Chapter 7
Global Convergent Dynamics of Delayed Neural Networks

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7.1 Introduction

Artificial neural networks arise from the research of the configuration and function of the brain. As pointed out in [79], the brain can be regarded as a complex nonlinear parallel information processing system with a concept of neuron as a basic functional unit. Compared with modern computer, the processing speed of a single neuron is 5–6 times slower than that of a single silicic logic gate but the brain has a processing speed 10^9 times faster than any computer due to a huge quantity of synapses that interconnect neurons. Based on this viewpoint, scientists proposed a network model to describe the function and state of the brain called neural networks. In short, a neural network is a computing network that accomplishes given tasks by connecting a large number of simple computing units. The most important characteristic of neural networks is the ability to learn. Reference [75] defined learning as the process through which the neural network adjusts its parameters using information of its circumstances via a simulating process. Many learning algorithms have been proposed in the past decades, for example, error-correction learning, Hebbian learning [52], competitive learning [47], and Boltzmann learning [1].

In particular, [2] proposes a definition of artificial neural network. Neural network is a large-scale parallel distributed processing system, which can learn and employ knowledge and satisfies that (1) knowledge is obtained by learning (learning algorithm); (2) knowledge is stored in the interconnection weights of the network. Since neural networks have many advantages, for instance, the ability to solve nonlinear problems, adaptability, fault tolerance, and mass computability, they have been one of the focal research topics for the last 50–60 years.

References [53, 36, 37] proposed multi-layered neuronal perceptron model which can approximate any continuous function. This model can be formulated as

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$$f(x_1, x_2, \ldots, x_n) \approx \sum_{i=1}^{n} C_i g\left(\sum_{j=1}^{m} \xi_{ij} x_j + \theta_i\right), \tag{7.1}$$

where x_i denotes the state variable of neuron i and $g(\,\cdot\,)$ is a certain nonlinear activation function. Furthermore, [22, 23] proved that this model can approximate any nonlinear function and operator. This is the theoretical basis of neural networks.

References [35, 47] proposed a competitive and cooperative model to generate self-organized and self-adaptive neural networks, which can be modeled as an ODE system:

$$\frac{dx_i}{dt} = a_i(x_i)\Big[-d_i(x_i) + \sum_{j=1}^{n} t_{ij} g_j(x_j) + I_i\Big], \quad i = 1, \ldots, n,$$

which is named Cohen–Grossberg neural network and widely used in pattern recognition, signal processing, and associative memory. Here, $x_i(t)$ denotes the state variable of the i-th neuron, $d_i(\,\cdot\,)$ represents the self-inhibition function with which the i-th neuron will reset its potential to the resting state in isolation when disconnected from the network, t_{ij} denotes the strength of j-th neuron on the i-th neuron, $g_i(\,\cdot\,)$ denotes the activation function of i-th neuron, I_i denotes the external input to the i-th neuron, and $a_i(\,\cdot\,)$ denotes the amplification function of the i-th neuron.

References [54, 57] developed a computing method using recurrent networks based on energy functions, which is called Hopfield neural network:

$$\frac{dx_i}{dt} = -d_i x_i + \sum_{j=1}^{n} t_{ij} g_j(x_j) + I_i, \quad i = 1, \ldots, n,$$

which has been applied to solve some combinatorial optimization problems such as the traveling salesman problem.

As pointed out by [51], the common characteristic is that each neural network model can be regarded as a class of nonlinear signal-flow graphs. As indicated in Fig. 7.1, x_i denotes the state of neuron i, $y_i = \phi_i(x_i)$ denotes the output of neuron i by a nonlinear activation function $\phi_i(\,\cdot\,)$, t_{ij} denotes the weight of interconnection from neuron j to i, and I_i is the external input. Hence, neural networks are in fact a class of nonlinear dynamical systems due to the nonlinearity of the activations. The computation developed from neural networks is a self-adaptive distributed method based on a learning algorithm. The key point of success of an algorithm lies on whether the dynamical flow converges to a given equilibrium or manifold. So, dynamical analysis of neural networks is the first step for the expected applications.

In practice, time delays inevitably occur due to the finite switching speed of the amplifiers and communication time. Moreover, to process moving images, one must introduce time delays in the signals transmitted among the cells [25]. Neural networks with time delays have much more complicated dynamics due to the

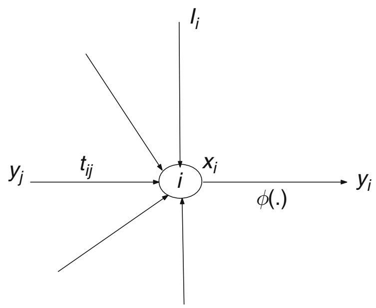

Fig. 7.1 Signal-flow graph

incorporation of delays. These neural networks can be modeled by the following delayed differential equations:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t-\tau_{ij})\right) + I_i, \quad i = 1, \ldots, n, \tag{7.2}$$

where b_{ij} denotes the delayed feedback of the j-th neuron on the i-th neuron and τ_{ij} denotes the transmission delay from neuron j to i. If the activation functions concerned with delayed or without delayed terms are the same, i.e., $f_j = g_j$, $j = 1, \ldots, n$, then this model can be formulated as

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij})) + I_i, \quad i = 1, \ldots, n.$$

One can see that this model contains cellular neural networks [32, 33] as a special case. If $\tau_{ij} = \tau$ is uniform, it has the following form:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau)) + I_i, \quad i = 1, \ldots, n.$$

Also, the delayed Cohen–Grossberg neural networks can be written as

$$\frac{dx_i(t)}{dt} = a_i(x_i)\left[-d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau)) + I_i \right], \quad i = 1, \ldots, n, \tag{7.3}$$

a special form.

Research of delayed neural networks with varying self-inhibitions, interconnection weights, and inputs is an important issue, because in many real-world applications, self-inhibitions, interconnection weights, and inputs vary with time. Thus, we also study the delayed neural networks with a more general form, which is first introduced in [26]:

$$\frac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^{n} \int_0^{\infty} f_j(x_j(t-s))d_sK_{ij}(s) \\ +I_i(t), \quad i = 1, \dots, n,$$

where $d_sK_{ij}(t,s)$, $i,j = 1,\dots,n$, are Lebesgue–Stieltjes measures with respect to s, which denotes the delayed terms. For example, if $d_sK_{ij}(t,s)$ has the form $b_{ij}\delta_{\tau_{ij}}(t-s)ds$, one obtains (7.2). More details about the descriptions of the models will be discussed in the following sections.

In this chapter, we study the global convergent dynamics of a class of delayed neural networks. The models are rather general, including Hopfield neural networks, Cohen–Grossberg networks, cellular neural networks, as well as the case of discontinuous activation functions. The purpose of this chapter is not only to present the existing results but also to illustrate the methodologies used in obtaining and proving these results. These methodologies could be utilized or extended in analysis of global convergent dynamics of other models or general delayed differential systems.

Two mathematical problems must be solved. One is the existence of a static orbit: an equilibrium, a periodic orbit, or an almost periodic orbit. Ordinarily, this can be investigated by the fixed point theory. In addition, in this chapter we use novel methods. We study the system of the derivative of the delayed Hopfield neural networks instead and conclude that the global exponential stability of the derivative can lead the global exponential stability of the intrinsic neural networks. Moreover, the existence of periodic or almost periodic orbits can be handled by regarding it as a clustering orbit of any trajectory. The second problem is the stability of such a static orbit. This is investigated by designing a suitable Lyapunov functional. We should point out that it is not the theorems but the ideas of Lyapunov and Lyapunov–Krasovskii stability theory that is used to prove global stability. The main results and proofs in this chapter come from our recent literature [19–21, 30, 63, 65–68].

We organize this chapter as follows. In Sect. 7.2, we discuss the stability of delayed neural networks. We study the periodicity and almost periodicity in Sect. 7.3. In Sect. 7.4, we investigate the convergence analysis of delayed neural networks with discontinuous activation functions. We present reviews of literature on this topic and compare them with the results in Sect. 7.5.

We first present the notation used in this chapter. $\|\cdot\|$ denotes the norm of a vector in some sense. In particular, $\|v\|_2$ for a vector $v = (v_1,\dots,v_n)^{\top}$ denotes the 2-norm, i.e., $\|v\|_2 = \sqrt{\sum_{i=1}^{n}|v_i|^2}$ and $\|v\|_1 = \sum_{i=1}^{n}|v_i|$. For some positive vector $\xi = (\xi_1,\dots,\xi_n)^{\top}$, we denote $\|v\|_{\{\xi,\infty\}} = \max_i \xi_i^{-1}|v_i|$ and $\|v\|_{\{\xi,1\}} = \sum_{i=1}^{n}\xi_i|v_i|$. The norm of a matrix is induced by the definition of the norm of vectors. $C([a,\,b],\mathbb{R}^n)$

denotes the class of continuous functions from $[a,b]$ to $\mathbb{R}^n$. The norm of $x(\cdot) \in C([a,b],\mathbb{R}^n)$ is denoted by $\|x(\cdot)\| = \max_{a\le t\le b}\|x(t)\|$ for some vector norm $\|\cdot\|$. We write $a^+ = \max\{a,0\}$ for a real number a. The spectral set of a square matrix A is denoted by $\lambda(A)$. Among them, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum one, respectively, if all eigenvalues of A are real. For a matrix A, $A^\top$ denotes its transpose and A^s denotes its symmetric part, i.e., $A^s = (A+A^\top)/2$. For a matrix $A \in \mathbb{R}^{n,n}$, $A > 0$ denotes that A is positive definite, with similar definitions for the notations $A \ge 0$, $A < 0$, and $A \le 0$. For two matrices $A, B \in \mathbb{R}^{n,n}$, $A > B$ denotes $A - B > 0$; similarly with $A \ge B$, $A < B$, and $A \le B$. $\mathbb{R}^n_+$ denotes the first orthant, $\mathbb{R}^n_+ = \{x = (x_1\dots,x_n)^\top : x_i > 0,\ \forall\, i = 1,\dots,n\}$. For a matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n,n}$, $|A|$ denotes the matrix $(|a_{ij}|)_{i,j=1}^n$. Finally, $\mathrm{sign}(\cdot)$ denotes the signature function.

7.2 Stability of Delayed Neural Networks

In this section we will study the global stability of delayed neural networks. The basic mathematical method is the theory of functional differential equations. For more details, we refer interested readers to [49]. The study of stability of these differential systems contains two main contents: (1) existence of an equilibrium and (2) global attractivity of this equilibrium as done in previous literature. We study the delayed Hopfield neural network (7.2) and the delayed Cohen–Grossberg neural network (7.3) and prove that under several assumptions, diagonal dominant conditions can lead the global stability.

7.2.1 Preliminaries

Before presenting the main results, we provide a brief review of necessary theoretical preliminaries.

7.2.1.1 Functional Differential Equations (FDE)

Delayed neural networks can be modeled as a class of functional differential equations, which have the following general forms:

$$\frac{dx}{dt} = f(x_t). \tag{7.4}$$

Here, $x(t) \in \mathbb{R}^n$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau,0]$, where $\tau > 0$ can even be infinite, $f(\cdot)$ is a function in $C([-\tau,0],\mathbb{R}^n)$. A solution of the system (7.4) with initial condition $\phi \in C([-\tau,0],\mathbb{R}^n)$ is a smooth $x(t)$ satisfying (1) $x(\theta) = \phi(\theta)$ for all $\theta \in [-\tau,0]$ and (2) (7.4) holds for all $t \ge 0$. As pointed out in [49], local Lipschitz continuity of $f(\cdot)$ can guarantee the existence and uniqueness of the solution of the system (7.4). In addition, if the solution is bounded, then the solution exists for the whole time interval.

Stability of (7.4) is with respect to an equilibrium. An *equilibrium* $x^* \in \mathbb{R}^n$ is a solution of the equation

$$f(x^*) = 0, \tag{7.5}$$

i.e., $x_t(\cdot)$ is picked as a constant function. Stability is then defined as follows.

Definition 7.1 Equation (7.4) is said to be globally stable if for any initial condition $\phi \in C([-\tau, 0], \mathbb{R}^n)$, the corresponding solution $x(t)$ satisfies $\lim_{t\to\infty} x(t) = x^*$. Moreover, if there exist some $M > 0$ and $\epsilon > 0$ such that $\|x(t) - x^*\| \leq M \exp(-\epsilon t)$ for all $t \geq 0$, (7.4) is said to be globally exponentially stable. If there exist some $M > 0$ and $\gamma > 0$ such that $\|x(t) - x^*\| \leq M t^{-\gamma}$ for all $t \geq 0$, (7.4) is said to be globally stable in power rate.

In this chapter, we use Lyapunov functional methods to study the global stability of the equilibrium. Actually, we do not directly cite Lyapunov stability theorem for FDEs but use the underlying idea. We design a suitable functional which is zero if and only if $x_t = x^*$, give conditions to guarantee that it decreases through the system, and directly prove that the Lyapunov functional converges to zero. Also, we use the idea of Lyapunov–Krasovskii theory instead of the theorem, which can be cited as the following simple lemma:

Lemma 7.2 *Let $x(t)$ be a solution of the system (7.4) with the initial time $t_0 > 0$ and $\phi(t) = \|x_t(\cdot)\|$. If at each t^* with $\phi(t^*) = \|x(t^*)\|$, we have*

$$\frac{d\|x(t)\|}{dt} \Big|_{t=t^*} \leq -\eta\phi(t^*) + M(t^*) \tag{7.6}$$

for some positive continuous function $M(t^*)$, then $\phi(t) \leq \max\{M(t)/\eta, \phi(t_0)\}$ for all $t \geq t_0$.

Proof We prove it by discussing the following two cases.

Case 1: $\phi(t_0) \leq M(t_0)/\eta$. We can prove $\phi(t) \leq M(t)/\eta$ for all $t \geq t_0$. In fact, if there exists some $t_1 > t_0$ such that $\phi(t_1) = M(t_1)$ for the first time, then $\phi(t)$ is non-increasing at t_1. Otherwise, if $\phi(t)$ is strictly increasing at t_1, then $\phi(t_1) = \|x(t_1)\|$ and $\|x(t)\|$ is strictly increasing at t_1, which by (7.6) is impossible. Hence, $\phi(t)$ will never increase beyond $M(t)$.

Case 2: $\phi(t_0) > M(t_0)/\eta$. Then, $\phi(t)$ is decreasing in a small right neighborhood of t_0. If at some $t_1 > T_0$, $\phi(t_1) \leq M(t_1)/\eta$, then it reduces to Case 1. Otherwise, $\phi(t)$ keeps decreasing.

In both cases, it can be concluded that $\phi(t) \leq \max\{M(t)/\eta, \phi(t_0)\}$. □

7.2.1.2 Matrix Theory

A matrix $T \in \mathbb{R}^{n,n}$ is said to be *Lyapunov diagonally stable* (LDS) if there exists a positive definite diagonal matrix $D \in \mathbb{R}^{n,n}$ such that $DT + T^\top D$ is positive definite.

Lemma 7.3 (See Lemma 2 in [41]) *Let D and G be positive definite diagonal matrices and $T \in \mathbb{R}^{n,n}$. If $DG^{-1} - T$ is LDS, then for any positive definite diagonal matrix $\bar{D} \geq D$ and nonnegative definite diagonal matrix $0 \leq K \leq G$, we have $det(\bar{D} - TK) \neq 0$.*

A nonsingular matrix $C \in \mathbb{R}^{n,n}$ with $c_{ij} \leq 0$, $i,j = 1,\ldots,m$, $i \neq j$, is said to be an M-matrix if all elements of C^{-1} are nonnegative.

Lemma 7.4 ([11]) *Let $C = (c_{ij}) \in \mathbb{R}^{n,n}$ be a nonsingular matrix with $c_{ij} \leq 0$, $i,j = 1,\ldots,n$, $i \neq j$. Then the following statements are equivalent.*

1. *C is an M-matrix;*
2. *All the successive principal minors of C are positive;*
3. *$C^\top$ is an M-matrix;*
4. *The real parts of all eigenvalues are positive;*
5. *There exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^\top$ with $\xi_i > 0$, $i = 1,\ldots,n$, such that every component of $\xi^\top C$ is positive, or every component of $C\xi$ is positive;*
6. *C is LDS;*
7. *For any two diagonal matrices $P = \mathrm{diag}\{p_1, p_2, \ldots, p_n\}$, $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\}$, where $p_i > 0$, $q_i > 0$, $i = 1,\ldots,n$, PCQ is an M-matrix.*

The following lemma states the Schur Complement.

Lemma 7.5 (Schur Complement [13]) *The following Linear Matrix Inequality (LMI)*

$$\begin{bmatrix} Q(x) & S(x) \\ S^\top(x) & R(x) \end{bmatrix} > 0,$$

where $Q(x) = Q^\top(x), R(x) = R^\top(x)$, and $S(x)$ depend affinely on x, is equivalent to

$$R(x) > 0 \text{ and } Q(x) - S(x)R^{-1}(x)S^\top(x) > 0.$$

7.2.1.3 Nonlinear Complimentary Problems

To discuss the existence and uniqueness of the equilibrium, we give a brief review on Nonlinear Complementarity Problem (NCP).

Definition 7.6 For a continuous function $f(x) = (f_1(x),\ldots,f_n(x))^\top : \mathbb{R}^n_+ \to \mathbb{R}^n$, an NCP is to find x_i, $i = 1,\ldots,n$, satisfying

$$x_i \geq 0, \quad f_i(x) - I_i \geq 0, \quad x_i(f_i(x) - I_i) = 0 \quad \text{for all } i = 1,\ldots,n. \tag{7.7}$$

Define a function $F(x) : \mathbb{R}^n \to \mathbb{R}^n$

$$F(x) = f(x^+) + x^-,$$

where

$$x_i^+ = \begin{cases} x_i, x_i \geq 0 \\ 0, \text{ otherwise}, \end{cases} \quad x_i^- = \begin{cases} x_i, x_i \leq 0 \\ 0, \text{ otherwise} \end{cases} \quad \text{for } i = 1, \ldots, n.$$

The following lemma gives a sufficient and necessary condition for the solvability of a NCP.

Lemma 7.7 (Theorem 2.3 in [76]) *The NCP* (7.7) *has a unique solution for every* $I \in \mathbb{R}^n$ *if and only if* $F(x)$ *is norm-coercive, i.e.,*

$$\lim_{\|x\| \to \infty} \|F(x)\| = \infty,$$

and $F(x)$ *is locally one-to-one.*

7.2.1.4 Descriptions of Activations

The activation functions in these models are assumed to be Lipschitz continuous.

Definition 7.8 A continuous function $g(x) = (g_1(x_1), \ldots, g_n(x_n))^\top : \mathbb{R}^n \to \mathbb{R}^n$ is said to belong to the function class $H_1\{G_1, \ldots, G_n\}$ for some positive numbers $G_1, \ldots, G_n$ if $|g_i(\xi) - g_i(\zeta)| \leq G_i |\xi - \zeta|$ for all $\xi, \zeta \in \mathbb{R}$ and $i = 1, \ldots, n$. If, in addition, each $g_i(\cdot)$ is monotonously increasing, then g is said to belong to the function class $H_2\{G_1, \ldots, G_n\}$.

7.2.2 Delayed Hopfield Neural Networks

In this section we study the following delayed differential system:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij})) + I_i, \quad i = 1, \ldots, n. \tag{7.8}$$

Different from the ordinary way to handle this topic, we do not first prove the existence of the equilibrium but derive it with global stability. Instead of directly studying the system (7.8), we consider its derivative system with respect to $\dot{x}$ and prove that under several diagonal dominant conditions, $\dot{x}$ converges to zero exponentially. This in fact implies that $x(t)$ converges to some equilibrium globally exponentially. This idea comes from [18–20] and can be summarized in the following theorems.

Theorem 7.9 *Suppose that* $g(x) = (g_1(x), \ldots, g_n(x))^\top \in H_2\{G_1, \ldots, G_n\}$ *and* $f(x) = (f_1(x), \ldots, f_n(x))^\top \in H_1\{F_1, \ldots, F_n\}$. *If there are positive constants* $\xi_1, \ldots, \xi_n$ *such that*

$$-\xi_j d_j + [\xi_j a_{jj} + \sum_{i \neq j} \xi_i |a_{ij}|]^+ G_j + \sum_{i=1}^{n} \xi_i |b_{ij}| F_j < 0, \quad j = 1, \ldots, n, \tag{7.9}$$

then the system (7.8) is globally exponentially stable.

Proof According to the condition, there exists some $\alpha > 0$ such that

$$\xi_j(-d_j + \alpha) + [\xi_j a_{jj} + \sum_{i \neq j} \xi_i |a_{ij}|]^+ G_j + \sum_{i=1}^{n} \xi_i |b_{ij}| e^{\alpha \tau_{ij}} F_j \leq 0$$

for all $j = 1, \ldots, n$. Let $v_i(t) = \dot{x}_i(t)$ and $y(t) = e^{\alpha t} v(t)$. Then, for almost every $t \geq 0$, we have

$$\begin{aligned} \frac{dy_i(t)}{dt} &= (-d_i + \alpha) y_i(t) + \sum_{j=1}^{n} a_{ij} g_j'(x_j(t)) y_j(t) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j'(x_j(t - \tau_{ij})) e^{\alpha \tau_{ij}} y_j(t - \tau_{ij}), \quad i = 1, \ldots, n. \end{aligned} \tag{7.10}$$

Define the following candidate Lyapunov functional

$$L(t) = \sum_{i=1}^{n} \xi_i |y_i(t)| + \sum_{i,j=1}^{n} \xi_i |b_{ij}| \int_{t-\tau_{ij}}^{t} e^{\alpha(s+\tau_{ij})} |f_j'(x_j(s))| |v_j(s)| ds. \tag{7.11}$$

Differentiating $L(t)$ gives

$$\begin{aligned} \dot{L}(t) &= \sum_{i=1}^{n} \xi_i \mathrm{sign}\{y_i(t)\} \Big\{ (-d_i + \alpha) y_i(t) + \sum_{j=1}^{n} a_{ij} g_j'(x_j(t)) y_j(t) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j'(x_j(t - \tau_{ij})) e^{\alpha \tau_{ij}} y_j(t - \tau_{ij}) \Big\} + \sum_{i,j=1}^{n} \xi_i |b_{ij}| e^{\alpha(t+\tau_{ij})} \\ &\quad |f_j'(x_j(t))| |v_j(t)| - \sum_{i,j=1}^{n} \xi_i |b_{ij}| |f_j'(x_j(t - \tau_{ij}))| e^{\alpha t} |v_j(t - \tau_{ij})| \\ &\leq \sum_{j=1}^{n} \left\{ \xi_j(-d_j + \alpha) + \left[\xi_j a_{jj} + \sum_{i \neq j} \xi_i |a_{ij}| \right]^+ G_j + \sum_{i=1}^{n} \xi_i |b_{ij}| e^{\alpha \tau_{ij}} F_j \right\} |y_j(t)| \\ &\leq 0. \end{aligned}$$

Therefore, $L(t)$ is bounded and $\sum_{i=1}^{n} \xi_i |\dot{x}_i(t)| = O(e^{-\alpha t})$. By Cauchy convergence principle, there exists an equilibrium point $x^* = (x_1^*, \ldots, x_1^*)^\top$, such that

$$\sum_{i=1}^{n}\xi_i|x_i(t)-x_i^*|=O(e^{-\alpha t}). \tag{7.12}$$

Uniqueness of the equilibrium point can be proved by defining another candidate Lyapunov functional

$$L(t)=\sum_{i=1}^{n}\xi_i|x_i(t)-x_i^*|+\sum_{i,j=1}^{n}|b_{ij}|\xi_i\int_{t-\tau_{ij}}^{t}e^{\alpha(s+\tau_{ij})}|f_j(x_j(s))-f_j(x_j^*)|ds$$

and differentiating it similarly as done above. This completes the proof. $\square$

Another result comes from another Lyapunov functional for $y(t)$.

Theorem 7.10 *Suppose that* $g(x)=(g_1(x),\ldots,g_n(x))^\top\in H_2\{G_1,\ldots,G_n\}$ *and* $f(x)=(f_1(x),\ldots,f_n(x))^\top\in H_1\{F_1,\ldots,F_n\}$. *If there are positive constants* $\xi_1,\ldots,\xi_n$ *such that*

$$-\xi_i d_i+\xi_i a_{ii}^{+}G_i+\sum_{j=1,j\neq i}^{N}\xi_j|a_{ij}|G_j+\sum_{j=1}^{n}\xi_j|b_{ij}|F_j<0,\quad j=1,\ldots,n, \tag{7.13}$$

then the system (7.8) is globally exponentially stable.

Proof Let $v_i(t)$ and $y(t)$ be defined in the same way as in the proof of Theorem 7.9. Define

$$\|y(t)\|_{\{\xi,\infty\}}=\max_{i=1,\ldots,n}\xi_i^{-1}\|y_i(t)\|,\ \varphi(t)=\sup_{0\le s<\tau}\|y(t-s)\|_{\{\xi,\infty\}}.$$

Denoting $i_0=i_0(t)$ by $\xi_{i_0}^{-1}|y_{i_0}(t)|=\|y(t)\|_{\{\xi,\infty\}}$, we have

$$\begin{aligned}
&\xi_{i_0}\frac{d\|y(t)\|_{\{\xi,\infty\}}}{dt}=\mathrm{sign}(x_{i_0}(t))\frac{dy_{i_0}}{dt}\\
&=\mathrm{sign}(x_{i_0}(t))\Big\{-(d_{i_0}-\alpha)y_{i_0}+a_{i_0i_0}g_{i_0}^{'}(x_{i_0}(t))y_{i_0}(t)\\
&+\sum_{j=1,j\neq i_0}^{n}g_j^{'}(y_j)(t)y_j(t)+\sum_{j=1}^{n}b_{ij}f_j^{'}(x_j(t-\tau_{ij}))y_j(t-\tau_{ij})e^{\alpha\tau_{ij}}\Big\}\\
&\le[-(d_{i_0}-\alpha)\xi_{i_0}+a_{i_0i_0}^{+}G_{i_0}\xi_{i_0}]\xi_{i_0}^{-1}|y_{i_0}(t)|+\sum_{j=1,j\neq i_0}^{N}|a_{i_0j}|G_j\xi_j\xi_j^{-1}|y_j(t)|\\
&+\sum_{j=1}^{N}F_j\xi_j|b_{i_0j}|\xi_j^{-1}|y_j(t-\tau_{i_0j})|e^{\alpha\tau_{ij}}.
\end{aligned}$$

If $\varphi(t)$ is strictly monotone increasing at $t = t^*$, then $\varphi(t^*) = \|x(t^*)\|_{\{\xi,\infty\}}$ and we have

$$\xi_{i_0}\frac{d\|y(t)\|_{\{\xi,\infty\}}}{dt} \leq \Bigg\{ -(d_{i_0}-\alpha)\xi_{i_0} + a^+_{i_0i_0}G_{i_0}\xi_{i_0} + \sum_{j=1,j\neq i_0}^{n} |a_{i_0j}|G_j\xi_j \\ + \sum_{j=1}^{n} F_j\xi_j|b_{i_0j}|e^{\alpha\tau_{i_0j}}\Bigg\}\|y(t)\|_{\{\xi,\infty\}} \leq 0,$$

which implies that $\|y(t)\|_{\{\xi,\infty\}}$ is bounded according to Lemma 7.2, i.e., $\max_i \xi_i^{-1}|\dot{x}_i(t)| = O(e^{-\alpha t})$. By the Cauchy convergence principle, there exists an equilibrium point $x^* = (x_1^*,\ldots,x_n^*)^\top$ such that $\max_i \xi_i^{-1}|x_i(t) - x_i^*| = O(e^{-\alpha t})$. The uniqueness of the equilibrium point can be proved by arguments similar to those used in the proof of the previous theorem. □

A direct corollary can be obtained in the M-matrix term.

Corollary 7.11 *Suppose that* $g(x) = (g_1(x),\ldots,g_n(x))^\top \in H_1\{G_1,\ldots,G_n\}$ *and* $f(x) = (f_1(x),\ldots,f_n(x))^\top \in H_1\{F_1,\ldots,F_n\}$. *Let* $G = diag\{G_1,\ldots,G_n\}$ *and* $F = diag\{F_1,\ldots,F_n\}$. *If* $-D+|A|G+|B|F$ *is a M-matrix, then the system (7.8) is globally exponentially stable.*

So far we have studied the exponential stability of delayed Hopfield neural networks with constant delays. However, in many cases the time delays are temporally variant. Then the delayed system can be formulated as

$$\frac{dx_i(t)}{dt} = -d_ix_i(t) + \sum_{j=1}^{n} a_{ij}g_j(x_j(t)) \\ + \sum_{j=1}^{n} b_{ij}f_j(x_j(t-\tau_{ij}(t))) + I_i, \quad i = 1,\ldots,n. \tag{7.14}$$

For the case of bounded delays, i.e., $\tau_{ij}(t) \leq \tau$ for all $i,j = 1,\ldots,n$ and $t \geq 0$, the method in the proof of Theorem 7.10 can be used and the same results can be obtained. However, the case of unbounded delays needs further investigation. It should be pointed out that most of the literature is concerned with stability of delayed neural networks with unbounded delays, which always assumes $\dot{\tau}_{ij}(t) < 1$. Reference [30] presented a novel analysis with a weaker assumption $\tau_{ij}(t) < t$, which includes $\dot{\tau}_{ij} < 1$ as a special case. The result can be summarized as follows.

Theorem 7.12 *Suppose* $\tau_{ij}(t) \leq \mu t$ *for some* $0 < \mu < 1$ *and all* $t \geq 0$, $g(\cdot) \in H_1\{G_1,\ldots,G_n\}$, *and* $f(\cdot) \in H_1\{F_1,\ldots,F_n\}$. *If there are positive constants* $\xi_1,\ldots,\xi_n$ *such that*

$$-\xi_id_i + \sum_{j=1}^{n}\xi_j|a_{ij}|G_j + \sum_{j=1}^{n}\xi_j|b_{ij}|F_j < 0, \quad i = 1,\ldots,n, \tag{7.15}$$

then the system (7.14) has a unique equilibrium x^* which is globally stable in power rate, i.e., there exists some $\gamma > 0$ such that

$$\|x(t) - x^*\| = O(t^{-\gamma})$$

Proof Under the condition (7.15), according to the results in Theorem 7.10, there exists an equilibrium point $x^* = (x_1^*, \ldots, x_n^*)^\top$ for the system (7.14). Moreover, there exists a scalar $\gamma > 0$ and a sufficiently large T, such that for all $t > T$,

$$\left(-d_i + \frac{\gamma}{t}\right)\xi_i + \sum_{j=1}^{n} \xi_j |a_{ij}| G_j + (1-\mu)^{-\gamma} \sum_{j=1}^{n} \xi_j |b_{ij}| F_j < 0, \qquad i = 1, \ldots, n. \tag{7.16}$$

We always assume $t > T$ afterward.

Let $x(t)$ be a solution of the system (7.8). Define $z(t) = t^{\gamma}(x(t) - x^*)$ and

$$M_2(t) = \sup_{s \le t} \|z(s)\|_{\{\xi,\infty\}}. \tag{7.17}$$

We will prove that $M_2(t)$ is bounded. For any t_0 with $\|z(t_0)\|_{\{\xi,\infty\}} = M_2(t_0)$, letting $i_{t_0} = i_{t_0}(t_0)$ be such an index that $|\xi_{i_{t_0}}^{-1} z_{i_{t_0}}(t_0)| = \|z(t_0)\|_{\{\xi,\infty\}}$, we have

$$\begin{aligned}
\left\{\frac{d|z_{i_{t_0}}(t)|}{dt}\right\}_{t=t_0} &= \mathrm{sign}\{z_{i_{t_0}}(t_0)\}\xi_{i_{t_0}} \left(-d_{i_{t_0}} + \frac{\gamma}{t_0}\right) \xi_{i_{t_0}}^{-1} z_{i_{t_0}}(t_0) \\
&\quad + \mathrm{sign}\{z_{i_{t_0}}(t_0)\} t_0^{\gamma} \left\{ \sum_{j=1}^{n} a_{i_{t_0}j} \Big[g_j(u_j(t_0)) - g_j(v_j^*) \Big] \right. \\
&\quad \left. + \sum_{j=1}^{n} b_{i_{t_0}j} \Big[f_j\Big(u_j(t_0 - \tau_{i_{t_0}j}(t_0))\Big) - f_j(v_j^*) \Big] \right\} \\
&\le \left\{ \xi_{i_{t_0}} \left(-d_{i_{t_0}} + \frac{\gamma}{t_0}\right) + \sum_{j=1}^{n} \xi_j |a_{i_{t_0}j}| G_j \right\} \|z(t_0)\|_{\{\xi,\infty\}} \\
&\quad + \sum_{j=1}^{n} \xi_j |b_{i_{t_0}j}| F_j \left[\frac{t_0}{t_0 - \tau_{i_{t_0}j}(t_0)} \right]^{\gamma} \xi_j^{-1} |z_j\Big(t_0 - \tau_{i_{t_0}j}(t_0)\Big)| \\
&\le \left\{ \xi_{i_{t_0}} \left(-d_{i_{t_0}} + \frac{\gamma}{t_0}\right) + \sum_{j=1}^{n} \xi_j |a_{i_{t_0}j}| G_j \right. \\
&\quad \left. + \sum_{j=1}^{n} \xi_j |b_{i_{t_0}j}| F_j \left\{ \frac{t_0}{t_0 - \tau_{i_{t_0}j}(t_0)} \right\}^{\gamma} \right\} M_2(t_0)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{\xi_{i_{t_0}}\left(-d_{i_{t_0}}+\frac{\gamma}{t_0}\right)\right.\\
&\quad \left.+\sum_{j=1}^{n}\xi_j|a_{i_{t_0}j}|G_j+(1-\mu)^{-\gamma}\sum_{j=1}^{n}\xi_j|b_{i_{t_0}j}|F_j\right\}M_2(t_0)\\
&< 0.
\end{aligned}
$$

By Lemma 7.2, we can conclude that $M_2(t)$ is bounded, which implies that $\|u(t) - v^*\|_{\{\xi,\infty\}} = O(t^{-\gamma})$, which completes the proof. □

We give a numerical example to verify the theoretical results. We consider the following system

$$
\dot{x}(t) = -5x(t) + x(t - \tau(t)), \tag{7.18}
$$

where $\tau(t) \leq \mu t$, with $\mu = 0.5$. The power convergence is shown in Fig. 7.2. The slope of the straight line is approximately -2.3221, which means that $x(t) \approx O(t^{-2.3217})$. The theoretical result is $x(t) \approx O(t^{-\gamma})$, where $\gamma \approx -\frac{\log 5}{\log(1-\mu)} = -\frac{\log 5}{\log(0.5)} \approx 2.3219$, which agrees well with the numerical result.

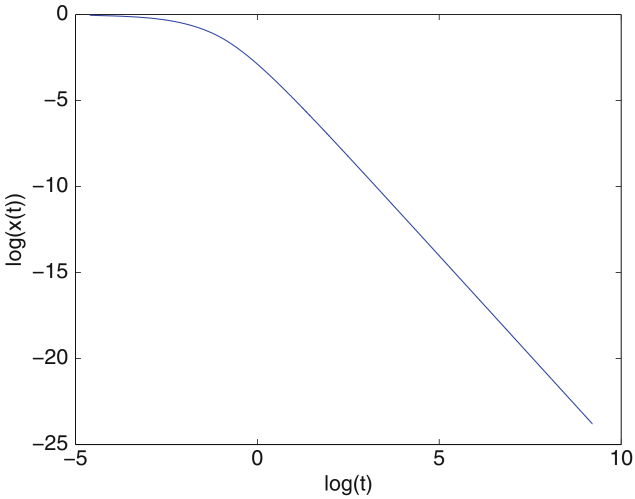

Fig. 7.2 Illustration of power stability. Slope of the straight line is -2.3221

7.2.3 Delayed Cohen–Grossberg Competitive and Cooperative Networks

We consider delayed Cohen–Grossberg neural networks with a uniform delay, which can be formalized as follows:

$$\frac{dx_i(t)}{dt} = a_i(x_i(t))\left[-d_i(x_i) + \sum_{j=1}^{n} a_{ij}g_j(x(t)) \ + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau)) + I_i\right], \quad i = 1,\dots,n. \tag{7.19}$$

This model is very general, and includes a large class of existing neural field and evolution models. For instance, assuming that $a_i(\rho) = 1$ for all $\rho \in \mathbb{R}$ and $i = 1,\dots,n$, then it is the famous Hopfield neural network, which can be written as

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau)) + I_i, \ i = 1,\dots,n$$

with $d_i(\rho) = d_i\rho$ for given $d_i > 0, i = 1,\dots,n$. It also includes the famous Volterra–Lotka competitive-cooperation equations:

$$\frac{dx_i}{dt} = A_i x_i\left(I_i - \sum_{j=1}^{n} a_{ij}x_j\right), \ i = 1,\dots,n$$

with $a_i(\rho) = A_i\rho$, for all $\rho > 0$ and given $A_i > 0$, and $g_i(\rho) = \rho$, $i = 1,\dots,n$. Most existing results in the literature are based on the assumption that the amplifier function $a_i(\cdot)$ is always *positive* (see [29, 71, 82]). But in the original papers [35, 46, 47] this model was proposed as a kind of competitive-cooperation dynamical system for decision rules, pattern formation, and parallel memory storage. Here, the state of the neuron x_i might be the population size, activity, or concentration, etc., of the i-th species in the system, which is always nonnegative for all time. To guarantee the positivity of the states, one should assume $a_i(\rho) > 0$ for all $\rho > 0$ and $a_i(0) = 0$ for all $i = 1,\dots,n$.

The purpose of this section is to study the convergent dynamics of the delayed Cohen–Grossberg neural networks without assuming the strict positivity of $a_i(\cdot)$, symmetry of the connection matrix, or boundedness of the activation functions, but with considering a time delay. Hereby, we focus our study of the dynamical behavior on the first orthant: $\mathbb{R}^n_+ = \{(x_1,\dots,x_n)^\top \in \mathbb{R}^n\colon x_i \geq 0, \ i = 1,\dots,n\}$ and introduce the concept of $\mathbb{R}^n_+$-global stability, which means that all trajectories are initiated in the first orthant $\mathbb{R}^n_+$ instead of the whole space $\mathbb{R}^n$. We point out that an asymptotically stable nonnegative equilibrium is closely related to the solution of a Nonlinear Complementary Problem (NCP). Based on the Linear Matrix Inequality (LMI) technique (for more details on LMI, see [13]) and NCP theory (for more details on NCP, we refer to [76]), we give a sufficient condition for existence and uniqueness of nonnegative equilibrium. Moreover, the $\mathbb{R}^n_+$-global asymptotic stability and exponential stability of the equilibrium are investigated. The main results of this section comes from [67].

Let $x(t) = (x_i(t), x_2(t),\dots,x_n(t))^\top$, $d(x) = (d_i(x_i), d_2(x_2),\dots,d_n(x_n))^\top$, $g(x) = (g_1(x_1), g_2(x_2),\dots,g_n(x_n))^\top$, $a(x) = \mathrm{diag}\{a_1(x_1), a_2(x_2),\dots,a_n(x_n)\}$, $A = (a_{ij})^n_{i,j=1}$,

$B = (b_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n,n}$, and $I = (I_1, I_2, \ldots, I_n)^\top$. Then, the system (7.19) can be rewritten in the matrix form:

$$\frac{dx(t)}{dt} = a(x)\Big[-d(x) + Ag(x(t)) + Bg(x(t-\tau)) + I\Big]. \tag{7.20}$$

For the amplifier and activation functions, we give the following assumptions:

(i) $a(\,\cdot\,) \in \mathcal{A}_1$; that is, every $a_i(\rho)$ is continuous with $a_i(0) = 0$, and $a_i(\rho) > 0$, whenever $\rho > 0$;
(ii) $a(\,\cdot\,) \in \mathcal{A}_2$; that is, $a(\,\cdot\,) \in \mathcal{A}_1$, and for any $\epsilon > 0$, $\int_0^\epsilon d\rho/a_i(\rho) = +\infty$ for all $i = 1, \ldots, n$;
(iii) $a(\,\cdot\,) \in \mathcal{A}_3$; that is, $a(\,\cdot\,) \in \mathcal{A}_1$, and for any $\epsilon > 0$, $\int_\epsilon^\infty \rho\, d\rho/a_i(\rho) = +\infty$ for all $i = 1, \ldots, n$;
(iv) $a(\cdot) \in \mathcal{A}_4$; that is, $a(\cdot) \in \mathcal{A}_1$, and for any $\epsilon > 0$, $\int_0^\epsilon \rho\, d\rho/a_i(\rho) < +\infty$ for all $i = 1, \ldots, n$;
(v) $d(\cdot) \in \mathcal{D}$; that is, $d_i(\cdot)$ is continuous and satisfies $[d_i(\xi) - d_i(\zeta)]/(\xi - \zeta) \geq D_i$, for all $\xi \neq \zeta$, where D_i are positive constants, $i = 1, \ldots, n$, and $g(\,\cdot\,)$ belongs to $H_2\{G_1, \ldots, G_n\}$ for some $G_i > 0$, $i = 1, \ldots, n$.

First, we define positive solutions componentwise.

Definition 7.13 A solution $x(t)$ of the system (7.20) is said to be a positive solution if for every positive initial condition $\phi(t) > 0$, $t \in [-\tau, 0]$, the trajectory $x(t) = (x_1(t), \ldots, x_n(t))^\top$ satisfies $x_i(t) > 0$ for all $t \geq 0$ and $i = 1, \ldots, n$.

Lemma 7.14 (Positive Solution) *If $a(\cdot) \in \mathcal{A}_2$, then the solution of the system (7.20) is a positive solution.*

Proof Assume that the initial value $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))^\top$ satisfies $\phi_i(t) > 0$ for $i = 1, \ldots, n$ and $t \in [-\tau, 0]$. Suppose for some $t_0 > 0$ and some index i_0, $x_{i_0}(t_0) = 0$. Then, the assumption $a(\,\cdot\,) \in \mathcal{A}_2$ leads

$$\begin{aligned}&\int_0^{t_0}\Big[-d_i(x_i(t)) + \sum_{j=1}^n a_{ij}g_j(x_j(t)) + \sum_{j=1}^n b_{ij}g_j(x_j(t-\tau)) + I_i\Big]dt\\ &= \int_0^{t_0}\frac{\dot{x}_i(t)dt}{a_i(x_i(t))} = -\int_0^{\phi_i(0)}\frac{d\rho}{a_i(\rho)} = -\infty,\end{aligned}$$

which is impossible due to the continuity of $x_i(\,\cdot\,)$ on $[0,\, t_0]$. Hence, $x_i(t) \neq 0$ for all $t \geq 0$ and $i = 1, \ldots, n$. This implies that $x_i(t) > 0$ for all $t \geq 0$ and $i = 1, \ldots, n$. $\square$

By this lemma we can actually concentrate on the first orthant $\mathbb{R}^n_+$. If $a(\cdot) \in \mathcal{A}_1$, then any equilibrium in $\mathbb{R}^n_+$ of the system (7.19) is a solution of the equations

$$x_i[f_i(x) - I_i] = 0,\ i = 1, \ldots, n, \tag{7.21}$$

where $f_i(x) = d_i(x_i) - \sum_{j=1}^{n} (a_{ij} + b_{ij})g_j(x_j)$, $i = 1, \ldots, n$. Even though (7.21) might possess multiple solutions, we can show that an asymptotically stable nonnegative equilibrium is just a solution of a nonlinear complementary problem (NCP).

Proposition 7.15 *Suppose $a(\cdot) \in \mathcal{A}_1$. If $x^* = (x_i^*, \ldots, x_n^*)^\top \in \mathbb{R}_+^n$ is an asymptotically stable equilibrium of the system (7.20), then it must be a solution of the following nonlinear complementary problem (NCP):*

$$x_i^* \geq 0, \quad f_i(x^*) - I_i \geq 0, \quad x_i^*(f_i(x^*) - I_i) = 0, \quad i = 1, \ldots, n, \tag{7.22}$$

where $f_i(x) = d_i(x_i) - \sum\limits_{j=1}^{n} (a_{ij} + b_{ij})g_j(x_j)$, $i = 1, \ldots, n$.

Proof Suppose that $x^* \in \mathbb{R}_+^n$ is an asymptotically stable equilibrium of the system (7.20). Then $x_i^* > 0$ or $x_i^* = 0$. In case $x_i^* > 0$, we have $f_i(x_i^*) - I_i = 0$. If $x_i^* = 0$, we claim that $f_i(x_i^*) - I_i \geq 0$. Otherwise, if $f_{i_0}(x^*) - I_{i_0} < 0$ for some index i_0, then $\dot{x}_{i_0}(t) = a_i(x_{i_0}(t))[-f_{i_0}(x_{i_0}(t)) + I_{i_0}] > (1/2)a_i(x_{i_0}(t))[-f_{i_0}(x^*) + I_{i_0}] > 0$ when $x_{i_0}(t)$ is sufficiently close to x^*, which implies that $x_{i_0}(t)$ will never converge to 0. Therefore, x^* is unstable. □

Thus, we can propose a definition of a nonnegative equilibrium of the system (7.20).

Definition 7.16 x^* is said to be a nonnegative equilibrium of the system (7.20) in the NCP sense, if x^* is the solution of the Nonlinear Complementarity Problem (NCP) (7.22); moreover, if $x_i^* > 0$, for all $i = 1, \ldots, n$, then x^* is said to be a positive equilibrium of system (7.20). In this case, x^* must satisfy

$$d(x^*) - (A + B)g(x^*) + I = 0,\ x_i^* > 0, \quad i = 1, \ldots, n,$$

where $\mathbf{0} = (0, \ldots, 0)^\top \in \mathbb{R}^n$.

Definition 7.17 A nonnegative equilibrium x^* of the system (7.20) in the NCP sense is said to be $\mathbb{R}_+^n$-globally asymptotically stable if for any positive initial condition $\phi_i(t) > 0$, $t \in [-\tau, 0]$ and $i = 1, \ldots, n$, the trajectory $x(t)$ of the system (7.20) satisfies $\lim\limits_{t\to\infty} x(t) = x^*$. Moreover, if there exist constants $M > 0$ and $\epsilon > 0$ such that

$$\|x(t) - x^*\| \leq Me^{-\epsilon t}, \quad t \geq 0,$$

then x^* is said to be $\mathbb{R}_+^n$-exponentially stable.

So, we discuss the existence and uniqueness of the nonnegative equilibrium in the NCP sense.

Theorem 7.18 (Existence and Uniqueness of Nonnegative Equilibrium) *Suppose $a(\cdot) \in \mathcal{A}_2$, $d(\cdot) \in \mathcal{D}$, and $g(\cdot) \in H_2\{G_1, \ldots, G_n\}$ for $G_i > 0$, $i = 1, \ldots, n$.*

Let $D = diag\{D_1, \ldots, D_n\}$ and $G = diag\{G_1, \ldots, G_n\}$. If there exists a positive definite diagonal matrix $P = diag\{P_1, P_2, \ldots, P_n\}$ such that

$$\left\{P[DG^{-1} - (A+B)]\right\}^s > 0, \tag{7.23}$$

then for each $I \in \mathbb{R}^n$, there exists a unique nonnegative equilibrium of the system (7.20) in the NCP sense.

The proof is given in the Appendix.

The following corollary is a direct consequence of Theorem 7.18.

Corollary 7.19 *Suppose $a(\cdot) \in \mathcal{A}_2$, $d(\cdot) \in \mathcal{D}$, and $g(\cdot) \in H_2\{G_1, \ldots, G_n\}$ for $G_i > 0$, $i = 1, \ldots, n$. Let $D = diag\{D_1, \ldots, D_n\}$ and $G = diag\{G_1, \ldots, G_n\}$. If there exist a positive definite diagonal matrix P and a positive definite symmetric matrix Q such that*

$$\begin{bmatrix} 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -B^\top P & Q \end{bmatrix} > 0, \tag{7.24}$$

then for each $I \in \mathbb{R}^n$, there exists a unique nonnegative equilibrium for the system (7.20) in the NCP sense.

Let x^* be the nonnegative equilibrium of the system (7.20) in the NCP sense and $y(t) = x(t) - x^*$. Thus, the system (7.19) can be rewritten as

$$\frac{dy_i(t)}{dt} = a_i^*(y_i(t))\left[-d_i^*(y_i(t)) + \sum_{j=1}^n a_{ij} g_j^*(y_j(t)) + \sum_{j=1}^n b_{ij} g_j^*(y_j(t-\tau)) + J_i\right]$$

or in matrix form

$$\frac{dy(t)}{dt} = a^*(y(t))\left[-d^*(y(t)) + Ag^*(y(t)) + Bg^*(y(t-\tau)) + J\right], \tag{7.25}$$

where for $i = 1, \ldots, n$, $a_i^*(s) = a_i(s + x_i^*)$, $a^*(y) = diag\{a_1^*(y_1), \ldots, a_n^*(y_n)\}$, $d_i^*(s) = d_i^*(s + x_i^*) - d_i^*(x_i^*)$, $d^*(y) = [d_1^*(y_1), \ldots, d_n^*(y_n)]^\top$, $g_i^*(s) = g_i^*(s + x_i^*) - g_i^*(x_i^*)$, $g^*(y) = [g_1^*(y_1), \ldots, g_n^*(y_n)]^\top$, and

$$J_i = \begin{cases} -d_i(x_i^*) + \sum_{j=1}^n (a_{ij} + b_{ij}) g_j(x_j^*) + I_i & x_i^* = 0 \\ 0 & x_i^* > 0 \end{cases} \qquad J = (J_1, \ldots, J_n)^\top.$$

Since x^* is the nonnegative equilibrium of (7.20) in the NCP sense, i.e., the solution of NCP (7.7), $J_i \leq 0$ holds for all $i = 1, \ldots, n$ which implies that $g_i^*(y_i(t)) J_i \leq 0$ for all $i = 1, \ldots, n$ and $t \geq 0$.

Theorem 7.20 ($\mathbb{R}^n_+$-Global Asymptotic Stability of the Nonnegative Equilibrium) *Suppose $a(\cdot) \in \mathcal{A}_2 \bigcap \mathcal{A}_3 \bigcap \mathcal{A}_4$, $d(\cdot) \in \mathcal{D}$, and $g(\cdot) \in H_2\{G_1, \ldots, G_n\}$ for $G_i > 0$, $i = 1, \ldots, n$. Let $D = diag\{D_1, \ldots, D_n\}$ and $G = diag\{G_1, \ldots, G_n\}$. If there exist a positive definite diagonal matrix $P = diag\{P_1, \ldots, P_n\}$ and a positive definite symmetric matrix Q such that*

$$\begin{bmatrix} 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -B^\top P & Q \end{bmatrix} > 0, \tag{7.26}$$

then the unique nonnegative equilibrium x^* for the system (7.20) in the NCP sense is $\mathbb{R}^n_+$-globally asymptotically stable.

Proof Without loss of generality, we assume $x_i^* = 0$, $i = 1, 2, \ldots, p$ and $x_i^* > 0$, $i = p+1, \ldots, n$ for some integer p. By the assumptions $\mathcal{A}_3$ and $\mathcal{A}_4$, it can be seen that

$$\int_0^{y_i(t)} \frac{\rho d\rho}{a_i^*(\rho)} < +\infty, \quad \int_0^{+\infty} \frac{\rho d\rho}{a_i^*(\rho)} = +\infty, \quad \int_0^{y_i(t)} \frac{g_i^*(\rho) d\rho}{a_i^*(\rho)} < +\infty$$

for $i = 1, \ldots, n$ and $t \geq 0$. By inequality (7.26), there exists $\beta > 0$ such that

$$Z = \begin{bmatrix} 2\beta D & -\beta A & -\beta B \\ -\beta A^\top & 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -\beta B^\top & -B^\top P & Q \end{bmatrix} > 0.$$

Let

$$V(t) = 2\beta \sum_{i=1}^n \int_0^{y_i(t)} \frac{\rho d\rho}{a_i^*(\rho)} + 2 \sum_{i=1}^n P_i \int_0^{y_i(t)} \frac{g_i^*(\rho) d\rho}{a_i^*(\rho)} + \int_{t-\tau}^t g^{*\top}(y(s)) Q g^*(y(s)) ds.$$

It is easy to see that $V(t)$ is positive definite and radially unbounded. Noting $g_i^*(y_i(t)) J_i \leq 0$, we have

$$\begin{aligned}
\frac{dV(t)}{dt} &= 2\beta \sum_{i=1}^n y_i(t) \Big[-d_i^*(y_i(t)) + \sum_{j=1}^n a_{ij} g_j^*(y_j(t)) + \sum_{j=1}^n b_{ij} g_j^*(y_j(t-\tau)) + J_j \Big] \\
&\quad + 2 \sum_{i=1}^n P_i g_i^*(y_i(t)) \Big[-d_i^*(y_i(t)) + \sum_{j=1}^n a_{ij} g_j^*(y_j(t)) + \sum_{j=1}^n g_j^*(y_j(t-\tau)) + J_j \Big] \\
&\quad + g^{*\top}(y(t)) Q g^*(y(t)) - g^{*\top}(y(t-\tau)) Q g^*(y(t-\tau)) \\
&\leq -2\beta \Big[y^\top(t) D y(t) - y^\top(t) A g^*(y(t)) - y^\top(t) B g^*(y(t-\tau)) \Big] \\
&\quad -2 \Big[g^{*\top}(y(t)) P D G^{-1} g^*(y(t)) - g^{*\top}(y(t)) P B g^*(y(t))
\end{aligned}$$

$$
\begin{aligned}
&\quad -g^{*\top}(y(t))PBg^*(y(t-\tau))\Big] \\
&\quad +g^{*\top}(y(t))Qg^*(y(t)) - g^{*\top}(y(t-\tau))Qg^*(y(t-\tau)) \\
&= -[y^\top(t), g^{*\top}(y(t)), g^{*\top}(y(t-\tau))]Z \begin{bmatrix} y(t) \\ g^*(y(t)) \\ g^*(y(t-\tau)) \end{bmatrix} \le -\delta y^\top(t)y(t),
\end{aligned}
$$

where $\delta = \lambda_{\min}(Z) > 0$. Therefore, $\lim_{t\to\infty} \|y(t)\|_2 = 0$. This completes the proof. □

In the following, we present a numerical example to verify the theoretical results obtained above and compare the convergent dynamics of the Cohen–Grossberg neural systems with an amplification function which is always positive versus an amplification function which is only positive in the first orthant. A result for positive amplification function was provided in [28, 69].

Theorem 7.21 *Suppose that* $g \in H_2\{G_1, G_2, ..., G_n\}$ *and there exists* $\alpha > 0$ *such that* $a_i(\rho) > \alpha$ *for any* $\rho \in \mathbb{R}$ *and* $i = 1, \ldots, n$. *If there exist a positive definite diagonal matrix P and a positive definite matrix Q such that inequality (7.26) holds, then for each* $I \in \mathbb{R}^n$, *the system (7.20) has a unique equilibrium point that is globally exponentially stable.*

Consider the dynamical behavior of the following two systems:

$$
\begin{cases}
\frac{dx_1(t)}{dt} = x_1(t)\Big[- 6x_1(t) + 2g(x_1(t)) - g(x_2(t)) \\
\qquad\qquad +3g(x_1(t-2)) + g(x_2(t-2)) + I_1 \Big] \\
\frac{dx_2(t)}{dt} = x_2(t)\Big[- 6x_2(t) - 2g(x_1(t)) \\
\qquad\qquad +3g(x_2(t)) + \frac{1}{2}g(x_1(t-2)) + 2g(x_2(t-2)) + I_2 \Big],
\end{cases}
\tag{7.27}
$$

$$
\begin{cases}
\frac{du_1(t)}{dt} = \frac{1}{|u_1(t)|+1}\Big[- 6u_1(t) + 2g(u_1(t)) \\
\qquad\qquad -g(u_2(t)) + 3g(u_1(t-2)) + g(u_2(t-2)) + I_1 \Big] \\
\frac{du_2(t)}{dt} = \frac{1}{|u_2(t)|+1}\Big[- 6u_2(t) - 2g(u_1(t)) \\
\qquad\qquad +3g(u_2(t)) + \frac{1}{2}g(u_1(t-2)) + 2g(u_2(t-2)) + I_2 \Big],
\end{cases}
\tag{7.28}
$$

where $g(\rho) = (1/2)(\rho + \arctan(\rho))$ and $I = (I_1, I_2)^\top$ is the constant input that will be determined below. Furthermore,

$$
D = 6 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}.
$$

By the Matlab LMI and Control Toolbox, we obtain

$$P = \begin{bmatrix} 0.2995 & 0 \\ 0 & 0.3298 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.0507 & 0.3258 \\ 0.3258 & 0.9430 \end{bmatrix}.$$

The eigenvalues of

$$Z = \begin{bmatrix} 2PDG^{-1} - PA - A^{\top}P - Q & -PB \\ -B^{\top}P & Q \end{bmatrix}$$

are 2.6490, 1.1343, 0.5302, and 0.0559, which implies that Z is positive definite. By Theorem 7.20, for any $I \in \mathbb{R}^2$, the system (7.27) has a unique nonnegative equilibrium x^* in the NCP sense which is $\mathbb{R}^2_+$-globally asymptotically stable. By Theorem 7.21, for any $I \in \mathbb{R}^2$, system (7.28) has a unique equilibrium x^0, which is globally asymptotically stable in $\mathbb{R}^2$.

In case $I = (1, 0.1)^{\top}$, the equilibria of the system (7.27) are $(0,0)^{\top}$, $(0.7414, 0)^{\top}$, $(0, 0.0992)^{\top}$, and $(0.7414, -0.7062)^{\top}$. Among them, $x^* = (0.7414, 0)^{\top}$ is the nonnegative equilibrium of the system (7.27) in the NCP sense and $x^0 = (0.7414, -0.7062)$ is the unique equilibrium of the system (7.28). Pick initial condition $\phi_1(t) = (7/2)(\cos(t) + 1)$ and $\phi_2(t) = e^{-t}$, for $t \in [-2, 0]$. Figure 7.3 shows that the solution of the system (7.27) converges to $x^* = (0.7414, 0)^{\top}$, while the solution of the system (7.28) converges to $x^0 = (0.7414, -0.7062)$

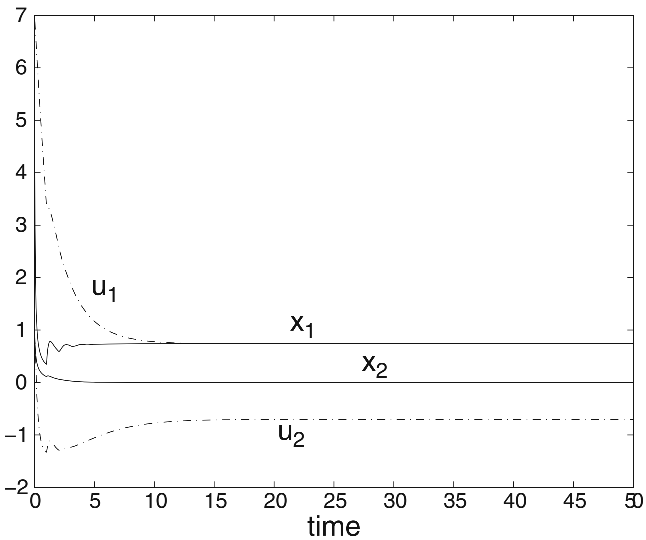

Fig. 7.3 Dynamical behavior of systems (7.27) and (7.28) with $I = (1, 0.1)^{\top}$

7.3 Periodicity and Almost Periodicity of Delayed Neural Networks

In this section, we discuss a large class of delayed neural networks with time-varying inhibitions, interconnection weights, and inputs which can be periodic or almost periodic. We will prove that under several diagonal dominant conditions, the periodic or almost periodic system has at least one periodic or almost periodic solution, respectively, which is globally stable. Moreover, the equilibrium of the delayed neural networks with constant coefficients can be regarded as a periodic orbit with arbitrary period.

We consider a rather general delayed system,

$$\begin{aligned}\frac{du_i}{dt} &= -d_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(u_j(t)) \\ &\quad + \sum_{j=1}^{n} \int_0^{\infty} f_j(u_j(t-\tau_{ij}-s))d_sK_{ij}(t,s) + I_i(t), \quad i = 1,\dots,n \end{aligned} \tag{7.29}$$

or

$$\begin{aligned}\frac{du_i}{dt} &= -d_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(u_j(t)) \\ &\quad + \sum_{j=1}^{n} \int_0^{\infty} f_j(u_j(t-\tau_{ij}(t)-s))d_sK_{ij}(t,s) + I_i(t),\ i = 1,\dots,n, \end{aligned} \tag{7.30}$$

where for any fixed t, $d_sK_{ij}(t,s)$ are Lebesgue–Stieltjes measures with respect to s.

This model contains many delayed recurrent neural network models as special cases. For example, if $d_sK_{ij}(t,t_k) = b_{ij}^k(t)$ for $0 < t_1 < \cdots t_m < \infty$ and $d_sK_{ij}(t,s) = 0$ for $s \neq t_k$, we obtain the following system with multi-discrete delays,

$$\begin{aligned}\frac{du_i}{dt} &= -d_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(u_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij}^k(t)f_j(u_j(t-\tau_{ij}(t)-t_k)) + I_i(t), \quad i = 1,2,\dots,n. \end{aligned} \tag{7.31}$$

Instead, if $d_sK_{ij}(t,s) = b_{ij}(t)k_{ij}(s)ds$, then we have the following system with distributed delays,

$$\frac{du_i(t)}{dt} = -d_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(u_j(t))$$
$$+ \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} k_{ij}(s)f_j(u_j(t-\tau_{ij}(t)-s))ds + I_i(t),\ i=1,2,\ldots,n. \tag{7.32}$$

From [49], one can see that if the activation functions $g_i(\cdot)$ and $f_i(\cdot)$ are Lipschitz continuous, then the system has a unique solution for any bounded continuous initial condition.

Periodicity and almost periodicity are defined as follows.

Definition 7.22 A vector-valued function $x(t)$: $\mathbb{R} \to \mathbb{R}^n$ is said to be periodic if there exists $\omega > 0$ such that $x(t+\omega) = x(t)$ for all $t \in \mathbb{R}$. In this case, ω is called the period of $x(t)$. The function $x(t)$ is said to be almost periodic on $\mathbb{R}$ if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, such that for any interval with length $l(\epsilon)$, there exists a number $\omega = \omega(\epsilon)$ in this interval such that $\|x(t+\omega) - x(t)\| < \epsilon$ for all $t \in \mathbb{R}$.

The key problem of this section is to prove the existence of a periodic or almost periodic solution. Different from the existing literature, which uses Mawhin coincidence degree theory (see [44]), we use two methods to prove existence. The first method is to regard the periodic solution as a fixed point of a Poincaré–Andronov map [63]. A basic result is the famous Brouwer fixed point theorem [60].

Lemma 7.23 *A continuous map T over a compact subset Ω of a Banach space such that $T(\Omega) \subset \Omega$ has at least one fixed point, namely, there exists $\omega^* \in \Omega$ such that $T(\omega^*) = \omega^*$.*

The second method is to regard the periodic or almost periodic solution as a limit of a solution of (7.29). See [27].

The global stability of such periodic or almost periodic solutions is studied by Lyapunov and Lyapunov–Krasovskii methods.

7.3.1 Delayed Periodic Hopfield Neural Networks

Considering the system (7.30), we give the following hypotheses.

$\mathcal{B}_1$:

(1) $d_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$, $I_i(t)$, $\tau_{ij}(t)$:$\mathbb{R}^+ \to \mathbb{R}$ are continuous functions, and $d_sK_{ij}(t,s)$ is continuous in the sense that $\lim_{h\to\infty}\int_0^{\infty} |d_sK_{ij}(t+h,s) - d_sK_{ij}(t,s)| = 0$ for all $i,j = 1,\ldots,n$; they are all periodic functions with period $\omega > 0$, i.e., $d_i(t+\omega) = d_i(t)$, $a_{ij}(t) = a_{ij}(t+\omega)$, $b_{ij}(t) = b_{ij}(t+\omega)$, $I_i(t) = I_i(t+\omega)$, $\tau_{ij}(t+\omega) = \tau_{ij}(t)$, and $dK_{ij}(t+\omega,s) = d_sK_{ij}(t,s)$ for all $t > 0$ and $i,j = 1,\ldots,n$.

(2) $g(\cdot) \in H_2\{G_1,\ldots,G_n\}$ and $f(\cdot) \in H_1\{F_1,\ldots,F_n\}$ for some positive constants G_i and F_i, $i = 1,\ldots,n$;

(3) the initial condition $x(\theta) = \phi(\theta), \theta \in (-\infty, 0]$ satisfies that $\phi \in C((-\infty, 0], \mathbb{R}^n)$ is bounded.

The following result is concerned with the Poincaré–Andronov map.

Theorem 7.24 *Suppose that the hypotheses $\mathcal{B}_1$ above are satisfied. If there exist positive constants $\xi_1, \xi_2, \ldots, \xi_n$ such that for all $\omega \geq t > 0$,*

$$-\xi_i d_i(t) + \sum_{j=1}^{n} \xi_j G_j |a_{ij}(t)| + \sum_{j=1}^{n} \xi_j F_j \int_0^{\infty} |d_s K_{ij}(t,s)| < 0, \quad i = 1, 2, \ldots, n, \tag{7.33}$$

then the system (7.30) has at least one ω-periodic solution $v(t)$. In addition, if there exists a constant α such that for all $\omega \geq t > 0$,

$$\begin{aligned} &-\xi_i(d_i(t) - \alpha) + \sum_{j=1}^{n} \xi_j G_j |a_{ij}(t)| \\ &+ \sum_{j=1}^{n} \xi_j F_j e^{\alpha \tau_{ij}} \int_0^{\infty} e^{\alpha s} |d_s K_{ij}(t,s)| \leq 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{7.34}$$

then for any solution $x(t) = (x_1(t), \ldots, x_n(t))$ of (7.30),

$$\|x(t) - v(t)\| = O(e^{-\alpha t}) \quad t \to \infty. \tag{7.35}$$

Proof Pick a constant M satisfying $M > J/\eta$, where

$$J = \max_i \max_t \left\{ \sum_{j=1}^{n} |a_{ij}(t)| C_j + \sum_{j=1}^{n} D_j \int_0^{\infty} |d_s K_{ij}(t,s)| + |I_i(t)| \right\}$$

and let $C = C((-\infty, 0], \mathbb{R}^n)$ be the Banach space with norm

$$\|\phi\| = \sup_{\{-\infty < \theta \leq \omega\}} \|\phi(\theta)\|_{\{\xi,\infty\}}.$$

Denote

$$\Omega = \{x(\theta) \in C : \|x(\theta)\| \leq M, \|\dot{x}(\theta)\| \leq N\},$$

where $N = (\alpha+\beta+\gamma)M + c$, $\alpha = \max_i \sup_t |d_i(t)|\xi_i^{-1}$, $\beta = \max_{i,j} \sup_t |a_{ij}(t)|\xi_i^{-1} G_j$, $\gamma = \max_{i,j} \sup_t \int_0^{\infty} |d_s K_{ij}(t,s)| F_j \xi_i^{-1}$, and $c = \max_i \sup_{0 \leq t < \omega} |I_i(t)|\xi_i^{-1}$. It is easy to check that Ω is a convex compact set.

Now, define a map T from Ω to C by

$$T{:}\phi(\theta) \to x(\theta+\omega,\phi),$$

where $x(t) = x(t,\phi)$ is the solution of the system (7.29) with the initial condition $x_i(\theta) = \phi_i(\theta)$, for $\theta \in (-\infty, 0]$ and $i = 1, \ldots, n$.

In the following, we will prove that $T\Omega \subset \Omega$, i.e., if $\phi \in \Omega$, then $x \in \Omega$. To do that, we define the following function

$$M(t) = \sup_{s\in(-\infty,0]} \|x(t+s)\|_{\{\xi,\infty\}}.$$

It is easy to see that $\|x(t)\|_{\{\xi,\infty\}} \leq M(t)$. Therefore, what we need to do is to prove $M(t) \leq M$ for all $t > 0$.

Assume that $t_0 \geq 0$ is the smallest value such that $\|x(t_0)\|_{\{\xi,\infty\}} = M(t_0) = M$, and $\|x(t)\|_{\{\xi,\infty\}} \leq M$ *if* $t < t_0$. Let i_0 be an index such that $\xi_{i_0}^{-1}|x_{i_0}(t)| = \|x(t_0)\|_{\{\xi,\infty\}}$. Then, direct calculations give

$$\begin{aligned}
\left\{\frac{d|x_{i_0}(t)|}{dt}\right\}_{t=t_0} &\leq \text{sign}(x_{i_0}(t_0))\Bigg\{-d_{i_0}(t_0)x_{i_0}(t_0) + \sum_{j=1}^n a_{i_0j}(t_0)g_j(x_j) \\
&\quad + \sum_{j=1}^n \int_0^\infty f_j(x_j(t_0 - \tau_{i_0j}(t_0) - s))d_sK_{i_0j}(t_0,s) + I_{i_0}(t_0)\Bigg\} \\
&\leq \left[-d_{i_0}\xi_{i_{t_0}} + \sum_{j=1}^n |a_{i_0j}(t_0)|G_j\xi_j\right]\|x(t_0)\|_{\{\xi,\infty\}} \\
&\quad + \sum_{j=1}^n F_j\xi_j \int_0^\infty \|x(t_0 - \tau_{i_0j}(t_0) - s)\|_{\{\xi,\infty\}}|d_sK_{i_0j}(t_0,s)| + J \\
&\leq \Bigg[-d_{i_0}\xi_{i_0} + \sum_{j=1}^n |a_{i_0j}(t_0)|G_j\xi_j \\
&\quad + \sum_{j=1}^n F_j\xi_j \int_0^\infty |d_sK_{i_0j}(t_0,s)|\Bigg]M(t_0) + J \\
&\leq -\eta M(t_0) + J = -\eta M + J < 0,
\end{aligned}$$

which implies that $\|x(t)\|_{\{\xi,\infty\}}$ can never exceed M. Thus, $\|x(t)\|_{\{\xi,\infty\}} \leq M(t) \leq M$ for all $t > t_0$. Moreover, it is easy to see that $\|\dot{x}(\theta+\omega)\| \leq N$. Therefore, $T\Omega \subset \Omega$. By Lemma 7.23, there exists $\phi^* \in \Omega$ such that $T\phi^* = \phi^*$. Hence $x(t,\phi^*) = x(t, T\phi^*)$, i.e., $x(t,\phi^*) = x(t+\omega,\phi^*)$, which is an ω-periodic solution of the system (7.30).

Now, we prove that inequality (7.34) leads to the global attractivity of the periodic solution. Let $\bar{x}(t) = [x(t) - v(t)]$ and $z(t) = e^{\alpha t}\bar{x}(t)$. We have

$$\frac{dz_i(t)}{dt} = -(d_i(t)-\alpha)z_i(t) + e^{\alpha t}\Bigg\{\sum_{j=1}^{n} a_{ij}(t)\Big[g_j(x_j(t)) - g_j(v_j(t))\Big]$$
$$+\sum_{j=1}^{n}\int_0^\infty \Big[f_j(x_j(t-\tau_{ij}(t)-s)) - f_j(v_j(t-\tau_{ij}(t)-s))\Big]d_sK_{ij}(t,s)\Bigg\}.$$

Therefore,

$$\begin{aligned}\frac{d|z_i(t)|}{dt} &\le -(d_i(t)-\alpha)|z_i(t)| + \sum_{j=1}^{n}|a_{ij}(t)|G_j|z_j(t)| \\ &\quad + \sum_{j=1}^{n} F_j e^{\alpha\tau_{ij}(t)}\int_0^\infty e^{\alpha s}|z_j(t-\tau_{ij}(t)-s)||d_sK_{ij}(t,s)| \\ &\le \Big[-\xi_i(d_i(t)-\alpha) + \sum_{j=1}^{n}\xi_j|a_{ij}(t)|G_j\Big]\|z(t)\|_{\{\xi,\infty\}} \\ &\quad + \sum_{j=1}^{n}\xi_j F_j e^{\alpha\tau_{ij}(t)}\int_0^\infty e^{\alpha s}|z_j(t-\tau_{ij}(t)-s)||d_sK_{ij}(t,s)|.\end{aligned}$$

By the same approach used before, we can prove that $z(t)$ is bounded. That is, $\bar{x}(t) = O(e^{-\alpha t})$. This completes the proof of the theorem. □

7.3.2 Delayed Periodic Cohen–Grossberg Competitive and Cooperative Neural Networks

In this section, we investigate the following delayed Cohen–Grossberg neural network:

$$\begin{aligned}\frac{dx_i(t)}{dt} = -a_i(x_i(t))\Bigg[& d_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t)) \\ & -\sum_{j=1}^{n}\int_0^\infty f_j(x_j(t-s))d_sK_{ij}(t,s) + I_i(t)\Bigg], \quad i=1,\dots,n, \end{aligned} \tag{7.36}$$

where $x_i(t)$ denotes the state variable of neuron i, all coefficients satisfy the condition $\mathcal{B}_1$, $d(x) = (d_1(x_1),\dots,d_n(x_n))^\top \in \mathcal{D}$ as defined in Sect. 7.2.3, and the amplification functions $a_i(\cdot)$, $i=1,\dots,n$, might satisfy some of the assumptions $\mathcal{A}_{1-4}$ defined in Sect. 7.2.3. The initial condition is $x_i(\theta) = \phi_i(\theta)$, $\theta \in (-\infty, 0]$ for some continuous bounded positive functions $\phi_i(\cdot) \in C(-\infty, 0]$. The main results come from [21].

By Lemma 7.14, the assumption $\mathcal{A}_2$ implies that for positive, bounded, and continuous initial conditions, the trajectory of the system (7.36) is always positive. Moreover, we can obtain its boundedness.

Lemma 7.25 *Assume the hypotheses $\mathcal{B}_1$, and suppose further that $a(\cdot) \in \mathcal{A}_2$ and $d(\cdot) \in \mathcal{D}$. If there exist constants $\xi_i > 0$, $i = 1, \dots, n$, such that for all $i = 1, 2, \dots, n$ and $0 \le t < \omega$,*

$$- \gamma_i \xi_i + \sum_{j=1}^{n} |c_{ij}(t)| G_j \xi_j + \sum_{j=1}^{n} F_j \xi_j \int_0^{\infty} |d_s K_{ij}(t,s)| < 0, \tag{7.37}$$

then any solution $x(t)$ of the system (7.36) is bounded.

Proof First, by Lemma 7.14, any solution of (7.36) under positive initial conditions is globally positive. Since $c_{ij}(t)$ are continuous and periodic with period ω, $d_s K_{ij}(t,s)$ are ω-periodic with respect to t, and there exists a constant $\eta > 0$ with

$$\eta = \min_i \min_{0 \le t < \omega} \left\{ \gamma_i \xi_i - \sum_{j=1}^{n} |c_{ij}(t)| G_j \xi_j - \sum_{j=1}^{n} F_j \xi_j \int_0^{\infty} |d_s K_{ij}(t,s)| \right\}.$$

So, we have

$$-\gamma_i \xi_i + \sum_{j=1}^{n} |c_{ij}(t)| G_j \xi_j + \sum_{j=1}^{n} F_j \xi_j \int_0^{\infty} |d_s K_{ij}(t,s)| \le -\eta < 0.$$

Let $M(t) = \max_{s \le t} \|x(s)\|_{\{\xi,\infty\}}$. Clearly, $M(t)$ is nondecreasing and $\|x(t)\|_{\{\xi,\infty\}} \le M(t)$. Denote

$$H = \sup_{0 < t \le \omega} \max_i \left\{ |d_i(0)| + |I_i^*| + \sum_{j=1}^{n} c_{ij}^* |g_j(0)| + \sum_{j=1}^{n} |f_j(0)| \int_0^{\infty} |d_s K_{ij}(t,s)| \right\}.$$

Now, we can prove that $M(t) \le \max\{M(0), H/\eta\}$. For any $t_0 \ge 0$ with $M(t_0) = \|x(t_0)\|_{\{\xi,\infty\}}$, let i_0 be the index with $\|x(t_0)\|_{\{\xi,\infty\}} = |x_{i_0}(t_0)| \xi_{i_0}^{-1}$. Note that the assumptions imply that, for $i = 1, \dots, n$,

$$|g_i(s)| \le G_i |s| + |g_i(0)|, \ |f_i(s)| \le F_i |s| + |f_i(0)|, \ s \in \mathbb{R},$$

and

$$\operatorname{sign}(s) d_i(s) \ge \gamma_i |s| + \operatorname{sign}(s) d_i(0), \ s \in \mathbb{R}.$$

Hence, we have

$$\begin{aligned}
\left\{\frac{d}{dt}|x_{i_0}(t)|\right\}_{t=t_0} &= a_{i_0}(x_{i_0}(t_0))\mathrm{sign}(x_{i_0}(t_0))[-b_{i_0}(x_{i_0}(t_0)) \\
&\quad + \sum_{j=1}^{n} c_{i_0 j}(t_0) g_j(x_j(t_0)) \\
&\quad + \sum_{j=1}^{n} \int_0^{\infty} f_j(x_j(t_0 - s)) d_s K_{i_0 j}(t_0, s) + I_{i_0}(t_0)] \\
&\le a_{i_0}(x_{i_0}(t_0)) \Big[-\gamma_{i_0} \xi_{i_0} |x_{i_0}(t_0)| \xi_{i_0}^{-1} + \sum_{j=1}^{n} |c_{i_0 j}(t_0)| G_j \xi_j |x_j(t_0)| \xi_j^{-1} \\
&\quad + |b_{i_0}(0)| + |I_{i_0}(t_0)| + \sum_{j=1}^{n} F_j \xi_j \int_0^{\infty} |x_j(t_0 - s)| \xi_j^{-1} |d_s K_{i_0 j}(t_0, s)| \\
&\quad + \sum_{j=1}^{n} |c_{i_0 j}(t)| |g_j(0)| + \sum_{j=1}^{n} |f_j(0)| \int_0^{\infty} |d_s K_{i_0 j}(t_0, s)| \Big] \\
&\le a_{i_0}(x_{i_0}(t_0)) \Bigg\{ \Bigg[-\gamma_{i_0} \xi_{i_0} + \sum_{j=1}^{n} |c_{i_0 j}(t_0)| G_j \xi_j \\
&\quad + \sum_{j=1}^{n} F_j \xi_j \int_0^{\infty} |d_s K_{i_0 j}(t_0, s)| \Bigg] \|x(t_0)\|_{\{\xi,\infty\}} + H \Bigg\} \\
&\le a_{i_0}(x_{i_0}(t_0))(-\eta \|x(t_0)\|_{\{\xi,\infty\}} + H) = a_{i_0}(x_{i_0}(t_0))(-\eta M(t_0) + H).
\end{aligned}$$

This implies $M(t) \le \max\{M(t_0), H/\eta\}$ according to Lemma 7.2. So, $x(t)$ is bounded. This completes the proof. □

Thus, we can give the main result of this section.

Theorem 7.26 *Assume the hypotheses $\mathcal{B}_1$, and suppose further that $a(\cdot) \in \mathcal{A}_2$ and $d(\cdot) \in \mathcal{D}$. If there exist constants $\zeta_i > 0$, $i = 1, 2, \ldots, n$, such that for all $i = 1, 2, \ldots, n$ and $0 \le t < \omega$,*

$$-\zeta_i \gamma_i + \sum_{j=1}^{n} |c_{ji}(t)| \zeta_j G_i + \sum_{j=1}^{n} \zeta_j F_i \int_0^{\infty} |dK_{ji}(s)| < 0, \tag{7.38}$$

then the system (7.36) has a nonnegative periodic solution with period ω which is globally asymptotically stable.

Proof First, inequality (7.38) implies that inequality (7.37) holds owing to the M-matrix theory (see Lemma 7.4). By Lemma 7.14, any solution of the system (7.36) with a positive, bounded, and continuous initial condition is globally positive. Let

$$-\lambda = \max_i \sup_{0\le t<\omega} \left\{ -\zeta_i\gamma_i + \sum_{j=1}^n |c_{ji}(t)|\zeta_j G_i + \sum_{j=1}^n \zeta_j F_i \int_0^\infty |dK_{ji}(s)| \right\}.$$

The conditions stated in the theorem implies that $\lambda > 0$.

For a specific positive solution $x(t)$ of system (7.36), let $u_i(t) = x_i(t+\omega) - x_i(t)$, and $v_i(t) = \int_{x_i(t)}^{x_i(t+\omega)} 1/a_i(\rho)\,d\rho$, $i = 1, 2, \ldots, n$. Note that $a_i(\cdot)$ is continuous, $a_i(\rho) > 0$ when $\rho > 0$, and x_i is positive and bounded, thus $\int_{x_i(t)}^{x_i(t+\omega)} 1/a_i(\rho)\,d\rho$ exists. By the mean-value theorem for integrals, $v_i(t) = 1/a_i(\xi)(x_i(t+\omega) - x_i(t)) = (1/a_i(\xi))u_i(t)$, where $\xi \in [\min\{x_i(t), x_i(t+\omega)\}, \max\{x_i(t), x_i(t+\omega)\}]$. Since $a_i(x) > 0$ when $x > 0$, we have $\mathrm{sign}(v_i(t)) = \mathrm{sign}(u_i(t))$.

Direct calculations give

$$\begin{aligned}
\frac{dv_i(t)}{dt} &= \frac{1}{a_i(x_i(t+\omega))}\left\{\frac{dx_i(s)}{ds}\right\}_{s=t+\omega} - \frac{1}{a_i(x_i(t))}\left\{\frac{dx_i(s)}{ds}\right\}_{s=t} \\
&= -d_i(x_i(t+\omega)) + \sum_{j=1}^n c_{ij}(t+\omega) g_j(x_j(t+\omega)) \\
&\quad + \sum_{j=1}^n \int_0^\infty f_j(x_j(t+\omega-s)) d_s K_{ij}(t+\omega, s) - I_i(t+\omega) \\
&\quad - \left[-d_i(x_i(t)) + \sum_{j=1}^n c_{ij}(t) g_j(x_j(t)) + \sum_{j=1}^n \int_0^\infty f_j(x_j(t-s)) d_s K_{ij}(t,s) - I_i(t) \right] \\
&= -\left(d_i(x_i(t+\omega)) - d_i(x_i(t)) + \sum_{j=1}^n c_{ij}(t)(g_j(x_j(t+\omega)) - g_j(x_j(t))) \right) \\
&\quad + \sum_{j=1}^n \int_0^\infty (f_j(x_j(t+\omega)) - f_j(x_j(t))) d_s K_{ij}(t,s),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}|v_i(t)| &= \mathrm{sign}(v_i(t)) \Bigg\{ -(d_i(x_i(t+\omega)) - d_i(t)) \\
&\qquad + \sum_{j=1}^n c_{ij}(t)(g_j(x_j(t+\omega)) - g_j(x_j(t))) \\
&\qquad + \sum_{j=1}^n \int_0^\infty (f_j(x_j(t+\omega)) - f_j(x_j(t))) d_s K_{ij}(t,s) \Bigg\} \\
&\le -\gamma_i |u_i(t)| + \sum_{j=1}^n |c_{ij}(t)| G_j |u_j(t)| + \sum_{j=1}^n \int_0^\infty F_j |u_j(t-s)||dK_{ij}(t,s)|.
\end{aligned}$$

Define

$$L(t) = \sum_{i=1}^{n} \zeta_i |v_i(t)| + \sum_{i,j=1}^{n} \zeta_i F_j \int_0^{\infty} \int_{t-s}^{t} |u_j(\rho)| d\rho |dK_{ij}(s)|.$$

Differentiating $L(t)$ along the trajectory $x(t)$ of system (7.36) gives

$$\begin{aligned}
\frac{dL(t)}{dt} &\le \sum_{i=1}^{n} \zeta_i \left[-\gamma_i |u_i(t)| + \sum_{j=1}^{n} |c_{ij}(t)| G_j |u_j(t)| + \sum_{j=1}^{n} \int_0^{\infty} F_j |u_j(t-s)| |dK_{ij}(s)| \right] \\
&\quad + \sum_{i,j=1}^{n} \zeta_i F_j \left[\int_0^{\infty} |u_j(t)| |dK_{ij}(s)| - \int_0^{\infty} |u_j(t-s)| |dK_{ij}(s)| \right] \\
&= \sum_{i=1}^{n} \left[-\zeta_i \gamma_i + \sum_{j=1}^{n} |c_{ji}(t)| \zeta_j G_i \right] |u_i(t)| + \sum_{i,j=1}^{n} \zeta_j F_i \int_0^{\infty} |u_i(t-s)| |dK_{ji}(s)| \\
&\quad + \sum_{i,j=1}^{n} \zeta_j F_i \int_0^{\infty} |u_i(t)| |dK_{ji}(s)| - \sum_{i,j=1}^{n} \zeta_j F_i \int_0^{\infty} |u_i(t-s)| |dK_{ji}(s)| \\
&= \sum_{i=1}^{n} \left[-\zeta_i \gamma_i + \sum_{j=1}^{n} |c_{ji}(t)| \zeta_j G_i + \sum_{j=1}^{n} \zeta_j F_i \int_0^{\infty} |dK_{ji}(s)| \right] |u_i(t)| \le -\lambda \|u(t)\|_1.
\end{aligned}$$

Since $L(t) \ge 0$, integrating both sides of (7.39) from 0 to ∞ gives

$$\int_0^{\infty} \sum_{i=1}^{n} |u_i(t)| dt \le \frac{1}{\lambda} L(0) < +\infty, \tag{7.39}$$

which implies

$$\sum_{n=1}^{\infty} \int_0^{\omega} \|x(t+n\omega) - x(t+(n-1)\omega)\|_1 dt < +\infty.$$

By the Cauchy convergence principle, we have that $x(t+n\omega)$ converges in $L^1[0,\omega]$ as $n \to \infty$. Since $x(t)$ is bounded, $a_i(x_i(t)),\ i = 1, 2, \ldots, n$, are also bounded and $x(t)$ is uniformly continuous. Then, the sequence $\{x(t+n\omega)\}$ is uniformly bounded and equicontinuous. Thus, by the Arzéla–Ascoli theorem, there exists a subsequence $\{x(t+n_k\omega)\}$ converging on any compact set of $\mathbb{R}$. Denote its limit by $x^*(t)$. We have that $x^*(t)$ is also the limit of $\{x(t+n\omega)\}$ in $L^1[0,\omega]$, i.e.,

$$\lim_{n\to\infty} \int_0^{\infty} \|x(t+n\omega) - x^*(t)\| dt = 0.$$

Then, we have that $\|x(t+n\omega)-x^*(t)\| \to 0$ uniformly on $[0,\omega]$. Similarly, $\|x(t+n\omega)-x^*(t)\| \to 0$ uniformly on any compact set of $\mathbb{R}$.

We next prove that $x^*(t)$ is a nonnegative periodic solution with period ω. Since

$$x^*(t+\omega) = \lim_{n\to\infty} x(t+(n+1)\omega) = \lim_{n\to\infty} x(t+n\omega) = x^*(t),$$

we have that $x^*(t)$ is periodic with period ω. Then, replacing $x(t)$ with $x(t+n_k\omega)$ in system (7.36) and letting $k\to\infty$ give

$$\frac{dx_i^*(t)}{dt} = -a_i(x_i^*(t))\left[d_i(x_i^*(t)) - \sum_{j=1}^n c_{ij}(t)g_j(x_j^*(t))\right.$$
$$\left. - \sum_{j=1}^n \int_0^\infty f_j(x_j^*(t-s))d_sK_{ij}(t,s) + I_i(t)\right],\ i=1,\dots,n.$$

Hence, $x^*(t)$ is a solution of the system (7.36). Let $t = t_1 + n\omega$, where $0 \le t_1 < \omega$. Then, $\|x(t)-x^*(t)\| = \|x(t_1+n\omega)-x^*(t_1)\|$. The uniform convergence of $\{x(t+n\omega)\}$ on $[0,\omega]$ implies that

$$\lim_{t\to\infty} \|x(t)-x^*(t)\| = 0. \tag{7.40}$$

Finally, we prove that any positive solution of the system (7.36) converges to $x^*(t)$. Suppose that $y(t)$ is another positive solution of system (7.36) and let $u_i(t) = y_i(t) - x_i(t)$, $v_i(t) = \int_{x_i(t)}^{y_i(t)} 1/a_i(\rho)d\rho$, $i = 1,\dots,n$. The same arguments above yield $\lim_{t\to\infty}\|y(t)-x(t)\| = 0$. In conjunction with (7.40), we conclude that $\lim_{t\to\infty}\|x(t)-x^*(t)\| = 0$, completing the proof. □

7.3.3 Delayed Almost Periodic Hopfield Neural Networks

In this section, we investigate the dynamical system (7.30) with almost periodic coefficients. The main results come from [65]. At this stage, we give the following set of hypotheses.

$\mathcal{B}_2$:

(1) The activation functions g,f satisfy $g(\cdot) \in H_2\{G_1,G_2,\dots,G_n\}$ and $f(\cdot) \in H_1\{F_1,F_2,\dots,F_n\}$ for some positive constants;
(2) $d_i(t)$, $a_{ij}(t)$, $\tau_{ij}(t)$, and $I_i(t)$ are continuous, $d_i(t) \ge d_i > 0$ and $\tau_{ij} \ge 0$ for $i,j = 1,2,\dots,n$;
(3) For any $s\in\mathbb{R}$, $K_{ij}(t,s)$: $t \mapsto K_{ij}(t,s)$ is continuous in the same sense as in $\mathcal{B}_1$, and for any $t\in\mathbb{R}$, $dK_{ij}(t,s)$: $s\mapsto dK_{ij}(t,s)$ is a Lebesgue–Stieltjes measure, for all $i,j = 1,2,\dots,n$;

(4) For any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number ω for which $|d_i(t+\omega) - d_i(t)| < \epsilon$, $|a_{ij}(t+\omega) - a_{ij}(t)| < \epsilon$, $|I_i(t+\omega) - I_i(t)| < \epsilon$, $|\tau_{ij}(t+\omega) - \tau_{ij}(t)| < \epsilon$, and $\int_0^\infty |dK_{ij}(t+\omega, s) - dK_{ij}(t,s)| < \epsilon$ for all $i, j = 1.2. \dots, n$ and $t \in \mathbb{R}$.
(5) $|dK_{ij}(t,s)| \le |dK_{ij}(s)|$, and for some $\epsilon > 0$, $\int_0^\infty e^{\epsilon s}|dK_{ij}(s)| < \infty$.

It can be seen that under Item 4 in this assumption, $d_i(t)$, $a_{ij}(t)$, $I_i(t)$, and $\tau_{ij}(t)$ are almost periodic functions. Therefore, they are all bounded. We also denote $|a^*_{ij}| = \sup_{\{t\in\mathbb{R}\}} |a_{ij}(t)|$, $|b^*_{ij}| = \sup_{\{t\in\mathbb{R}\}} |b_{ij}(t)|$, $|I^*_i| = \sup_{\{t\in\mathbb{R}\}} |I_i(t)|$, $\tau^*_{ij} = \sup_{\{t\in\mathbb{R}\}} \tau_{ij}(t)$, $i, j = 1, \dots, n$, which are surely finite due their almost periodicity.

Before stating the main result, we need several lemmas for the proof of the main theorem.

Lemma 7.27 *Suppose that the hypotheses $\mathcal{B}_2$ are satisfied. If there exist $\xi_i > 0$, $i = 1, \dots, n$, and $\eta > 0$ such that*

$$-d_i(t)\xi_i + \sum_{j=1}^n |a_{ij}(t)|G_j\xi_j + \sum_{j=1}^n F_j\xi_j \int_0^\infty |dK_{ij}(t,s)| < -\eta < 0 \tag{7.41}$$

for all $t > 0$ and $i = 1, \dots, n$, then any solution $x(t)$ of the system (7.29) is bounded.

Proof Define $M(t) = \max_{s\le t} \|x(s)\|_{\{\xi,\infty\}}$. It is obvious that $\|x(t)\|_{\{\xi,\infty\}} \le M(t)$, and $M(t)$ is nondecreasing. We will prove that $M(t) \le \max\{M(0), (2/\eta)\hat{I}\}$, where

$$\hat{I} = \max_i \left\{ |I^*_i| + \sum_{j=1}^n \left[|a^*_{ij}||g_j(0)| + |b^*_{ij}||f_j(0)| \right] \right\}.$$

Fix t_0 such that $\|x(t_0)\|_{\{\xi,\infty\}} = M(t_0) = \max_{s\le t_0} \|x(s)\|_{\{\xi,\infty\}}$. In this case, let i_{t_0} be such an index that $\xi^{-1}_{i_{t_0}}|x_{i_{t_0}}(t_0)| = \|x(t_0)\|_{\{\xi,\infty\}}$. Then, noting that $|g_j(s)| \le G_j|s| + |g_j(0)|$ and $|f_j(s)| \le F_j|s| + |f_j(0)|$ for $j = 1, \dots, n$ and $s \in \mathbb{R}$, we have

$$\begin{aligned}
\left\{\frac{d}{dt}|x_{i_{t_0}}(t)|\right\}_{t=t_0} &= \operatorname{sign}(x_{i_{t_0}}(t_0))\Big[- d_{i_{t_0}}(t_0)x_{i_{t_0}}(t_0) + \sum_{j=1}^n a_{i_{t_0}j}(t_0)g_j(x_j(t_0)) \\
&\quad + \sum_{j=1}^n \int_0^\infty f_j(x_j(t_0 - \tau_{i_{t_0}j}(t_0) - s))dK_{i_{t_0}j}(t_0,s) + I_{i_{t_0}}(t_0)\Big] \\
&\le -d_{i_{t_0}}(t_0)|x_{i_{t_0}}(t_0)|\xi^{-1}_{i_{t_0}}\xi_{i_{t_0}} + \sum_{j=1}^n |a_{i_{t_0}j}(t_0)|G_j|x_j(t_0)|\xi_j^{-1}\xi_j \\
&\quad + \sum_{j=1}^n F_j\xi_j \int_0^\infty |x_j(t_0 - \tau_{i_{t_0}j}(t_0) - s)|\xi_j^{-1}|dK_{i_{t_0}j}(t_0,s)| + |I_{i_{t_0}}(t_0)|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}|a_{i_0j}(t)||g_j(0)|+|b_{i_0j}(t)||f_j(0)| \\
&\leq -d_{i_{t_0}}(t_0)\xi_{i_{t_0}}+\sum_{j=1}^{n}\left[|a_{i_{t_0}j}(t_0)|G_j\xi_j+F_j\xi_j\int_0^{\infty}|dK_{i_{t_0}j}(t_0,s)|\|x(t_0)\|_{\{\xi,\infty\}}\right]+\hat{I} \\
&\leq -\eta\|x(t_0)\|_{\{\xi,\infty\}}+\hat{I}=-\eta M(t_0)+\hat{I}, \qquad (7.42)
\end{aligned}
$$

which implies $M(t)\leq\max\{M(0),(2/\eta)\hat{I}\}$ for all $t>0$ according to Lemma 7.2. This proves that $x(t)$ is bounded. The lemma is proved. □

Lemma 7.28 *Suppose that the hypotheses $\mathcal{B}_2$ are satisfied. If there exist $\xi_i>0$, $i=1,2,\ldots,n$, $\beta>0$, and $\eta>0$ such that for all $t>0$,*

$$
-d_i(t)\xi_i+\sum_{j=1i}^{n}|a_{ij}(t)|G_j\xi_j+\sum_{j=1}^{n}F_j\xi_je^{\beta\tau_{ij}^*}\int_0^{\infty}e^{\beta s}|dK_{ij}(t,s)|<-\eta, \qquad (7.43)
$$

then for any $\epsilon>0$, there exist $T>0$ and $l=l(\epsilon)>0$, such that every interval $[\alpha,\alpha+l]$ contains at least one number ω for which the solution $x(t)$ of system (7.30) satisfies

$$
\|x(t+\omega)-x(t)\|_{\{\xi,\infty\}}\leq\epsilon \quad \textit{for all } t>T. \qquad (7.44)
$$

Proof Let

$$
\begin{aligned}
\epsilon_i(\omega,t)&=-[d_i(t+\omega)-d_i(t)]x_i(t+\omega)+\sum_{j=1}^{n}[a_{ij}(t+\omega)-a_{ij}(t)]g_j(x_j(t+\omega)) \\
&+\sum_{j=1}^{n}\int_0^{\infty}[f_j(x_j(t-\tau_{ij}(t+\omega)+\omega-s))-f_j(x_j(t-\tau_{ij}(t)+\omega-s))]dK_{ij}(t+\omega,s) \\
&+\sum_{j=1}^{n}\int_0^{\infty}f_j(x_j(t-\tau_{ij}(t)+\omega-s))d[K_{ij}(t+\omega,s)-K_{ij}(t,s)]+[I_i(t+\omega)-I_i(t)].
\end{aligned}
$$

Lemma 7.27 tells that $x(t)$ is bounded. Thus, the right side of (7.30) is also bounded, which implies that $x(t)$ is uniformly continuous. Therefore, by the fourth item in assumption $\mathcal{B}_2$, for any $\epsilon>0$, there exists $l=l(\epsilon)>0$ such that every interval $[\alpha,\alpha+l]$, $\alpha\in\mathbb{R}$, contains an ω for which $|\epsilon_i(\omega,t)|\leq(1/2)\eta\epsilon$, for all $t\in\mathbb{R}$ and $i=1,2,\ldots,n$.

Denote $z_i(t) = x_i(t+\omega) - x_i(t)$. We have

$$\frac{dz_i(t)}{dt} = -d_i(t)z_i(t) + \sum_{j=1}^{n} a_{ij}(t)[g_j(x_j(t+\omega)) - g_j(x_j(t))]$$
$$+\sum_{j=1}^{n}\int_0^\infty \left[f_j(x_j(t+\omega-\tau_{ij}(t)-s)) - f_j(x_j(t-\tau_{ij}(t)-s))\right] dK_{ij}(t,s)$$
$$+\epsilon_i(\omega,t).$$

Let i_t be such an index that $\xi_{i_t}^{-1}|z_{i_t}(t)| = \|z(t)\|_{\{\xi,\infty\}}$. Differentiating $e^{\beta s}|z_{i_t}(s)|$ gives

$$\frac{d}{ds}\left\{e^{\beta s}|z_{i_t}(s)|\right\}\Bigg|_{s=t} = \beta e^{\beta t}|z_{i_t}(t)| + e^{\beta t}\mathrm{sign}(z_{i_t}(t))\Bigg\{ - d_{i_t}(t)z_{i_t}(t)$$
$$+\sum_{j=1}^{n} a_{i_tj}(t)\Bigg[g_j(x_j(t+\omega)) - g_j(x_j(t))\Bigg]$$
$$+\sum_{j=1}^{n}\int_0^\infty \Bigg[f_j(x_j(t+\omega-\tau_{i_tj}(t)-s)) - f_j(x_j(t-\tau_{i_tj}(t)-s))\Bigg]dK_{i_tj}(t,s)$$
$$+\epsilon_{i_t}(\omega,t)\Bigg\}$$
$$\leq e^{\beta t}\Bigg\{ - [d_{i_t}(t)-\beta]|z_{i_t}(t)|\xi_{i_t}^{-1}\xi_{i_t} + \sum_{j=1}^{n}|a_{i_tj}(t)|G_j|z_j(t)|\xi_j^{-1}\xi_j$$
$$+\sum_{j=1}^{n}F_j\xi_j\int_0^\infty |z_j(t-\tau_{i_tj}(t)-s)|\xi_j^{-1}e^{-\beta(\tau_{i_tj}(t)+s)}e^{\beta(s+\tau_{ij}^*)}|dK_{i_tj}(t,s)|\Bigg\}$$
$$+\frac{1}{2}\eta\epsilon e^{\beta t}.$$

Using arguments similar to those in the proof of Lemma 7.27, let

$$\Psi(t) = \max_{s\leq t}\left\{e^{\beta s}\|z(s)\|_{\{\xi,\infty\}}\right\}. \tag{7.45}$$

For any $t_0 > 0$ with $\Psi(t_0) = e^{\beta t_0}\|z(t_0)\|_{\{\xi,\infty\}}$, we have $d\{e^{\beta t}|z_{i_t}(t)|\}/dt|_{t=t_0} \leq -\eta\Psi(t_0) + \frac{1}{2}\eta\epsilon e^{\beta t}$. From Lemma 7.2, this implies that there must exist $T > 0$ such that $\|z(t)\|_{\{\xi,\infty\}} \leq \epsilon$ for all $t > T$. □

Thus, we obtain the main theorem.

Theorem 7.29 *Suppose that the hypotheses $\mathcal{B}_2$ are satisfied. If there exist $\xi_i > 0$, $i = 1, 2, \ldots, n$, $\beta > 0$, and $\eta > 0$ such that the inequality*

$$-[d_i(t)-\beta]\xi_i+\sum_{j=1}^{n}|a_{ij}(t)|G_j\xi_j+\sum_{j=1}^{n}F_j\xi_j e^{\beta\tau_{ij}^*}\int_0^\infty e^{\beta s}|dK_{ij}(t,s)|<-\eta,$$
$$i=1,\dots,n \tag{7.46}$$

holds for all $t>0$*, then the system (7.30) has a unique almost periodic solution* $v(t)=(v_1(t),\dots,v_n(t))^\top$*, and for any solution* $x(t)=(x_1(t),\dots,x_n(t))^\top$ *of (7.30), one has*

$$\|x(t)-v(t)\|=O(e^{-\beta t}). \tag{7.47}$$

Proof $\epsilon_{i,k}(t)$ is defined as in the proof of Lemma 7.28. From the hypotheses $\mathcal{B}_2$ and the boundedness of $u(t)$, we can select a sequence $\{t_k\}\to\infty$ such that $|\epsilon_{i,k}(t)|\le 1/k$ for all i,t. Since $\{x(t+t_k)\}_{k=1}^{\infty}$ are uniformly bounded and equicontinuous, by the Arzela–Ascoli lemma and the diagonal selection principle, we can select a subsequence t_{k_j} of t_k, such that $x(t+t_{k_j})$ (for convenience, we still denote by $x(t+t_k)$) uniformly converges to a continuous function $v(t)=[v_1(t),v_2(t),\dots,v_n(t)]^\top$ on any compact subset of $\mathbb{R}$.

Now, we prove $v(t)$ is a solution of system (7.30). In fact, by Lebesgue dominated convergence theorem, for any $t>0$ and $\delta t\in\mathbb{R}$, we have

$$\begin{aligned}
&v_i(t+\delta t)-v_i(t)=\lim_{k\to\infty}\Big[u_i(t+\delta t+t_k)-u_i(t+t_k)\Big]\\
&=\lim_{k\to\infty}\int_t^{t+\delta t}\Big\{-d_i(\sigma+t_k)u_i(\sigma+t_k)+\sum_{j=1}^{n}a_{ij}(\sigma+t_k)g_j(u_j(\sigma+t_k))\\
&\quad+\sum_{j=1}^{n}\int_0^\infty f_j(u_j(\sigma+t_k-\tau_{ij}(\sigma+t_k)-s))dK_{ij}(\sigma+t_k,s)+I_i(\sigma+t_k)\Big\}d\sigma\\
&=\int_t^{t+\delta t}\Big\{-d_i(\sigma)v_i(\sigma)+\sum_{j=1}^{n}a_{ij}(\sigma)g_j(v_j(\sigma))\\
&\quad+\sum_{j=1}^{n}\int_0^\infty f_j(v_j(\sigma-\tau_{ij}(\sigma)-s))dK_{ij}(\sigma,s)+I_i(\sigma)\Big\}d\sigma+\lim_{k\to\infty}\int_t^{t+\delta t}\epsilon_{i,k}(s)d\sigma\\
&=\int_t^{t+\delta t}\Big\{-d_i(\sigma)v_i(\sigma)+\sum_{j=1}^{n}a_{ij}(\sigma)g_j(v_j(\sigma))\\
&\quad+\sum_{j=1}^{n}\int_0^\infty f_j(v_j(\sigma-\tau_{ij}(\sigma)-s))dK_{ij}(\sigma,s)+I_i(\sigma)\Big\}d\sigma,
\end{aligned}$$

which implies

$$\frac{dv_i}{dt} = -d_i(t)v_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(u_j(t))$$
$$+ \int_0^{\infty} f_j(u_j(t - \tau_{ij}(t) - s))dK_{ij}(t,s) + I_i(t),$$

i.e., $v(t)$ is a solution of the system (7.30).

Second, we prove that $v(t)$ is an almost periodic function. By Lemma 7.28, for any $\epsilon > 0$, there exist $T > 0$ and $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number ω for which $|x_i(t+\omega) - x_i(t)| \le \epsilon$, for all $t > T$. Then we can find a sufficient large $K \in N$ such that for any $k > K$ and all $t > 0$, we have $|x_i(t + t_k + \omega) - x_i(t + t_k)| \le \epsilon$. Let $k \to \infty$, we have $|v_i(t+\omega) - v_i(t)| \le \epsilon$, for all $t > 0$. In other words, $v(t)$ is an almost periodic function.

Finally, we prove that every solution $x(t)$ of the system (7.30) converges to $v(t)$ exponentially with rate β.

Denote $y(t) = x(t) - v(t)$. We have

$$\frac{dy_i(t)}{dt} = -d_i(t)y_i(t) + \sum_{j=1}^{n} a_{ij}(t)\left[g_j(x_j(t)) - g_j(v_j(t))\right]$$
$$+ \sum_{j=1}^{n} \int_0^{\infty} \left[f_j(x_j(t - \tau_{ij}(t) - s)) - f_j(v_j(t - \tau_{ij}(t) - s))\right] dK_{ij}(t,s).$$

Let i_t be an index such that $|y_{i_t}(t)| = \xi_{i_t}\|y(t)\|_{\{\xi,\infty\}}$. Differentiating $e^{\beta s}|y_{i_t}(s)|$, we have

$$\frac{d}{ds}\left\{e^{\beta s}|y_{i_t}(s)|\right\}\Big|_{s=t} = \beta e^{\beta t}|y_{i_t}(t)| + e^{\beta t}\mathrm{sign}(y_{i_t}(t))\Big\{ - d_{i_t}(t)y_{i_t}(t)$$
$$+ \sum_{j=1}^{n} a_{i_t j}(t)[g_j(x_j(t)) - g_j(v_j(t))]$$
$$+ \sum_{j=1}^{n} [f_j(x_j(t - \tau_{i_t j}(t) - s)) - f_j(v_j(t - \tau_{i_t j}(t) - s))]dK_{i_t j}(t,s)\Big\}$$
$$\le e^{\beta t}\Big\{ - [d_{i_t} - \beta]|y_{i_t}(t)|\xi_{i_t}^{-1}\xi_{i_t} + \sum_{j=1}^{n} |a_{i_t j}(t)|G_j|y_j(t)|\xi_j^{-1}\xi_j$$
$$+ \sum_{j=1}^{n} F_j\xi_j \int_0^{\infty} |y_j(t - \tau_{i_t j}(t) - s)|\xi_j^{-1} e^{-\beta(s+\tau_{i_t j}(t))} e^{\beta(s+\tau_{i_t j}^*)}|dK_{i_t j}(t,s)|\Big\} \tag{7.48}$$

Define $\Delta(t) = \max_{s \le t}\{e^{\beta s}\|y(s)\|_{\{\xi,\infty\}}\}$. Fix t_0 such that $\Delta(t_0) = e^{\beta t_0}\|y(t_0)\|_{\{\xi,\infty\}}$. Inequality (7.48) becomes $d\{e^{\beta t}|y_{i_0}(t)|\}/dt_{t=t_0} \le -\eta\Delta(t_0) \le 0$. By Lemma 7.2, this implies that $\Delta(t) \leq \Delta(0)$ for all $t \geq 0$ and $\|y(t)\|_{\{\xi,\infty\}} \leq \Delta(0)e^{-\beta t}$. In other words, $\|x(t) - v(t)\|_{\{\xi,\infty\}} \leq \Delta(0)e^{-\beta t}$. The theorem is proved. □

Since periodic functions are a special case of almost periodic functions, the results in this section can easily be used to obtain the criterion guaranteeing the existence of a periodic trajectory and its global stability for the case when coefficients are all periodic with a uniform period. Hence, the following theorem is a direct consequence of Theorem 7.29.

Theorem 7.30 *Suppose that the hypotheses* $\mathcal{B}_1$ *are satisfied. If there exist positive constants* $\xi_1, \ldots, \xi_n$ *and* $\beta > 0$ *such that*

$$-\xi_i[d_i - \beta] + \sum_{j=1}^{n} |a_{ij}(t)|\xi_j G_j + \sum_{j=1}^{n} \xi_i F_j e^{\beta\tau_{ij}^*} \int_0^\infty e^{\beta s}|d\bar{K}_{ij}(s)| < 0, \quad i = 1, \ldots, n, \tag{7.49}$$

then the system (7.29) has a unique periodic solution $v(t) = (v_1(t), \ldots, v_n(t))^\top$, *and for any solution* $x(t) = (x_1(t), \ldots, x_n(t))^\top$ *of (7.29), one has* $|x(t) - v(t)| = O(e^{-\beta t})$ *as* $t \to \infty$.

Moreover, consider the following system with constant coefficients:

$$\frac{du_i}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} \int_0^\infty f_j(u_j(t - \tau_{ij} - s)) d_s K_{ij}(s) + I_i, \quad i = 1, \ldots, n, \tag{7.50}$$

where $d_s K_{ij}(s)$ denotes the Lebesgue–Stieltjes measures, $i,j = 1, \ldots, n$. Since a constant can be regarded as a function with arbitrary period, we have the following result.

Theorem 7.31 *Suppose* $g(\cdot) \in H_1\{G_1, \ldots, G_n\}$ *and* $f(\cdot) \in H_1\{F_1, \ldots, F_n\}$. *If there are positive constants* $\xi_1, \ldots, \xi_n$ *and* $\beta > 0$ *such that*

$$-\xi_i[d_i - \beta] + \sum_{j=1}^{n} |a_{ij}|\xi_j G_j + \sum_{j=1}^{n} \xi_j F_j e^{\beta\tau_{ij}} \int_0^\infty e^{\beta s}|dK_{ij}(s)|F_j < 0, \; i = 1, \ldots, n, \tag{7.51}$$

then the system (7.50) is globally exponentially stable.

7.4 Delayed Neural Network with Discontinuous Activations

So far, all discussions and results have been based on the assumption that the activation functions are Lipschitz continuous. As pointed in [40], a brief review on some common neural network models reveals that neural networks with discontinuous activation functions are of importance and arise frequently in practice. For example, consider the classical Hopfield neural networks with graded response neurons (see [54]). The standard assumption is that the activations are used in the high-gain limit where they closely approach discontinuous and comparator functions. As shown in [54, 57], the high-gain hypothesis is crucial to make negligible the connection to the neural network energy function of the term depending on neuron self-inhibitions, and to favor binary output formation, as in a hard comparator function like sign(s).

A conceptually analogous model based on hard comparators are discrete-time neural networks discussed in [50]. Another important example concerns the class of neural networks introduced in [59] to solve linear and nonlinear programming problems. Those networks exploit constraint neurons with diode-like input–output activations. Again, in order to guarantee satisfaction of the constraints, the diodes are required to possess a very high slope in the conducting region, i.e., they should approximate the discontinuous characteristic of an ideal diode (see [31]). When dealing with dynamical systems possessing high-slope nonlinear elements, it is often advantageous to model them with a system of differential equations with discontinuous right-hand side, rather than studying the case where the slope is high but of finite value (see [85]).

In this section, we consider the following delayed dynamical system:

$$\frac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^{n} \int_0^\infty g_j(x_j(t-s))d_sK_{ij}(t,s) + I_i(t),\ i = 1,\dots,n, \tag{7.52}$$

with discontinuous activations g_j for both delayed and undelayed terms. A special form with a uniform discrete delay is

$$\frac{dx(t)}{dt} = -Dx(t) + Ag(x(t)) + Bg(x(t-\tau)) + I, \tag{7.53}$$

when rewritten in matrix form. We introduce the concept of a solution in the Filippov sense for the system (7.52) and prove its existence by the idea introduced in [48]. We construct a sequence of delayed systems in which the activations have high slope and converge to the discontinuous activations. First, we prove that under diagonal dominance conditions, the sequence of solutions has at least a subsequence converging to a solution of the system (7.52) with discontinuous activations by a well-known diagonal selection argument. Second, we consider the system (7.53). Without assuming the boundedness and the continuity of the neuron activations, we present sufficient conditions for the global stability of neural networks with time delay based on linear matrix inequalities and discuss their convergence. Third, we discuss the system (7.52) with almost periodic coefficients. We use the Lyapunov functional method to obtain an asymptotically almost periodic solution which leads to the existence of an almost periodic solution [84]. We also use the Lyapunov functional to obtain the global exponential stability of this almost periodic solution. Furthermore, from the proof of the existence and uniqueness of the solution, we can conclude that each solution sequence of the system with high-slope activations which converge to the discontinuous activations will actually converge to the unique solution of the system (7.52) with discontinuous activations in the Filippov sense. The main results come from [64, 66].

7.4.1 Preliminaries

In this section, we introduce the definitions and lemmas on nonsmooth and variational analysis, report some definitions and existing results on differential inclusions, and based on those results, give the mathematical description for the generalized neural network model to be studied.

7.4.1.1 Nonsmooth Analysis of Single-Valued Functions

Here, we introduce some necessary definitions and lemma on nonsmooth and variational analysis. We refer interested readers to [34, 80] for more details on these topics.

A single-valued function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be *strictly continuous* at $\bar{x} \in \mathbb{R}^n$ if the value $\mathrm{lip} f(\bar{x}) := \limsup_{x,x' \to \bar{x},\, x \neq x'} |f(x) - f(x')|/\|x - x'\|$ is finite. If f is strictly continuous at each $\bar{x} \in \mathbb{R}^n$, then f is said to be *strictly continuous in* $\mathbb{R}^n$. A strictly continuous function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be *(Clarke) regular* at $x \in \mathbb{R}^n$ if there exists the usual *one-sided directional derivative* $f'(x, v) = \lim_{\rho \searrow 0} [f(x + \rho v) - f(x)]/\rho$ for all $v \in \mathbb{R}^n$ and it equals to the *generalized directional derivative* $f^o(x, v) = \limsup_{y \to x, t \searrow 0} [f(y + tv) - f(y)]/t$. f is said to be *regular in* $\mathbb{R}^n$ if f is regular on each $x \in \mathbb{R}^n$. For a strictly continuous function $f:\mathbb{R}^n \to \mathbb{R}$, the *Clarke's generalized gradient* of f at $x \in \mathbb{R}^n$, which can be used to handle gradient flow on nonsmooth functions, can be written as

$$\partial f = \{p \in \mathbb{R}^n : f^o(x, v) \geq \langle p, v\rangle,\ \forall\, v \in \mathbb{R}^n\}.$$

A point $x_0 \in \mathbb{R}^n$ is said to be a *critical point* of f if $0 \in \partial f(x_0)$, and $\mathrm{crit}(f)$ denotes the set of critical points of f.

The following chain rule for nonsmooth functions is very important for later arguments.

Lemma 7.32 (Chain Rule, Theorem 2.3.9 in [34]) *If $x(t):\mathbb{R}^+ \to \mathbb{R}^n$ is locally absolutely continuous and a single-valued function $f:\mathbb{R}^n \to \mathbb{R}$ is strictly continuous and regular in $\mathbb{R}^n$, then the derivative $\frac{d}{dt} f(x(t))$ exists for almost all $t \geq 0$ and*

$$\frac{d}{dt} f(x(t)) = \langle p, \dot{x}(t)\rangle,\ \text{for all } p \in \partial f(x(t)),$$

for almost all $t \geq 0$.

7.4.1.2 Set-Valued Map

We introduce some definitions and lemmas for set-valued and variational analysis. We refer the interested readers to [5, 80] for more details.

Suppose $E \subset \mathbb{R}^n$. Then $x \mapsto F(x)$ is called a set-valued map from $E \hookrightarrow \mathbb{R}^n$, if to each point x of a set $E \subset \mathbb{R}^n$, there corresponds a non-empty set $F(x) \subset \mathbb{R}^n$. A set-valued map F with non-empty values is said to be *upper semicontinuous* (u.s.c. for short) at $x_0 \in E$, if for any open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$. $F(x)$ is said to have closed (convex, compact) image, if for each $x \in E$, $F(x)$ is closed (convex, compact).

7.4.1.3 Description of the Solution of the Model

Consider the following system:

$$\frac{dx}{dt} = f(x), \tag{7.54}$$

where $f(\cdot)$ is not continuous. Reference [39] proposed the following definition of the solution for the system (7.54).

Definition 7.33 Let ϕ be a set-valued map given by

$$\phi(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{co}\left[f(\overline{\mathcal{O}}(x,\delta) - N) \right], \tag{7.55}$$

where $\overline{co}(E)$ is the closure of the convex hull of some set E, $\overline{\mathcal{O}}(x,\delta) = \{y \in \mathbb{R}^n : \|y - x\| \leq \delta\}$, and $\mu(N)$ is the Lebesgue measure of the set N. A solution of the Cauchy problem for (7.54) with initial condition $x(0) = x_0$ is an absolutely continuous function $x(t)$, $t \in [0,T)$, which satisfies $x(0) = x_0$, and the differential inclusion

$$\frac{dx}{dt} \in \phi(x), \qquad \text{a.e. } t \in [0,T). \tag{7.56}$$

Furthermore, [4, 6, 48] have proposed the following functional differential inclusion with memory:

$$\frac{dx}{dt}(t) \in F(t, A(t)x), \tag{7.57}$$

where $F : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \mapsto \mathbb{R}^n$ is a given set-valued map, and

$$[A(t)x](\theta) = x_t(\theta) = x(t+\theta). \tag{7.58}$$

Inspired by these works, we denote $\overline{co}[g_i(s)] = [g_i^-(s), g_i^+(s)]$ and $\overline{co}[g(x)] = \overline{co}[g_1(x_1)] \times \overline{co}[g_2(x_2)] \times \cdots \times \overline{co}[g_n(x_n)]$, where $\times$ denotes the Cartesian product.

The set-valued map $\overline{co}[g(x)]$ is always u.s.c., convex, and compact. Thus, we can define solution of the system (7.52) in the Filippov sense as follows.

Definition 7.34 For a continuous function $\phi(\theta) = (\phi_1(\theta), \ldots, \phi_n(\theta))^\top$ and a measurable function $\lambda(\theta) = (\lambda_1(\theta), \ldots, \lambda_n(\theta))^\top \in \overline{co}[g(\phi(\theta))]$ for almost all $\theta \in (-\infty, 0]$, an absolute continuous function $x(t) = x(t, \phi, \lambda) = (x_1(t), \ldots, x_n(t))^\top$ associated with a measurable function $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))^\top$ is said to be a solution of the Cauchy problem for the system (7.52) on $[0, T)$ (T might be ∞) with initial value $(\phi(\theta), \lambda(\theta))$, $\theta \in (-\infty, 0]$, if

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) & \\ + \int_0^\infty \gamma_j(t-s)d_sK_{ij}(t,s) + I_i(t) & \text{a.e. } t \in [0,T), \\ \gamma_i(t) \in \overline{co}[g_i(x_i(t))] & \text{a.e. } t \in [0,T), \\ x_i(\theta) = \phi_i(\theta) & \theta \in (-\infty, 0], \\ \gamma_i(\theta) = \lambda_i(\theta) & \text{a.e. } \theta \in (-\infty, 0], \end{cases} \tag{7.59}$$

for all $i = 1, \ldots, n$.

The solution of the system (7.53) can be defined in the same way.

7.4.1.4 Set-up of Discontinuous Activations

We summarize the set-up of the model with the following assumptions.

$\mathcal{C}_1$: Every $g_i(\cdot)$ is nondecreasing and local Lipschizian, except on a set of isolated points $\{\rho_k^i\}$. More precisely, for each $i = 1, \ldots, n$, $g_i(\cdot)$ is nondecreasing and continuous except on a set of isolated points $\{\rho_k^i\}$, where the right and left limits $g_i^+(\rho_k^i)$ and $g_i^-(\rho_k^i)$ satisfy $g_i^+(\rho_k^i) > g_i^-(\rho_k^i)$. In each compact set of $\mathbb{R}$, $g_i(\cdot)$ has only finite number of discontinuities. Moreover, ordering the set of discontinuities as $\{\rho_k^i \colon \rho_{k+1}^i > \rho_k^i,\ k \in \mathbb{Z}\}$, there exist positive constants $G_{i,k} > 0$, $i = 1, \ldots, n$, $k \in \mathbb{Z}$, such that $|g_i(\xi) - g_i(\zeta)| \le G_{i,k}|\xi - \zeta|$ for all $\xi, \zeta \in (\rho_k^i, \rho_{k+1}^i)$.

$\mathcal{C}_2$: The initial condition $\phi(\theta) \in C((-\infty, 0], \mathbb{R}^n)$ is bounded, and $\lambda(\theta)$ is measurable and essentially bounded.

7.4.1.5 Viability

Here, we give the conditions guaranteeing the existence of Filippov solution in the sense (7.59) for the system (7.52). Similar to the idea proposed in [48], the solution of the system (7.52) in the sense (7.59) can be regarded as an approximation of the solutions of delayed neural networks with high-slope activations. This is the main idea of proving the existence and almost periodicity of the solution. More precisely, define a family of functions Ξ containing $f(x) = [f_1(x_1), f_2(x_2), \ldots, f_n(x_n)]^\top \in C(\mathbb{R}^n, \mathbb{R}^n)$ and satisfying the following properties: (1) every $f_i(\cdot)$ is monotonically nondecreasing, for $i = 1, 2, \ldots, n$; (2) every $f_i(\cdot)$ is uniformly locally bounded, i.e., for any compact set $Z \subset \mathbb{R}^n$, there exists a constant $M > 0$ independent of

f such that $|f_i(x)| \le M$ for all $x \in Z$ and $i = 1, \dots, n$; (3) every $f_i(\cdot)$ is locally Lipschitz continuous, i.e., for any compact set $Z \subset \mathbb{R}^n$, there exists $\lambda > 0$ such that $|f_i(\xi) - f_i(\zeta)| \le \lambda|\xi - \zeta|$ for all $\xi, \zeta \in Z$, and $i = 1, 2, \dots, n$. For any $f \in \Xi$, by the theory given in [49], the following system:

$$\begin{cases} \frac{du_i^f}{dt}(t) = -d_i(t)u_i^f(t) + \sum\limits_{j=1}^{n} a_{ij}(t)\sigma_j^f(t) + \sum\limits_{j=1}^{n} \int_0^\infty \sigma_j^f(t-s)d_sK_{ij}(t,s) + I_i(t) \\ u_i^f(\theta) = \phi_i(\theta),\ \theta \in (-\infty, 0] \\ \sigma_i^f(\theta) = \begin{cases} \lambda_i(\theta), & \theta \le 0 \\ f_i(u_i^f(\theta)), & \theta \ge 0 \end{cases} \quad i = 1, \dots, n \end{cases} \tag{7.60}$$

admits a unique solution $u_f(t) = (u_1(t), u_2(t), \dots, u_n(t))^\top$ on $[0, T)$, where T might be ∞.

First, we prove that the solutions $u^f(t)$ are uniformly bounded with respect to $f \in \Xi$.

Lemma 7.35 *Suppose that the assumptions $\mathcal{C}_{1,2}$ and $\mathcal{B}_2$ hold. If there exist constants $\xi_i > 0$, $i = 1, \dots, n$, and $\delta > 0$ such that $d_i(t) \ge \delta$ and*

$$\xi_i a_{ii}(t) + \sum_{j=1, j\ne i}^{n} \xi_j |a_{ji}(t)| + \sum_{j=1}^{n} \xi_j \int_0^\infty e^{\delta s}|d\bar{K}_{ji}(s)| < 0 \tag{7.61}$$

for all $t \ge 0$ and $i = 1, \dots, n$, then the solutions $u^f(t)$ are uniformly bounded with respect to $f \in \Xi$. That is, there exists $M = M(\phi, \lambda) > 0$, which is independent of $f \in \Xi$, such that $\|u^f(t)\|_{\{\xi,1\}} \le M$ for all $f \in \Xi$ and $t \ge 0$. Consequently, the existence interval of $u^f(t)$ can be extended to $[0, \infty)$.

Proof Let

$$V^f(t) = \sum_{i=1}^{n} \xi_i \left|u_i^f(t)\right| e^{\delta t} + \sum_{i,j=1}^{n} \xi_i \int_0^\infty \int_{t-s}^{t} \left|\sigma_j^f(\theta)\right| e^{\delta(s+\theta)} d\theta |d\bar{K}_{ij}(s)|.$$

Differentiating yields

$$\begin{aligned} \frac{d}{dt}V^f(t) &= \sum_{i=1}^{n} \delta e^{\delta t}\xi_i \left|u_i^f(t)\right| + \sum_{i=1}^{n} \xi_i e^{\delta t} \mathrm{sign}\left(u_i^f(t)\right) \Bigg\{ -d_i(t)u_i^f(t) \\ &+ a_{ii}(t)f_i\left(u_i^f(t)\right) + \sum_{j=1, j\ne 1}^{n} a_{ij}(t)f_j\left(u_j^f(t)\right) + \sum_{j=1}^{n} \int_0^\infty \sigma_j^f(t-s)d_sK_{ij}(t,s) \Bigg\} \\ &+ \sum_{i=1}^{n} \xi_i e^{\delta t} \mathrm{sign}\left(u_i^f(t)\right) I_i(t) + \sum_{i,j=1}^{n} \xi_i \left|f_j\left(u_j^f(t)\right)\right| e^{\delta t} \int_0^\infty e^{\delta s}|d\bar{K}_{ij}(s)| \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i,j=1}^{n}\xi_j e^{\delta t}\int_0^{\infty}\left|\sigma_j^f(t-s)\right||\bar{K}_{ij}(s)| \\
\le&\sum_{i=1}^{n}\xi_i\left|u_i^f(t)\right|e^{\delta t}(-d_i(t)+\delta)+\sum_{i=1}^{n}e^{\delta t}\left|f_i\left(u_i^f(t)\right)\right|\Bigg\{a_{ii}(t)\xi_i \\
&+\sum_{j=1,j\neq i}^{n}|a_{ji}(t)|\xi_j+\sum_{j=1}^{n}\xi_j\int_0^{\infty}e^{\delta s}|d\bar{K}_{ji}(s)|\Bigg\}+e^{\delta t}\hat{I}\le e^{\delta t}\hat{I},
\end{aligned}
$$

where $\hat{I}=\sup_{t\ge 0}\|I(t)\|_{\{\xi,1\}}<+\infty$. It follows that

$$
\begin{aligned}
\|u^f(t)\|_{\{\xi,1\}}&\le e^{-\delta t}V^f(t)=e^{-\delta t}\left[\int_0^t\dot{V}^f(s)ds+V^f(0)\right] \\
&\le e^{-\delta t}\int_0^t e^{\delta s}\hat{I}ds+e^{-\delta t}V^f(0) \\
&\le\frac{\hat{I}}{\delta}(1-e^{-\delta t})+e^{-\delta t}V^f(0)<\frac{\hat{I}}{\delta}+V^f(0).
\end{aligned}
$$

Noting that $V^f(0)$ is independent of $f\in\Xi$, we obtain the uniform boundedness of the solutions $u^f(t)$ by letting $M=\hat{I}/\delta+V^f(0)$. Moreover, $f(\cdot)$ is locally Lipschitz continuous, and we conclude that the existence interval of the solution $u^f(t)$ can be extended to the infinite interval $[0,+\infty)$ according to the results given in [49]. This lemma is proved. □

Now, for any sequence $\{g^m(x)=(g_1^m(x_1),\dots,g_n^m(x_n))^\top\}_{m\in\mathbb{N}}\in\Xi$ satisfying

$$
\lim_{m\to\infty}d_H(\mathrm{Graph}(g^m(K)),\overline{co}[g(K)])=0,\ \textit{for all } K\subset\mathbb{R}^n, \tag{7.62}
$$

where $d_H(\cdot,\cdot)$ denotes the Hausdorff metric on $\mathbb{R}^n$; we construct a sequence of delayed systems with high-slope continuous activations as follows:

$$
\begin{aligned}
\frac{du_i^m(t)}{dt}=&-d_i(t)u_i^m(t)+\sum_{j=1}^{n}a_{ij}(t)\sigma_j^m(t) \\
&+\sum_{j=1}^{n}\int_0^{\infty}\sigma_j^m(t-s)d_sK_{ij}(t,s)+I_i(t),\ i=1,\dots,n,
\end{aligned}
\tag{7.63}
$$

where $u_i^m(\theta)=\phi_i(\theta)$, $\theta\in(-\infty,0]$, and

$$
\sigma_j^m(\theta)=\begin{cases}\lambda_j(\theta), & \theta\le 0\\ g_j^m(u_j(\theta)), & \theta>0\end{cases}.
$$

For instance, let $\{\rho_{k,i}\}$ be the set of discontinuous points of $g_i(\cdot)$. Pick a strictly decreasing sequence $\{\delta_{k,i,m}\}$ with $\lim_{m\to\infty}\delta_{k,i,m} = 0$ and define $I_{k,i,m} = [\rho_{k,i} - \delta_{k.i.m}, \rho_{k,i} + \delta_{k,i,m}]$ such that $I_{k_1,i,m}\bigcap I_{k_2,i,m} = \emptyset$ for every $k_1 \neq k_2$. Then, define functions $g_i^m(\cdot)$ as follows:

$$g_i^m(s) = \begin{cases} g_i(s) & s \notin \bigcup_{k\in\mathbb{Z}} I_{k,i,m}, \\ \dfrac{g_i(\rho_{k,i}+\delta_{k,i,m}) - g_i(\rho_{k,i}-\delta_{k,i,m})}{2\delta_{k,i,m}}[s-\rho_{k,i}-\delta_{k,i,m}] & \\ +g_i(\rho_{k,i}+\delta_{k,i,m}) & s \in I_{k,i,m}. \end{cases}$$

It can be seen that the sequence $\{g^m(\cdot)\}_{m\in\mathbb{N}} \subset \Xi$ satisfies condition (7.62).

We point out that the solution sequence of the system sequence (7.63) converges to a solution of the system (7.52) in the sense (7.59).

Lemma 7.36 *Suppose the assumptions $\mathcal{C}_{1,2}$ and $\mathcal{B}_2$ are satisfied. If the condition (7.61) holds, then for each initial value pair (ϕ, λ), the system (7.52) has a solution in the sense of (7.59) on the whole time interval* $[0, \infty)$.

Proof Lemma 7.35 states that all solutions $\{u^m(t)\}_{m\in\mathbb{N}}$ are uniformly bounded, which implies that $\{\dot{u}^m(t)\}_{m\in\mathbb{N}}$ is uniformly essentially bounded. By the Arzela–Ascoli lemma and the diagonal selection principle, we can select a subsequence of $\{u^m(t)\}_{m\in\mathbb{N}}$ (still denoted by $u^m(t)$) such that $u^m(t)$ converges uniformly to a continuous function $u(t)$ on any compact interval of $\mathbb{R}$. Since $\{\dot{u}^m(t)\}_{m\in\mathbb{N}}$ is uniformly essentially bounded, $u(t)$ is Lipschitz continuous on $[0, T]$ for any $T > 0$. This implies that $\dot{u}(t)$ exists for almost all $t \in [0, T]$ and is bounded almost everywhere in $[0, T]$.

We claim that $\{\dot{u}^m(t)\}_{m\in\mathbb{N}}$ weakly converges to $\dot{u}(t)$ on the space $L^\infty([0, T], \mathbb{R}^n)$.

In fact, since $C_0^\infty([0, T], \mathbb{R}^n$ is dense in the Banach space $L^1([0, T], \mathbb{R}^n)$ and is the conjugate space $L^\infty([0, T], \mathbb{R}^n)$, for each $p(t) \in C_0^\infty([0, T], \mathbb{R}^n)$, we have

$$\int_0^\top \langle \dot{u}^m(t) - \dot{u}(t), p(t)\rangle dt = -\int_0^\top \langle \dot{p}(t), u^m(t) - u(t)\rangle dt.$$

By the uniform essential boundedness of $\{\dot{u}^m(t)\}_{m\in\mathbb{N}}$ and the Lebesgue dominated convergence theorem, we conclude that $\{\dot{u}^m(t)\}_{m\in\mathbb{N}}$ weakly converges to $\dot{u}(t)$ on the space $L^\infty([0, T], \mathbb{R}^n)$.

By Mazur's convexity theorem (see p. 120–123 in [83]), for any m, we can find a finite number of constants $\alpha_l^m \geq 0$ satisfying $\sum_{l=m}^\infty \alpha_l^m = 1$, such that $\lim_{m\to\infty} y^m(t) = u(t)$, uniformly on $[0, T]$, $\lim_{m\to\infty} \dot{y}^m(t) = \dot{u}(t)$, a.e. $t \in [0, T]$, where $y^m(t) = \sum_{l=m}^\infty \alpha_l^m u^l(t)$. Let $\eta_j^m(t) = \sum_{l=m}^\infty \alpha_l^m \sigma_j^l(u_j(t))$. Then,

$$\dot{y}_i^m(t) = -d_i(t)y_i^m(t) + \sum_{j=1}^n a_{ij}(t)\eta_j^m(t) + \sum_{j=1}^n \int_0^\infty \eta_j^m(t-s)d_sK_{ij}(t,s) + I_i(t)$$

for $i = 1, \ldots, n$.

Let $\varphi^m(t) = \int_0^t \eta^m(s)\,ds$, which is absolutely continuous and has uniformly essentially bounded derivative. By the same arguments, we can find $\gamma^m(t) = \sum_{l=m}^{\infty} \beta_l^m \eta^l(t)$ such that $\lim_{m\to\infty} \gamma^m(t) = \gamma(t)$ for almost every $t \in (-\infty, T]$ and $\gamma(t)$ is measurable.

Now, denoting $z^m(t) = \sum\limits_{l=m}^{\infty} \beta_l^m y^m(t)$, we have

$$\dot{z}_i^m(t) = -d_i(t)z_i^m(t) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j^m(t) + \sum_{j=1}^{n} \int_0^{\infty} \gamma_j^m(t-s)d_sK_{ij}(t,s) + I_i(t),\ i = 1, \ldots, n. \tag{7.64}$$

Letting $m \to \infty$, by the Lebesgue dominated convergence theorem, we obtain

$$\dot{u}_i(t) = -d_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) + \sum_{j=1}^{n} \int_0^{\infty} \gamma_j(t-s)d_sK_{ij}(t,s) + I_i(t),\ i = 1, \ldots, n,$$

for a.e. $t \in [0,T]$. It remains to prove $\gamma(t) \in \overline{co}[g(u(t))]$ on $t \in [0,T]$. Since $u^m(t)$ converges to $u(t)$ uniformly with respect to $t \in [0,T]$ and $\overline{co}[g(\cdot)]$ is an upper-semi-continuous set-valued map, for any $\epsilon > 0$, there exists $N > 0$ such that $g^m(u^m(t)) \in \mathcal{O}(\overline{co}[g(u(t))], \epsilon)$ for all $m > N$ and $t \in [0,T]$. Noting that $\overline{co}[g(u(t))]$ is convex and compact, we conclude that $\gamma^m(t) \in \mathcal{O}(\overline{co}[g(u(t))], \epsilon)$, which implies $\gamma(t) \in \mathcal{O}(\overline{co}[g(u(t))], \epsilon)$ for any $t \in [0,T]$. Because of the arbitrariness of ϵ, we conclude that $\gamma(t) \in \overline{co}[g(u(t))]$, $t \in [0,T]$. Since T is also arbitrary, the solution can be extended to $[0,\infty)$. This completes the proof. □

Similar arguments yield existence of solutions for the system (7.53). The Filippov solution of the system (7.53) with discontinuous activation functions can be described as

$$\frac{dx}{dt}(t) = -Dx(t) + A\alpha(t) + B\alpha(t-\tau) + I, \quad \text{for almost all } t, \tag{7.65}$$

where the output $\alpha(t)$ is measurable and satisfies $\alpha(t) \in \overline{co}[g(x(t))]$ for almost all t.

Lemma 7.37 *Suppose the assumptions $\mathcal{C}_{1,2}$ satisfied. If there exist $P = diag\{P_1, P_2, \ldots, P_n\}$ with $P_i > 0$, and a positive definite symmetric matrix Q such that*

$$Z = \begin{bmatrix} -PA - A^{\top}P - Q & -PB \\ -B^{\top}P & Q \end{bmatrix} > 0, \tag{7.66}$$

then the system (7.53) has a solution $x(t) = (x_1(t) \dots, x_n(t))^\top$ *for* $t \in [0, \infty)$.

The details of the proof can be found in [64].

7.4.2 Stability of Equilibrium

In this section, we study the global stability of the system (7.53) in the sense (7.65). The main results come from [64]. Here, the equilibrium of such system is defined as follows:

Definition 7.38 (Equilibrium) x^* is said to be an equilibrium of the system (7.53) if there exists $\alpha^* \in \overline{co}[g(x^*)]$ such that

$$0 = -Dx^* + A\alpha^* + B\alpha^* + I.$$

Definition 7.39 An equilibrium x^* of the system (7.53) is said to be globally asymptotically stable if for any solution $x(t)$ of (7.65), whose existence interval is $[0, +\infty)$, we have

$$\lim_{t \to \infty} x(t) = x^*.$$

Moreover, $x(t)$ is said to be globally exponentially asymptotically stable, if there exist constants $\epsilon > 0$ and $M > 0$, such that

$$\|x(t) - x^*\| \le Me^{-\epsilon t}.$$

We first investigate the existence of an equilibrium point. For this purpose, consider the differential inclusion

$$\frac{dy}{dt} \in -Dy(t) + T\overline{co}[g(y(t))] + I, \tag{7.67}$$

where $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^\top$, D, $\overline{co}[g(\,\cdot\,)]$, and I are the same as those in the system (7.53). We have the following result.

Lemma 7.40 (Theorem 2 in [64]) *Suppose that* $g(\,\cdot\,)$ *satisfies the assumption* $\mathcal{C}_1$. *If there exists a positive definite diagonal matrix* P *such that* $-PT - T^\top P$ *is positive definite, then there exists an equilibrium point of system (7.67), i.e., there exist* $y^* \in \mathbb{R}^n$ *and* $\alpha^* \in \overline{co}[g(y^*)]$, *such that*

$$0 = -Dy^* + T\alpha^* + I.$$

See Appendix B for the proof.

By Lemma 7.40, we can prove the following theorem.

Theorem 7.41 *If there exist a positive definite diagonal matrix $P = \mathrm{diag}\{P_1, P_2, \ldots, P_n\}$ and a positive definite symmetric matrix Q such that*

$$\begin{bmatrix} -PA - A^{\top}P - Q & -PB \\ -B^{\top}P & Q \end{bmatrix} > 0, \tag{7.68}$$

then there exists an equilibrium point of system (7.65).

Proof By the Schur Complement Theorem (Lemma 7.5), inequality (7.68) is equivalent to $-(PA+A^{\top}P) > PBQ^{-1}B^{\top}P+Q$. By the inequality $[Q^{-\frac{1}{2}}B^{\top}P-Q^{\frac{1}{2}}]^{\top}[Q^{-\frac{1}{2}}B^{\top}P-Q^{\frac{1}{2}}] \geq 0$, one has $PBQ^{-1}B^{\top}P+Q \geq PB+B^{\top}P$. Then, the inequality (7.68) becomes $-P(A+B)-(A+B)^{\top}P > 0$. By Lemma 7.40, there exist an equilibrium point $x^* \in \mathbb{R}^n$ and $\alpha^* \in \overline{co}[g(x^*)]$ such that

$$0 = -Dx^* + (A+B)\alpha^* + I, \tag{7.69}$$

which implies that α^* is an equilibrium point of system (7.65). □

Suppose that $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\top}$ is an equilibrium point of the system (7.65), i.e., there exists $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)^{\top} \in \overline{co}[g(x)]$ such that (7.69) is satisfied. Let $u(t) = x(t) - x^*$ be a translation of $x(t)$ and $\gamma(t) = \alpha(t) - \alpha^*$ be a translation of $\alpha(t)$. Then $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^{\top}$ satisfies

$$\frac{du(t)}{dt} = -Du(t) + A\gamma(t) + B\gamma(t-\tau), \quad \text{a.e. } t \in \mathbb{R},$$

where $\gamma(t) \in \overline{co}[g^*(u(t))]$, $g_i^*(s) = g_i(s+x_i^*) - \gamma_i^*$, $i = 1, 2, \ldots, n$. To simplify, we still use $g_i(s)$ to denote $g_i^*(s)$. Therefore, in the following, instead of the system (7.65), we will investigate

$$\frac{du(t)}{dt} = -Du(t) + A\gamma(t) + B\gamma(t-\tau), \quad \text{a.e. } t \in \mathbb{R}, \tag{7.70}$$

where $\gamma(t) \in \overline{co}[g(u(t))]$, $g(\cdot) \in \bar{G}$, and $0 \in \overline{co}[g_i(0)]$, for all $i = 1, 2, \ldots, n$. It can be seen that the dynamical behavior of (7.65) is equivalent to that of (7.70). Namely, if there exists a solution $u(t)$ for (7.70), then $x(t) = u(t) + x^*$ must be a solution for (7.65); moreover, if all trajectories of (7.70) converge to the origin, then the equilibrium x^* must be globally stable for system (7.65) as defined in Definition 7.39.

Theorem 7.42 (Global Exponential Asymptotic Stability) *If the matrix inequality (7.68) and the assumptions $\mathcal{C}_{1,2}$ hold, then the system (7.53) is globally exponentially stable.*

Proof From the condition (7.68), we can find a sufficiently small $\epsilon > 0$ such that the matrix

$$Z_1 = \begin{bmatrix} -2D + \epsilon I & \epsilon A & \epsilon B \\ \epsilon A^\top & PA + A^\top P + Qe^{\epsilon\tau} & PB \\ \epsilon B^\top & B^\top P & -Q \end{bmatrix}$$

is negative definite. Let

$$V_3(t) = e^{\varepsilon t}u^\top(t)u(t) + 2\sum_{i=1}^{n} e^{\varepsilon t}P_i \int_0^{u_i(t)} g_i(\rho)\,d\rho + \int_{t-\tau}^{t} \gamma(s)^\top Q\gamma(s)e^{\varepsilon(s+\tau)}\,ds,$$

with $\gamma(t) = \alpha(t) - \alpha^*$. Notice that for $p_i(s) = \int_0^s g_i(\rho)\,d\rho$, we have $\partial_c p_i(s) = \{v \in \mathbb{R}: g_i^-(s) \le v \le g_i^+(s)\}$. Differentiating $V_3(t)$ by the chain rule (Lemma 7.32) gives

$$\begin{aligned} \frac{dV_3(t)}{dt} &= \varepsilon e^{\varepsilon t}u(t)^\top u(t) + 2e^{\varepsilon t}u^\top\Big[-Du + A\gamma(t) + B\gamma(t-\tau)\Big] \\ &\quad +2e^{\varepsilon t}\gamma(t)P\Big[-Du(t) + A\gamma(t) + B\gamma(t-\tau)\Big] \\ &\quad +\varepsilon e^{\varepsilon t}\sum_{i=1}^{n} P_i\int_0^{u_i} g_i(\rho)\,d\rho - e^{\varepsilon t}\gamma^\top(t-\tau)Q\gamma(t-\tau) \\ &\quad +e^{\varepsilon(t+\tau)}\gamma^\top(t)Q\gamma(t). \end{aligned} \tag{7.71}$$

Since $\varepsilon < \min_i d_i$, we have $\varepsilon\int_0^{u_i} g_i(\rho)\,d\rho \le \varepsilon u_i(t)\gamma_i(t) \le d_i u_i(t)\gamma_i(t)$ and

$$\frac{dV_3(t)}{dt} \le e^{\varepsilon t}[u^\top(t), \gamma^\top(t), \gamma^\top(t-\tau)]Z_1 \begin{bmatrix} u(t) \\ \gamma(t) \\ \gamma(t-\tau) \end{bmatrix} \le 0.$$

Then, $u(t)^\top u(t) \le V_3(0)e^{-\varepsilon t}$ and $\|u(t)\|_2 \le \sqrt{V_3(0)}e^{-\frac{\varepsilon}{2}t}$. That is, $\|x(t) - x^*\|_2 \le \sqrt{V_3(0)}e^{-\frac{\varepsilon}{2}t}$. This proves the theorem. □

In case $g(\,\cdot\,)$ is continuous, we have the following consequence.

Corollary 7.43 *If the condition (7.68) holds and $g_i(\,\cdot\,)$ is locally Lipschitz continuous, then there exist $\varepsilon > 0$ and $x^* \in \mathbb{R}^n$ such that for any solution $x(t)$ on $[0, \infty)$ of the system (7.53), there exist $M = M(\phi) > 0$ and $\epsilon > 0$ such that*

$$\|x(t) - x^*\| \le Me^{-\frac{\varepsilon}{2}t} \quad \textit{for all } t > 0.$$

If every x_i^* is a continuous point of the activation functions $g_i(\,\cdot\,)$, $i = 1, \ldots, n$, for the outputs we have $\lim_{t\to\infty} g_i(x_i(t)) = g_i(x_i^*)$. Instead, if for some i, x_i^* is a discontinuous point of the activation function $g_i(\cdot)$, we can prove that the outputs converge in measure.

Theorem 7.44 (Convergence in measure of output) *If the condition (7.68) holds and $g(\cdot) \in \bar{G}$, then the output $\alpha(t)$ of the system (7.65) converges to α^* in measure, i.e., for all $\epsilon > 0$ we have* $\lim_{t\to\infty} \mu\{t\colon |\alpha(t) - \alpha^*| \geq \epsilon\} = 0$

Proof The condition (7.68) implies that there exists $\epsilon > 0$ such that the matrix

$$Z_2 = \begin{bmatrix} -2D & \epsilon A & \epsilon B \\ \epsilon A^\top & PA + A^\top P + \epsilon I & PB \\ \epsilon B^\top & B^\top P & -Q \end{bmatrix} \tag{7.72}$$

is negative definite. Let

$$V_5(t) = u^\top(t)u(t) + 2\sum_{i=1}^n P_i \int_0^{u_i} g_i(\rho)\,d\rho + \int_{t-\tau}^t \gamma(s)^\top Q\gamma(s)\,ds,$$

with $\gamma(t) = \alpha(t) - \alpha^*$, and P, Q, and ϵ are those in the matrix inequality (7.72). Differentiate $V_5(t)$:

$$\begin{aligned} \frac{dV_5(t)}{dt} &= 2u^\top(t)\Big[-Du(t) + A\gamma(t) + B\gamma(t-\tau)\Big] + 2\gamma^\top(t)P\Big[-Du(t) + A\gamma(t) \\ &\quad +B\gamma(t-\tau)\Big] + \gamma^\top(t)Q\gamma(t) - \gamma^\top(t-\tau)Q\gamma(t-\tau) + \epsilon\gamma(t)^\top\gamma(t) \\ &\quad -\epsilon\gamma(t)^\top\gamma(t) \\ &= [u^\top(t), \gamma^\top(t), \gamma^\top(t-\tau)]Z_2 \begin{bmatrix} u(t) \\ \gamma(t) \\ \gamma(t-\tau) \end{bmatrix} - \epsilon\gamma^\top(t)\gamma(t) \\ &\leq -\epsilon\gamma^\top(t)\gamma(t) \end{aligned} \tag{7.73}$$

Then, $V_5(t) - V_5(0) \leq -\epsilon\int_0^t \gamma^\top(s)\gamma(s)\,ds$. Since $\lim_{t\to\infty} V_5(t) = 0$, we have $\int_0^\infty \gamma^\top(s)\gamma(s)\,ds \leq -(1/\epsilon)V_5(0)$. For any $\epsilon_1 > 0$, let $E_{\epsilon_1} = \{t \in [0,\infty)\colon \|\gamma(t)\| > \epsilon_1\}$. Then,

$$\frac{V_5(0)}{\epsilon} \geq \int_0^\infty \gamma^\top(s)\gamma(s)ds \geq \int_{E_{\epsilon_1}} \gamma^\top(s)\gamma(s) \geq \epsilon_1^2\mu(E_\epsilon)$$

Hence, $\mu(E_{\epsilon_1}) < \infty$. From Proposition 2 in [40], one can see that $\gamma(t)$, i.e., $\alpha(t) - \alpha^*$, converges to zero in measure. □

7.4.3 Convergence of Periodic and Almost Periodic Orbits

Consider the system (7.52)

$$\frac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^{n} \int_0^{\infty} g_j(x_j(t-s))d_sK_{ij}(t,s) + I_i(t),\ i = 1, \ldots, n, \tag{7.74}$$

with the almost periodic assumption $\mathcal{B}_2$. We study the almost periodicity of delayed neural networks. The main result stated below comes from [68].

Theorem 7.45 *Suppose the assumptions $\mathcal{C}_{1,2}$ and $\mathcal{B}_2$ are satisfied. Suppose further that there exist constants $\xi_i > 0$, $i = 1, \ldots, n$, and $\delta > 0$ such that $d_i(t) \geq \delta$ and*

$$\xi_i a_{ii}(t) + \sum_{j=1, j\neq i}^{n} \xi_j |a_{ji}(t)| + \sum_{j=1}^{n} \xi_j \int_0^{\infty} e^{\delta s} |d\bar{K}_{ji}(s)| < 0 \tag{7.75}$$

for all $t \geq 0$ and $i = 1, \ldots, n$. Then, (1) for every initial value (ϕ, λ), the system (7.74) has a unique solution in the sense of (7.59); (2) there exists a unique almost periodic solution $x^*(t)$ for the system (7.74), which is globally exponentially stable, that is, for any other solution $x(t)$ with initial condition (ϕ, λ), there exists a constant $M = M(\phi, \lambda) > 0$ such that

$$\|x(t) - x^*(t)\|_{\{\xi,1\}} \leq Me^{-\delta t}$$

for all $t \geq 0$.

Besides the viability proved in Lemma 7.36, we prove this theorem step by step. **Step 1.** We show that any solution of the system (7.74) in the sense (7.59) is asymptotically stable.

Lemma 7.46 *Suppose that the assumptions of Theorem 7.45 are satisfied. For any two solutions $x(t) = x(t, \phi, \lambda)$ and $v(t) = v(t, \psi, \chi)$ of the system (7.74) in the sense of (7.59) associated with the outputs $\gamma(t)$ and $\mu(t)$ and initial value pairs (ϕ, λ) and (ψ, χ), respectively, there exists a constant $M = M(\phi, \psi, \lambda, \chi)$ satisfying $M(\phi, \phi, \lambda, \lambda) = 0$ for all (ϕ, λ) such that*

$$\|x(t) - v(t)\|_{\{\xi,1\}} \leq Me^{-\delta t}, \quad t \geq 0.$$

Moreover, the solution of the system (7.74) in the sense (7.59) is unique.

Proof Let $x(t) = (x_1(t), \ldots, x_n(t))^{\top}$ be a solution of

$$\frac{d}{dt}x_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) + \sum_{j=1}^{n}\int_0^{\infty} \gamma_j(t-s)d_sK_{ij}(t,s) + I_i(t),$$

and $v(t) = (v_1(t), \ldots, v_n(t))^{\top}$ be a solution of

$$\frac{d}{dt}v_i(t) = -d_i(t)v_i(t) + \sum_{j=1}^{n} a_{ij}(t)\mu_j(t) + \sum_{j=1}^{n}\int_0^{\infty} \mu_j(t-s)d_sK_{ij}(t,s) + I_i(t).$$

Then,

$$\begin{aligned}\frac{d}{dt}\Big[x_i(t) - v_i(t)\Big] &= -d_i(t)\Big[x_i(t) - v_i(t)\Big] + \sum_{j=1}^{n} a_{ij}(t)\Big[\gamma_j(t) - \mu_j(t)\Big]\\ &\quad + \sum_{j=1}^{n}\int_0^{\infty}\Big[\gamma_j(t-s) - \mu_j(t-s)\Big]d_sK_{ij}(t,s),\ i = 1, \ldots, n.\end{aligned}$$

Let

$$\begin{aligned}L_1(t) &= \sum_{i=1}^{n} \xi_i |x_i(t) - v_i(t)| e^{\delta t}\\ &\quad + \sum_{i,j=1}^{n} \xi_j \int_0^{\infty}\int_{t-s}^{t} |\gamma_j(\theta) - \mu_j(\theta)| e^{\delta(s+\theta)} d\theta |d\bar{K}_{ij}(s)|\end{aligned}$$

and $M = M(\phi, \psi, \lambda, \chi) = L_1(0)$. By the chain rule (Lemma 7.32), differentiating the above expression gives

$$\begin{aligned}\frac{d}{dt}L_1(t) &= \sum_{i=1}^{n} \delta e^{\delta t}\xi_i |x_i(t) - v_i(t)| + \sum_{i=1}^{n} \xi_i e^{\delta t}\mathrm{sign}(x_i(t) - v_i(t))\\ &\quad \Big\{ - d_i(t)[x_i(t) - v_i(t)] + a_{ii}(t)[\gamma_i(t) - \mu_i(t)]\\ &\quad + \sum_{j=1, j\neq i}^{n} a_{ij}(t)[\gamma_j(t) - \mu_j(t)]\\ &\quad + \sum_{j=1}^{n}\int_0^{\infty}[\gamma_j(t-s) - \mu_j(t-s)]d_sK_{ij}(t,s)\Big\} + \sum_{i,j=1}^{n} \xi_i |\gamma_j(t) - \mu_j(t)|\\ &\quad e^{\delta t}\int_0^{\infty} e^{\delta s}|d\bar{K}_{ij}(s)| - \sum_{i,j=1}^{n} \xi_j e^{\delta t}\int_0^{\infty} |\gamma_j(t-s) - \mu_j(t-s)||\bar{K}_{ij}(s)|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} \xi_i |x_j(t) - v_j(t)| e^{\delta t}(-d_i(t) + \delta) + \sum_{i=1}^{n} e^{\delta t} |\gamma_j(t) - \mu_j(t)| \Bigg\{ a_{ii}(t)\xi_i \\
&+ \sum_{j=1, j\neq i}^{n} |a_{ji}(t)|\xi_j + \sum_{j=1}^{n} \xi_j \int_0^{\infty} e^{\delta s} |d\bar{K}_{ji}(s)| \Bigg\} \leq 0,
\end{aligned}
$$

which implies $\|x(t) - v(t)\|_{\{\xi,1\}} \leq L_1(0)e^{-\delta t} = M(\phi, \psi, \lambda, \chi)e^{-\delta t}$. It is clear that $M(\phi, \phi, \lambda, \lambda) = 0$. Therefore, the solution in unique. □

In Lemma 7.36, we have proved that some subsequence of $u^m(t)$ converges to the solution $u(t)$. In fact, we can prove that $u^m(t)$ itself converges to the solution $u(t)$.

Proposition 7.47 *Suppose that the assumptions of the Main Theorem 7.45 are satisfied. For any function sequence* $\{\tilde{g}^m(x) = (\tilde{g}_1^m(x_1), \ldots, \tilde{g}_n^m(x_n))^\top : m = 1, 2, \ldots\} \subset \Xi$ *satisfying the condition (7.62) on any compact set in* R^n*, let* $\tilde{u}^m(t) = [\tilde{u}_1^m(t), \ldots, \tilde{u}_n^m(t)]^\top$ *be the solution of the following system:*

$$
\begin{aligned}
\frac{d\tilde{u}_i^m}{dt} &= -d_i(t)\tilde{u}_i^m(t) + \sum_{j=1}^{n} a_{ij}(t)\tilde{g}_j(\tilde{u}_j^m(t)) \\
&\quad + \sum_{j=1}^{n} \int_0^{\infty} \tilde{\sigma}_j^m(t-s) d_s K_{ij}(t,s) + I_i(t),
\end{aligned}
$$

$$
\tilde{u}_i^m(\theta) = \phi_i(\theta),\ \theta \in [-\infty, 0],\ \tilde{\sigma}_i^m(\theta) = \begin{cases} \lambda_i(\theta), & \theta \leq 0 \\ \tilde{g}_i^m(\tilde{u}_i^m(\theta)), & \theta \geq 0 \end{cases}, \tag{7.76}
$$

for $i = 1, \ldots, n$*, and* $u(t) = u(t, \phi, \lambda)$ *be the solution of the delayed system (7.74) in the sense (7.59) with initial value* (ϕ, λ)*. Then,* $\tilde{u}^m(t)$ *uniformly converges to* $u(t)$ *on any finite time interval* $[0, T]$.

Proof First, we prove that $u^m(t)$ converges to the solution of the delayed system (7.74) in the sense (7.59) by reduction to absurdity. Assume that there exist $T > 0$, $\epsilon_0 \geq 0$, and a subsequence of integers $\{m_k\}_{k\in\mathbb{N}}$ such that

$$
\max_{t\in[0,T]} \|u^{m_k}(t) - u(t)\| \geq \epsilon_0. \tag{7.77}
$$

By the same arguments used in the proof of Lemma 7.36, we can select a subsequence $\{u^{m_{k_l}}\}_{l\geq 0}$ of $\{u^{m_k}\}_{k\geq 0}$, which converges to a solution $v(t) = v(t, \phi, \lambda)$ of the delayed system (7.74) in the sense (7.59) uniformly in any finite interval $[0, T]$ with the initial value (ϕ, λ). By Lemma 7.46, $u(t) = v(t)$, which leads a contradiction with inequality (7.77). This completes the proof. □

Remark 7.48 Proposition 7.47 indicates that the solution $v(t) = v(t, \phi, \lambda)$ of the delayed system (7.74) in the sense (7.59) does not depend on the choice of the sequence $\{g^m(x)\}_{m\in\mathbb{N}} \subset \Xi$ satisfying the condition (7.62).

The following lemma points out that any solution is asymptotically almost periodic [84].

Lemma 7.49 *Suppose that the assumptions of Theorem 7.45 are satisfied. Let $u(t,\phi,\lambda)$ be a solution of the system (7.74) in the sense of (7.59). For any $\epsilon > 0$, there exist $T > 0$ and $l = l(\epsilon)$ such that any interval $[\alpha, \alpha + l]$ contains an ω such that*

$$\|x(t+\omega) - x(t)\|_{\xi} \le \epsilon \quad \textit{for all } t \ge T.$$

Proof We introduce the following auxiliary functions

$$\begin{aligned}\epsilon_i(t,\omega) = {} & x_i(t+\omega)[d_i(t+\omega) - d_i(t)] + \sum_{j=1}^{n} \gamma_j(t+\omega)[a_{ij}(t+\omega) - a_{ij}(t)] \\ & + \int_0^{\infty} \sum_{j=1}^{n} \gamma_j(t+\omega-s) d[K_{ij}(t+\omega,s) - K_{ij}(t,s)] \\ & + I_i(t+\omega) - I_i(t) \end{aligned} \tag{7.78}$$

for $i = 1, \ldots, n$. From the assumption $\mathcal{C}_2$ and the boundedness of $x(t)$ and $\gamma(t)$, one can see that for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$ such that every interval $[\alpha, \alpha+l]$ contains at least one number ω with $\sum_{i=1}^{n} \xi_i |\epsilon_i(t,\omega)| < \delta\epsilon/2$ for all $t \ge 0$. Denote $z(t) = x(t+\omega) - x(t)$. Then,

$$\begin{aligned}\frac{dz_i(t)}{dt} = {} & -d_i(t)z_i(t) + \sum_{j=1}^{n} a_{ij}(t)[\gamma_j(t+\omega) - \gamma_j(t)] \\ & + \sum_{j=1}^{n} \int_0^{\infty} [\gamma_j(t+\omega-s) - \gamma_j(t-s)] d_s K_{ij}(t,s) + \epsilon_i(t,\omega). \end{aligned}$$

Let

$$L_2(t) = \sum_{i=1}^{n} \xi_i |z_i(t)| e^{\delta t} + \sum_{i,j=1}^{n} \xi_i \int_0^{\infty} \int_{t-s}^{t} |\gamma_j(\theta+\omega) - \gamma_j(\theta)| e^{\delta(\theta+s)} d\theta |d\bar{K}_{ij}(s)|.$$

Pick a sufficiently large T such that $e^{-\delta t} L_2(0) < \epsilon/2$ for all $t \ge T$. Differentiating $L_2(t)$ gives

$$\begin{aligned}\frac{dL_2(t)}{dt} = {} & \sum_{i=1}^{n} \xi_i \delta e^{\delta t} |z_i(t)| + \sum_{i=1}^{n} \xi_i e^{\delta t} \mathrm{sign}(z_i(t)) \Big\{ - d_i(t) z_i(t) \\ & + a_{ii}(t)[\gamma_i(t+\omega) - \gamma_i(t)] + \sum_{j=1, j\neq i} a_{ij}(t)[\gamma_j(t+\omega) - \gamma_j(t)] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\int_0^{\infty}[\gamma_j(t+\omega-s)-\gamma_j(t-s)]d_sK_{ij}(t,s)+\epsilon_i(t,\omega)\Big\}\\
&+\sum_{i,j=1}^{n}\xi_i e^{\delta t}|\gamma_j(t+\omega)-\gamma_j(t)|\int_0^{\infty}e^{\delta s}|d\bar{K}_{ij}(s)|\\
&-\sum_{i,j=1}^{n}\xi_i e^{\delta t}\int_0^{\infty}|\gamma_j(t+\omega-s)-\gamma_j(t-s)||d\bar{K}_{ij}(s)|\\
\le&\sum_{i}^{n}\xi_i e^{\delta t}|z_i(t)|(-d_i(t)+\delta)+\sum_{i=1}^{n}|\gamma_j(t+\omega)-\gamma_j(t)|e^{\delta t}\Big\{\xi_i a_{ii}(t)\\
&+\sum_{j=1,j\ne i}\xi_j|a_{ji}(t)|+\sum_{j=1}^{n}\xi_j\int_0^{\infty}e^{\delta s}|d\bar{K}_{ji}(s)|\Big\}+\sum_{i=1}^{n}\xi_i e^{\delta t}|\epsilon_i(t,\omega)|\\
\le&\, e^{\delta t}\frac{\delta}{2}\epsilon,\quad \text{a.e. } t\ge T.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{i=1}^{n}\xi_i|z_i(t)|&\le e^{-\delta}L_2(t)=e^{-\delta}\left[L_2(0)+\int_0^t\dot{L}_2(s)ds\right]\\
&\le e^{-\delta t}L_2(0)+e^{-\delta t}\int_0^t e^{\delta s}ds\frac{\delta}{2}\epsilon<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon
\end{aligned}
$$

for all $t\ge T$, which completes the proof. □

Step 2. Now, we are to prove that the system (7.74) has at least one almost periodic solution in the sense of (7.59).

Lemma 7.50 *Under the assumptions of Theorem 7.45, the system (7.74) has at least one almost periodic solution in the sense of (7.59).*

Proof Let $x(t)=x(t,\phi,\lambda)$ be a solution of system (7.59). Pick a sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfying $\lim_{k\to\infty}t_k=\infty$ and $\sup_{t\ge0}\sum_{i=1}^{n}\xi_i|\epsilon_i(t,t_k)|\le 1/k$, where $\epsilon_i(t,t_k)$, $i=1,\ldots,n$, are the auxiliary functions (7.78) defined in the proof of Lemma 7.49.

Let $x^k(t)=x(t+t_k)$ and $\gamma^k(t)=\gamma(t+t_k)$. It is clear that the sequence $\{x(t+t_k)\}_{k\in\mathbb{N}}$ is uniformly continuous and bounded. By the Arzela–Ascoli lemma and the diagonal selection principle, we can select a subsequence of $x(t+t_k)$ (still denoted by $x(t+t_k)$), which converges to some absolutely continuous function $x^*(t)$ uniformly on any compact interval $[0,T]$.

In the following, we will prove that $x^*(t)$ is an almost periodic solution of the system (7.74) in the sense of (7.59). First, we prove that $x^*(t)$ is a solution of the system (7.74) in the sense of (7.59). With the notations above, we have

$$\frac{dx_i(t+t_k)}{dt} = -d_i(t)x_i(t+t_k) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t+t_k)$$
$$+\sum_{j=1}^{n}\int_0^{\infty}\gamma_j(t+t_k-s)d_sK_{ij}(t,s) + I_i(t) + \epsilon_i(t,t_k),\ i=1,\ldots,n.$$

With the method used in the proof of Lemma 7.36, we can select a subsequence from $x(t+t_k)$ (still denoted by $x(t+t_k)$) and constants $\nu_l^k \geq 0$ with finite $\nu_l^k > 0$ satisfying $\sum_{l=k}^{\infty}\nu_l^k = 1$ such that (i) $v^k(t) = \sum_{l=k}^{\infty}\nu_l^k x(t+t_l)$ converges to a Lipschitz continuous function $x^*(t)$ uniformly on $[0,T]$, and $\{\dot{v}^k(t)\}$ converges to $\dot{v}^*(t)$ for almost all $t \in [0,T]$ and (ii) $\zeta^k(t) = \sum_{l=k}^{\infty}\nu_l^k\gamma(t+t_l)$ converges to a measurable function $\zeta(t)$ for almost all $t \in [0,T]$.

Moreover, for each k, we have

$$\frac{dv_i^k(t)}{dt} = -d_i(t)v_i^k(t) + \sum_{j=1}^{n} a_{ij}(t)\zeta_j^k(t)$$
$$+\sum_{j=1}^{n}\int_0^{\infty}\zeta_j^k(t-s)d_sK_{ij}(t,s) + I_i(t) + \bar{\epsilon}_i(t,k),\ i=1,\ldots,n,$$

where $\bar{\epsilon}_i(t,k) = \sum_{l=k}^{\infty}\nu_l^k\epsilon_i(t,t_k)$. Letting $k \to \infty$, we obtain

$$\frac{dx_i^*(t)}{dt} = -d_i(t)x_i^*(t) + \sum_{j=1}^{n} a_{ij}(t)\zeta_j(t)$$
$$+\sum_{j=1}^{n}\int_0^{\infty}\zeta_j(t-s)d_sK_{ij}(t,s) + I_i(t),\ i=1,\ldots,n.$$

Repeating the proof of Lemma 7.36, we can prove $\zeta(t) \in \overline{co}[g(x^*(t))]$, which means that $x^*(t)$ is a solution of the system (7.74) in the sense of (7.59).

Second, we prove that $x^*(t)$ is almost periodic. By Lemma 7.49, for any $\epsilon > 0$, there exist $K > 0$ and $l = l(\epsilon)$ such that each interval $[\alpha, \alpha + l]$ contains an ω such that

$$\|x(t+t_k+\omega) - x(t+t_k)\|_{\{\xi,1\}} < \epsilon$$

for all $k \geq K$ and $t \geq 0$. As $k \to \infty$, we conclude that $\|x^*(t+\omega) - x^*(t)\|_{\{\xi,1\}} < \epsilon$ for all $t \geq 0$. This implies that $x^*(t)$ is an almost periodic function. The proof is completed. $\square$

Now, we can prove the main Theorem 7.45.

Proof By Lemma 7.50, we know that there exists an almost periodic solution for the system (7.74) in the sense of (7.59). By Lemma 7.46, we have $\|x(t) - x^*(t)\|_{\{\xi,1\}} = O(e^{-\delta t})$.

Finally, we prove that the almost periodic solution of the system (7.74) is unique. In fact, suppose that $x^*(t)$ and $v^*(t)$ are two almost periodic solutions of the system (7.74). Applying Lemma 7.46 again, we have $\|v^*(t) - x^*(t)\|_{\{\xi,1\}} = O(e^{-\delta t})$. From [61], one can conclude that $v^*(t) = x^*(t)$. Therefore, the almost periodic solution of the system (7.74) is unique. This completes the proof. □

Since any periodic function can be regarded as an almost periodic function, all the results apply to periodic case. Now, replacing assumption $\mathcal{B}_2$ with $\mathcal{B}_1$, we have the following result.

Corollary 7.51 *Suppose that the discontinuous activations satisfy assumptions $\mathcal{C}_{1,2}$, and that the hypotheses $\mathcal{B}_1$ hold. Suppose further that there exist positive constants ξ_i, $i = 1, \ldots, n$, and $\delta > 0$ such that $d_i(t) \geq \delta$ and*

$$\xi_i a_{ii}(t) + \sum_{j=1, j\neq i}^{n} \xi_j |a_{ji}(t)| + \sum_{j=1}^{n} \xi_j \int_0^\infty e^{\delta s} |d\bar{K}_{ji}(s)| < 0$$

for all $t \geq 0$ and $i = 1, \ldots, n$. Then, (1) for each initial data with assumption $\mathcal{A}_3$, the system (7.74) has a unique solution in the sense of (7.59) and (2) there exists a unique periodic solution $x^*(t)$ for system (7.74), which is globally exponentially stable.

Furthermore, a constant can be regarded as a periodic function with any period. Therefore, for the delayed system

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} \int_0^\infty g_j(x_j(t-s)) d_s K_{ij}(s) + I_i, \quad i = 1, \ldots, n \end{aligned} \tag{7.79}$$

we have the following result.

Corollary 7.52 *Suppose that the discontinuous activations satisfy the assumptions $\mathcal{C}_{1,2}$, and suppose that there exist positive constants ξ_i, $i = 1, \ldots, n$, and $\delta > 0$ such that $d_i \geq \delta$ and*

$$\xi_i a_{ii} + \sum_{j=1, j\neq i}^{n} \xi_j |a_{ji}| + \sum_{j=1}^{n} \xi_j \int_0^\infty e^{\delta s} |d\bar{K}_{ji}(s)| \leq 0$$

for all $t \geq 0$ and $i = 1, \ldots, n$. Then, (1) for each initial data satisfying the stated assumptions, the system (7.79) has a unique solution in sense of (7.59) and (2) the system (7.79) has a unique equilibrium x^*, which is globally exponentially stable.

7.5 Review and Comparison of Literature

In the past decades, global stability analysis has been a focal topic in neural network theory and dynamical systems, with a large literature devoted to it. In this section, we give a brief review of selected papers and compare them with the results in this chapter.

The stability of equilibrium of delayed neural networks has been studied in many papers. For example, [9, 10, 14, 17, 18, 62, 74, 86] and many others. For more general functional differential equations, see the early works [49, 73] and others. The approach used in these papers consists of two steps: (1) prove the existence of the equilibrium and (2) prove its stability. In theorems in Sect. 7.2.2, we unify two types of delayed dynamical systems and investigate their dynamical behavior and global convergence. We consider the derivative of the state variable and prove that it converges to zero exponentially. This implies that the state trajectory converges to a certain equilibrium exponentially according to the Cauchy convergence principle.

Moreover, in most papers dealing with time-varying delays, the assumption of bounded delays is necessary, i.e., $\tau_{ij}(t) \leq \tau$ for all $i, j = 1, \ldots, n$ and $t \in \mathbb{R}$, or $\dot{\tau}_{ij}(t) \leq \mu$ for some $0 \leq \mu < 1$, which can guarantee exponential stability under some additional conditions. However, in this chapter, we have studied stability in the power rate, which is weaker than exponential rate, but under a milder condition for the unbounded delays, namely, $\tau_{ij}(t) \leq \mu t$ for some $0 \leq \mu < 1$.

As for the delayed Cohen–Grossberg neural network (7.19), there is also a large literature concerned with global stability. However, all the results obtained in these papers were based on the assumption that amplifier function $a_i(\cdot)$ is always *positive* (see [28, 29]; or even greater than some positive number $a_i(\cdot) \geq \underline{a}_i > 0$ (see [16, 71, 82]). In their original papers [35, 46, 47], the authors proposed this model as a kind of competitive-cooperation dynamical system for decision rules, pattern formation, and parallel memory storage. Hereby, each state of neuron x_i might be the population size, activity, or concentration, etc., of the i-th species in the system, which is nonnegative for all time. Theorem 7.20 gives a sufficient condition guaranteeing stability in the first orthant.

Periodicity and almost periodicity of delayed neural networks with time-varying coefficients have attracted much research attention [15, 24, 45, 72, 87, 88]. It should be pointed out that [87] studied the periodicity of delayed neural network via a L^p-norm-like Lyapunov functional and proved that among the sufficient conditions according to parameter $p \in [1, \infty]$, the condition vith L^1-norm-like Lyapunov functional would be the best one, i.e., the mildest condition. Most of these papers concerned with periodic delayed neural networks use the Mawhin coincidence degree theory [44]. We use two different methods to prove existence, as mentioned in Sect. 7.3.3. In [24, 55], the authors presented some results on almost periodic trajectories and their attractivity of shunting inhibitory cellular neural networks (CNNs) with delays. In [24], authors proved existence and attractivity of almost periodic solutions for CNNs with distributed delays and variable coefficients.

In the last few years, several papers have appeared studying neural networks with discontinuous activations. Reference [40] discussed the absolute stability of

Hopfield neural networks with bounded and discontinuous activations. Reference [64] proved the global convergence for Cohen–Grossberg neural networks with unbounded and discontinuous activations. Also, [42] studied the dynamics of delayed neural networks and [78] discussed periodic solutions of periodic delayed neural networks with discontinuous activations and periodic parameters. In all these papers, the authors use the solution in the Filippov sense to handle differential equations with discontinuous right-hand side. The concept of the solution in the sense of Filippov is useful in engineering applications. Since a Filippov solution is a limit of the solutions of a sequence of ordinary differential equations with continuous right-hand side, we can model a system which is near a discontinuous system and expect that the Filippov trajectories of the discontinuous system will be close to the real trajectories. This approach is of significance in many applications, for instance, variable structure control, nonsmooth analysis [4, 77, 85]. In fact, the solution in the Filippov sense satisfies the corresponding differential inclusion induced by the convex extension of discontinuity.

The generalized viability of differential inclusions was investigated in the textbooks [4, 6]. Periodicity and almost periodicity for differential inclusions or Filippov systems have been studied in the recent decades. Methodologically, the existence of a periodic solution of a differential inclusion or differential system with discontinuous right-hand side (despite that some researchers did not study the Filippov solution) can be proved by fixed point theory, i.e., the periodic boundary condition can be regarded as a fixed point of a certain evolution operator [12, 38, 56, 70, 72, 89]. Several authors constructed a sequence of differential systems with continuous right-hand sides having periodic solutions and proved that the solution sequence converges to a periodic solution of the original differential inclusion [43, 48]. As for stability, the first approximation was used to deal with the local asymptotical stability for periodic differential inclusions [81], and Lyapunov method was extended to handle the global stability [7, 8]. Furthermore, similar methods were utilized to study the almost periodic solution of almost periodic differential inclusions, especially with delays. See [3] and [58] for references.

Appendix

Proof of Theorem 7.18

Proof Let

$$\begin{aligned} f_i(x) &= d_i(x_i) - \sum_{j=1}^{n} (a_{ij} + b_{ij}) g_j(x_j), \quad i = 1, \ldots, n, \\ f(x) &= (f_1(x), \ldots, f_n(x))^\top, \\ F(x) &= f(x^+) + x^-, \end{aligned}$$

where x^+ and x^- are defined in Definition 7.6.

According to Lemma 7.7, we only need to prove that $F(x)$ is norm-coercive and locally univalent (one-to-one). First, we prove $F(x)$ is locally univalent. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Without loss of generality, by some rearrangement of the x_i, we can assume $x_i > 0$ if $i = 1, \ldots, p$, $x_i < 0$ if $i = p+1, \ldots, m$, and $x_i = 0$ if $i = m+1, \ldots, n$, for some integers $p \leq m \leq n$. Moreover, if $y \in \mathbb{R}^n$ is sufficiently close to $x \in \mathbb{R}^n$, without loss of generality, we can assume

$$\begin{cases} y_i > 0, \; i = 1, \ldots, p \\ y_i < 0, \; i = p+1, \ldots, m \\ y_i > 0, \; i = m+1, \ldots, m_1 \\ y_i < 0, \; i = m_1, \ldots, m_2 \\ y_i = 0, \; i = m_2+1, \ldots, n, \end{cases}$$

for some integers $m \leq m_1 \leq m_2 \leq n$. It can be seen that

$$(x_i^+ - y_i^+)(x_i^- - y_i^-) = 0, \quad i = 1, \ldots, n, \tag{7.80}$$

and

$$\begin{aligned} F(x) - F(y) &= d(x^+) - d(y^+) - (A+B)[g(x^+) - g(y^+)] + (x^- - y^-) \\ &= [\bar{D} - (A+B)K](x^+ - y^+) + (x^- - y^-), \end{aligned}$$

where $\bar{D} = diag\{\bar{d}_i, \ldots, \bar{d}_n\}$ and $K = diag\{K_1, \ldots, K_n\}$ with

$$\bar{d}_i = \begin{cases} \dfrac{d_i(x_i^+) - d_i(y_i^+)}{x_i^+ - y_i^+}, x_i^+ \neq y_i^+ \\ D_i, \qquad\qquad\qquad \text{otherwise} \end{cases}, \quad K_i = \begin{cases} \dfrac{g_i(x_i^+) - g_i(y_i^+)}{x_i^+ - y_i^+}, x_i^+ \neq y_i^+ \\ G_i \qquad\qquad\qquad \text{otherwise} \end{cases}.$$

Then, $\bar{d}_i \geq D_i$ and $K_i \leq G_i$ because $d(\cdot) \in \mathcal{D}$ and $g(\cdot) \in H_2\{G_1, \ldots, G_n\}$.

If $F(x) - F(y) = 0$, then we have

$$x^- - y^- = -[\bar{D} - (A+B)K](x^+ - y^+). \tag{7.81}$$

By (7.80), without loss of generality, we can assume

$$x^+ - y^+ = \begin{bmatrix} z_1 \\ 0 \end{bmatrix}, \quad x^- - y^- = \begin{bmatrix} 0 \\ z_2 \end{bmatrix},$$

where $z_1 \in \mathbb{R}^k$ and $z_2 \in \mathbb{R}^{n-k}$, for some integer k. Write

$$\bar{D} - (A+B)K = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where $R_{11} \in \mathbb{R}^{k,k}$, $R^{12} \in \mathbb{R}^{k,n-k}$, $R_{21} \in \mathbb{R}^{n-k,k}$, and $R_{22} \in \mathbb{R}^{n-k,n-k}$. The equation (7.81) can be rewritten as

$$\begin{bmatrix} 0 \\ z_2 \end{bmatrix} = - \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \end{bmatrix},$$

which implies $R_{11}z_1 = 0$. From Lemma 7.3, we can conclude that R_{11} is nonsingular, which implies $z_1 = 0$ and $x^+ = y^+$. Similarly, we can prove $x^- = y^-$. Therefore, $x = y$, which means that $F(x)$ is locally univalent.

Second, we will prove that $F(x)$ is norm-coercive. Suppose that there exists a sequence $\{x_m = (x_{m,1}, \ldots, x_{m,n})^\top\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} \|x_m\|_2 = \infty$. Then, there exists some index i such that $\lim_{m\to\infty} |d_i(x_{m,i}^+) + x_{m,i}^-| = \infty$, which implies that $\lim_{m\to\infty} \|g(x_m^+)\|_2 = \infty$.

Some simple algebraic manipulations lead to

$$\begin{aligned} g(x^+)^\top PF(x) &= \sum_{i=1}^{n} g_i(x_i^+)P_i d_i(x_i^+) - g(x^+)^\top P(A+B)g(x^+) + \sum_{i=1}^{n} g_i(x_i^+)P_i x_i^- \\ &\geq g(x^+)^\top \{P[DG^{-1} - (A+B)]\}^s g(x^+) \geq \alpha g(x^+)^\top g(x^+), \end{aligned}$$

where $\alpha = \lambda_{\min}(\{P[DG^{-1} - (A+B)]\}^s) > 0$. Therefore,

$$\|F(x_m)\|_2 \geq \alpha \|P\|_2^{-1} \|g(x_m^+)\|_2 \to \infty,$$

which implies that $F(x)$ is norm-coercive. Combining with Lemma 7.7 proves the theorem. □

Proof of Lemma 7.40

We will prove the existence of equilibrium of the system (7.53) under the assumption $\mathcal{C}_1$. We will prove existence of equilibrium for the system (7.53) by the Equilibrium Theorem [5]. First, we give some necessary definitions concerned with the equilibrium of a set-valued map.

Definition 7.53 For a convex subset K of $\mathbb{R}^n$, the tangent cone $T_K(x)$ to K at $x \in K$ is defined as

$$T_K(x) = \overline{\bigcup_{h>0}} \frac{K - x}{h}, \tag{7.82}$$

where $\overline{\bigcup}$ is the closure of the union set.

Proposition 7.54 *The necessary and sufficient condition for $v \in T_K(x)$ is that there exist $h_n \to 0^+$ and $v_n \to v$ as $n \to +\infty$, such that $x + h_n v_n \in K$ for all n. Moreover, if $x \in$* int(K)*, where* int(K) *is the set of the interior points of K, then $T_k(x) = \mathbb{R}^n$.*

Definition 7.55 (Viability Domain) Let $F{:}X \to X$ be a non-trivial set-valued map. We say that a subset $K \subset \mathrm{Dom}(F)$ is a viability domain of F, if for all $x \in K$, we have $F(x) \bigcap T_K(x) \neq \emptyset$ where $\mathrm{Dom}(F)$ is the domain of F.

Definition 7.56 (Equilibrium) x^* is said to be an equilibrium of a set-valued map $F(x)$ if $0 \in F(x^*)$.

The following theorem is used below.

Lemma 7.57 (Equilibrium Theorem) (See p. 84 in [5]) *Assume that X is a Banach space and $F{:}X \to X$ is an upper semicontinuous set-valued map with closed convex image. If $K \subset X$ is a convex compact viability domain of $F(x)$, then K contains an equilibrium x^* of $F(x)$, i.e., $0 \in F(x^*)$.*

Now, we use the Equilibrium Theorem to prove the existence of the equilibrium of the system (7.53).

Lemma 7.58 *Suppose $\mathcal{C}_1$ satisfied, and each $g_i(\cdot)$ is non-trivial, $P_i > 0$, for $i = 1, 2, \ldots, n$. Define*

$$\bar{V}(x) = \sum_{i=1} P_i \int_0^{x_i} g_i(\rho)\, d\rho. \tag{7.83}$$

For any $M > 0$, define $\Omega_M = \{x{:}\bar{V}(x) \le M\}$, $\partial\Omega_M = \{x{:}\bar{V}(x) = M\}$, and

$$K_1 = \left\{ v = (v_1, v_2, \ldots, v_n)^\top \in \mathbb{R}^n{:} \sum_{i=1}^n v_i P_i \gamma_i \le 0, \textit{ for all } \gamma_i \in \overline{co}[g_i(x_i)] \right\}. \tag{7.84}$$

Then $K_1 \subset T_{\Omega_M}(x)$ whenever $x \in \partial\Omega_M$.

Proof For each $x \in \partial\Omega_M$, i.e., $\bar{V}(x) = M$, and $v \in \mathrm{int}(K_1)$ satisfying $\sum_{i=1}^n v_i P_i \gamma_i < 0$ for all $\gamma_i \in \overline{co}[g_i(x_i)]$. Let $y_n = x + h_n v$, where $0 < h_n \to 0$, as $n \to +\infty$. We will prove that $\bar{V}(y_n) \le M$, namely, $y_n \in \Omega_M$.

Denote

$$\gamma_i^e = \begin{cases} g_i(x_i^+), & \text{if } v_i > 0 \\ g_i(x_i^-), & \text{if } v_i < 0 \\ \text{any value}, & \text{if } v_i = 0. \end{cases} \tag{7.85}$$

Then we have $\sum_{i=1}^n v_i P_i \gamma_i \le \sum\limits_{i=1}^n v_i P_i \gamma_i^e$ for all $\gamma_i \in \overline{co}[g_i(x_i)]$. Thus, let $\epsilon = -\sum\limits_{i=1}^n v_i P_i \gamma_i^e$, which is positive. We have

$$\bar{V}(y_n) - \bar{V}(x) = \sum_{i=1}^{n} P_i \int_{x_i}^{y_{n_i}} g_i(\rho)\,d\rho = \sum_{i=1}^{n} P_i \int_{x_i}^{x_i+h_n v_i} g_i(\rho)\,d\rho$$
$$= \left(\sum_{i=1}^{n} v_i P_i \gamma_i^e\right) h_n + o(h_n) = -\epsilon h_n + o(h_n). \tag{7.86}$$

If n is large enough, we obtain $\bar{V}(y_n) < \bar{V}(x) = M$, which implies $v \in T_{\Omega_M}(x)$, i.e., $\mathrm{int}(K_1) \subset T_{\Omega_M}(x)$. Since $T_{\Omega_M}(x)$ is closed, $K_1 \subset T_{\Omega_M}(x)$. □

Lemma 7.59 (Ky Fan Inequality [5]) *Let K be a compact convex subset in a Banach space X and $\varphi: X \times X \to \mathbb{R}$ be a function satisfying the following conditions:*

(1) For all $y \in K$, $x \mapsto \varphi(x, y)$ is lower semicontinuous;
(2) For all $x \in K$, $y \mapsto \varphi(x, y)$ is concave, i.e., for all $\lambda_i > 0$ satisfying $\sum_i^n \lambda_i = 1$ and $y_i \in K$,

$$\varphi\left(x, \sum_{i=1}^{n} \lambda_i y_i\right) \geq \sum_{i=1}^{n} \lambda_i \varphi(x, y_i); \tag{7.87}$$

(3) For all $y \in K$, $\varphi(y, y) \leq 0$.

Then, there exists $\bar{x} \in K$ such that, for all $y \in K$, $\varphi(\bar{x}, y) \leq 0$.

Theorem 7.60 *Assume $\mathcal{C}_1$ and let $-T$ be a Lyapunov diagonally stable (LDS) matrix. Then there exists an equilibrium x^* of system (7.53), i.e.,*

$$0 \in F(x^*), \tag{7.88}$$

where $F(x^*) = [-d(x^*) + T\,\overline{co}\,[g(x^*)] + J]$.

Proof Because $-T$ is LDS, there exists a diagonal matrix $P = diag\{P_1, P_2, \ldots, P_n\}$, with $P_i > 0$, $i = 1, 2, \ldots, n$, such that $(PT)^s < 0$. Let

$$\bar{V}(x) = \sum_{i=1} P_i \int_0^{x_i} g_i(\rho)\,d\rho. \tag{7.89}$$

Case 1: All $g_i(\cdot)$, $i = 1, 2, \ldots, n$, are non-trivial.

It is easy to see that Ω_M is a convex compact subset of $\mathbb{R}^n$. Let $\alpha = \min\lambda(\{-PT\}^s) > 0$, $I = \sum_{i=1}^n [1/(2\alpha)] P_i^2 J_i^2$, $l = \min_i D_i$, and $M_0 = I/l$. In the following, we will prove that if $M > M_0$, then Ω_M is a viability domain of $F(x)$.

In fact, if $x \in \mathrm{int}(\Omega_M)$, then $T_{\Omega_M}(x) = \mathbb{R}^n$ and it is easy to see that $F(x) \bigcap T_{\Omega_M}(x) = \emptyset$.

Now, we will prove that if $x \in \partial\Omega_M$, then $F(x) \bigcap T_{\Omega_M}(x) = \emptyset$. For this purpose, we define $\varphi(g_1, g_2): \overline{co}[g(x)] \times \overline{co}[g(x)] \mapsto \mathbb{R}$, as follows:

$$\varphi(g_1, g_2) = \sum_{i=1}^{n} g_{1,i} P_i \Big[-d_i(x_i) + \sum_{j=1}^{n} t_{ij} g_{2,j} + J_i \Big], \tag{7.90}$$

where $g_1 = (g_{1,1}, g_{1,2}, \ldots, g_{1,n})^\top$ and $g_2 = (g_{2,1}, g_{2,2}, \ldots, g_{2,n})^\top$. If we can find $g_2 \in \overline{co}[g(x)]$, such that $\varphi(g_1, g_2) \le 0$ for all $g_1 \in \overline{co}[g(x)]$, then by Lemma 7.57, we have $F(x) \bigcap T_{\Omega_M}(x) \ne \emptyset$.

It can be seen that for each $g_1 \in \overline{co}[g(x)]$, $g_2 \mapsto \varphi(g_1, g_2)$ is continuous; for each $g_2 \in \overline{co}[g(x)]$, $g_1 \mapsto \varphi(g_1, g_2)$ is concave. Moreover, let $f = (f_1, f_2, \ldots, f_n)^\top$, where $f_i \in \overline{co}[g_i(x)]$. Then it is easy to see that $f_i x_i \ge \int_0^{x_i} g_i(\rho)\, d\rho$, which implies

$$\begin{aligned}\varphi(f, f) &= -\sum_{i=1}^{n} f_i P_i \frac{d_i(x_i)}{x_i} x_i + f^\top P T f + f^\top P J \\ &\le -l f^\top P x - \alpha f^\top f + f^\top P J \le -l f^\top P x - \frac{\alpha}{2} f^\top f + I \\ &\le -lM + I \le 0. \end{aligned} \tag{7.91}$$

By Lemma 7.59, we can find $\bar{g} \in \overline{co}[g(x)]$ such that $\varphi(g, \bar{g}) \le 0$ for all $g \in \overline{co}[g(x)]$. Therefore, for each $x \in \Omega_M$, we have $F(x) \bigcap T_{\Omega_M}(x) \ne \emptyset$. According to Lemma 7.57, in this case, Ω_M contains an equilibrium of $F(x)$.

Case 2: There exist some indices i such that $g_i(s) = 0$ for all $s \in \mathbb{R}$.

Without loss of generalization, we can assume that $g_n(s) = 0$ for all $s \in \mathbb{R}$ and $g_1, \ldots, g_{n-1}$ are non-trivial. Considering $\tilde{x} = (x_1, x_2, \ldots, x_{n-1})^\top$, by the discussion in Case 1, there exists an equilibrium $\tilde{x}^* = (x_1^*, x_2^*, \ldots, x_{n-1}^*)^\top$, such that $0 \in -d_i(x_i^*) + \sum_{j=1}^{n-1} t_{ij} \overline{co}[g_j(x_j)] + J_i$ for $i = 1, \ldots, n-1$. That is, there exist $\gamma_i \in \overline{co}[g_i(x_i^*)]$, for $i = 1, 2, \ldots, n-1$, such that $0 = -d_i(x_i^*) + \sum_{j=1}^{n-1} t_{ij}\gamma_j + J_i$, $i = 1, 2, \ldots, n-1$.

It can also be seen that there exists x_n^* such that $-d_n(x_n^*) + \sum_{j=1}^{n-1} t_{nj}\gamma_j + J_n = 0$. Therefore, $x^* = (\tilde{x}^*, x_n^*)^\top$ is an equilibrium of $F(x)$. The theorem is proved. □

References

1. D. H. Ackley, G. E. Hinton, and T. J. Sejnowski. A learning algorithm for Boltzmann machines. *Cognit. Sci.*, 9: 147–169 1985.
2. I. Aleksander and H. Morton An Introduction to Neural Computing. Lodon, Chapman & Hall 1990.
3. J. Andres. Almost-periodic and bounded solutions of Carathéodory differential inclusions. *Differential Integral Equations*, 12(6): 887–912, 1999.
4. J. P. Aubin and A. Cellina. *Differential Inclusions*. Berlin, Springer-Verlag, 1984.
5. J. P. Aubin and H. Frankowska. *Set-valued Analysis*. Boston, Birhauser, 1990.
6. J. P. Aubin. *Viability Theory*. Boston, Birhauser, 1991.
7. A. Bacciotti, et.al. Discontinuous ordinary differential equations and stabilization. Tesi di Dottorato di Ricerca in Mathematica, Universita degli STUDI di Firenze, 2000.
8. A. Bacciotti. Generalized solutions of differential inclusions and stability. *Ital. J. Pure Appl. Math.*, 17: 183–192, 2005.

9. J. Belair. Stability in A Model of Delayed Neural Networks. *J. Dynam. Differential Equations*, 5: 607–623, 1993.
10. J. Belair, S. A. Campbell, and P. Driessche, Van Den. Stability and delay-induced oscillations in a neural network model. *SIAM J. Appl. Math.*, 56: 245–255, 1996.
11. A. Berman. Completely positive matrices. New Jersey, World Scientific Publishing, 2003.
12. R. Bader and W. Kryszewski. On the solution sets of differential inclusions and the periodic problem in Banach space. *Nonlinear Analysis*, 54: 707–754, 2003.
13. S. Boyd et al. *Linear Matrix Inequalities in System and Control Theory. Philadelphia, SIAM*, 1994.
14. J. Cao. On exponential stability and periodic solution of CNNs with delay. *Phys. Lett.* A, 267(5–6): 312–318, 2000.
15. J. Cao. New results concerning exponential stability and periodic solutions of delayed cellular neural networks. *Phys. Lett. A*, 307: 136–147, 2003.
16. J. Cao and J. Liang. Boundedness and stability for Cohen-Grossberg neural network with time-varying delays. *J. Math. Anal. Appl.*, 296: 665–685, 2004.
17. Y. J. Cao and Q. H. Wu. A note on stability of analog neural networks with time delays. *IEEE Trans. Neural Networks*, 7: 1533–1535, 1996.
18. T. Chen. Convergence of delayed dynamical systems. *Neural Proc. Lett.*, 10(3): 267–271, 1999.
19. T. Chen. Global exponential stability of delayed hopfield neural networks. *Neural Networks*, 14(8): 977–980, 2001.
20. T. Chen and S. Amari. Stability of asymmetric hopfield neural networks. *IEEE T. Neural Networks*, 12(1): 159–163, 2001.
21. T. Chen and Y. Bai. Stability of Cohen-Grossberg neural networks with nonnegative periodic solutions. *Proceedings of International Joint Conference Neural Networks (IJCNN 2007)*, 242–247, 2007.
22. T. Chen and H. Chen. Universal approximation to nonlinear operators by neural networks and its applications to dynamical systems. *IEEE Trans. Neural Networks*, 6(4): 911–917, 1995.
23. T. Chen and H. Chen. Approxiamtion capability functions of several variables, nonlinear functionals and operators with radial basis function neural networks. *IEEE Trans. Neural Networks*, 6(4): 904–910, 1995.
24. A. Chen and J. Cao. Existence and attractivity of almost periodic solution for cellular neural networks with distributed delays and variable coefficients. *Appl. Math. Comp.*, 134(1): 125–140, 2003.
25. P. P. Civalleri, L. M. Gilli, and L. Pabdolfi. On stability of cellular neural networks with delay. *IEEE Trans.Circuits Syst.*, 40: 157–164, 1993.
26. T. Chen and W. Lu. Stability analysis of dynamical neural networks. *Proceedings of International Conference Neural Networks and Signal Processing*, 14–17, 2003.
27. T. Chen, W. Lu, and G. R. Chen. Dynamical behaviors of a large class of general delayed neural networks. *Neural Comput.*, 17: 949–968, 2005.
28. T. Chen and L. Rong. Delay-independent stability analysis of Cohen-Grossberg neural networks. *Phys. Lett. A*, 317: 436–449, 2003.
29. T. Chen and L. Rong. Robust global exponential stability of Cohen-Grossberg neural networks with time delays. *IEEE Trans. Neural Networks*, 15(1): 203–206, 2004.
30. T. Chen and L. Wang. Power-rate stability of dynamical systems with unbounded time-varying delays. *IEEE T. CAS-II: Express Briefs*, 54(8): 705–709, (2007).
31. L. O. Chua, C. A. Desoer, and E. S. Kuh. *Linear and Nonlinear Circuits*. New York, Macgraw-Hill, 1987.
32. L. O. Chua and L. Yang. Cellular neural networks: Theory. *IEEE Trans.Circuits Syst.*, 35: 1257–1272, 1988.
33. L. O. Chua and L. Yang. Cellular neural networks: Application. *IEEE Trans.Circuits Syst.*, 35: 1273–1290, 1988.
34. F. H. Clarke. *Optimization and Nonsmooth Analysis. Pliladelphia, SIAM* 1983.

35. M. A. Cohen and S. Grossberg. Absolute stability and global pattern formation and parallel memory storage by competitive neural networks. *IEEE Trans. on Man, Syst. Cybern.*, 13: 815–826, 1983.
36. G. Cybenko. *Approximation by Superpositions of a Sigmoidal Function*. Urbana, University of ILLinois, 1988.
37. G. Cybenko. Approximation by superpositions of a sigmoidal function. *Math. Cont., Sig., Sys.*, 2: 303–314, 1989.
38. B. C. Dhage. Existence of extremal solutions for discontinuous functional integral equations. *Appl. Math. Lett.*, 19: 881–886, 2006.
39. A. F. Filippov. Classical solution of differential equations with multivalued right-hand side. *SIAM J. Control*, 5(4): 609–621, 1967.
40. M. Forti and P. Nistri. Global convergence of neural networks with discontinuous neuron activations. *IEEE Trans. Circuits Syst.I*, 50(11): 1421–1435, 2003.
41. M. Forti and A. Tesi. New condition for global stability of neural networks with application to linear and quadratic programming problems. *IEEE Trans. Circiuts Syst.-1*, 42: 354–366, 1995.
42. M. Forti, P. Nistri, and D. Papini. Global exponential stability and global convergence of delayed neural networks with infinite gain. *IEEE Trans. Neural Networks*, 16(6): 1449–1463, 2005.
43. M. E. Filippakis and N. S. Papageorgiou. Periodic solutions for differential inclusions in $\mathbb{R}^{\mathbb{N}}$. *Arch. Math. (Brno)*, 42: 115–123, 2006.
44. R. E. Gaines and J. L. Mawhin. *Coincidence Degree and Nonlinear Differential Equations. Lecture Notes on Mathematics*, Berlin, Springer, 568: 10–35, (1977).
45. K. Gopalsamy and S. Sariyasa. Time delays and stimulus-dependent pattern formation in periodic environments in isolated neurons. *IEEE Trans. Neural Networks*, 13(2): 551–563, 2001.
46. S. Grossberg. Biological competition: Decision rules, pattern formation, and oscillations. *Proc. Natl. Acd. Sci. USA*, 77(4): 2338–2342, 1980.
47. S. Grossberg. Nonlinear neural networks: principles, Mechanisms, and architectures. *Neural Networks*, 1: 17–61, 1988.
48. G. Haddad. Monotone viable trajectories for functional differential inclusions. *J. Diff. Equ.*, 42: 1–24, 1981.
49. J. K. Hale. *Theory of Functional Differential Equations*. New York, Springer-Verlag 1977.
50. H. Harrer, J. A. Nossek, and R. Stelzl. An analog implementation of discrete-time neural networks. *IEEE Trans. Neural Networks*, 3: 466–476, 1992.
51. S. Haykin. *Neural Networks: A Comprehensive Foundation*. New York, Macmillan Publishing Company, 1994.
52. D. O. Hebb. *The Orgnization of Behavior: A Neuropsuchological Theory*. New York, Wiley, 1949.
53. R. Hecht-Nielsen. *Neurocomputing, Reading*. MA, Addison-Wesley, 1990.
54. J. J. Hopfield. Neurons with graded response have collective computational properties like thoseof of two-stage neurons. *Proc. Nat. Acad. Sci -Biol.*, 81: 3088–3092, 1984.
55. X. Huang and J. Cao. Almost periodic solution of shunting inhibitory cellular neural networks with time-varying delays. *Phys. Lett. A*, 314(3): 222–231, 2003.
56. S. Hu and N. S. Papageorgiou. On the existence of perioidic solutions for nonconvex-valued differential inclusions in $\mathbb{R}^N$. *Proc. Amer. Math. Soc.*, 123(10): 3043–3050 1995.
57. J. J. Hopfield, and D. W. Tank. Computing neural circuits: a model. *Science*, 233: 625–633 1986.
58. A. G. Ivanov. On the equivalence of differential inclusions and controlled almost periodic systems. (Russian) *Differ. Uravn.* 33(7): 876–884, 1997; translation in *Differ. Equ.*, 33(7): 879–887, 1997.
59. M. P. Kennedy and L. O. Chua. Neural networks for nonlinear programming. *IEEE Trans. Circuits Syst.-1*, 35: 554–562, 1988.
60. W. A. Kirk and B. Sims. *Handbook of Metric Fixed Point Theory*. Berlin, NY, Springer-Verlag, 2001.

61. B. M. Levitan and V. V. Zhikov. *Almost Periodic Functions and Differential Equations*. New York, Cambriadge University Press, 1982.
62. H. Lu. On stability of nonlinear continuous-time neural networks with delays. *Neural Networks*, 13(10): 1135–1144, 2000.
63. W. Lu and T. Chen. On periodic dynamical systems. *Chin. Ann. Math.*, 25B(4): 455–462, 2004.
64. W. Lu and T. P. Chen. Dynamical behaviors of Cohen-Grossberg neural networks with discontinuous activation functions. *Neural Networks*, 18: 231–242, 2005.
65. W. Lu and T. Chen. Global exponential stability of almost periodic trajectory of a large class of delayed dynamical systems. *Sci. China A: Math.*, 48(8): 1015–1026, 2005.
66. W. Lu and T. P. Chen. Dynamical Behaviors of delayed neural network systems with discontinuous activation functions. *Neural Comput.*, 18(3): 683–708, 2006.
67. W. Lu and T. Chen. $\mathbb{R}^n_+$ global stability of Cohen-Grossberg neural network system with nonnegative equilibrium. *Neural Networks*, 20: 714–722, 2007.
68. W. Lu and T. Chen. Almost periodic dynamics of a class of delayed neural networks with discontinuous activations. *Neural Comput.*, 20: 1065–1090, 2008.
69. W. Lu, L. Rong, and T. Chen. Global convergence of delayed neural networks systems. *Int. J. Neural Sys.*, 13(3): 193–204, 2003.
70. D. Li and P. E. Kloeden. On the dynamics of nonautonomous periodic general dynamical systems and differential inclusions. *J. Diff. Equ.*, 224: 1–38, 2006.
71. X. Liao, C. Li, and K. Wong. Criteria for exponential stability of Cohen-Grossberg neural networks. *Neural Networks*, 17: 1401–1414, 2004.
72. G Li and X. Xue. On the existence of periodic solutions for differential inclusions. *J. Math. Anal. Appl.*, 276: 168–183, 2002.
73. R. K. Miller. Asymptotic behavior of nonlinear delayed-differential equations. *J. Differ. Equ.*, 1: 293–305, 1995.
74. C. M. Marcus and R. M. Westervelt. Stability of analog neural networks with *delay. Phys. Rev. A.*, 39: 347–359, 1989.
75. J. M. Mendel and K. S. Fu. *Adaptive, Learning, and Pattern Recognition Systems: Theory and Applications*. New York, Academic Press, 1970.
76. N. Megiddo and M. Kojima. On the existence and uniqueness of solutions in nonlinear complementarity theory. *Math. Prog.*, 12: 110–130, 1977.
77. B. E. Paden and S. S. Sastry. Calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulator. *IEEE Trans. Circuits Syst.*, 34: 73–82, 1987.
78. D. Papini and V. Taddei. Global exponential stability of the periodic solution of the delayed neural networks with discontinuous activations. *Phys. Lett. A.*, 343: 117–128, 2005.
79. C. S. Ramóny. Histologie du système nerveux de l'homme et des vertébrés. Paris: Maloine; Edition Francaise Revue: Tome I (1952); Tome II (1955); Madrid: Consejo Superior de Inverstigaciones Cientificas 2001.
80. T. Rockafellar and R.J.-B. Wets. *Variational Analysis*. Berlin Heidelberg, Springer-Verlag, 1998.
81. G. V. Smirnov. Weak asymptotic stability at first approximation for periodic differential inclusions. *NoDEA*, 2: 445–461, 1995.
82. L. Wang and X. Zou. Exponential stability of Cohen-Grossberg neural networks. *Neural Networks*, 15: 415–422, 2002.
83. K. Yoshida. *Functional Analysis. Grundlehren der Mathematicchen Wissenschaften*. New York: Springer-Verlag, 1978.
84. T. Yosizawa. *Stability Theory and The Existence of Periodic Solutions and Almost Periodic Solutions*. New York, Springer-Verlag, 1975.
85. V. I. Utkin. *Sliding Modes and Their Applications in Variable Structure Systems*. Moskow, MIR Publishers, 1978.

86. J. Zhang and X. Jin. Global stability analysis in delayed hopfield neural models. *Neural Networks*, 13(7): 745–753, 2000.
87. Y. Zheng and T. Chen. Global exponential stability of delayed periodic dynamical systems. *Phys. Lett. A*, 322(5–6): 344–355, 2004.
88. J. Zhou, Z. Liu, and G. Chen. Dynamics of delayed periodic neural networks. *Neural Networks*, 17(1): 87–101, 2004.
89. A. V. Zuev. On periodic solutions of ordinary differential equations with discontinuous right-hand side. *Math. Notes*, 79(4): 518–527, 2006.