Cauchy sequence argument. □

Let us give a first elementary application of the above construction to the proof of a Poincaré inequality on Bernoulli space. Using (1.2.3) we have

$$\begin{aligned} \mathrm{Var}\,(F) &= \mathbb{E}[|F - \mathbb{E}[F]|^2] \\ &= \mathbb{E}\left[\left(\sum_{k=0}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k\right)^2\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{\infty} (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])^2\right] \\ &\leq \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}[|D_k F|^2 \mid \mathcal{F}_{k-1}]\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{\infty} |D_k F|^2\right], \end{aligned}$$

hence

$$\mathrm{Var}\,(F) \leq \|DF\|^2_{L^2(\Omega\times\mathbb{N})}.$$

More generally the Clark formula implies the following.

Corollary 1.7.4. *Let $a \in \mathbb{N}$ and $F \in L^2(\Omega)$. We have*

$$F = \mathbb{E}[F \mid \mathcal{F}_a] + \sum_{k=a+1}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k, \tag{1.7.3}$$

and

$$\mathbb{E}[F^2] = \mathbb{E}[(\mathbb{E}[F \mid \mathcal{F}_a])^2] + \mathbb{E}\left[\sum_{k=a+1}^{\infty} (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])^2\right]. \qquad (1.7.4)$$

Proof. From Proposition 1.2.3 and the Clark formula (1.7.1) of Proposition 1.7.1 we have

$$\mathbb{E}[F \mid \mathcal{F}_a] = \mathbb{E}[F] + \sum_{k=0}^{a} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k,$$

which implies (1.7.3). Relation (1.7.4) is an immediate consequence of (1.7.3) and the isometry property of J. □

As an application of the Clark formula of Corollary 1.7.4 we obtain the following predictable representation property for discrete-time martingales.

Proposition 1.7.5. *Let $(M_n)_{n\in\mathbb{N}}$ be a martingale in $L^2(\Omega)$ with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. There exists a predictable process $(u_k)_{k\in\mathbb{N}}$ locally in $L^2(\Omega\times\mathbb{N})$, (i.e. $u(\cdot)\mathbf{1}_{[0,N]}(\cdot) \in L^2(\Omega \times \mathbb{N})$ for all $N > 0$) such that*

$$M_n = M_{-1} + \sum_{k=0}^{n} u_k Y_k, \qquad n \in \mathbb{N}. \qquad (1.7.5)$$

Proof. Let $k \geq 1$. From Corollaries 1.6.3 and 1.7.4 we have:

$$\begin{aligned} M_k &= \mathbb{E}[M_k \mid \mathcal{F}_{k-1}] + \mathbb{E}[D_k M_k \mid \mathcal{F}_{k-1}] Y_k \\ &= M_{k-1} + \mathbb{E}[D_k M_k \mid \mathcal{F}_{k-1}] Y_k, \end{aligned}$$

hence it suffices to let

$$u_k = \mathbb{E}[D_k M_k \mid \mathcal{F}_{k-1}], \qquad k \geq 0,$$

to obtain

$$M_n = M_{-1} + \sum_{k=0}^{n} M_k - M_{k-1} = M_{-1} + \sum_{k=0}^{n} u_k Y_k.$$

□

1.8 Divergence Operator

The divergence operator δ is introduced as the adjoint of D. Let $\mathcal{U} \subset L^2(\Omega \times \mathbb{N})$ be the space of processes defined as

$$\mathcal{U} = \left\{ \sum_{k=0}^{n} J_k(f_{k+1}(*, \cdot)), \quad f_{k+1} \in \ell^2(\mathbb{N})^{\circ k} \otimes \ell^2(\mathbb{N}), \quad k = 0, \ldots, n, \ n \in \mathbb{N} \right\}.$$

We refer to Section 9.7 in the appendix for the definition of the tensor product $\ell^2(\mathbb{N})^{\circ k} \otimes \ell^2(\mathbb{N})$, $k \geq 0$.

Definition 1.8.1. *Let $\delta : \mathcal{U} \to L^2(\Omega)$ be the linear mapping defined on $\mathcal{U}$ as*

$$\delta(u) = \delta(J_n(f_{n+1}(*, \cdot))) = J_{n+1}(\tilde{f}_{n+1}), \qquad f_{n+1} \in \ell^2(\mathbb{N})^{\circ n} \otimes \ell^2(\mathbb{N}),$$

for $(u_k)_{k\in\mathbb{N}}$ of the form

$$u_k = J_n(f_{n+1}(*, k)), \qquad k \in \mathbb{N},$$

where $\tilde{f}_{n+1}$ denotes the symmetrization of f_{n+1} in $n+1$ variables, i.e.

$$\tilde{f}_{n+1}(k_1, \ldots, k_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} f_{n+1}(k_1, \ldots, k_{k-1}, k_{k+1}, \ldots, k_{n+1}, k_i).$$

From Proposition 1.5.3, $\mathcal{S}$ is dense in $L^2(\Omega)$, hence $\mathcal{U}$ is dense in $L^2(\Omega \times \mathbb{N})$.

Proposition 1.8.2. *The operator δ is adjoint to D:*

$$\mathbb{E}[\langle DF, u\rangle_{\ell^2(\mathbb{N})}] = \mathbb{E}[F\delta(u)], \quad F \in \mathcal{S}, \ u \in \mathcal{U}.$$

Proof. We consider $F = J_n(f_n)$ and $u_k = J_m(g_{m+1}(*, k))$, $k \in \mathbb{N}$, where $f_n \in \ell^2(\mathbb{N})^{\circ n}$ and $g_{m+1} \in \ell^2(\mathbb{N})^{\circ m} \otimes \ell^2(\mathbb{N})$. We have

$$\begin{aligned}
&\mathbb{E}[\langle D.J_n(f_n), J_m(g_{m+1}(*, \cdot))\rangle_{\ell^2(\mathbb{N})}] \\
&= n\mathbb{E}[\langle J_{n-1}(f_n(*, \cdot)), J_m(g_m(*, \cdot))\rangle_{\ell^2(\mathbb{N})}] \\
&= n\mathbb{E}[\langle J_{n-1}(f_n(*, \cdot)\mathbf{1}_{\Delta_n}(*, \cdot)), J_m(g_m(*, \cdot))\rangle_{\ell^2(\mathbb{N})}] \\
&= n!\mathbf{1}_{\{n-1=m\}} \sum_{k=0}^{\infty} \mathbb{E}[J_{n-1}(f_n(*, k)\mathbf{1}_{\Delta_n}(*, k))J_m(g_{m+1}(*, k))] \\
&= n\mathbf{1}_{\{n-1=m\}} \sum_{k=0}^{\infty} \langle \mathbf{1}_{\Delta_n}(*, k)f_n(*, k), g_{m+1}(*, k)\rangle_{\ell^2(\mathbb{N}^{n-1})} \\
&= n!\mathbf{1}_{\{n=m+1\}} \langle \mathbf{1}_{\Delta_n} f_n, g_{m+1}\rangle_{\ell^2(\mathbb{N}^n)} \\
&= n!\mathbf{1}_{\{n=m+1\}} \langle \mathbf{1}_{\Delta_n} f_n, \tilde{g}_{m+1}\rangle_{\ell^2(\mathbb{N}^n)} \\
&= \mathbb{E}[J_n(f_n)J_m(\tilde{g}_{m+1})] \\
&= \mathbb{E}[F\delta(u)].
\end{aligned}$$

□

The next proposition shows that δ coincides with the stochastic integral operator J on the square-summable predictable processes.

Proposition 1.8.3. *The operator* δ *can be extended to* $u \in L^2(\Omega \times \mathbb{N})$ *with*

$$\delta(u) = \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k - \delta(\varphi D u), \tag{1.8.1}$$

provided all series converges in $L^2(\Omega)$*, where* $(\varphi_k)_{k\in\mathbb{N}}$ *appears in the structure equation* (1.4.1). *We also have for all* $u, v \in \mathcal{U}$*:*

$$\mathbb{E}[|\delta(u)|^2] = \mathbb{E}[\|u\|^2_{\ell^2(\mathbb{N})}] + \mathbb{E}\left[\sum_{k,l=0}^{\infty} D_k u_l D_l u_k\right]. \tag{1.8.2}$$

Proof. Using the expression (1.3.3) of $u_k = J_n(f_{n+1}(*, k))$ we have

$$\begin{aligned}
\delta(u) &= J_{n+1}(\tilde{f}_{n+1}) \\
&= \sum_{(i_1,\ldots,i_{n+1})\in\Delta_{n+1}} \tilde{f}_{n+1}(i_1,\ldots,i_{n+1}) Y_{i_1}\cdots Y_{i_{n+1}} \\
&= \sum_{k=0}^{\infty} \sum_{(i_1,\ldots,i_n)\in\Delta_n} \tilde{f}_{n+1}(i_1,\ldots,i_n,k) Y_{i_1}\cdots Y_{i_n} Y_k \\
&\quad -n \sum_{k=0}^{\infty} \sum_{(i_1,\ldots,i_{n-1})\in\Delta_{n-1}} \tilde{f}_{n+1}(i_1,\ldots,i_{n-1},k,k) Y_{i_1}\cdots Y_{i_{n-1}} |Y_k|^2 \\
&= \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k |Y_k|^2 \\
&= \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k - \sum_{k=0}^{\infty} \varphi_k D_k u_k Y_k.
\end{aligned}$$

By polarization, orthogonality and density it suffices to take $u = g J_n(f^{\circ n})$, $f, g \in \ell^2(\mathbb{N})$, and to note that

$$\begin{aligned}
\|\delta(u)\|^2_{L^2(\Omega)} &= \|J_{n+1}(\mathbf{1}_{\Delta_{n+1}} f^{\circ n} \circ g)\|^2_{L^2(\Omega)} \\
&= \frac{1}{(n+1)^2} \left\| \sum_{i=0}^{n} J_{n+1}(f^{\otimes i} \otimes g \otimes f^{\otimes(n-i)} \mathbf{1}_{\Delta_{n+1}}) \right\|^2_{L^2(\Omega)} \\
&= \frac{1}{(n+1)^2} ((n+1)!(n+1)\|f\|^{2n}_{\ell^2(\mathbb{N})} \|g\|^2_{\ell^2(\mathbb{N})} \\
&\quad +(n+1)!n(n+1)\|f\|^{2n-2}_{\ell^2(\mathbb{N})} \langle f, g\rangle^2_{\ell^2(\mathbb{N})}) \\
&= n!\|f\|^{2n}_{\ell^2(\mathbb{N})} \|g\|^2_{\ell^2(\mathbb{N})} + (n-1)!n^2 \|f\|^{2n-2}_{\ell^2(\mathbb{N})} \langle f, g\rangle^2_{\ell^2(\mathbb{N})} \\
&= \|u\|^2_{L^2(\Omega\times\mathbb{N})} + \mathbb{E}\left[\langle g, DJ_n(f^{\circ n})\rangle_{\ell^2(\mathbb{N})} \langle g, DJ_n(f^{\circ n})\rangle_{\ell^2(\mathbb{N})}\right]
\end{aligned}$$

$$= \|u\|^2_{L^2(\Omega\times\mathbb{N})} + \mathbb{E}\left[\sum_{k,l=0}^{\infty} g(k)g(l)D_l J_n(f^{\circ n})D_k J_n(f^{\circ n})\right]$$

$$= \mathbb{E}[\|u\|^2_{\ell^2(\mathbb{N})}] + \mathbb{E}\left[\sum_{k,l=0}^{\infty} D_k u_l D_l u_k\right].$$

□

From the above argument the Skorohod isometry can also be written as

$$\|\delta(u)\|^2_{L^2(\Omega)} = \mathbb{E}[\|u\|^2_{\ell^2(\mathbb{N})}] + \mathbb{E}\left[\sum_{k,l=0}^{\infty} D_k u_k D_l u_l\right],$$

however this formulation does not lead to a well defined expression in the continuous time limit of Chapter 4.
In particular, (1.8.1) implies the following divergence formula

Corollary 1.8.4. *For* $u \in L^2(\Omega \times \mathbb{N})$ *an* $F \in L^2(\Omega)$ *we have*

$$\delta(Fu) = F\delta(u) - \langle u, DF\rangle_{\ell^2(\mathbb{N})} - \delta(\varphi(\cdot)u(\cdot)D.F), \tag{1.8.3}$$

provided all series converge in $L^2(\Omega)$*.*

In the symmetric case $p_k = q_k = 1/2$ we have $\varphi_k = 0$, $k \in \mathbb{N}$, and

$$\delta(u) = \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k.$$

Moreover, (1.8.2) can be rewritten as a Weitzenböck type identity, cf. Section 7.6 for details:

$$\|\delta(u)\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{k,l=0}^{\infty} \|D_k u(l) - D_l u(k)\|^2_{L^2(\Omega)} = \|u\|^2_{L^2(\Omega\times\mathbb{N})} + \|Du\|^2_{L^2(\Omega\times\mathbb{N}^2)}. \tag{1.8.4}$$

The last two terms in the right hand side of (1.8.1) vanish when $(u_k)_{k\in\mathbb{N}}$ is predictable, and in this case the Skorohod isometry (1.8.2) becomes the Itô isometry as shown in the next proposition.

Corollary 1.8.5. *If* $(u_k)_{k\in\mathbb{N}}$ *satisfies* $D_k u_k = 0$*, i.e.* u_k *does not depend on* X_k*,* $k \in \mathbb{N}$*, then* $\delta(u)$ *coincides with the (discrete time) stochastic integral*

$$\delta(u) = \sum_{k=0}^{\infty} Y_k u_k, \tag{1.8.5}$$

provided the series converges in $L^2(\Omega)$. *If moreover* $(u_k)_{k\in\mathbb{N}}$ *is predictable and square-summable we have the isometry*

$$\mathbb{E}[\delta(u)^2] = \mathbb{E}\left[\|u\|^2_{\ell^2(\mathbb{N})}\right], \tag{1.8.6}$$

and $\delta(u)$ *coincides with* $J(u)$ *on the space of predictable square-summable processes.*

1.9 Ornstein-Uhlenbeck Semi-Group and Process

The Ornstein-Uhlenbeck operator L is defined as $L = \delta D$, i.e. L satisfies

$$LJ_n(f_n) = nJ_n(f_n), \qquad f_n \in \ell^2(\mathbb{N})^{\circ n}.$$

Proposition 1.9.1. *For any* $F \in \mathcal{S}$ *we have*

$$LF = \delta DF = \sum_{k=0}^{\infty} Y_k(D_k F) = \sum_{k=0}^{\infty} \sqrt{p_k q_k} Y_k(F_k^+ - F_k^-),$$

Proof. Note that $D_k D_k F = 0$, $k \in \mathbb{N}$, and use Relation (1.8.1) of Proposition 1.8.3. □

Note that L can be expressed in other forms, for example

$$LF = \sum_{k=0}^{\infty} \Delta_k F,$$

where

$$\begin{aligned}\Delta_k F &= (\mathbf{1}_{\{X_k=1\}} q_k (F(\omega) - F(\omega_-^k)) - \mathbf{1}_{\{X_k=-1\}} p_k (F(\omega_+^k) - F(\omega)))\\ &= F - (\mathbf{1}_{\{X_k=1\}} q_k F(\omega_-^k) + \mathbf{1}_{\{X_k=-1\}} p_k F(\omega_+^k))\\ &= F - \mathbb{E}[F \mid \mathcal{F}_k^c], \quad k \in \mathbb{N},\end{aligned}$$

and $\mathcal{F}_k^c$ is the σ-algebra generated by

$$\{X_l \ : \ l \neq k, \ l \in \mathbb{N}\}.$$

Let now $(P_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{tL})_{t\in\mathbb{R}_+}$ denote the semi-group associated to L and defined as

$$P_t F = \sum_{n=0}^{\infty} \mathrm{e}^{-nt} J_n(f_n), \qquad t \in \mathbb{R}_+,$$

on $F = \sum_{n=0}^{\infty} J_n(f_n) \in L^2(\Omega)$. The next result shows that $(P_t)_{t\in\mathbb{R}_+}$ admits an integral representation by a probability kernel. Let $q_t^N : \Omega \times \Omega \to \mathbb{R}_+$ be defined by

$$q_t^N(\tilde{\omega}, \omega) = \prod_{i=0}^{N} (1 + \mathrm{e}^{-t} Y_i(\omega) Y_i(\tilde{\omega})), \qquad \omega, \tilde{\omega} \in \Omega, \quad t \in \mathbb{R}_+.$$

Lemma 1.9.2. *Let the probability kernel $Q_t(\tilde{\omega}, d\omega)$ be defined by*

$$\mathbb{E}\left[\frac{dQ_t(\tilde{\omega}, \cdot)}{d\mathbb{P}} \Big| \mathcal{F}_N \right](\omega) = q_t^N(\tilde{\omega}, \omega), \qquad N \geq 1, \quad t \in \mathbb{R}_+.$$

For $F \in L^2(\Omega, \mathcal{F}_N)$ we have

$$P_t F(\tilde{\omega}) = \int_{\Omega} F(\omega) Q_t(\tilde{\omega}, d\omega), \qquad \tilde{\omega} \in \Omega, \quad n \geq N. \tag{1.9.1}$$

Proof. Since $L^2(\Omega, \mathcal{F}_N)$ has finite dimension 2^{N+1}, it suffices to consider functionals of the form $F = Y_{k_1} \cdots Y_{k_n}$ with $0 \leq k_1 < \cdots < k_n \leq N$. By Relation (1.4.5) we have for $\omega \in \Omega$, $k \in \mathbb{N}$:

$$\begin{aligned}
&\mathbb{E}\left[Y_k(\cdot)(1 + \mathrm{e}^{-t} Y_k(\cdot) Y_k(\omega)) \right] \\
&= p_k \sqrt{\frac{q_k}{p_k}} \left(1 + \mathrm{e}^{-t} \sqrt{\frac{q_k}{p_k}} Y_k(\omega) \right) - q_k \sqrt{\frac{p_k}{q_k}} \left(1 - \mathrm{e}^{-t} \sqrt{\frac{p_k}{q_k}} Y_k(\omega) \right) \\
&= \mathrm{e}^{-t} Y_k(\omega),
\end{aligned}$$

which implies, by independence of the sequence $(X_k)_{k\in\mathbb{N}}$,

$$\begin{aligned}
\mathbb{E}[Y_{k_1} \cdots Y_{k_n} q_t^N(\omega, \cdot)] &= \mathbb{E}\left[Y_{k_1} \cdots Y_{k_n} \prod_{i=1}^{N} (1 + \mathrm{e}^{-t} Y_{k_i}(\omega) Y_{k_i}(\cdot)) \right] \\
&= \prod_{i=1}^{N} \mathbb{E}\left[Y_{k_i}(\cdot)(1 + \mathrm{e}^{-t} Y_{k_i}(\omega) Y_{k_i}(\cdot)) \right] \\
&= \mathrm{e}^{-nt} Y_{k_1}(\omega) \cdots Y_{k_n}(\omega) \\
&= \mathrm{e}^{-nt} J_n(\tilde{\mathbf{1}}_{\{(k_1, \ldots, k_n)\}})(\omega) \\
&= P_t J_n(\tilde{\mathbf{1}}_{\{(k_1, \ldots, k_n)\}})(\omega) \\
&= P_t(Y_{k_1} \cdots Y_{k_n})(\omega).
\end{aligned}$$

□

Consider the Ω-valued stationary process

$$(X(t))_{t\in\mathbb{R}_+} = ((X_k(t))_{k\in\mathbb{N}})_{t\in\mathbb{R}_+}$$

with independent components and distribution given by

$$\mathbb{P}(X_k(t) = 1 \mid X_k(0) = 1) = p_k + e^{-t} q_k, \tag{1.9.2}$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0) = 1) = q_k - e^{-t} q_k, \tag{1.9.3}$$

$$\mathbb{P}(X_k(t) = 1 \mid X_k(0) = -1) = p_k - e^{-t} p_k, \tag{1.9.4}$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0) = -1) = q_k + e^{-t} p_k, \tag{1.9.5}$$

$k \in \mathbb{N}$, $t \in \mathbb{R}_+$.

Proposition 1.9.3. *The process* $(X(t))_{t\in\mathbb{R}_+} = ((X_k(t))_{k\in\mathbb{N}})_{t\in\mathbb{R}_+}$ *is the Ornstein-Uhlenbeck process associated to* $(P_t)_{t\in\mathbb{R}_+}$*, i.e. we have*

$$P_t F = \mathbb{E}[F(X(t)) \mid X(0)], \qquad t \in \mathbb{R}_+, \tag{1.9.6}$$

for F *bounded and* $\mathcal{F}_n$*-measurable on* Ω*,* $n \in \mathbb{N}$*.*

Proof. By construction of $(X(t))_{t\in\mathbb{R}_+}$ in Relations (1.9.2)-(1.9.5) we have

$$\mathbb{P}(X_k(t) = 1 \mid X_k(0)) = p_k \left(1 + e^{-t} Y_k(0) \sqrt{\frac{q_k}{p_k}}\right),$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0)) = q_k \left(1 - e^{-t} Y_k(0) \sqrt{\frac{p_k}{q_k}}\right),$$

where $Y_k(0)$ is defined by (1.4.6), i.e.

$$Y_k(0) = \frac{q_k - p_k + X_k(0)}{2\sqrt{p_k q_k}}, \qquad k \in \mathbb{N},$$

thus

$$d\mathbb{P}(X_k(t)(\tilde{\omega}) = \epsilon \mid X(0))(\omega) = \left(1 + e^{-t} Y_k(\omega) Y_k(\tilde{\omega})\right) d\mathbb{P}(X_k(\tilde{\omega}) = \epsilon),$$

$\varepsilon = \pm 1$. Since the components of $(X_k(t))_{k\in\mathbb{N}}$ are independent, this shows that the law of $(X_0(t), \ldots, X_n(t))$ conditionally to $X(0)$ has the density $q_t^n(\tilde{\omega}, \cdot)$ with respect to $\mathbb{P}$:

$$\begin{aligned} d\mathbb{P}(X_0(t)(\tilde{\omega}) = \epsilon_0, \ldots, X_n(t)(\tilde{\omega}) = \epsilon_n \mid X(0))(\tilde{\omega}) \\ = q_t^n(\tilde{\omega}, \omega) d\mathbb{P}(X_0(\tilde{\omega}) = \epsilon_0, \ldots, X_n(\tilde{\omega}) = \epsilon_n). \end{aligned}$$

Consequently we have

$$\mathbb{E}[F(X(t)) \mid X(0) = \tilde{\omega}] = \int_\Omega F(\omega) q_t^N(\tilde{\omega}, \omega) \mathbb{P}(d\omega), \tag{1.9.7}$$

hence from (1.9.1), Relation (1.9.6) holds for $F \in L^2(\Omega, \mathcal{F}_N)$, $N \geq 0$. □

The independent components $X_k(t)$, $k \in \mathbb{N}$, can be constructed from the data of $X_k(0) = \epsilon$ and an independent exponential random variable τ_k via the following procedure. If $\tau_k > t$, let $X_k(t) = X_k(0) = \epsilon$, otherwise if $\tau_k < t$, take $X_k(t)$ to be an independent copy of $X_k(0)$. This procedure is illustrated in the following equalities:

$$\begin{aligned} \mathbb{P}(X_k(t) = 1 \mid X_k(0) = 1) &= \mathbb{E}[\mathbf{1}_{\{\tau_k > t\}}] + \mathbb{E}[\mathbf{1}_{\{\tau_k < t\}} \mathbf{1}_{\{X_k = 1\}}] \\ &= \mathrm{e}^{-t} + p_k(1 - \mathrm{e}^{-t}), \end{aligned} \tag{1.9.8}$$

$$\begin{aligned} \mathbb{P}(X_k(t) = -1 \mid X_k(0) = 1) &= \mathbb{E}[\mathbf{1}_{\{\tau_k < t\}} \mathbf{1}_{\{X_k = -1\}}] \\ &= q_k(1 - \mathrm{e}^{-t}), \end{aligned} \tag{1.9.9}$$

$$\begin{aligned} \mathbb{P}(X_k(t) = -1 \mid X_k(0) = -1) &= \mathbb{E}[\mathbf{1}_{\{\tau_k > t\}}] + \mathbb{E}[\mathbf{1}_{\{\tau_k < t\}} \mathbf{1}_{\{X_k = -1\}}] \\ &= \mathrm{e}^{-t} + q_k(1 - \mathrm{e}^{-t}), \end{aligned} \tag{1.9.10}$$

$$\begin{aligned} \mathbb{P}(X_k(t) = 1 \mid X_k(0) = -1) &= \mathbb{E}[\mathbf{1}_{\{\tau_k < t\}} \mathbf{1}_{\{X_k = 1\}}] \\ &= p_k(1 - \mathrm{e}^{-t}). \end{aligned} \tag{1.9.11}$$

The operator $L^2(\Omega \times \mathbb{N}) \to L^2(\Omega \times \mathbb{N})$ which maps $(u_k)_{k\in\mathbb{N}}$ to $(P_t u_k)_{k\in\mathbb{N}}$ is also denoted by P_t. As a consequence of the representation of P_t given in Lemma 1.9.2 we obtain the following bound.

Lemma 1.9.4. *For $F \in \mathrm{Dom}\,(D)$ we have*

$$\|P_t u\|_{L^\infty(\Omega, \ell^2(\mathbb{N}))} \leq \|u\|_{L^\infty(\Omega, \ell^2(\mathbb{N}))}, \qquad t \in \mathbb{R}_+, \quad u \in L^2(\Omega \times \mathbb{N}).$$

Proof. As a consequence of the representation formula (1.9.7) we have $\mathbb{P}(d\tilde{\omega})$-a.s.:

$$\begin{aligned} \|P_t u\|^2_{\ell^2(\mathbb{N})}(\tilde{\omega}) &= \sum_{k=0}^\infty |P_t u_k(\tilde{\omega})|^2 \\ &= \sum_{k=0}^\infty \left(\int_\Omega u_k(\omega) Q_t(\tilde{\omega}, d\omega) \right)^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{\infty} \int_{\Omega} |u_k(\omega)|^2 Q_t(\tilde{\omega}, d\omega) \\
&= \int_{\Omega} \|u\|^2_{\ell^2(\mathbb{N})}(\omega) Q_t(\tilde{\omega}, d\omega) \\
&\leq \|u\|^2_{L^\infty(\Omega, \ell^2(\mathbb{N}))}.
\end{aligned}
$$

□

1.10 Covariance Identities

In this section we state the covariance identities which will be used for the proof of deviation inequalities in the next section. The covariance $\mathrm{Cov}\,(F, G)$ of $F, G \in L^2(\Omega)$ is defined as

$$
\begin{aligned}
\mathrm{Cov}\,(F, G) &= \mathbb{E}[(F - \mathbb{E}[F])(G - \mathbb{E}[G])] \\
&= \mathbb{E}[FG] - \mathbb{E}[F]\mathbb{E}[G].
\end{aligned}
$$

Proposition 1.10.1. *For all $F, G \in L^2(\Omega)$ such that $\mathbb{E}[\|DF\|^2_{\ell^2(\mathbb{N})}] < \infty$ we have*

$$
\mathrm{Cov}\,(F, G) = \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}\left[D_k G \mid \mathcal{F}_{k-1}\right] D_k F\right]. \tag{1.10.1}
$$

Proof. This identity is a consequence of the Clark formula (1.7.1):

$$
\begin{aligned}
\mathrm{Cov}\,(F, G) &= \mathbb{E}[(F - \mathbb{E}[F])(G - \mathbb{E}[G])] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k\right)\left(\sum_{l=0}^{\infty} \mathbb{E}[D_l G \mid \mathcal{F}_{l-1}] Y_l\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] \mathbb{E}[D_k G \mid \mathcal{F}_{k-1}]\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{E}[\mathbb{E}[D_k G \mid \mathcal{F}_{k-1}] D_k F \mid \mathcal{F}_{k-1}]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}[D_k G \mid \mathcal{F}_{k-1}] D_k F\right],
\end{aligned}
$$

and of its extension to $G \in L^2(\Omega)$ in Corollary 1.7.3. □

A covariance identity can also be obtained using the semi-group $(P_t)_{t \in \mathbb{R}_+}$.

Proposition 1.10.2. *For any* $F, G \in L^2(\Omega)$ *such that*

$$\mathbb{E}\left[\|DF\|^2_{\ell^2(\mathbb{N})}\right] < \infty \qquad \textit{and} \qquad \mathbb{E}\left[\|DG\|^2_{\ell^2(\mathbb{N})}\right] < \infty,$$

we have

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\sum_{k=0}^{\infty}\int_0^{\infty} \mathrm{e}^{-t}(D_kF)P_tD_kGdt\right]. \tag{1.10.2}$$

Proof. Consider $F = J_n(f_n)$ and $G = J_m(g_m)$. We have

$$\begin{aligned}
&\mathrm{Cov}\,(J_n(f_n), J_m(g_m)) = \mathbb{E}\,[J_n(f_n)J_m(g_m)]\\
&= \mathbf{1}_{\{n=m\}} n! \langle f_n, g_n \mathbf{1}_{\Delta_n}\rangle_{\ell^2(\mathbb{N}^n)}\\
&= \mathbf{1}_{\{n=m\}} n! n \int_0^{\infty} \mathrm{e}^{-nt} dt \langle f_n, g_n \mathbf{1}_{\Delta_n}\rangle_{\ell^2(\mathbb{N}^n)}\\
&= \mathbf{1}_{\{n-1=m-1\}} n! n \int_0^{\infty} \mathrm{e}^{-t} \sum_{k=0}^{\infty} \langle f_n(*,k), \mathrm{e}^{-(n-1)t} g_n(*,k)\mathbf{1}_{\Delta_n}(*,k)\rangle_{\ell^2(\mathbb{N}^{n-1})} dt\\
&= nm\mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-t} \sum_{k=0}^{\infty} J_{n-1}(f_n(*,k)\mathbf{1}_{\Delta_n}(*,k)) \mathrm{e}^{-(m-1)t} J_{m-1}(g_m(*,k)\mathbf{1}_{\Delta_m}(*,k)) dt\right]\\
&= nm\mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-t} \sum_{k=0}^{\infty} J_{n-1}(f_n(*,k)\mathbf{1}_{\Delta_n}(*,k)) P_t J_{m-1}(g_m(*,k)\mathbf{1}_{\Delta_m}(*,k)) dt\right]\\
&= \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-t} \sum_{k=0}^{\infty} D_k J_n(f_n) P_t D_k J_m(g_m) dt\right].
\end{aligned}$$

□

By the relations (1.9.8)-(1.9.11) the covariance identity (1.10.2) shows that

$$\begin{aligned}
&\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\sum_{k=0}^{\infty}\int_0^{\infty} \mathrm{e}^{-t} D_kFP_tD_kGdt\right]\\
&= \mathbb{E}\left[\int_0^1 \sum_{k=0}^{\infty} D_kFP_{(-\log\alpha)}D_kGd\alpha\right]\\
&= \int_0^1 \int_{\Omega\times\Omega} \sum_{k=0}^{\infty} D_kF(\omega)D_kG((\omega_i \mathbf{1}_{\{\tau_i<-\log\alpha\}} + \omega_i' \mathbf{1}_{\{\tau_i<-\log\alpha\}})_{i\in\mathbb{N}}) d\alpha \mathbb{P}(d\omega)\mathbb{P}(d\omega')\\
&= \int_0^1 \int_{\Omega\times\Omega} \sum_{k=0}^{\infty} D_kF(\omega)D_kG((\omega_i \mathbf{1}_{\{\xi_i<\alpha\}} + \omega_i' \mathbf{1}_{\{\xi_i>\alpha\}})_{i\in\mathbb{N}}) \mathbb{P}(d\omega)\mathbb{P}(d\omega')d\alpha,
\end{aligned} \tag{1.10.3}$$

where $(\xi_i)_{i\in\mathbb{N}}$ is a family of independent identically distributed (i.i.d.) random variables, uniformly distributed on $[0,1]$. Note that the marginals of

$(X_k, X_k \mathbf{1}_{\{\xi_k<\alpha\}} + X'_k \mathbf{1}_{\{\xi_i>\alpha\}})$ are identical when X'_k is an independent copy of X_k. Letting

$$\phi_\alpha(s,t) = \mathbb{E}\left[\mathrm{e}^{isX_k} \mathrm{e}^{it(X_k + \mathbf{1}_{\{\xi_k<\alpha\}}) + it(X'_k + \mathbf{1}_{\{\xi_k>\alpha\}})} \right],$$

we have the relation

$$\begin{aligned} \mathrm{Cov}\,(\mathrm{e}^{isX_k}, \mathrm{e}^{itX_k}) &= \phi_1(s,t) - \phi_0(s,t) \\ &= \int_0^1 \frac{d\phi_\alpha}{d\alpha}(s,t) d\alpha. \end{aligned}$$

Next we prove an iterated version of the covariance identity in discrete time, which is an analog of a result proved in [56] for the Wiener and Poisson processes.

Theorem 1.10.3. *Let $n \in \mathbb{N}$ and $F, G \in L^2(\Omega)$. We have*

$$\begin{aligned} &\mathrm{Cov}\,(F,G) \qquad (1.10.4) \\ &= \sum_{d=1}^n (-1)^{d+1} \mathbb{E}\left[\sum_{\{1\le k_1<\cdots<k_d\}} (D_{k_d}\cdots D_{k_1}F)(D_{k_d}\cdots D_{k_1}G) \right] \\ &\quad + (-1)^n \mathbb{E}\left[\sum_{\{1\le k_1<\cdots<k_{n+1}\}} (D_{k_{n+1}}\cdots D_{k_1}F) \mathbb{E}\left[D_{k_{n+1}}\cdots D_{k_1}G \mid \mathcal{F}_{k_{n+1}-1} \right] \right]. \end{aligned}$$

Proof. Take $F = G$. For $n = 0$, (1.10.4) is a consequence of the Clark formula. Let $n \ge 1$. Applying Lemma 1.7.4 to $D_{k_n}\cdots D_{k_1}F$ with $a = k_n$ and $b = k_{n+1}$, and summing on $(k_1, \ldots, k_n) \in \Delta_n$, we obtain

$$\begin{aligned} &\mathbb{E}\left[\sum_{\{1\le k_1<\cdots<k_n\}} (\mathbb{E}[D_{k_n}\cdots D_{k_1}F \mid \mathcal{F}_{k_n-1}])^2 \right] \\ &= \mathbb{E}\left[\sum_{\{1\le k_1<\cdots<k_n\}} \mid D_{k_n}\cdots D_{k_1}F|^2 \right] \\ &\quad - \mathbb{E}\left[\sum_{\{1\le k_1<\cdots<k_{n+1}\}} \left(\mathbb{E}\left[D_{k_{n+1}}\cdots D_{k_1}F \mid \mathcal{F}_{k_{n+1}-1} \right] \right)^2 \right], \end{aligned}$$

which concludes the proof by induction and polarization. □

As a consequence of Theorem 1.10.3, letting $F = G$ we get the variance inequality

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} \mathbb{E}\left[\|D^k F\|^2_{\ell^2(\Delta_k)}\right] \leq \mathrm{Var}\,(F) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} \mathbb{E}\left[\|D^k F\|^2_{\ell^2(\Delta_k)}\right],$$

since

$$\begin{aligned}
&\mathbb{E}\left[\sum_{\{1\leq k_1<\cdots<k_{n+1}\}} (D_{k_{n+1}}\cdots D_{k_1}F)\mathbb{E}\left[D_{k_{n+1}}\cdots D_{k_1}F \mid \mathcal{F}_{k_{n+1}-1}\right]\right] \\
&= \mathbb{E}\left[\sum_{\{1\leq k_1<\cdots<k_{n+1}\}} \right. \\
&\qquad \left. \mathbb{E}\left[(D_{k_{n+1}}\cdots D_{k_1}F)\mathbb{E}\left[D_{k_{n+1}}\cdots D_{k_1}F \mid \mathcal{F}_{k_{n+1}-1}\right] \mid \mathcal{F}_{k_{n+1}-1}\right]\right] \\
&= \mathbb{E}\left[\sum_{\{1\leq k_1<\cdots<k_{n+1}\}} \left(\mathbb{E}\left[D_{k_{n+1}}\cdots D_{k_1}F \mid \mathcal{F}_{k_{n+1}-1}\right]\right)^2\right] \\
&\geq 0,
\end{aligned}$$

see Relation (2.15) in [56] in continuous time. In a similar way, another iterated covariance identity can be obtained from Proposition 1.10.2.

Corollary 1.10.4. *Let $n \in \mathbb{N}$ and $F, G \in L^2(\Omega, \mathcal{F}_N)$. We have*

$$\begin{aligned}
\mathrm{Cov}\,(F,G) &= \sum_{d=1}^{n} (-1)^{d+1} \mathbb{E}\left[\sum_{\{1\leq k_1<\cdots<k_d\leq N\}} (D_{k_d}\cdots D_{k_1}F)(D_{k_d}\cdots D_{k_1}G)\right] \\
&\quad +(-1)^n \int_{\Omega\times\Omega} \sum_{\{1\leq k_1<\cdots<k_{n+1}\leq N\}} D_{k_{n+1}}\cdots D_{k_1}F(\omega) D_{k_{n+1}}\cdots D_{k_1}G(\omega') \\
&\quad q_t^N(\omega,\omega')\mathbb{P}(d\omega)\mathbb{P}(d\omega'). \qquad (1.10.5)
\end{aligned}$$

Using the tensorization property

$$\begin{aligned}
\mathrm{Var}\,(FG) &= \mathbb{E}[F^2]\mathrm{Var}\,(G) + (\mathbb{E}[G])^2 \mathrm{Var}\,(F) \\
&\leq \mathbb{E}[F^2 \mathrm{Var}\,(G)] + \mathbb{E}[G^2 \mathrm{Var}\,(F)]
\end{aligned}$$

of the variance for independent random variable F, G, most of the identities in this section can be obtained by tensorization of elementary one dimensional covariance identities.

The following lemma is an elementary consequence of the covariance identity proved in Proposition 1.10.1.

Lemma 1.10.5. *Let $F, G \in L^2(\Omega)$ such that*

$$\mathbb{E}[D_k F|\mathcal{F}_{k-1}] \cdot \mathbb{E}[D_k G|\mathcal{F}_{k-1}] \geq 0, \qquad k \in \mathbb{N}.$$

Then F and G are non-negatively correlated:

$$\mathrm{Cov}\,(F, G) \geq 0.$$

According to the next definition, a non-decreasing functional F satisfies $D_k F \geq 0$ for all $k \in \mathbb{N}$.

Definition 1.10.6. *A random variable $F : \Omega \to \mathbb{R}$ is said to be non-decreasing if for all $\omega_1, \omega_2 \in \Omega$ we have*

$$\omega_1(k) \leq \omega_2(k), \qquad k \in \mathbb{N}, \qquad \Rightarrow\; F(\omega_1) \leq F(\omega_2).$$

The following result is then immediate from Proposition 1.6.2 and Lemma 1.10.5, and shows that the FKG inequality holds on Ω. It can also be obtained from Proposition 1.10.2.

Proposition 1.10.7. *If $F, G \in L^2(\Omega)$ are non-decreasing then F and G are non-negatively correlated:*

$$\mathrm{Cov}\,(F, G) \geq 0.$$

Note however that the assumptions of Lemma 1.10.5 are actually weaker as they do not require F and G to be non-decreasing.

1.11 Deviation Inequalities

In this section, which is based on [59], we recover a deviation inequality of [19] in the case of Bernoulli measures, using covariance representations instead of the logarithmic Sobolev inequalities to be presented in Section 1.12. The method relies on a bound on the Laplace transform $L(t) = \mathbb{E}[\mathrm{e}^{tF}]$ obtained via a differential inequality and Chebychev's inequality.

Proposition 1.11.1. *Let $F \in L^1(\Omega)$ be such that $|F_k^+ - F_k^-| \leq K$, $k \in \mathbb{N}$, for some $K \geq 0$, and $\|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} < \infty$. Then*

$$\begin{aligned}\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-\frac{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}{K^2}\, g\left(\frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}\right)\right)\\ &\leq \exp\left(-\frac{x}{2K}\log\left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}\right)\right),\end{aligned}$$

with $g(u) = (1+u)\log(1+u) - u$, $u \geq 0$.

Proof. Although D_k does not satisfy a derivation rule for products, from Proposition 1.6.4 we have

$$\begin{aligned} D_k \mathrm{e}^F &= \mathbf{1}_{\{X_k=1\}}\sqrt{p_kq_k}(\mathrm{e}^F - \mathrm{e}^{F_k^-}) + \mathbf{1}_{\{X_k=-1\}}\sqrt{p_kq_k}(\mathrm{e}^{F_k^+} - \mathrm{e}^F) \\ &= \mathbf{1}_{\{X_k=1\}}\sqrt{p_kq_k}\mathrm{e}^F(1-\mathrm{e}^{-\frac{1}{\sqrt{p_kq_k}}D_kF}) + \mathbf{1}_{\{X_k=-1\}}\sqrt{p_kq_k}\mathrm{e}^F(\mathrm{e}^{\frac{1}{\sqrt{p_kq_k}}D_kF}-1) \\ &= -X_k\sqrt{p_kq_k}\mathrm{e}^F(\mathrm{e}^{-\frac{X_k}{\sqrt{p_kq_k}}D_kF}-1), \end{aligned}$$

hence

$$D_k \mathrm{e}^F = X_k\sqrt{p_kq_k}\mathrm{e}^F(1-\mathrm{e}^{-\frac{X_k}{\sqrt{p_kq_k}}D_kF}), \tag{1.11.1}$$

and since the function $x \mapsto (\mathrm{e}^x - 1)/x$ is positive and increasing on $\mathbb{R}$ we have:

$$\begin{aligned} \frac{\mathrm{e}^{-sF}D_k\mathrm{e}^{sF}}{D_kF} &= -\frac{X_k\sqrt{p_kq_k}}{D_kF}\left(\mathrm{e}^{-s\frac{X_k}{\sqrt{p_kq_k}}D_kF}-1\right) \\ &\leq \frac{\mathrm{e}^{sK}-1}{K}, \end{aligned}$$

or in other terms:

$$\begin{aligned} \frac{\mathrm{e}^{-sF}D_k\mathrm{e}^{sF}}{D_kF} &= \mathbf{1}_{\{X_k=1\}}\frac{\mathrm{e}^{sF_k^- - F_k^+}-1}{F_k^- - F_k^+} + \mathbf{1}_{\{X_k=-1\}}\frac{\mathrm{e}^{sF_k^+ - F_k^-}-1}{F_k^+ - F_k^-} \\ &\leq \frac{\mathrm{e}^{sK}-1}{K}. \end{aligned}$$

We first assume that F is a bounded random variable with $\mathbb{E}[F] = 0$. From Proposition 1.10.2 applied to F and e^{sF}, noting that since F is bounded,

$$\begin{aligned} \mathbb{E}\left[\|De^{sF}\|^2_{\ell^2(\mathbb{N})}\right] &\leq C_K\mathbb{E}[e^{2sF}]\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \\ &< \infty, \end{aligned}$$

for some $C_K > 0$, we have

$$\begin{aligned} \mathbb{E}[F\mathrm{e}^{sF}] &= \mathrm{Cov}\,(F, \mathrm{e}^{sF}) \\ &= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-v}\sum_{k=0}^\infty D_k\mathrm{e}^{sF}P_vD_kFdv\right] \\ &\leq \left\|\frac{\mathrm{e}^{-sF}D\mathrm{e}^{sF}}{DF}\right\|_\infty \mathbb{E}\left[\mathrm{e}^{sF}\int_0^\infty \mathrm{e}^{-v}\|DFP_vDF\|_{\ell^1(\mathbb{N})}dv\right] \end{aligned}$$

$$
\begin{aligned}
&\le \frac{e^{sK}-1}{K}\mathbb{E}\left[e^{sF}\|DF\|_{\ell^2(\mathbb{N})}\int_0^\infty e^{-v}\|P_vDF\|_{\ell^2(\mathbb{N})}dv\right]\\
&\le \frac{e^{sK}-1}{K}\mathbb{E}\left[e^{sF}\right]\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}\int_0^\infty e^{-v}dv\\
&\le \frac{e^{sK}-1}{K}\mathbb{E}\left[e^{sF}\right]\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))},
\end{aligned}
$$

where we also applied Lemma 1.9.4 to $u = DF$.
In the general case, letting $L(s) = \mathbb{E}[e^{s(F-\mathbb{E}[F])}]$, we have

$$
\begin{aligned}
\log(\mathbb{E}[e^{t(F-\mathbb{E}[F])}]) &= \int_0^t \frac{L'(s)}{L(s)}ds\\
&\le \int_0^t \frac{\mathbb{E}[(F-\mathbb{E}[F])e^{s(F-\mathbb{E}[F])}]}{\mathbb{E}[e^{s(F-\mathbb{E}[F])}]}ds\\
&\le \frac{1}{K}\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}\int_0^t (e^{sK}-1)ds\\
&= \frac{1}{K^2}(e^{tK}-tK-1)\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))},
\end{aligned}
$$

$t \ge 0$. We have for all $x \ge 0$ and $t \ge 0$:

$$
\begin{aligned}
\mathbb{P}(F-\mathbb{E}[F]\ge x) &\le e^{-tx}\mathbb{E}[e^{t(F-\mathbb{E}[F])}]\\
&\le \exp\left(\frac{1}{K^2}(e^{tK}-tK-1)\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}-tx\right),
\end{aligned}
$$

The minimum in $t \ge 0$ in the above expression is attained with

$$
t = \frac{1}{K}\log\left(1+\frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}\right),
$$

hence

$$
\begin{aligned}
&\mathbb{P}(F-\mathbb{E}[F]\ge x)\\
&\le \exp\left(-\frac{1}{K}\left(x+\frac{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}{K}\right)\log\left(1+\frac{Kx}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}\right)-\frac{x}{K}\right)\\
&\le \exp\left(-\frac{x}{2K}\log\left(1+xK\|DF\|^{-2}_{L^\infty(\Omega,\ell^2(\mathbb{N}))}\right)\right),
\end{aligned}
$$

where we used the inequality

$$\frac{u}{2}\log(1+u) \leq (1+u)\log(1+u) - u, \qquad u \in \mathbb{R}_+.$$

If $K = 0$, the above proof is still valid by replacing all terms by their limits as $K \to 0$. Finally if F is not bounded the conclusion holds for $F_n = \max(-n, \min(F, n))$, $n \geq 1$, and $(F_n)_{n\in\mathbb{N}}$, $(DF_n)_{n\in\mathbb{N}}$, converge respectively almost surely and in $L^2(\Omega \times \mathbb{N})$ to F and DF, with $\|DF_n\|^2_{L^\infty(\Omega,L^2(\mathbb{N}))} \leq \|DF\|^2_{L^\infty(\Omega,L^2(\mathbb{N}))}$. □

In case $p_k = p$ for all $k \in \mathbb{N}$, the conditions

$$|D_k F| \leq \beta, \quad k \in \mathbb{N}, \quad \text{and} \quad \|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \leq \alpha^2,$$

give

$$\begin{aligned}\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-\frac{\alpha^2 pq}{\beta^2} g\left(\frac{x\beta}{\alpha^2\sqrt{pq}}\right)\right)\\ &\leq \exp\left(-\frac{x\sqrt{pq}}{2\beta}\log\left(1+\frac{x\beta}{\alpha^2\sqrt{pq}}\right)\right),\end{aligned}$$

which is Relation (13) in [19]. In particular if F is $\mathcal{F}_N$-measurable, then

$$\begin{aligned}\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-Ng\left(\frac{x\sqrt{pq}}{\beta N}\right)\right)\\ &\leq \exp\left(-\frac{x\sqrt{pq}}{\beta}\left(\log\left(1+\frac{x\sqrt{pq}}{\beta N}\right)-1\right)\right).\end{aligned}$$

Finally we show a Gaussian concentration inequality for functionals of $(S_n)_{n\in\mathbb{N}}$, using the covariance identity (1.10.1). We refer to [17], [18], [61], [75], for other versions of this inequality.

Proposition 1.11.2. *Let $F \in L^1(\Omega)$ be such that*

$$\left\| \sum_{k=0}^{\infty} \frac{1}{2(p_k \wedge q_k)} |D_k F| \|D_k F\|_\infty \right\|_\infty \leq K^2.$$

Then

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2K^2}\right), \qquad x \geq 0. \tag{1.11.2}$$

Proof. Again we assume that F is a bounded random variable with $\mathbb{E}[F] = 0$. Using the inequality

$$|e^{tx} - e^{ty}| \leq \frac{t}{2}|x-y|(e^{tx} + e^{ty}), \qquad x, y \in \mathbb{R}, \tag{1.11.3}$$

we have

$$\begin{aligned}
|D_k e^{tF}| &= \sqrt{p_k q_k} |e^{tF_k^+} - e^{tF_k^-}| \\
&\leq \frac{1}{2}\sqrt{p_k q_k} t |F_k^+ - F_k^-| (e^{tF_k^+} + e^{tF_k^-}) \\
&= \frac{1}{2} t |D_k F| (e^{tF_k^+} + e^{tF_k^-}) \\
&\leq \frac{t}{2(p_k \wedge q_k)} |D_k F| \mathbb{E}\left[e^{tF} \mid X_i, \ i \neq k \right] \\
&= \frac{1}{2(p_k \wedge q_k)} t \mathbb{E}\left[e^{tF} |D_k F| \mid X_i, \ i \neq k \right].
\end{aligned}$$

Now Proposition 1.10.1 yields

$$\begin{aligned}
\mathbb{E}[F e^{tF}] &= \mathrm{Cov}\,(F, e^{sF}) \\
&= \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] D_k e^{tF}] \\
&\leq \sum_{k=0}^{\infty} \|D_k F\|_\infty \mathbb{E}\left[|D_k e^{tF}| \right] \\
&\leq \frac{t}{2} \sum_{k=0}^{\infty} \frac{1}{p_k \wedge q_k} \|D_k F\|_\infty \mathbb{E}\left[\mathbb{E}\left[e^{tF} |D_k F| \mid X_i, \ i \neq k \right] \right] \\
&= \frac{t}{2} \mathbb{E}\left[e^{tF} \sum_{k=0}^{\infty} \frac{1}{p_k \wedge q_k} \|D_k F\|_\infty |D_k F| \right] \\
&\leq \frac{t}{2} \mathbb{E}[e^{tF}] \left\| \sum_{k=0}^{\infty} \frac{1}{p_k \wedge q_k} |D_k F| \|D_k F\|_\infty \right\|_\infty .
\end{aligned}$$

This shows that

$$\begin{aligned}
\log(\mathbb{E}[e^{t(F - \mathbb{E}[F])}]) &= \int_0^t \frac{\mathbb{E}[(F - \mathbb{E}[F]) e^{s(F - \mathbb{E}[F])}]}{\mathbb{E}[e^{s(F - \mathbb{E}[F])}]} ds \\
&\leq K^2 \int_0^t s ds \\
&= \frac{t^2}{2} K^2,
\end{aligned}$$

hence

$$\begin{aligned}
e^x \mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \mathbb{E}[e^{t(F - \mathbb{E}[F])}] \\
&\leq e^{t^2 K^2 / 2}, \qquad t \geq 0,
\end{aligned}$$

and

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \mathrm{e}^{\frac{t^2}{2}K^2 - tx}, \quad t \geq 0.$$

The best inequality is obtained for $t = x/K^2$.
Finally if F is not bounded the conclusion holds for $F_n = \max(-n, \min(F, n))$, $n \geq 0$, and $(F_n)_{n\in\mathbb{N}}$, $(DF_n)_{n\in\mathbb{N}}$, converge respectively to F and DF in $L^2(\Omega)$, resp. $L^2(\Omega \times \mathbb{N})$, with $\|DF_n\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \leq \|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}$. □

In case $p_k = p$, $k \in \mathbb{N}$, we obtain

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{px^2}{\|DF\|^2_{\ell^2(\mathbb{N},L^\infty(\Omega))}}\right).$$

Proposition 1.11.3. *We have* $\mathbb{E}[\mathrm{e}^{\alpha|F|}] < \infty$ *for all* $\alpha > 0$, *and* $\mathbb{E}[\mathrm{e}^{\alpha F^2}] < \infty$ *for all* $\alpha < 1/(2K^2)$.

Proof. Let $\lambda < c/e$. The bound (1.11.2) implies

$$\begin{aligned}
\mathbb{E}\left[e^{\alpha|F|}\right] &= \int_0^\infty \mathbb{P}(e^{\alpha|F|} \geq t)dt \\
&= \int_{-\infty}^\infty \mathbb{P}(\alpha|F| \geq y)e^y dy \\
&\leq 1 + \int_0^\infty \mathbb{P}(\alpha|F| \geq y)e^y dy \\
&\leq 1 + \int_0^\infty \exp\left(-\frac{(\mathbb{E}[|F|] + y/\alpha)^2}{2K^2}\right) e^y dy \\
&< \infty,
\end{aligned}$$

for all $\alpha > 0$. On the other hand we have

$$\begin{aligned}
\mathbb{E}[e^{\alpha F^2}] &= \int_0^\infty \mathbb{P}(e^{\alpha F^2} \geq t)dt \\
&= \int_{-\infty}^\infty \mathbb{P}(\alpha F^2 \geq y)e^y dy \\
&\leq 1 + \int_0^\infty \mathbb{P}(|F| \geq (y/\alpha)^{1/2})e^y dy \\
&\leq 1 + \int_0^\infty \exp\left(-\frac{(\mathbb{E}[|F|] + (y/\alpha)^{1/2})^2}{2K^2}\right) e^y dy \\
&< \infty,
\end{aligned}$$

provided $2K^2\alpha < 1$. □

1.12 Logarithmic Sobolev Inequalities

The logarithmic Sobolev inequalities on Gaussian space provide an infinite dimensional analog of Sobolev inequalities, cf. e.g. [77]. On Riemannian path space [22] and on Poisson space [6], [151], martingale methods have been successfully applied to the proof of logarithmic Sobolev inequalities. Here, discrete time martingale methods are used along with the Clark predictable representation formula (1.7.1) as in [46], to provide a proof of logarithmic Sobolev inequalities for Bernoulli measures. Here we are only concerned with modified logarithmic Sobolev inequalities, and we refer to [127], Theorem 2.2.8 and references therein, for the standard version of the logarithmic Sobolev inequality on the hypercube under Bernoulli measures.

The entropy of a random variable $F > 0$ is defined by

$$\mathrm{Ent}\,[F] = \mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F],$$

for sufficiently integrable F.

Lemma 1.12.1. *The entropy has the tensorization property, i.e. if F, G are sufficiently integrable independent random variables we have*

$$\mathrm{Ent}\,[FG] = \mathbb{E}[F\mathrm{Ent}\,[G]] + \mathbb{E}[G\mathrm{Ent}\,[F]]. \qquad (1.12.1)$$

Proof. We have

$$\begin{aligned}
&\mathrm{Ent}\,[FG] = \mathbb{E}[FG\log(FG)] - \mathbb{E}[FG]\log \mathbb{E}[FG] \\
&= \mathbb{E}[FG(\log F + \log G)] - \mathbb{E}[F]\mathbb{E}[G](\log \mathbb{E}[F] + \log \mathbb{E}[G]) \\
&= \mathbb{E}[G]\mathbb{E}[F\log F] + \mathbb{E}[F]\mathbb{E}[G\log G)] - \mathbb{E}[F]\mathbb{E}[G](\log \mathbb{E}[F] + \log \mathbb{E}[G]) \\
&= \mathbb{E}[F\mathrm{Ent}\,[G]] + \mathbb{E}[G\mathrm{Ent}\,[F]].
\end{aligned}$$

□

In the next proposition we recover the modified logarithmic Sobolev inequality of [19] using the Clark representation formula in discrete time.

Theorem 1.12.2. *Let $F \in \mathrm{Dom}\,(D)$ with $F > \eta$ a.s. for some $\eta > 0$. We have*

$$\mathrm{Ent}\,[F] \le \mathbb{E}\left[\frac{1}{F}\|DF\|^2_{\ell^2(\mathbb{N})}\right]. \qquad (1.12.2)$$

Proof. Assume that F is $\mathcal{F}_N$-measurable and let $M_n = \mathbb{E}[F \mid \mathcal{F}_n]$, $0 \le n \le N$. Using Corollary 1.6.3 and the Clark formula (1.7.1) we have

$$M_n = M_{-1} + \sum_{k=0}^{n} u_k Y_k, \qquad 0 \le n \le N,$$

with $u_k = \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}]$, $0 \le k \le n \le N$, and $M_{-1} = \mathbb{E}[F]$. Letting $f(x) = x\log x$ and using the bound

$$f(x+y) - f(x) = y \log x + (x+y) \log \left(1 + \frac{y}{x}\right)$$
$$\leq y(1 + \log x) + \frac{y^2}{x},$$

we have:

$$\begin{aligned}
\mathrm{Ent}\,[F] &= \mathbb{E}[f(M_N)] - \mathbb{E}[f(M_{-1})] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} f(M_k) - f(M_{k-1})\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} f\left(M_{k-1} + Y_k u_k\right) - f(M_{k-1})\right] \\
&\leq \mathbb{E}\left[\sum_{k=0}^{N} Y_k u_k (1 + \log M_{k-1}) + \frac{Y_k^2 u_k^2}{M_{k-1}}\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} \frac{1}{\mathbb{E}[F \mid \mathcal{F}_{k-1}]} (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])^2\right] \\
&\leq \mathbb{E}\left[\sum_{k=0}^{N} \mathbb{E}\left[\frac{1}{F} |D_k F|^2 \mid \mathcal{F}_{k-1}\right]\right] \\
&= \mathbb{E}\left[\frac{1}{F} \sum_{k=0}^{N} |D_k F|^2\right].
\end{aligned}$$

where we used the Jensen inequality (9.3.1) and the convexity of $(u, v) \mapsto v^2/u$ on $(0, \infty) \times \mathbb{R}$, or the Schwarz inequality applied to

$$1/\sqrt{F} \qquad \text{and} \qquad (D_k F/\sqrt{F})_{k \in \mathbb{N}},$$

as in the Wiener and Poisson cases [22] and [6]. This inequality is extended by density to $F \in \mathrm{Dom}\,(D)$. □

By a one-variable argument, letting $\mathrm{d}f = f(1) - f(-1)$, we have

$$\begin{aligned}
\mathrm{Ent}\,[f] &= pf(1) \log f(1) + qf(-1) \log f(-1) - \mathbb{E}[f] \log \mathbb{E}[f] \\
&= p(\mathbb{E}[f] + q\mathrm{d}f) \log(\mathbb{E}[f] + q\mathrm{d}f) \\
&\quad + q(\mathbb{E}[f] - p\mathrm{d}f) \log(\mathbb{E}[f] - p\mathrm{d}f) - (pf(1) + qf(-1)) \log \mathbb{E}[f] \\
&= p\mathbb{E}[f] \log \left(1 + q\frac{\mathrm{d}f}{\mathbb{E}[f]}\right) + pq\mathrm{d}f \log f(1) \\
&\quad + q\mathbb{E}[f] \log \left(1 - p\frac{\mathrm{d}f}{\mathbb{E}[f]}\right) - qp\mathrm{d}f \log f(-1) \\
&\leq pq\mathrm{d}f \log f(1) - pq\mathrm{d}f \log f(-1) \\
&= pq\mathbb{E}\left[\mathrm{d}f \mathrm{d} \log f\right],
\end{aligned}$$

which, by tensorization, recovers the following L^1 inequality of [47], [29], and proved in [151] in the Poisson case. In the next proposition we state and prove this inequality in the multidimensional case, using the Clark representation formula, similarly to Theorem 1.12.2.

Theorem 1.12.3. *Let $F > 0$ be $\mathcal{F}_N$-measurable. We have*

$$\mathrm{Ent}\,[F] \leq \mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k \log F\right]. \qquad (1.12.3)$$

Proof. Let $f(x) = x \log x$ and

$$\Psi(x,y) = (x+y)\log(x+y) - x\log x - (1+\log x)y, \quad x,\ x+y > 0.$$

From the relation

$$\begin{aligned} Y_k u_k &= Y_k \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] \\ &= q_k \mathbf{1}_{\{X_k=1\}} \mathbb{E}[(F_k^+ - F_k^-) \mid \mathcal{F}_{k-1}] + p_k \mathbf{1}_{\{X_k=-1\}} \mathbb{E}[(F_k^- - F_k^+) \mid \mathcal{F}_{k-1}] \\ &= \mathbf{1}_{\{X_k=1\}} \mathbb{E}[(F_k^+ - F_k^-)\mathbf{1}_{\{X_k=-1\}} \mid \mathcal{F}_{k-1}] \\ &\quad + \mathbf{1}_{\{X_k=-1\}} \mathbb{E}[(F_k^- - F_k^+)\mathbf{1}_{\{X_k=1\}} \mid \mathcal{F}_{k-1}], \end{aligned}$$

we have, using the convexity of Ψ:

$$\begin{aligned} \mathrm{Ent}\,[F] &= \mathbb{E}\left[\sum_{k=0}^{N} f\left(M_{k-1} + Y_k u_k\right) - f(M_{k-1})\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} \Psi(M_{k-1}, Y_k u_k) + Y_k u_k (1 + \log M_{k-1})\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} \Psi(M_{k-1}, Y_k u_k)\right] \\ &= \mathbb{E}\Bigg[\sum_{k=0}^{N} p_k \Psi\left(\mathbb{E}[F \mid \mathcal{F}_{k-1}], \mathbb{E}[(F_k^+ - F_k^-)\mathbf{1}_{\{X_k=-1\}} \mid \mathcal{F}_{k-1}]\right) \\ &\quad + q_k \Psi\left(\mathbb{E}[F \mid \mathcal{F}_{k-1}], \mathbb{E}[(F_k^- - F_k^+)\mathbf{1}_{\{X_k=1\}} \mid \mathcal{F}_{k-1}]\right)\Bigg] \\ &\leq \mathbb{E}\left[\sum_{k=0}^{N} \mathbb{E}\left[p_k \Psi\left(F, (F_k^+ - F_k^-)\mathbf{1}_{\{X_k=-1\}}\right) + q_k \Psi\left(F, (F_k^- - F_k^+)\mathbf{1}_{\{X_k=1\}}\right) \Big| \mathcal{F}_{k-1}\right]\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} p_k \mathbf{1}_{\{X_k=-1\}} \Psi\left(F_k^-, F_k^+ - F_k^-\right) + q_k \mathbf{1}_{\{X_k=1\}} \Psi\left(F_k^+, F_k^- - F_k^+\right)\right] \end{aligned}$$

$$= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k \Psi(F_k^-, F_k^+ - F_k^-) + p_k q_k \Psi(F_k^+, F_k^- - F_k^+)\right]$$
$$= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k (\log F_k^+ - \log F_k^-)(F_k^+ - F_k^-)\right]$$
$$= \mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k \log F\right].$$

□

The application of Theorem 1.12.3 to e^F gives the following inequality for $F > 0$, $\mathcal{F}_N$-measurable:

$$\begin{aligned}\mathrm{Ent}\,[\mathrm{e}^F] &\leq \mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k \mathrm{e}^F\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k \Psi(\mathrm{e}^{F_k^-}, \mathrm{e}^{F_k^+} - \mathrm{e}^{F_k^-}) + p_k q_k \Psi(\mathrm{e}^{F_k^+}, \mathrm{e}^{F_k^-} - \mathrm{e}^{F_k^+})\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k \mathrm{e}^{F_k^-}((F_k^+ - F_k^-)\mathrm{e}^{F_k^+ - F_k^-} - \mathrm{e}^{F_k^+ - F_k^-} + 1)\right. \\ &\qquad \left. + p_k q_k \mathrm{e}^{F_k^+}((F_k^- - F_k^+)\mathrm{e}^{F_k^- - F_k^+} - \mathrm{e}^{F_k^- - F_k^+} + 1)\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} p_k \mathbf{1}_{\{X_k=-1\}} \mathrm{e}^{F_k^-}((F_k^+ - F_k^-)\mathrm{e}^{F_k^+ - F_k^-} - \mathrm{e}^{F_k^+ - F_k^-} + 1)\right. \\ &\qquad \left. + q_k \mathbf{1}_{\{X_k=1\}} \mathrm{e}^{F_k^+}((F_k^- - F_k^+)\mathrm{e}^{F_k^- - F_k^+} - \mathrm{e}^{F_k^- - F_k^+} + 1)\right] \\ &= \mathbb{E}\left[\mathrm{e}^F \sum_{k=0}^{N} \sqrt{p_k q_k}|Y_k|(\nabla_k F \mathrm{e}^{\nabla_k F} - \mathrm{e}^{\nabla_k F} + 1)\right]. \qquad (1.12.4)\end{aligned}$$

This implies

$$\mathrm{Ent}\,[\mathrm{e}^F] \leq \mathbb{E}\left[\mathrm{e}^F \sum_{k=0}^{N} (\nabla_k F \mathrm{e}^{\nabla_k F} - \mathrm{e}^{\nabla_k F} + 1)\right]. \qquad (1.12.5)$$

As noted in [29], Relation (1.12.3) and the Poisson limit theorem yield the L^1 inequality of [151]. More precisely, letting $M_n = (n + X_1 + \cdots + X_n)/2$, $F = \varphi(M_n)$ and $p_k = \lambda/n$, $k \in \mathbb{N}$, , $n \geq 1$, $\lambda > 0$, we have, from Proposition 1.6.2,

$$\sum_{k=0}^{n} D_k F D_k \log F$$
$$= \frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)(n-M_n)(\varphi(M_n+1)-\varphi(M_n))\log(\varphi(M_n+1)-\varphi(M_n))$$
$$+\frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)M_n(\varphi(M_n)-\varphi(M_n-1))\log(\varphi(M_n)-\varphi(M_n-1)),$$

in the limit as n goes to infinity we obtain

$$\mathrm{Ent}\,[\varphi(U)] \le \lambda\mathbb{E}[(\varphi(U+1)-\varphi(U))(\log\varphi(U+1)-\log\varphi(U))],$$

where U is a Poisson random variable with parameter λ. In one variable we have, still letting $\mathrm{d}f = f(1)-f(-1)$,

$$\begin{aligned}
\mathrm{Ent}\,[\mathrm{e}^f] &\le pq\mathbb{E}\left[\mathrm{d}\mathrm{e}^f \mathrm{d}\log\mathrm{e}^f\right]\\
&= pq(\mathrm{e}^{f(1)}-\mathrm{e}^{f(-1)})(f(1)-f(-1))\\
&= pq\mathrm{e}^{f(-1)}((f(1)-f(-1))\mathrm{e}^{f(1)-f(-1)}-\mathrm{e}^{f(1)-f(-1)}+1)\\
&\quad +pq\mathrm{e}^{f(1)}((f(-1)-f(1))\mathrm{e}^{f(-1)-f(1)}-\mathrm{e}^{f(-1)-f(1)}+1)\\
&\le q\mathrm{e}^{f(-1)}((f(1)-f(-1))\mathrm{e}^{f(1)-f(-1)}-\mathrm{e}^{f(1)-f(-1)}+1)\\
&\quad +p\mathrm{e}^{f(1)}((f(-1)-f(1))\mathrm{e}^{f(-1)-f(1)}-\mathrm{e}^{f(-1)-f(1)}+1)\\
&= \mathbb{E}\left[\mathrm{e}^f(\nabla f\mathrm{e}^{\nabla f}-\mathrm{e}^{\nabla f}+1)\right],
\end{aligned}$$

where ∇_k is the gradient operator defined in (1.6.4). This last inequality is not comparable to the optimal constant inequality

$$\mathrm{Ent}\,[\mathrm{e}^F] \le \mathbb{E}\left[\mathrm{e}^F\sum_{k=0}^{N} p_k q_k(|\nabla_k F|\mathrm{e}^{|\nabla_k F|}-\mathrm{e}^{|\nabla_k F|}+1)\right], \tag{1.12.6}$$

of [19] since when $F_k^+ - F_k^- \ge 0$ the right-hand side of (1.12.6) grows as $F_k^+\mathrm{e}^{2F_k^+}$, instead of $F_k^+\mathrm{e}^{F_k^+}$ in (1.12.5). In fact we can prove the following inequality which improves (1.12.2), (1.12.3) and (1.12.6).

Theorem 1.12.4. *Let F be $\mathcal{F}_N$-measurable. We have*

$$\mathrm{Ent}\,[\mathrm{e}^F] \le \mathbb{E}\left[\mathrm{e}^F\sum_{k=0}^{N} p_k q_k(\nabla_k F\mathrm{e}^{\nabla_k F}-\mathrm{e}^{\nabla_k F}+1)\right]. \tag{1.12.7}$$

Clearly, (1.12.7) is better than (1.12.6), (1.12.4) and (1.12.3). It also improves (1.12.2) from the bound

$$x\mathrm{e}^x-\mathrm{e}^x+1 \le (\mathrm{e}^x-1)^2, \quad x\in\mathbb{R},$$

which implies

$$\mathrm{e}^F(\nabla F \mathrm{e}^{\nabla F} - \mathrm{e}^{\nabla F} + 1) \le \mathrm{e}^F(\mathrm{e}^{\nabla F} - 1)^2 = \mathrm{e}^{-F}|\nabla \mathrm{e}^F|^2.$$

By the tensorization property (1.12.1), the proof of (1.12.7) reduces to the following one dimensional lemma.

Lemma 1.12.5. *For any* $0 \le p \le 1$, $t \in \mathbb{R}$, $a \in \mathbb{R}$, $q = 1 - p$,

$$\begin{aligned} & pt\mathrm{e}^t + qa\mathrm{e}^a - \left(p\mathrm{e}^t + q\mathrm{e}^a\right)\log\left(p\mathrm{e}^t + q\mathrm{e}^a\right) \\ & \le pq\left(q\mathrm{e}^a\left((t-a)\mathrm{e}^{t-a} - \mathrm{e}^{t-a} + 1\right) + p\mathrm{e}^t\left((a-t)\mathrm{e}^{a-t} - \mathrm{e}^{a-t} + 1\right)\right). \end{aligned}$$

Proof. Set

$$\begin{aligned} g(t) = & \, pq\left(q\mathrm{e}^a\left((t-a)\mathrm{e}^{t-a} - \mathrm{e}^{t-a} + 1\right) + p\mathrm{e}^t\left((a-t)\mathrm{e}^{a-t} - \mathrm{e}^{a-t} + 1\right)\right) \\ & -pt\mathrm{e}^t - qa\mathrm{e}^a + \left(p\mathrm{e}^t + q\mathrm{e}^a\right)\log\left(p\mathrm{e}^t + q\mathrm{e}^a\right). \end{aligned}$$

Then

$$g'(t) = pq\left(q\mathrm{e}^a(t-a)\mathrm{e}^{t-a} + p\mathrm{e}^t\left(-\mathrm{e}^{a-t} + 1\right)\right) - pt\mathrm{e}^t + p\mathrm{e}^t\log(p\mathrm{e}^t + q\mathrm{e}^a)$$

and $g''(t) = p\mathrm{e}^t h(t)$, where

$$h(t) = -a - 2pt - p + 2pa + p^2 t - p^2 a + \log(p\mathrm{e}^t + q\mathrm{e}^a) + \frac{p\mathrm{e}^t}{p\mathrm{e}^t + q\mathrm{e}^a}.$$

Now,

$$\begin{aligned} h'(t) &= -2p + p^2 + \frac{2p\mathrm{e}^t}{p\mathrm{e}^t + q\mathrm{e}^a} - \frac{p^2\mathrm{e}^{2t}}{(p\mathrm{e}^t + q\mathrm{e}^a)^2} \\ &= \frac{pq^2(\mathrm{e}^t - \mathrm{e}^a)(p\mathrm{e}^t + (q+1)\mathrm{e}^a)}{(p\mathrm{e}^t + q\mathrm{e}^a)^2}, \end{aligned}$$

which implies that $h'(a) = 0$, $h'(t) < 0$ for any $t < a$ and $h'(t) > 0$ for any $t > a$. Hence, for any $t \neq a$, $h(t) > h(a) = 0$, and so $g''(t) \ge 0$ for any $t \in \mathbb{R}$ and $g''(t) = 0$ if and only if $t = a$. Therefore, g' is strictly increasing. Finally, since $t = a$ is the unique root of $g' = 0$, we have that $g(t) \ge g(a) = 0$ for all $t \in \mathbb{R}$. □

This inequality improves (1.12.2), (1.12.3), and (1.12.6), as illustrated in one dimension in Figure 1.1, where the entropy is represented as a function of $p \in [0, 1]$ with $f(1) = 1$ and $f(-1) = 3.5$:
The inequality (1.12.7) is a discrete analog of the sharp inequality on Poisson space of [151]. In the symmetric case $p_k = q_k = 1/2$, $k \in \mathbb{N}$, we have

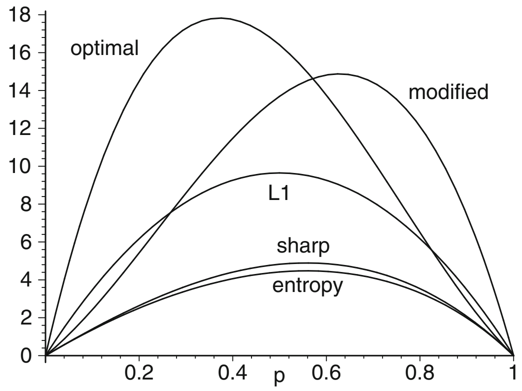

Fig. 1.1 Graph of bounds on the entropy as a function of $p \in [0,1]$

$$
\begin{aligned}
\mathrm{Ent}\,[\mathrm{e}^F] &\leq \mathbb{E}\left[\mathrm{e}^F \sum_{k=0}^{N} p_k q_k (\nabla_k F \mathrm{e}^{\nabla_k F} - \nabla_k F + 1)\right] \\
&= \frac{1}{8}\mathbb{E}\left[\sum_{k=0}^{N} \mathrm{e}^{F_k^-}((F_k^+ - F_k^-)\mathrm{e}^{F_k^+ - F_k^-} - \mathrm{e}^{F_k^+ - F_k^-} + 1)\right. \\
&\qquad \left. + \mathrm{e}^{F_k^+}((F_k^- - F_k^+)\mathrm{e}^{F_k^- - F_k^+} - \mathrm{e}^{F_k^- - F_k^+} + 1)\right] \\
&= \frac{1}{8}\mathbb{E}\left[\sum_{k=0}^{N} (\mathrm{e}^{F_k^+} - \mathrm{e}^{F_k^-})(F_k^+ - F_k^-)\right] \\
&= \frac{1}{2}\mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k \mathrm{e}^F\right],
\end{aligned}
$$

which improves on (1.12.3).
Similarly the sharp inequality of [151] can be recovered by taking $F = \varphi(M_n)$ in

$$
\begin{aligned}
&\mathrm{Ent}\,[\mathrm{e}^F] \leq \mathbb{E}\left[\mathrm{e}^F \sum_{k=0}^{N} p_k q_k (\nabla_k F \mathrm{e}^{\nabla_k F} - \nabla_k F + 1)\right] \\
&= \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)\mathbb{E}\left[M_n \mathrm{e}^{\varphi(M_n)}\right. \\
&\quad \left. \times ((\varphi(M_n) - \varphi(M_n - 1))\mathrm{e}^{\varphi(M_n) - \varphi(M_n - 1)} - \mathrm{e}^{\varphi(M_n) - \varphi(M_n - 1)} + 1)\right] \\
&\quad + \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)\mathbb{E}\left[(n - M_n)\mathrm{e}^{\varphi(M_n)}\right. \\
&\quad \left. \times ((\varphi(M_n + 1) - \varphi(M_n))\mathrm{e}^{\varphi(M_n + 1) - \varphi(M_n)} - \mathrm{e}^{\varphi(M_n + 1) - \varphi(M_n)} + 1)\right],
\end{aligned}
$$

which, in the limit as n goes to infinity, yields

$$\mathrm{Ent}\,[\mathrm{e}^{\varphi(U)}] \leq \lambda \mathbb{E}[\mathrm{e}^{\varphi(U)}((\varphi(U+1)-\varphi(U))\mathrm{e}^{\varphi(U+1)-\varphi(U)} - \mathrm{e}^{\varphi(U+1)-\varphi(U)} + 1)],$$

where U is a Poisson random variable with parameter λ.

1.13 Change of Variable Formula

In this section we state a discrete-time analog of Itô's change of variable formula which will be useful for the predictable representation of random variables and for option hedging.

Proposition 1.13.1. *Let $(M_n)_{n\in\mathbb{N}}$ be a square-integrable martingale and $f : \mathbb{R}\times\mathbb{N} \to \mathbb{R}$. We have*

$$\begin{aligned} f(M_n,n) &= f(M_{-1},-1)+\sum_{k=0}^{n} D_k f(M_k,k)Y_k+\sum_{k=0}^{n} \mathbb{E}[f(M_k,k)-f(M_{k-1},k-1) \mid \mathcal{F}_{k-1}] \\ &\quad +\sum_{k=0}^{n} f(M_{k-1},k)-f(M_{k-1},k-1). \end{aligned} \tag{1.13.1}$$

Proof. By Proposition 1.7.5 there exists square-integrable process $(u_k)_{k\in\mathbb{N}}$ such that

$$M_n = M_{-1} + \sum_{k=0}^{n} u_k Y_k, \qquad n \in \mathbb{N}.$$

We write

$$\begin{aligned} f(M_n,n)-f(M_{-1},-1) &= \sum_{k=0}^{n} f(M_k,k)-f(M_{k-1},k-1) \\ &= \sum_{k=0}^{n} f(M_k,k)-f(M_{k-1},k)+f(M_{k-1},k)-f(M_{k-1},k-1) \\ &= \sum_{k=0}^{n} \sqrt{\frac{p_k}{q_k}}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f(M_{k-1},k)\right)Y_k \\ &\quad +\frac{p_k}{q_k}\mathbf{1}_{\{X_k=-1\}}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f(M_{k-1},k)\right) \\ &\quad +\mathbf{1}_{\{X_k=-1\}}\left(f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)-f(M_{k-1},k)\right) \\ &\quad +\sum_{k=0}^{n} f(M_{k-1},k)-f(M_{k-1},k-1) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \sqrt{\frac{p_k}{q_k}} \left(f\left(M_{k-1} + u_k \sqrt{\frac{q_k}{p_k}}, k \right) - f(M_{k-1}, k) \right) Y_k \\
&\quad + \sum_{k=0}^{n} \frac{1}{q_k} \mathbf{1}_{\{X_k=-1\}} \mathbb{E}[f(M_k, k) - f(M_{k-1}, k) \mid \mathcal{F}_{k-1}] \\
&\quad + \sum_{k=0}^{n} f(M_{k-1}, k) - f(M_{k-1}, k-1).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
& f(M_n, n) \\
&= f(M_{-1}, -1) - \sum_{k=0}^{n} \sqrt{\frac{q_k}{p_k}} \left(f\left(M_{k-1} - u_k \sqrt{\frac{p_k}{q_k}}, k \right) - f(M_{k-1}, k) \right) Y_k \\
&\quad + \sum_{k=0}^{n} \frac{1}{p_k} \mathbf{1}_{\{X_k=1\}} \mathbb{E}[f(M_k, k) - f(M_{k-1}, k) \mid \mathcal{F}_{k-1}] \\
&\quad + \sum_{k=0}^{n} f(M_{k-1}, k) - f(M_{k-1}, k-1).
\end{aligned}
$$

Multiplying each increment in the above formulas respectively by q_k and p_k and summing on k we get

$$
\begin{aligned}
& f(M_n, n) = f(M_{-1}, -1) + \sum_{k=0}^{n} f(M_k, k) - f(M_{k-1}, k-1) \\
&= f(M_{-1}, -1) + \sum_{k=0}^{n} q_k (f(M_k, k) - f(M_{k-1}, k-1)) \\
&\quad + \sum_{k=0}^{n} p_k (f(M_k, k) - f(M_{k-1}, k-1)) \\
&= f(M_{-1}, -1) + \sum_{k=0}^{n} \sqrt{p_k q_k} \left(f\left(M_{k-1} + u_k \sqrt{\frac{q_k}{p_k}}, k \right) - f(M_{k-1}, k) \right) Y_k \\
&\quad - \sum_{k=0}^{n} \sqrt{p_k q_k} \left(f\left(M_{k-1} - u_k \sqrt{\frac{p_k}{q_k}}, k \right) - f(M_{k-1}, k) \right) Y_k \\
&\quad + \sum_{k=0}^{n} \mathbb{E}[f(M_k, k) \mid \mathcal{F}_{k-1}] - f(M_{k-1}, k-1) \\
&\quad + \sum_{k=0}^{n} f(M_{k-1}, k) - f(M_{k-1}, k-1) \\
&= f(M_{-1}, -1)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{n}\sqrt{p_kq_k}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)\right)Y_k \\
&+\sum_{k=0}^{n}\mathbb{E}[f(M_k,k)\mid\mathcal{F}_{k-1}]-f(M_{k-1},k-1) \\
&+\sum_{k=0}^{n}f(M_{k-1},k)-f(M_{k-1},k-1).
\end{aligned}
$$

□

Note that in (1.13.1) we have

$$
D_kf(M_k,k)=\sqrt{p_kq_k}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)\right),
$$

$k \in \mathbb{N}$.
On the other hand, the term

$$
\mathbb{E}[f(M_k,k)-f(M_{k-1},k-1)\mid\mathcal{F}_{k-1}]
$$

is analog to the generator part in the continuous time Itô formula, and can be written as

$$
p_kf\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)+q_kf\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)-f\left(M_{k-1},k-1\right).
$$

When $p_n = q_n = 1/2$, $n \in \mathbb{N}$, we have

$$
\begin{aligned}
f(M_n,n)=f(M_{-1},-1)&+\sum_{k=0}^{n}\frac{f\left(M_{k-1}+u_k,k\right)-f\left(M_{k-1}-u_k,k\right)}{2}Y_k \\
&+\sum_{k=0}^{n}\frac{f\left(M_{k-1}+u_k,k\right)-f\left(M_{k-1}-u_k,k\right)-2f\left(M_{k-1}-u_k,k\right)}{2} \\
&+\sum_{k=0}^{n}f(M_{k-1},k)-f(M_{k-1},k-1).
\end{aligned}
$$

The above proposition also provides an explicit version of the Doob decomposition for supermartingales. Naturally if $(f(M_n,n))_{n\in\mathbb{N}}$ is a martingale we have

$$
\begin{aligned}
f(M_n, n) &= f(M_{-1}, -1) \\
&\quad + \sum_{k=0}^{n} \sqrt{p_k q_k} \left(f\left(M_{k-1} + u_k \sqrt{\frac{q_k}{p_k}}, k \right) - f\left(M_{k-1} - u_k \sqrt{\frac{p_k}{q_k}}, k \right) \right) Y_k \\
&= f(M_{-1}, -1) + \sum_{k=0}^{n} D_k f(M_k, k) Y_k.
\end{aligned}
$$

In this case the Clark formula, the martingale representation formula Proposition 1.7.5 and the change of variable formula all coincide. In this case, we have in particular

$$
\begin{aligned}
D_k f(M_k, k) &= \mathbb{E}[D_k f(M_n, n) \mid \mathcal{F}_{k-1}] \\
&= \mathbb{E}[D_k f(M_k, k) \mid \mathcal{F}_{k-1}], \qquad k \in \mathbb{N}.
\end{aligned}
$$

If F is an $\mathcal{F}_N$-measurable random variable and f is a function such that

$$
\mathbb{E}[F \mid \mathcal{F}_n] = f(M_n, n), \qquad -1 \le n \le N,
$$

we have $F = f(M_N, N)$, $\mathbb{E}[F] = f(M_{-1}, -1)$ and

$$
\begin{aligned}
F &= \mathbb{E}[F] + \sum_{k=0}^{n} \mathbb{E}[D_k f(M_N, N) \mid \mathcal{F}_{k-1}] Y_k \\
&= \mathbb{E}[F] + \sum_{k=0}^{n} D_k f(M_k, k) Y_k \\
&= \mathbb{E}[F] + \sum_{k=0}^{n} D_k \mathbb{E}[f(M_N, N) \mid \mathcal{F}_k] Y_k.
\end{aligned}
$$

Such a function f exists if $(M_n)_{n\in\mathbb{N}}$ is Markov and $F = h(M_N)$. In this case, consider the semi-group $(P_{k,n})_{0\le k<n\le N}$ associated to $(M_n)_{n\in\mathbb{N}}$ and defined by

$$
(P_{k,n} h)(x) = \mathbb{E}[h(M_n) \mid M_k = x].
$$

Letting $f(x, n) = (P_{n,N} h)(x)$ we can write

$$
\begin{aligned}
F &= \mathbb{E}[F] + \sum_{k=0}^{n} \mathbb{E}[D_k h(M_N) \mid \mathcal{F}_{k-1}] Y_k \\
&= \mathbb{E}[F] + \sum_{k=0}^{n} D_k (P_{k,N} h(M_k)) Y_k.
\end{aligned}
$$

1.14 Option Hedging

In this section we give a presentation of the Black-Scholes formula in discrete time, or Cox-Ross-Rubinstein model, see e.g. [45], [74], [125], or §15-1 of [149] as an application of the Clark formula.

In order to be consistent with the notation of the previous sections we choose to use the time scale $\mathbb{N}$, hence the index 0 is that of the first random value of any stochastic process, while the index -1 corresponds to its deterministic initial value.

Let $(A_k)_{k\in\mathbb{N}}$ be a riskless asset with initial value A_{-1}, and defined by

$$A_n = A_{-1}\prod_{k=0}^{n}(1+r_k), \qquad n\in\mathbb{N},$$

where $(r_k)_{k\in\mathbb{N}}$, is a sequence of deterministic numbers such that $r_k > -1$, $k\in\mathbb{N}$. Consider a stock price with initial value S_{-1}, given in discrete time as

$$S_n = \begin{cases} (1+b_n)S_{n-1} \text{ if } X_n = 1, \\ \\ (1+a_n)S_{n-1} \text{ if } X_n = -1, \quad n\in\mathbb{N}, \end{cases}$$

where $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are sequences of deterministic numbers such that

$$-1 < a_k < r_k < b_k, \qquad k\in\mathbb{N}.$$

We have

$$S_n = S_{-1}\prod_{k=0}^{n}\sqrt{(1+b_k)(1+a_k)}\left(\frac{1+b_k}{1+a_k}\right)^{X_k/2}, \quad n\in\mathbb{N}.$$

Consider now the discounted stock price given as

$$\begin{aligned}\tilde{S}_n &= S_n\prod_{k=0}^{n}(1+r_k)^{-1} \\ &= S_{-1}\prod_{k=0}^{n}\left(\frac{1}{1+r_k}\sqrt{(1+b_k)(1+a_k)}\left(\frac{1+b_k}{1+a_k}\right)^{X_k/2}\right), \quad n\in\mathbb{N}.\end{aligned}$$

If $-1 < a_k < r_k < b_k$, $k\in\mathbb{N}$, then $(\tilde{S}_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq -1}$ under the probability $\mathbb{P}^*$ given by

$$p_k = \frac{r_k - a_k}{b_k - a_k}, \quad q_k = \frac{b_k - r_k}{b_k - a_k}, \quad k\in\mathbb{N}.$$

In other terms, under $\mathbb{P}^*$ we have

$$\mathbb{E}^*[S_{n+1} \mid \mathcal{F}_n] = (1 + r_{n+1})S_n, \qquad n \geq -1,$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$. Recall that under this probability measure there is absence of arbitrage and the market is complete. From the change of variable formula Proposition 1.13.1 or from the Clark formula (1.7.1) we have the martingale representation

$$\tilde{S}_n = S_{-1} + \sum_{k=0}^{n} Y_k D_k \tilde{S}_k = S_{-1} + \sum_{k=0}^{n} \tilde{S}_{k-1} \sqrt{p_k q_k} \frac{b_k - a_k}{1 + r_k} Y_k.$$

Definition 1.14.1. *A portfolio strategy is represented by a pair of predictable processes $(\eta_k)_{k\in\mathbb{N}}$ and $(\zeta_k)_{k\in\mathbb{N}}$ where η_k, resp. ζ_k represents the numbers of units invested over the time period $(k, k+1]$ in the asset S_k, resp. A_k, with $k \geq 0$.*

The value at time $k \geq -1$ of the portfolio $(\eta_k, \zeta_k)_{0\leq k\leq N}$ is defined as

$$V_k = \zeta_{k+1} A_k + \eta_{k+1} S_k, \qquad k \geq -1, \tag{1.14.1}$$

and its discounted value is defined as

$$\tilde{V}_n = V_n \prod_{k=0}^{n} (1 + r_k)^{-1}, \qquad n \geq -1. \tag{1.14.2}$$

Definition 1.14.2. *A portfolio $(\eta_k, \zeta_k)_{k\in\mathbb{N}}$ is said to be self-financing if*

$$A_k(\zeta_{k+1} - \zeta_k) + S_k(\eta_{k+1} - \eta_k) = 0, \qquad k \geq 0.$$

Note that the self-financing condition implies

$$V_k = \zeta_k A_k + \eta_k S_k, \qquad k \geq 0.$$

Our goal is to hedge an arbitrary claim on Ω, i.e. given an $\mathcal{F}_N$-measurable random variable F we search for a portfolio $(\eta_k, \zeta_k)_{0\leq k\leq n}$ such that the equality

$$F = V_N = \zeta_N A_N + \eta_N S_N \tag{1.14.3}$$

holds at time $N \in \mathbb{N}$.

Proposition 1.14.3. *Assume that the portfolio $(\eta_k, \zeta_k)_{0\leq k\leq N}$ is self-financing. Then we have the decomposition*

$$V_n = V_{-1} \prod_{k=0}^{n} (1 + r_k) + \sum_{i=0}^{n} \eta_i S_{i-1} \sqrt{p_i q_i} (b_i - a_i) Y_i \prod_{k=i+1}^{n} (1 + r_k). \tag{1.14.4}$$

Proof. Under the self-financing assumption we have

$$\begin{aligned}
V_i - V_{i-1} &= \zeta_i(A_i - A_{i-1}) + \eta_i(S_i - S_{i-1}) \\
&= r_i\zeta_i A_{i-1} + (a_i\mathbf{1}_{\{X_i=-1\}} + b_i\mathbf{1}_{\{X_i=1\}})\eta_i S_{i-1} \\
&= \eta_i S_{i-1}(a_i\mathbf{1}_{\{X_i=-1\}} + b_i\mathbf{1}_{\{X_i=1\}} - r_i) + r_i V_{i-1} \\
&= \eta_i S_{i-1}((a_i - r_i)\mathbf{1}_{\{X_i=-1\}} + (b_i - r_i)\mathbf{1}_{\{X_i=1\}}) + r_i V_{i-1} \\
&= (b_i - a_i)\eta_i S_{i-1}(-p_i\mathbf{1}_{\{X_i=-1\}} + q_i\mathbf{1}_{\{X_i=1\}}) + r_i V_{i-1} \\
&= \eta_i S_{i-1}\sqrt{p_i q_i}(b_i - a_i)Y_i + r_i V_{i-1}, \qquad i \in \mathbb{N},
\end{aligned}$$

by Relation (1.4.5), hence for the discounted portfolio we get:

$$\begin{aligned}
\tilde{V}_i - \tilde{V}_{i-1} &= \prod_{k=1}^{i}(1+r_k)^{-1}V_i - \prod_{k=1}^{i-1}(1+r_k)^{-1}V_{i-1} \\
&= \prod_{k=1}^{i}(1+r_k)^{-1}(V_i - V_{i-1} - r_i V_{i-1}) \\
&= \eta_i S_{i-1}\sqrt{p_i q_i}(b_i - a_i)Y_i \prod_{k=1}^{i}(1+r_k)^{-1}, \qquad i \in \mathbb{N},
\end{aligned}$$

which successively yields (1.14.4). □

As a consequence of (1.14.4) and (1.14.2) we immediately obtain

$$\tilde{V}_n = \tilde{V}_{-1} + \sum_{i=0}^{n} \eta_i S_{i-1}\sqrt{p_i q_i}(b_i - a_i)Y_i \prod_{k=0}^{i}(1+r_k)^{-1}, \tag{1.14.5}$$

$n \geq -1$. The next proposition provides a solution to the hedging problem under the constraint (1.14.3).

Proposition 1.14.4. *Given $F \in L^2(\Omega, \mathcal{F}_N)$, let*

$$\eta_n = \frac{1}{S_{n-1}\sqrt{p_n q_n}(b_n - a_n)}\mathbb{E}^*[D_n F \mid \mathcal{F}_{n-1}] \prod_{k=n+1}^{N}(1+r_k)^{-1}, \tag{1.14.6}$$

$0 \leq n \leq N$, *and*

$$\zeta_n = A_n^{-1}\left(\prod_{k=n+1}^{N}(1+r_k)^{-1}\mathbb{E}^*[F \mid \mathcal{F}_n] - \eta_n S_n\right), \tag{1.14.7}$$

$0 \leq n \leq N$. Then the portfolio $(\eta_k, \zeta_k)_{0 \leq k \leq n}$ is self financing and satisfies

$$\zeta_n A_n + \eta_n S_n = \prod_{k=n+1}^{N} (1+r_k)^{-1} \mathbb{E}^*[F \mid \mathcal{F}_n],$$

$0 \leq n \leq N$, in particular we have $V_N = F$, hence $(\eta_k, \zeta_k)_{0 \leq k \leq N}$ is a hedging strategy leading to F.

Proof. Let $(\eta_k)_{-1 \leq k \leq N}$ be defined by (1.14.6) and $\eta_{-1} = 0$, and consider the process $(\zeta_n)_{0 \leq n \leq N}$ defined by

$$\zeta_{-1} = \frac{\mathbb{E}^*[F]}{S_{-1}} \prod_{k=0}^{N} (1+r_k)^{-1} \quad \text{and} \quad \zeta_{k+1} = \zeta_k - \frac{(\eta_{k+1} - \eta_k) S_k}{A_k},$$

$k = -1, \ldots, N-1$. Then $(\eta_k, \zeta_k)_{-1 \leq k \leq N}$ satisfies the self-financing condition

$$A_k(\zeta_{k+1} - \zeta_k) + S_k(\eta_{k+1} - \eta_k) = 0, \qquad -1 \leq k \leq N-1.$$

Let now

$$V_{-1} = \mathbb{E}^*[F] \prod_{k=0}^{N} (1+r_k)^{-1}, \quad \text{and} \quad V_n = \zeta_n A_n + \eta_n S_n, \qquad 0 \leq n \leq N,$$

and

$$\tilde{V}_n = V_n \prod_{k=0}^{n} (1+r_k)^{-1}, \qquad -1 \leq n \leq N.$$

Since $(\eta_k, \zeta_k)_{-1 \leq k \leq N}$ is self-financing, Relation (1.14.5) shows that

$$\tilde{V}_n = \tilde{V}_{-1} + \sum_{i=0}^{n} Y_i \eta_i S_{i-1} \sqrt{p_i q_i} (b_i - a_i) \prod_{k=1}^{i} (1+r_k)^{-1}, \tag{1.14.8}$$

$-1 \leq n \leq N$. On the other hand, from the Clark formula (1.7.1) and the definition of $(\eta_k)_{-1 \leq k \leq N}$ we have

$$\begin{aligned}
&\mathbb{E}^*[F \mid \mathcal{F}_n] \prod_{k=0}^{N} (1+r_k)^{-1} \\
&= \mathbb{E}^* \left[\mathbb{E}^*[F] \prod_{k=0}^{N} (1+r_k)^{-1} + \sum_{i=0}^{N} Y_i \mathbb{E}^*[D_i F \mid \mathcal{F}_{i-1}] \prod_{k=0}^{N} (1+r_k)^{-1} \middle| \mathcal{F}_n \right] \\
&= \mathbb{E}^*[F] \prod_{k=0}^{N} (1+r_k)^{-1} + \sum_{i=0}^{n} Y_i \mathbb{E}^*[D_i F \mid \mathcal{F}_{i-1}] \prod_{k=0}^{N} (1+r_k)^{-1}
\end{aligned}$$

$$= \mathbb{E}^*[F] \prod_{k=0}^{N} (1+r_k)^{-1} + \sum_{i=0}^{n} Y_i \eta_i S_{i-1} \sqrt{p_i q_i} (b_i - a_i) \prod_{k=1}^{i} (1+r_k)^{-1}$$
$$= \tilde{V}_n$$

from (1.14.8). Hence

$$\tilde{V}_n = \mathbb{E}^*[F \mid \mathcal{F}_n] \prod_{k=0}^{N} (1+r_k)^{-1}, \qquad -1 \le n \le N,$$

and

$$V_n = \mathbb{E}^*[F \mid \mathcal{F}_n] \prod_{k=n+1}^{N} (1+r_k)^{-1}, \qquad -1 \le n \le N.$$

In particular we have $V_N = F$. To conclude the proof we note that from the relation $V_n = \zeta_n A_n + \eta_n S_n$, $0 \le n \le N$, the process $(\zeta_n)_{0 \le n \le N}$ coincides with $(\zeta_n)_{0 \le n \le N}$ defined by (1.14.7). □

Note that we also have

$$\zeta_{n+1} A_n + \eta_{n+1} S_n = \mathbb{E}^*[F \mid \mathcal{F}_n] \prod_{k=n+1}^{N} (1+r_k)^{-1}, \qquad -1 \le n \le N.$$

The above proposition shows that there always exists a hedging strategy starting from

$$\tilde{V}_{-1} = \mathbb{E}^*[F] \prod_{k=0}^{N} (1+r_k)^{-1}.$$

Conversely, if there exists a hedging strategy leading to

$$\tilde{V}_N = F \prod_{k=0}^{N} (1+r_k)^{-1},$$

then $(\tilde{V}_n)_{-1 \le n \le N}$ is necessarily a martingale with initial value

$$\tilde{V}_{-1} = \mathbb{E}^*[\tilde{V}_N] = \mathbb{E}^*[F] \prod_{k=0}^{N} (1+r_k)^{-1}.$$

When $F = h(\tilde{S}_N)$, we have $\mathbb{E}^*[h(\tilde{S}_N) \mid \mathcal{F}_k] = f(\tilde{S}_k, k)$ with

$$f(x,k) = \mathbb{E}^* \left[h \left(x \prod_{i=k+1}^{n} \frac{\sqrt{(1+b_k)(1+a_k)}}{1+r_k} \left(\frac{1+b_k}{1+a_k} \right)^{X_k/2} \right) \right].$$

The hedging strategy is given by

$$\begin{aligned}\eta_k &= \frac{1}{S_{k-1}\sqrt{p_k q_k}(b_k - a_k)} D_k f(\tilde{S}_k, k) \prod_{i=k+1}^{N} (1 + r_i)^{-1} \\ &= \frac{\prod_{i=k+1}^{N}(1 + r_i)^{-1}}{S_{k-1}(b_k - a_k)} \left(f\left(\tilde{S}_{k-1}\frac{1 + b_k}{1 + r_k}, k\right) - f\left(\tilde{S}_{k-1}\frac{1 + a_k}{1 + r_k}, k\right)\right),\end{aligned}$$

$k \geq -1$. Note that η_k is non-negative (i.e. there is no short-selling) when f is an increasing function, e.g. in the case of European options we have $f(x) = (x - K)^+$.

1.15 Notes and References

This chapter is a revision of [113] with some additions, and is mainly based on [59] and [115]. It is included for the sake of consistency and for the role it plays as an introduction to the next chapters. Other approaches to discrete-time stochastic analysis include [53], [54], [48], [78] and [89]; see [8] for an approach based on quantum probability. Deviation inequalities and logarithmic Sobolev inequalities are treated in [19], [46], [59]. We also refer to [5], [17], [18], [61], [75], for other versions of logarithmic Sobolev inequalities in discrete settings. See [74], §15-1 of [149], and [125], for other derivations of the Black-Scholes formula in the discrete time Cox-Ross-Rubinstein model.