General Form of Probabilities on IF-Sets

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Abstract. The paper has two aims. First, a review of various definitions of probabilities on Atanassov IF-sets, and corresponding representation theorems. Secondly, a new representation theorem is proved for so-called φ-probabilities including a large variety of special cases.

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1 IF-Events

According to Atanassov ([1]), an IF-set is a mapping

$$A = (\mu_A, \nu_A),$$

defined on an non-empty set Ω to $[0,1]^2$ (i.e. $\mu_A : \Omega \to [0,1], \nu_A : \Omega \to [0,1]$), such that

$$\mu_A(x) + \nu_A(x) \leq 1$$

for any $x \in \Omega$. An IF-set $A = (\mu_A, \nu_A)$ is called an IF-event, if $\mu_A, \nu_A : \Omega \to [0,1]$ are measurable mappings with respect to the given σ-algebra $\mathcal{S}$ of subsets of Ω. Recall that μ_A is called a membership function, ν_A the non-membership function. Therefore it is natural to define

$$A = (\mu_A, \nu_A) \leq B = (\mu_B, \nu_B)$$

if and only if

$$\mu_A \leq \mu_A, \nu_A \geq \nu_B.$$

Denote by $\mathcal{F}$ the family of all IF-events. With respect to the preceding definition it is easy to see that $(0_\Omega, 1_\Omega)$ is the smallest element of $\mathcal{F}$, $(1_\Omega, 0_\Omega)$ is the greatest element of $\mathcal{F}$.

Probability of an IF-event A was defined first constructively (see [7,6]) as a compact interval, lately axiomatically (see [11,12,13]) as a mapping

$$\mathcal{P} : \mathcal{F} \to \mathcal{J}$$

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where $\mathcal{J}$ is the family of all compact subintervals $[c, d]$ of [0,1] in the real line. Since

$$\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)],$$

we obtain two real functions

$$\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to [0, 1].$$

These functions will be called states (analogously as in quantum structures [5]). Evidently any solution of the problem of IF - states on $\mathcal{F}$ leads naturally to a solution of the problem of IF - probabilities.

Similarly as in the classical or quantum case resp., IF - state $m : \mathcal{F} \to [0, 1]$ can be defined as a normalized, additive and continuous mapping. The first condition is clear:

$$(1) \quad m((0_\Omega, 1_\Omega)) = 0, m((1_\Omega, 0_\Omega)) = 1.$$

Similarly continuity is determine uniquely:

$$(2) \quad A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A).$$

Here $A_n \nearrow A$ is equivalent with two convergences of real functions:

$$\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A.$$

Of course, the additivity cannot be determined uniquely. First it was defined by the help of Lukasiewicz connectives.

2 Lukasiewicz States

If $f, g : \Omega \to [0, 1]$ are two fuzzy sets. then the Lukasiewicz connectives are

$$f \oplus_L g - \min(f + g, 1), f \odot_L g - \max(f + g = 1, 0).$$

Therefore for $A, B \in \mathcal{F}$ we define

$$A \oplus_L B = (\mu_A \oplus_L \mu_B, \nu_A \odot_L \nu_B),$$

$$A \odot_L B = (\mu_A \odot_L \mu_B, \nu_A \oplus_L \nu_B).$$

Hence L-additivity means the implication

$$(3) \quad A \odot_L B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus_L B) = m(A) + m(B).$$

A mapping $m : \mathcal{F} \to [0, 1]$ is called L-state if the conditions (1), (2) and (3) hold. The first important result was the representation theorem ([13]):

Theorem 1. *Let $m : \mathcal{F} \to [0,1]$ be an L-state and there exists a probability measure $P : \mathcal{S} \to [0,1]$ and a function $f : [0,1]^2 \to [[0,1]$ such that*

$$(*) \qquad m(A) = f(\int_\Omega \mu_A dP, \int_\Omega \nu_A dP).$$

Then f is linear, hence there exists $\alpha \in [0,1]$ such that

$$m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$$

for any $A \in \mathcal{F}$.

Recently L. Ciungu proved that the assumption (*) can be omitted (see [3]).

Theorem 2. *Let $m : \mathcal{F} \to [0,1]$ be an L-state. Then there are probability measures $P, Q : \mathcal{S} \to [0,1]$ and $\alpha \in [0,1]$ such that*

$$m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dQ)$$

for any $A \in \mathcal{F}$.

The second important result is in the following theorem ([14,15,4]).

Theorem 3. *To any $\mathcal{F}$ with an L-state $m : \mathcal{F} \to [0,1]$ there exists an MV algebra M with a state $\mu : M \to [0,1]$ such that $(\mathcal{F}, m)$ and (M, μ) are isomorphic.*

Theorem 3 gives possibility to use well developed probability theory on MV algebras (see [16]). Of course, later there appeared alternative definitions using alternative connectives and corresponding states.

1. M-probability with

$$A \oplus_M B = (\max(\mu_A, \mu_B), \min(nu_A, \nu_B))$$
$$A \odot_M B = (\min(\mu_A, \mu_B), \max(nu_A, \nu_B)).$$

(see [8,15,4]).

2. Q-probability with

$$A \oplus_Q B = (\sqrt{\mu_A^2 + \mu_B^2}, 1 - \sqrt{(1-\nu_A)^2 + (1-\nu_B)^2}),$$
$$A \odot_Q B = (\min(\mu_A + \mu_B, 1), \max(\nu_A + \nu_B - 1, 0)).$$

(see [2]).

3. P-probability with

$$A \oplus_P B = (\mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B),$$
$$A \odot_P B = (\mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B).$$

(see [2]).

The aim of this paper is to prove a representation theorem for so-called φ-states. Here $\varphi : [0,1] \to [0,1]$ is an increasing bijection such that $\varphi(u) \leq u (u \in [0,1])$. Following [9] we shall consider the following pair of connectives

$$A \oplus_\varphi B = (\varphi^{-1}(\min(\varphi(\mu_A) + \varphi(\mu_B), 1)), 1 - \varphi^{-1}(\min(\varphi(1-\nu_A) + \varphi(1-\nu_B), 1))),$$
$$A \odot B = (\max(\mu_A + \mu_B - 1, 0), \min(\nu_A + \nu_B, 1)).$$

3 φ -States

φ-state is a mapping $m : \mathcal{F} \to [0,1]$ such that

$$(i) \quad m((0_\Omega, 1_\Omega)) = 0, m((1_\Omega, 0_\Omega)) = 1;$$

$$(ii) \quad A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus_\varphi B) = m(A) + m(B);$$

$$(iii) \quad A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A).$$

Evidently any L-state is a φ-state, where $\varphi(u) = u, u \in [0,1]$. Similarly any Q-state is a φ-state, where $\varphi(u) = u^2$. Moreover, in [9] the Yager state was considered where $\varphi(u) = u^n(n \in N$;for $n = 1$ we obtain the Lukasiewicz state, for $n = 2$ the Yager state). In [9] also the cases $\varphi(u) = 2^u$ and $\varphi(u) = logu$ were considered.

In [10] Renčová introduced the notion of the strong additivity using the operation

$$A \odot_\varphi B = ((\varphi(\mu_A) + \varphi(\mu_B) - 1) \vee 0, (\varphi(1 - \nu_A) + \varphi(1 - \nu_B)) \wedge 1).$$

The state m is strongly additive, if

$$A \odot_\varphi B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus_\varphi B) = m(A) + m(B).$$

Since $\varphi(u) \leq u$ for any $u \in [0,1]$, it is not difficult to show that

$$A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow A \odot_\varphi B = (0_\Omega, 1_\Omega).$$

Therefore any strongly φ-additive state is φ-additive. Renčová using Theorem 1 proved the representation theorem for strongly φ-additive states:

$$m(A) = (1 - \alpha) \int_\Omega \varphi(\mu_A) dP + \alpha \int_\Omega \varphi(1 - \nu_A) dP.$$

In the following theorem the same result will be proved for arbitrary φ-state, of course again with the following additional assumption:

$$(**) \quad m(A) = f(\int_\Omega \varphi(\mu_A) dP, \int_\Omega \varphi(1 - \nu_A) dQ),$$

where $P, Q : \mathcal{S} \to [0,1]$ are some probability measures and $f : [0,1]^2 \to [0,1]$ is a continuous function.

Theorem 4. *To any φ-state m satisfying the condition (**) there exists $\alpha \in [0,1]$ such that*

$$m(A) = (1 - \alpha) \int_\Omega \varphi(\mu_A) dP + \alpha \int_\Omega \varphi(1 - \nu_A) dQ$$

for any $A \in \mathcal{F}$.

Proof. First we see that

$$0 = m((0_\Omega, 1_\Omega)) = f(\int_\Omega \varphi(0_\Omega)dP, \int_\Omega \varphi(1 - 1_\Omega)dQ) = f(0,0)$$

hence $f(0,0) = 0$. Similarly

$$1 = m((1_\Omega, 0_\Omega)) = f(\int_\Omega \varphi(1_\Omega)dP, \int_\Omega \varphi(1 - 0_\Omega)dQ) = f(1,1),$$

hence $f(1,1) = 1$. Let $A \odot B = (0_\Omega, 1_\Omega)$. It means $\mu_A + \mu_B \leq 1, \nu_A + \nu_B \geq 1$, hence $(1 - \nu_A) + (1 - \nu_B) \leq 1$. Therefore

$$\varphi(\mu_A) + \varphi(\mu_B) \leq 1, \varphi(1 - \nu_A) + \varphi(1 - \nu_B) \leq 1,$$

$$A \oplus_\varphi B = (\varphi^{-1}(\varphi(\mu_A) + \varphi(\mu_B)), 1 - \varphi^{-1}(\varphi(1 - \nu_A) + \varphi(1 - \nu_B))).$$

Put

$$\int_\Omega \varphi(\mu_A)dP = u_1, \int_\Omega \varphi(1 - \nu_A)dQ = u_2,$$

$$\int_\Omega \varphi(\mu_B)dP = v_1, \int_\Omega \varphi(1 - \nu_B)dQ = u_2.$$

Then

$$m(A) = f(u_1, u_2), m(B) = f(v_1, v_2),$$

$$m(A \oplus_\varphi B) = f(\int_\Omega \varphi(\varphi^{-1}(\varphi(\mu_A) + \varphi(\mu_B)))dP, \int_\Omega \varphi(1 - 1 + \varphi^{-1}(\varphi(1 - \nu_A) + \varphi(1 - \nu_B)))dQ) =$$

$$= f(\int_\Omega \varphi(\mu_A)dP + \int_\Omega \varphi(\mu_B)dP, \int_\Omega \varphi(1 - \nu_A)dQ + \int_\Omega \varphi(1 - \nu_B)dQ) =$$

$$= f(u_1 + v_1, u_2 + v_2),$$

hence we have obtained the identity

$$(1) \quad f(u_1 + v_1, u_2 + v_2) = f(u_1, u_2) + f(v_1 + v_2).$$

Putting $A = B$ and using induction we obtain

$$(2) \quad f(kx) = kf(x)$$

for any $k \in N$ such that $kx \in [0,1]^2$. Let $\frac{p}{q} \in Q$ with $\frac{p}{q}x \in [0,1]^2$. Then

$$f(x) = f(\frac{1}{q}x) + ...f(\frac{1}{q}x) = qf(\frac{1}{q}x),$$

hence

$$f(\frac{1}{q}x) = \frac{1}{q}f(x),$$

and

$$(3) \quad \frac{p}{q}f(x) = f(\frac{p}{q}x).$$

Since f is continuous, we obtain the identity

$$(4) \quad f(ax) = af(x), a \in [0,1], x \in [0,1]^2, ax \in [0,1]^2.$$

We have seen that

$$f(0,0) = 0, f(1,1) = 1.$$

Put $f(0,1) = \alpha$. By (1) we obtain

$$1 = f(1,1) = f(0,1) + f(1,0) = \alpha + f(1,0),$$

hence

$$f(1,0) = 1 - \alpha.$$

Finally

$$m(A) = f(u_1, u_2) = f(u_1, 0) + f(0, u_2) =$$
$$= u_1 f(1,0) + u_2 f(0,1) = (1-\alpha)u_1 + \alpha u_2 =$$
$$= (1-\alpha)\int_\Omega \varphi(\mu_A)dP + \alpha \int_\Omega \varphi(1-\nu_A)dQ.$$

□

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References

1. Atanassov, K.: Intuitionistic Fuzzy Sets: Theory and Applications. Physica Verlag, New York (1999)
2. Atanassov, K.: Riečan, B: On two new types of probability theory on IF-events. Notes on IFS (2007)
3. Ciungu, L.: Riečan, B.: General for of probabilities on IF-sets. Information Science (submitted)
4. Čunderlíková, K., Riečan, B.: Intuitionistic fuzzy probability theory. Edited volume on Intuitionistic Fuzzy Sets: recent Advances of the series Studies on Fuzziness and Soft Computing. Springer, Heidelberg (2008)
5. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer, Dordrecht (2000)
6. Gerstenkorn, T., Manko, J.: Probabilities of intuitionistic fuzzy events. In: Hryniewicz, O., et al. (eds.) Issues in Intelligent Systems: Paradigms, pp. 63–68. EXIT, Warszawa (2005)
7. Grzegorzewski, P., Mrowka, E.: Probability of intuitionistic fuzzy events. In: Grzegorzewski, P., et al. (eds.) Soft Methods in Probability, Statistics and data Analysis, pp. 105–115. Physica Verlag, New York (2002)
8. Krachounov, M.: Intuitionistic probability and intuitionistic fuzzy sets. In: El-Darzi, et al. (eds.) First Intern. Workshop on IFS, pp. 714–717. Univ. Of Westminster, London (2006)

9. Renčová, M.: On the φ-probability and φ-observables. Fuzzy Sets and Systems (submitted)
10. Renčová, M.: General form of strongly additive φ-probability. In: Magdalenca, L., et al. (eds.) Proc. IPMU 2008, pp. 1671–1674 (2008)
11. Riečan, B.: A descriptive definition of the probability on intuitionistic fuzzy sets. In: Wagenecht, M., Hampet, R. (eds.) Proc. EUSFLAT 2003, Zittau-Goerlitz Univ. Appl. Sci, Dordrecht, pp. 263–266 (2000)
12. Riečan, B.: Representation of probabilities on IFS events. In: López-Diaz, M., et al. (eds.) Advances in Soft Computing, Soft Methodology and Random Information Systems, pp. 243–246. Springer, Berlin (2004)
13. Riečan, B.: On a problem of Radko Mesiar: general form of IF - probabilities. Fuzzy sets and Systems 157, 1485–1490 (2006)
14. Riečan, B.: On the entropy of IF dynamical systems. In: Proc. IWIFSGN 2005, Warszawa, September 16, 2005, pp. 328–336 (2005)
15. Riečan, B.: Probability theory on IF events. In: Aguzzoli, S., Ciabattoni, A., Gerla, B., Manara, C., Marra, V. (eds.) ManyVal 2006. LNCS, vol. 4460, pp. 290–308. Springer, Heidelberg (2007)
16. Riečan, B.: - Mundici, D.: Probability on MV-algebras. In: Pap, E. (ed.) Hanbook on Measure Theory, pp. 869–909. Elsevier, Amsterdam (2002)
17. Riečan, B., Neubrunn, T.: Integral, Measure, and Ordering. Kluwer, Dordrecht (1997)