On the Rate of Structural Change in Scale Spaces

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Abstract. We analyze the rate in which image details are suppressed as a function of the regularization parameter, using first order Tikhonov regularization, Linear Gaussian Scale Space and Total Variation image decomposition. The squared L^2-norm of the regularized solution and the residual are studied as a function of the regularization parameter. For first order Tikhonov regularization it is shown that the norm of the regularized solution is a convex function, while the norm of the residual is not a concave function. The same result holds for Gaussian Scale Space when the parameter is the variance of the Gaussian, but may fail when the parameter is the standard deviation. Essentially this imply that the norm of regularized solution can not be used for global scale selection because it does not contain enough information. An empirical study based on synthetic images as well as a database of natural images confirms that the squared residual norms contain important scale information.

Keywords: Regularization, Tikhonov Regularization, Scale Space, TV, Total Variation, Geometric Structure, Texture.

1 Introduction

Images contain a mix of different type of information - from fine scale stochastic textures to large scale geometric structures. Image regularization can be viewed as approximating the observed original image with a simpler image, where simpler is defined by the regularization (prior) term and the regularization parameter λ. Here an image is considered to be simpler if it is smoother (or piece-wise smoother). Regularization can also be viewed as decomposing the observed image into a regularized (smooth) component and a small scale texture/noise component (called the residual, because it is the difference between the regularized solution and the observed image). By increasing the regularization parameter λ smoother and smoother approximations are generated. The rate in which image details are suppressed as a function of the regularization parameter depends on the image content and regularization method. The image residual contains the details that are suppressed during the regularization and the norm of the residual is a measurement of the amount of details that are suppressed. The norm of the residual as a function of the regularization parameter gives important information about the image content. For images containing small scale structure a lot of details are suppressed even for small λ and the norm of the residual will be large for small λ. For images containing solely large scale geometric structures few details will be suppressed for small λ and

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the norm of the residual will be small. The rate in which details are suppressed can be viewed as the derivative of the norm of the residual with respect to the regularization parameter, and reveals the amount of details that are suppressed if the regularization parameter increases.

First order Tikhonov regularization, Gaussian linear scale space (which is equivalent to infinite order Tikhonov regularization [1]) and Total Variation image decomposition are studied. The squared L^2-norm of the regularized solution and the residual are studied as functions of the regularization parameter. Of special interest is the convexity/concavity of those norms viewed as functions, because it relates to the possibility that the rate in which details are suppressed can increase/decrease. In section 2, first order Tikhonov regularization is revisited and it is shown that the norm of the regularized solution is a convex function, while the norm of the residual is not a concave function. In section 3, linear Gaussian Scale Space is revisited, and it is shown that the norm of the regularized solution is convex as a function of the Gaussian variance, or equivalently diffusion time, but may fail to be convex when the parameter is the Gaussian standard deviation. The squared norm of the residual is in general not a concave function of its parameter. In section 4, Total Variation (TV) image decomposition is revisited. In section 5 experimental results are presented, the norm of the Sinc function, synthetic image containing image structures at different scales and natural images are studied.

These studies tend to show that the square residual norm contains scale information, particularly at values where local convexity/concavity behavior changes.

1.1 Related Work

Characterization of images by analyzing the behavior of the norm of the regularized solution and the residual as functions of the regularization parameter has not received much research attention. Sporring and Weickert [2, 3] view images as distributions of light quanta and use information theory to study the structure of images in scale space. The entropy of an image as a function of the scale (in scale-space) is analyzed and shown to be an increasing function of the scale. The result holds both for linear Gaussian scale space and non-linear scale-space. Furthermore the derivative of the entropy with respect to the scale is shown, empirically, to be a good texture descriptor. The derivative of the scale-space entropy function with respect to the scale is a global measure of how much the entropy of an image changes at different scale. Where Sporring and Weickert studies monotone functions of images across scale, we study norms of the scale space image and residual.

Buades et.al [4] introduced the concept of Method Noise in denoising. The Method Noise is the image details that are removed in the denoising - i.e. the residual image - and the content is used for comparing denoising methods. The residual image has often been used for determine the optimal regularization parameter. (See Thompson et.al [5] for a classical study.) Selection of the optimal stopping time for diffusion filter was studied by Mrazek and Navara [6], which also relate to the Lyapunov functionals studied by Weickert [7].

1.2 Convexity, Fourier Transforms, Power Spectra

Recall that a function $f(x)$ defined on a convex set C is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $0 \leq \lambda \leq 1$ and for all $x, y \in C$. If $f(x)$ is convex on a convex set C then $-f(x)$ is said to be *concave* on C. When $f(x)$ is twice-differentiable, a necessary and sufficient condition for convexity is

$$\forall x \in C, \quad f''(x) \geq 0 \tag{1}$$

(in the multidimensional case a the Hessian matrix is positive semi-definite). Two elementary facts will be used in the sequel: 1) let $h(\lambda)$ be a function of the form

$$h(\lambda) = \int d(\lambda, x) s(x) dx \tag{2}$$

where $d(\lambda, x)$ is convex in λ and $s(x) \geq 0$ then $h(\lambda)$ is convex. 2) Assume that $f(x) = h(g(x))$ where $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$. Then

- if h is convex and non-decreasing and g is convex, then f is convex,
- if h is convex and non-increasing and g is concave, then f is concave.

The Fourier transform of a function f is denoted with $\hat{f}$. Parseval's theorem asserts that this is an isometry of L^2: $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ where

$$\|f(x,y)\|_2^2 = \iint |f(x,y)|^2 dxdy. \tag{3}$$

The frequency domain variables are denoted $(\omega_x, \omega_y) =: \omega$. The power spectrum function of a function f is the function $\omega \mapsto |\hat{f}(\omega)|^2$. f is said to follow a (α-)power law if $|\hat{f}(\omega)| \sim C/|\omega|^\alpha$, where C and α are some constants. It is well known that the power spectra computed over a large ensemble of natural image approximate a power law in spatial frequencies with α around 1.7 or at least in $(0, 2)$ [8,9].

We use often implicitly the following classical result from Calculus. Let $B := B(0,1)$ the unit ball of $\mathbb{R}^n$ and B^c its complement. Let g a positive function defined on $\mathbb{R}^n$. Assume that $g \sim \|x\|^{-\alpha}$ in B (resp B^c). Then $\int_B g\,dx < \infty$ if and only if $\alpha < n$ (resp. $\int_{B^c} g\,dx < \infty$ if and only if $\alpha > n$).

Finally, to conclude this paragraph, given a regularization, the functions $s(\lambda)$ and $r(\lambda)$ will denote the squared L^2-norm of respectively the the regularized solution and of the residual as a function of the regularization parameter λ.

2 Tikhonov Regularization

The first order Tikhonov regularization is defined as the minimizer of the energy functional

$$E_\lambda[f] = \iint (f-g)^2 + \lambda|\nabla f|^2 dxdy \tag{4}$$

where g is the observed data and λ is the regularization parameter. The energy functional is composed of two terms: the data fidelity term $\|f-g\|_2^2$ and the regularization term $\|\nabla f\|_2^2$. Note that Wiener filter can be regarded as a Tikhonov regularization method applied to the Fourier domain. Thanks to Parseval's theorem all calculation can be performed in the Fourier domain where this energy becomes

$$\hat{E}_\lambda[f] = \iint (\hat{f}-\hat{g})^2 - \lambda(\omega_x^2\hat{f}^2 + \omega_y^2\hat{f}^2)d\omega_x d\omega_y. \tag{5}$$

Using the Calculus of Variations, a necessary condition for a function f to minimize the functional (4) is given by its Euler-Lagrange equation: $(f-g)-\lambda\Delta f = 0$. In the Fourier domain, it becomes

$$\hat{f} - \hat{g} + \lambda(\omega_x^2\hat{f} + \omega_y^2\hat{f}) = 0 \quad \text{i.e } \hat{f} = \frac{\hat{g}}{1+\lambda|\omega|^2} \tag{6}$$

that is, the original signal multiplied with the filter function $F(\lambda,\omega) = \frac{1}{1+\lambda|\omega|^2}$ which is a non-increasing convex function w.r.t λ (for $\lambda \geq 0$). Set $d(\lambda,\omega) = F(\lambda,\omega)^2$. It is important to remark that defining the regularization in frequency domain by $\lambda \to F(\lambda,\omega)\hat{g}(\omega)$ extends Tikhonov regularization beyond the case where $g \in W^{1,2}(\mathbb{R}^2)$, the Sobolev space of L^2 functions with L^2 weak derivatives, which is the natural space for Tikhonov regularization as defined by minimization of (4). Indeed, the corresponding function $s(\lambda)$ is given by

$$s(\lambda) = \|F(\lambda,\omega)\hat{g}\|_2^2 = \iint d(\lambda,\omega)|\hat{g}|^2 d\omega. \tag{7}$$

This is the integral of the squared filter function times the power spectrum of the original signal g, and we have the following result:

Proposition 1. *The squared L^2-norm $s(\lambda)$ of the minimizer of the Tikhonov regularization functional as a function of the regularization parameter λ is, for non-trivial images, a monotonically decreasing convex function (for $\lambda \in (0,\infty)$), when it exists.*

If g follows an α-power law, then from the Calculus fact recalled in the previous section, $g \notin L^2(\mathbb{R}^n)$, however $s(\lambda)$, $s'(\lambda)$ and $s''(\lambda)$ exist and are finite for $\lambda > 0$ if and only if $\alpha \in (0,2)$ (which is the case for natural images). Both s' and s'' diverge for $\lambda \to 0^+$.

The square of a non-increasing convex function is a convex function, and from Section 1.2 we have the first part of the proposition. Now

$$d_\lambda(\lambda,\omega) = -\frac{2|\omega|^2}{(1+\lambda|\omega|^2)^3}, \quad d_{\lambda\lambda}(\lambda,\omega) = 6\frac{2|\omega|^4}{(1+\lambda|\omega|^2)^4}.$$

$s'(\lambda) = \iint d_\lambda(\lambda,\omega)|g|^2\,d\omega$ and $s''(\lambda) = \iint d_{\lambda\lambda}(\lambda,\omega)|g|^2\,d\omega$ and the rest of the proposition follows by elementary analysis. $\square$

Set $R(\lambda,\omega) = 1 - F(\lambda,\omega)$ and $e(\lambda,\omega) = R(\lambda,\omega)^2$. The Fourier image residual is $R(\lambda)\hat{g}$ and its squared norm is

$$r(\lambda) = \|R(\lambda,\omega)\hat{g}\|^2 = \iint e(\lambda,\omega)|\hat{g}|^2\,d\omega$$

An elementary calculation gives $e_\lambda(\lambda,\omega) = 2\lambda|\omega|^2/(1+\lambda|\omega|^2)^3$ and this function, is for λ fixed, bounded in ω while it satisfies

$$\forall\omega, \quad \lim_{\lambda\to 0+} e_\lambda(\lambda,\omega) \to 0, \ \lim_{\lambda\to\infty} e_\lambda(\lambda,\omega) \to 0$$

The same holds for $r'(\lambda)$ when it is finite and therefore by the mean value theorem, as it is positive, it must have a maximum and $r''(\lambda)$ must change sign and we can state the following:

Proposition 2. *Assume first that $g \in W^{1,2}(\mathbb{R}^2)$ is non trivial. Then, although $s(\lambda)$ is convex and decreasing, the squared norm residual $r(\lambda)$ of Tikhonov regularization, while increasing from 0 to $\|g\|_2^2$, is neither concave nor convex.*

Note that when g is a $\alpha-$power law with $\alpha \in (0,2)$, $g \notin L^2(\mathbb{R})$ while its regularization g_λ is when $\lambda > 0$, thus $g - g_\lambda \notin L^2(\mathbb{R}^2)$ and $r(\lambda) = \|g - g_\lambda\|_2^2 = +\infty$.

3 Linear Scale-Space and Regularization

Linear scale-space theory [10, 11, 12] deals with simplified coarse scale representation of an image g, generated by solving the diffusion (heat) equation with initial value g:

$$\frac{\partial f}{\partial t} = \triangle f, \quad f(-,0) = g(-) \tag{8}$$

where $\triangle = \partial_{xx} + \partial_{yy}$ is the Laplacian. Equivalently, this coarse scale representation can be obtained by convolution with a Gaussian kernel:

$$f_\sigma = g * G_\sigma, \quad G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{9}$$

and the link between the two formulations is given by $f_\sigma = f(-, 2\sigma^2)$. A third formulation of Linear Scale-Space is obtained as "infinite order" Tikhonov regularization, the 1-dimensional case was introduced by Nielsen *et al.* in [1]. In dimension 2, one defines for $\lambda > 0$

$$E[f] = \iint (f-g)^2 \, dxdy + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \iint \sum_{\ell=0}^{k} \left(\binom{k}{\ell} \frac{\partial^k f}{\partial x^\ell \partial y^{k-\ell}} \right)^2 dxdy \tag{10}$$

where $\binom{k}{\ell}$ is the (ℓ, k)-binomial coefficient. By a direct computation, its associated Euler-Lagrange equation is given by

$$f - g + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k}{k!} \triangle^k f = 0$$

where $\triangle^k$ is the k-th iterated Laplacian

$$\triangle^k = \underbrace{\triangle \circ \cdots \circ \triangle}_{k \text{ times}} = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{\partial^{2k}}{\partial x^{2\ell} \partial y^{2(k-\ell)}}.$$

Via Fourier Transform, the Laplacian operator becomes the multiplication by $-|\omega|^2$ operator and as in 1st order Tikhonov regularization, the solution is given by filtering:

$$\hat{f} = \frac{\hat{g}}{1 + \sum_{k=1}^{\infty} \frac{\lambda^k |\omega|^{2k}}{k!}} = e^{-\lambda|\omega|^2}\hat{g}. \tag{11}$$

The solution of the filtering problem for a given $\lambda > 0$ is the same as solving (8) with $t = \lambda$. By setting $\lambda = 2\sigma^2$ and applying the convolution theorem to (9) one gets the above equation. Using the Fourier formulation, the squared norm of the solution at λ of (11) $s(\lambda)$ the squared-norm residual $r(\lambda)$ are given by

$$s(\lambda) = \|e^{-\lambda|\omega|^2}\hat{g}\|_2^2 = \iint e^{-2\lambda|\omega|^2}|\hat{g}(\omega)|^2\, d\omega,$$

$$r(\lambda) = \|(1 - e^{-\lambda|\omega|^2})\hat{g}\|_2^2 = \iint \left(1 - e^{-\lambda|\omega|^2}\right)^2 |\hat{g}(\omega)|^2\, d\omega.$$

If one defines $d(\lambda, \omega) = e^{-2\lambda|\omega|^2}$ and $e(\lambda, \omega) = (1 - e^{\lambda|\omega|^2})$, they have with respect to convexity/concavity, the same properties as their Tikhonov counterpart defined in the previous section and one can state the following, in term of heat equation / Gaussian variance

Proposition 3

1. *The squared L^2-norm $s(t)$ of the solution of heat equation as a function of the diffusion "time" t (or equivalently the convolution by the Gaussian kernel in function of the kernel* variance*) is, for non-trivial images, a monotonically decreasing convex function (for $t \in (0, \infty)$), when it exists.*
2. *The squared norm residual $r(t)$ of the solution of the heat equation at time t, while increasing from 0 to $\|g\|_2^2$, is neither concave nor convex.*

If, instead of using the diffusion time / variance as parameter, one uses the *standard deviation* σ of the Gaussian kernel, the resulting solution squared norm function $s(\sigma)$, although increasing, may fail to be convex as the function $\sigma \mapsto e^{-\sigma^2|\omega|^2}$ is not convex in σ, this is a half Gaussian bell. A simple example showing the convexity failure is provided by the band limited function b whose Fourier transform is $\hat{b}(\omega) = 1$ if $|\omega| \le 1$ and $\hat{b}(\omega) = 0$ otherwise. A direct calculation gives

$$s(\sigma) = \frac{\pi}{\sigma}\left(1 - e^{-\sigma^2}\right)$$

which is neither convex nor concave. In the other hand, for a function g following a $\alpha-$power law with $\alpha < 2$, $s(\sigma)$, this seems to be convex (for instance if $\alpha = 0$, $s(\sigma) = \pi/\sigma^2$, if $\alpha = 1$, $s(\sigma) = \pi^{3/2}/\sigma^2$).

If,again, the power spectrum of the image g is following a power law in spatial frequencies, while its regularized L^2- norm is finite, the residual norm is not as the initial datum is not square-integrable.

4 Total Variation Image Decomposition

Bounded Variation image modeling was introduced in the seminal work of Rudin *et al.* in [13], where the following variational image denoising problem is considered. Given an image g and $\lambda > 0$, find the minimizer of the following energy

$$E(f; g, \lambda) = \int (g - f)^2 dxdy + \lambda \iint |\nabla f|\, dxdy \tag{12}$$

The regularized image f_λ can be interpreted as a denoised version of g, but also as the “geometric” content of g while the residual $\nu_\lambda = g - f_\lambda$ contains the “noise/fine texture” component. Several methods have been proposed to solve the above equation, by solving a regularized form of the Euler-Lagrange equation of the functional

$$f - g - \lambda\nabla \cdot \frac{\nabla g}{|\nabla g|} = 0$$

where $\nabla\cdot$ denote the divergence operator, but also for instance the non linear projection method of Chambolle ([14]), which we have used in this work. λ is a regularization parameter that determines the level of details that ends up in the (noise/texture) component ν_λ. As λ increases ν_λ will contain details of larger and larger scale, that will not appear in f_λ.

Again it is interesting so see how the image content changes as λ increases. The component v_λ is the residual of the regularization and contains the details that are suppressed in the cartoon component f_λ and we set

$$r(\lambda; g) = \|v_\lambda\|_2^2 = \|g - f_\lambda\|_2^2 \tag{13}$$

i.e. the squared L^2-norm of the residual image as a function of the regularization parameter λ. Related to the norm of the residual is the norm of the cartoon component as a function of λ

$$s(\lambda; u_0) = \|u_\lambda\|_2^2 \tag{14}$$

$s'(\lambda)$ encodes the rate in which details are suppressed in the cartoon component u_λ. Due to the high non linearity of the TV-regularization problem, there is no relatively simple expression for $s(\lambda)$, $r(\lambda)$ and their respective derivatives.

A norm study for the dual norm of the TV norm was done by Meyer in [15]. A more direct behavior for the 2-norm can be computed in a few cases. For instance Strong and Chan [16] showed that if g is the function $g(x) = 1$ if $x \in B(0,1)$ the unit disk, $g(x) = 0$ if $x \notin B(0,1)$, then its regularization has the form cg, where $c \in (0,1)$ is a constant, therefore attenuating the contrasts of the image.

In general situation, we cannot expect these type of simple results. We have instead decided to study the behavior of these functions experimentally on an image database.

5 Experiments

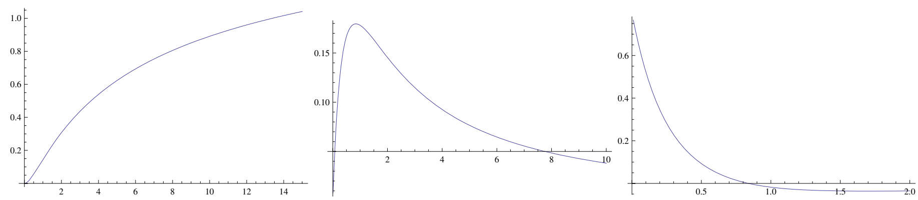
5.1 Sinc in Scale Space

Let $g(x) = \sin(x)/x$ be the Sinc function where $x \in [-\infty, \infty]$. The squared L_2 norm of the residual as a function of the regularization parameter is in the Tikhonov case

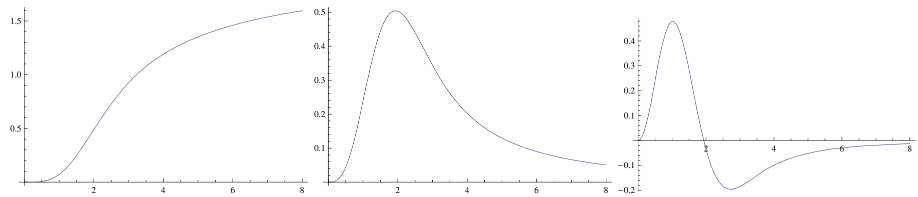
$$r(\lambda) = \int_{-1}^{1} (\frac{\lambda x^2}{1 + \lambda x^2})^2 dx \tag{15}$$

and in the scale space case

$$r(\sigma) = \int_{-1}^{1} (1 - e^{\frac{-\omega^2\sigma^2}{2}})^2 d\omega. \tag{16}$$



(a) Tikhonov regularization: Residual norm, first and second order derivative



(b) Scale Space: Residual norm, first and second order derivative

Fig. 1. The residual norm as a function of the regularization parameter for $g(x) = \frac{sin(x)}{x}$. The plots clearly indicate that residual norm function are, in both case, increasing functions, but not concave.

The result is presented in figure 1. The plots clearly indicate that the residual norm - in both cases -is not concave.

5.2 Black Squares with Added Gaussian Noise

The first experiment is done on an artificially generated 100×100 image containing four 3×3 black squares, one 20×20 black square and added Gaussian white noise with $\sigma^2 = 12$. The white background has intensity 125 and the black square 10, after the noise has been added the image is zero mean normalize.

In figure 2 the regularized and residual image are shown for increasing regularization using first order Tikhonov Regularization. As the small scale noise is suppressed, the large scale geometric structures are also smoothed out. The norm of the residual is an increasing function of the scale and it seems to be concave, and in fact it can be concave for the shown λ. However λ may be small at the inflection point.

In figure 3 the regularized and residual images are shown for increasing regularization using linear gaussian scale space. The results for the linear Gaussian scale-space is similar to the result using first order Tikhonov regularization.

In figure 4 the regularized and residual images are shown for increasing regularization using Total Variation image decomposition. The different structures are suppressed at using different λ while the large scale structures are well preserved. At $\lambda = 12$ the gaussian white noise is suppressed, and at $\lambda = 210$ is the small boxes remove and finally the large box is suppressed at $\lambda = 550$. The residual norm as a function of the regularization parameter is not a concave function of λ.

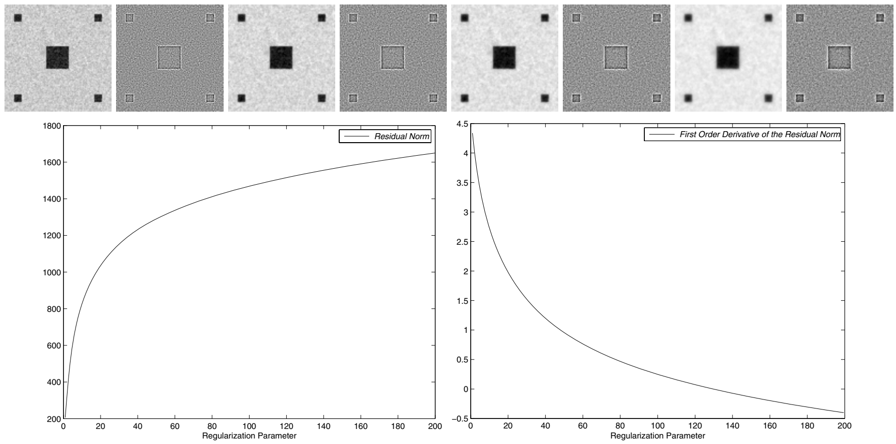


Fig. 2. Result for the squares and noise image using first order Tikhonov regularization. On the first row the regularized and the residual images for $\lambda = 3, 10, 20$ and 50 are shown. The plots contain the L^2-norm of the residual as a function the scale λ, followed by the first order derivative in log-scale.

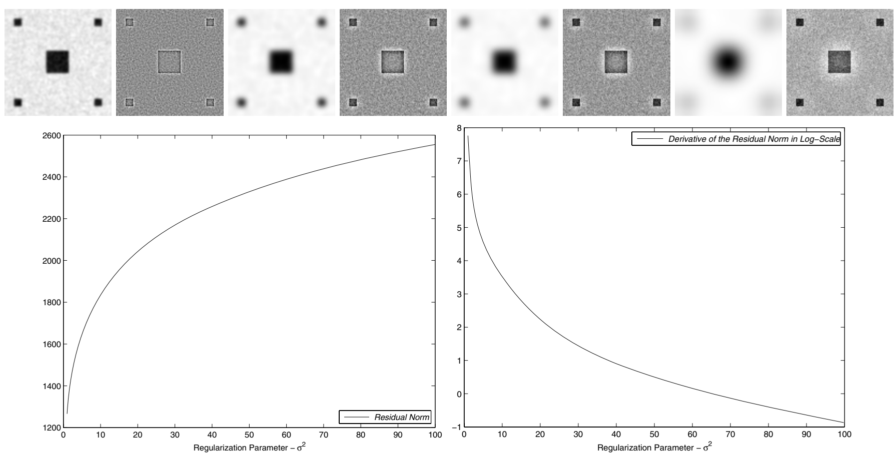


Fig. 3. Result for the squares and noise image using linear scale space. On the first row the regularized and the residual images for $\sigma^2 = 1, 7, 13$ and 64 are shown. The plots contain the L^2-norm of the residual as a function the scale σ, followed by the first order derivative in log-scale.

5.3 DIKU Multi Scale Image Sequence Database I

The newly collected DIKU Multi-Scale image sequence database [17], contains sequences of the same scene captured using varying focal length. The sequences contain both man-made structures and nature, the distance to the main objects in the scenes also show a large variation (from a few meters to a few kilometers).

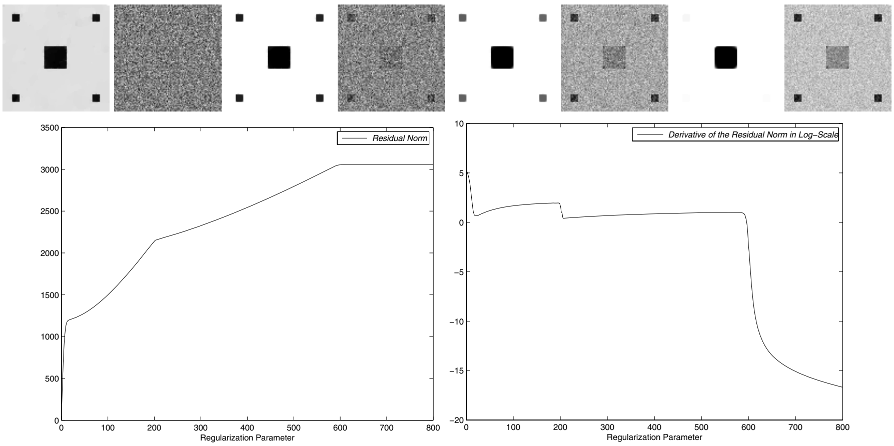


Fig. 4. Result for the squares and noise image using TV-decomposition. On the row regularized and the residual images for $\lambda = 12, 38, 100$ and 200 are shown. The plots contain the L^2-norm of the residual as a function the scale λ, followed by the first order derivative in log-scale. The residual norm seems to be a monotonically increasing non-concave function. The residual norm has three points of 'high' curvature: one at $\lambda = 12$ - the noise is suppressed - and $\lambda = 210$ - the small squares are suppressed, and $\lambda = 580$ - the large square is suppressed.

Each image has first been normalized by an affine intensity range change so that that the intensity range becomes $[0, 1]$, followed by subtracting the mean value (i.e. the mean intensity is 0 in each image).

The mean residual norm was computed on the normalized images in the database, using fixed scales $\sigma = 2^i$ where $i = 0, \cdots, 12$, using linear gaussian scale space. The result is a feature vector $\langle \bar{r}(0), \cdots, \bar{r}(12) \rangle$ containing

$$\bar{r}(i) = \frac{1}{N} \sum_{I \in F} r(i; I) \tag{17}$$

where F is the set of all N normalized images in the database.

The (signed) distance function $d(I_0)$ of a normalized image $I_0 \in F$ to the mean is defined as

$$d(I_0) = \sum_{i=0}^{12} r(i; I_0) - \bar{r}(i) \tag{18}$$

The (signed) distance to the mean has been computed for all images in the DIKU database. Images with large positive values have a larger than average residual and images with large negative values have a smaller than average residual.

The first row in figure 5 contains the 4 images with the largest positive distance to the mean, on the second row the 4 images with the largest negative distance to the mean. The image contents difference is striking and clearly indicate that the residual norm contains important contents information. The same experiment was performed using first order Tikhonov regularization with similar, but not identical, result.

Fig. 5. The top row show images where $f(\sigma)$ is much larger than the average and bottom row show images where $f(\sigma)$ is much smaller than the average. The contents difference is striking! The images in the first row contain small scale details (texture), while the images in the bottom row contain large scale geometric structures.

6 Conclusions

For square-integrable images, the squared L^2-norms of the regularized images in first order Tikhonov regularization and linear Gaussian Scale Space are, in general decreasing convex functions of the regularizing parameter. This may fail for Linear Scale space when Gaussian standard deviation is used as a parameter. Their squared residual norm are however not concave functions. For the the Total Variation regularization too, it is shown empirically that the squared norm of the residual is not concave.

This confirms that the squared norm of the residual may be an indicator of image structure, both for 1st order Tikhonov regularization, Gaussian Scale Space as well as Total variation regularization. The behavior of the latter will be studied further in future research.

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