Mixed Finite-/Infinite-Capacity Priority Queue with General Class-1 Service Times

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Abstract. This paper studies a single-server queue with two traffic classes in order to model Expedited Forwarding Per-Hop Behaviour in the Differentiated Services architecture. Generally, queueing models assume infinite queue capacity but in a DiffServ router the capacity for high priority traffic is often small to prevent this traffic from monopolizing the output link and hence causing starvation of other traffic. The presented model takes the exact (finite) high-priority queue capacity into account. Analytical formulas for system contents and packet delay of each traffic class are determined. This requires extensive use of the spectral decomposition theorem as the service time of a high-priority packet takes a general distribution, which complicates the analysis. Numerical examples indicate the considerable impact of the finite capacity on the system performance.

Keywords: Queueing Systems and Networks, Performance Modelling.

1 Introduction

In the nodes (routers, etc.) of computer networks, packets typically have to wait before being transmitted to the next node and queues are present in order to preserve waiting packets. Roughly two types of packets can be distinguished. Real-time traffic, such as Voice-over-IP, requires low delays but can endure a small amount of packet loss. Data traffic, such as file transfer, benefits from low packet loss but has less stringent delay characteristics.

Evidently, configuring the queue in order to allow both classes to meet their Quality of Service (QoS) requirements is of paramount importance. This is enabled by implementing Differentiated Services (DiffServ), a computer networking architecture in Internet Protocol (IP) networks that classifies packets [1]. It provides QoS differentiation between traffic classes by basing the order in which packets are transmitted on class-dependent priority rules. DiffServ defines the packet forwarding properties associated with a class of traffic by using Per-Hop Behaviors (PHBs). Obviously, implementation of DiffServ is particularly interesting in wireless networks and access networks, as these typically struggle to provide acceptable QoS because bandwidth is limited and/or variable.

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This paper considers a two-class priority queueing system representing a Diff-Serv implementation where real-time traffic (Expedited Forwarding PHB) has strict priority scheduling over data traffic (Default PHB). This is the most drastic scheduling algorithm, as data packets are only served if there are no real-time packets in the system. It thus minimizes the delay of the real-time packets. However, caution is required as these packets could occupy the server (almost) permanently, causing starvation of data traffic. This should be alleviated by controlling the amount of real-time traffic allowed into the system. Moreover, queueing a very large amount of real-time packets is useless anyway as they require small delays. These two observations emphasize the importance of limiting the capacity for real-time packets, without neglecting the packet loss constraints for these packets. On the other hand, the loss-sensitivity of data packets yields a capacity as large as practically feasible for these packets. Therefore, we can assume that the capacity for data packets is sufficiently large to be approximated by infinity but that the capacity for real-time packets should be modelled exactly. In the literature, priority queues have been discussed with various arrival and service processes. Analytic studies of queueing systems often assume infinite queue capacity facilitating mathematical analysis of the system.

From the former paragraph it indeed follows that we can assume that the capacity for data packets is sufficiently large to be approximated by infinity but that the capacity for real-time packets should be modelled as a finite number. The presented model is related to [2] where both queues are presumed to have infinite capacity and it is an extension of [3], where service of a packet was deterministically equal to a single slot for both classes. The current contribution introduces differentiation amongst packet sizes of both classes as the service time of a real-time packet takes a general distribution. This nontrivial extension leads to extensive use of the spectral decomposition theorem [4] in order to study the performance of our system. Finite queue capacity is considered in [5] as well, albeit by a different methodology, but only packet loss is investigated profoundly and delay is not analyzed at all. Assessing the impact of the finite real-time queue capacity is the main purpose of this contribution, as well as studying the effect of the general service times for real-time packets.

The remainder of this paper is organized as follows: first the model under consideration will be thoroughly described. In section 3, several performance measures for our system are determined analytically. Afterwards, the results are investigated in some (numerical) examples. The paper is concluded in section 5.

2 Model

This paper studies a discrete-time single-server two-class priority queueing system where class-1 (real-time) packets receive strict priority over class-2 (data) packets. Packets are handled in a First-In-First-Out (FIFO) manner within a class. We limit the capacity of the class-1 queue to N packets such that real-time packets that arrive at a full queue are dropped by the system. The system can hence contain up to $N+1$ class-1 packets simultaneously, N in the queue and 1 in the server. In contrast, the class-2 queue has infinite capacity. Time is divided into fixed-length slots and a packet can only enter the server at slot boundaries, even if arriving in an empty system.

Let s_i denote a generic random service time of a class-1 packet. Service of a class-2 packet takes a single slot (for convenience purposes), whereas service of a class-1 packet follows a general distribution with pgf $S(z)$ and mean value μ. When observing the system at the beginning of a slot this is after possible departures in the previous slot and before arrivals in the current slot.

We assume that for both classes the number of arrivals in consecutive slots form a sequence of independent and identically distributed (i.i.d.) random variables. We define $a_{i,k}$ as the number of class-i ($i = 1, 2$) packet arrivals during slot k. The arrivals of both classes are characterized by the joint probability mass function (pmf) $a(m, n) = \Pr[a_{1,k} = m, a_{2,k} = n]$ which allows us to take into account dependence between both classes. The partial probability generating function (pgf) of the number of class-2 arrivals in a slot with i ($0 \leq i \leq N$) and i or more class-1 arrivals are respectively denoted by $A_i(z)$ and $A_i^*(z)$. We establish

$$A_i(z) = \mathrm{E}[z^{a_{2,k}}\ 1\{a_{1,k} = i\}] = \sum_{j=0}^{\infty} a(i,j) z^j\ ,\ A_i^*(z) = \sum_{j=i}^{\infty} A_j(z)\ . \tag{1}$$

The indicator function $1\{.\}$ evaluates to 1 if its argument is true and to 0 if it is false. The mean number of class-1 and class-2 arrivals per slot are respectively expressed as

$$\bar{a}_1 = \sum_{i=1}^{\infty} i A_i(1)\ ,\ \bar{a}_2 = \frac{d}{dz} A_0^*(z)\bigg|_{z=1} = A_0^{*\prime}(1)\ . \tag{2}$$

The mean number of total arrivals is represented by $\bar{a}_T = \bar{a}_1 + \bar{a}_2$. Therefore, the arrival load is described as $\rho_T = \bar{a}_1 \mu + \bar{a}_2$.

3 Analysis

First, we review the spectral decomposition theorem for non-diagonalisable matrices as it will be used frequently in the remainder of this paper. The next subsection addresses the characterization of arrivals during a class-1 service. The system contents are obtained at so-called start-slots and non start-slots consecutively enabling identification of the system contents at the beginning of random slots. Finally, the packet delay is obtained for both classes.

3.1 Spectral Decomposition of Non-diagonalisable Matrices

Consider a square $m \times m$ matrix $\mathbf{A}$ and a scalar function f. The spectral decomposition theorem allows us to express the image of $\mathbf{A}$ under f by evaluating f (and its derivatives) in the eigenvalues of $\mathbf{A}$, see e.g. [4].

In this paper, the function f is typically a power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and the matrix $\mathbf{A}$ is non-diagonalisable. Such a matrix $\mathbf{A}$ cannot be reduced to a completely diagonal form by a similarity transform. However, any square matrix can be reduced to a form that is almost diagonal, called the Jordan normal form $\mathbf{J}$. Based on this reduction, it is possible to prove that the matrix $f(\mathbf{A})$ can be uniquely defined as

$$f(\mathbf{A}) = \sum_{j=1}^{s} \sum_{i=0}^{k_j-1} \frac{1}{i!} f^{(i)}(\lambda_j)\,(\mathbf{A} - \lambda_j \mathbf{I})^i\, \mathbf{G}_j\,. \tag{3}$$

In this expression, $\{\lambda_1, \ldots, \lambda_s\}$ $(s \leqslant m)$ are the eigenvalues of $\mathbf{A}$, k_j denotes the index of eigenvalue λ_j and $f^{(i)}$ is the ith derivative of f. Obviously, it is required that the function f and its derivatives exist in the eigenvalues, i.e.

$$\lambda_j \in \text{dom } f^{(i)}\,, \qquad j = 1, \ldots, s\,, i = 0, \ldots, k_j - 1\,. \tag{4}$$

The matrices $\mathbf{G}_j$ are called the constituents or spectral projectors of $\mathbf{A}$ belonging to the eigenvalue λ_j and have the following properties:

- $\mathbf{G}_j$ is idempotent, i.e. $\mathbf{G}_j^2 = \mathbf{G}_j$.
- $\mathbf{G}_1 + \mathbf{G}_2 + \ldots + \mathbf{G}_s = \mathbf{I}$, with $\mathbf{I}$ the $m \times m$ identity matrix.
- $\mathbf{G}_j \mathbf{G}_{j'} = \mathbf{0}$ whenever $j \neq j'$ $(1 \leqslant j, j' \leqslant s)$.

In general, the matrices $\mathbf{G}_j$ need to be calculated from the transformation matrix $\mathbf{P}$, for which $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. Specifically, if $\mathbf{P}$ is partitioned conformably as

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1} = \begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_s\end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_s \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \vdots \\ \mathbf{Q}_s \end{bmatrix}, \tag{5}$$

with $\mathbf{J}_j$ the Jordan segment corresponding with eigenvalue λ_j, then the projectors $\mathbf{G}_j$ are

$$\mathbf{G}_j = \mathbf{P}_j \mathbf{Q}_j \qquad (j = 1, \ldots, s)\,. \tag{6}$$

We also note that the columns of $\mathbf{P}_j$ span the space of the right eigenvectors of $\mathbf{A}$ corresponding to λ_j while the rows of $\mathbf{Q}_j$ span the space of its left eigenvectors.

This spectral decomposition theorem provides us with a very powerful tool from the computational point of view. Instead of having to evaluate the matrix power series $\sum_{n=0}^{\infty} f_n \mathbf{A}^n$ we now only need to evaluate the function f and its derivatives for scalar arguments and compute a finite number of matrix multiplications. The downside is that the eigenvalues of $\mathbf{A}$ have to be calculated, as well as the matrices $\mathbf{G}_j$. But once this is done, $f(\mathbf{A})$ can easily be calculated for any function f satisfying (4). In subsection 3.2, it will become clear that in our case these downsides are virtually non-existent as the eigenvalues and spectral projectors are surprisingly easy to obtain.

3.2 Arrivals During a Class-1 Service

Let $e_{i,k}$ represent the number of class-i arrivals during a class-1 service that starts in slot k. We have

$$e_{i,k} = \sum_{m=0}^{s_1-1} a_{i,k+m} \ . \tag{7}$$

Notice that the $e_{i,k}$ are i.i.d. as the $a_{i,k}$ are i.i.d. and independent of s_1. The partial pgfs of the number of class-2 arrivals during a class-1 service, during which i $(0 \leq i \leq N)$ and i or more class-1 packets arrive are respectively denoted by $E_i(z)$ and $E_i^*(z)$. We have

$$E_i(z) = \mathrm{E}[z^{e_{2,k}} \ 1\{e_{1,k} = i\}] \ , \ E_i^*(z) = \sum_{m=i}^{\infty} E_m(z) \ . \tag{8}$$

Obtaining these partial pgfs can be a tedious task. During each slot of a class-1 service, packets are added to the queue according to the $(N+1)\times(N+1)$ matrix

$$\mathbf{Y}(z) = \begin{bmatrix} A_0(z) & A_1(z) & \cdots & A_{N-1}(z) & A_N^*(z) \\ 0 & A_0(z) & \cdots & A_{N-2}(z) & A_{N-1}^*(z) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & A_0(z) & \vdots \\ 0 & \cdots & \cdots & 0 & A_0^*(z) \end{bmatrix} . \tag{9}$$

More precisely, given that the class-1 queue content (excluding the server) is $i-1$ during the previous slot, $\mathbf{Y}(1)_{ij}$ is the probability that it is $j-1$ in the current slot (this is the probability that $j-i$ class-1 packets are effectively allowed into the system), while $\mathbf{Y}(z)_{ij}$ is the partial pgf of the packets added to the class-2 queue.

The partial pgfs $E_i(z)$ and $E_i^*(z)$ are found as elements of the matrix $S\big(\mathbf{Y}(z)\big)$, which plays a crucial role. Using spectral decomposition, the latter is easily evaluated because of the special eigenstructure of $\mathbf{Y}(z)$. As this matrix has a triangular form, the eigenvalues simply are its diagonal elements. There are two distinct eigenvalues: $\lambda_1 = A_0^*(z)$, with index 1, and $\lambda_2 = A_0(z)$, with index N. The corresponding spectral projectors are shown to be independent of z and given by

$$\mathbf{G}_1 = \begin{bmatrix}\mathbf{0} \cdots \mathbf{0} \ \mathbf{e}\end{bmatrix} \ , \ \mathbf{G}_2 = \begin{bmatrix} \mathbf{I} & -\mathbf{e} \\ \mathbf{0}^T & 0 \end{bmatrix} \ . \tag{10}$$

Here $\mathbf{I}$ denotes the identity matrix of appropriate size, $\mathbf{x}^T$ is the transpose of vector $\mathbf{x}$ and $\mathbf{e}$ and $\mathbf{0}$ indicate the column vector of appropriate size with all elements equal to 1 and 0 respectively.

Spectral decomposition (3) yields

$$S\big(\mathbf{Y}(z)\big) = S\big(A_0^*(z)\big)\mathbf{G}_1 + \sum_{j=0}^{N-1} \frac{S^{(j)}\big(A_0(z)\big)}{j!}\big(\mathbf{Y}(z) - A_0(z)\mathbf{I}\big)^j \mathbf{G}_2 \ . \tag{11}$$

3.3 System Contents at the Beginning of Start-Slots

A start-slot is a slot where service of a packet can start. Note that a slot where the system is empty at the beginning of the slot is a start-slot as well. The class-i system contents at the beginning of start-slot l are denoted by $n_{i,l}$. The partial pgf of the class-2 system contents at the beginning of start-slot l that has class-1 system contents equal to i is denoted as

$$N_{i,l}(z) = \mathrm{E}[z^{n_{2,l}} \; 1\{n_{1,l} = i\}] \; . \tag{12}$$

The set $\{(n_{1,l}, n_{2,l}), l \geq 1\}$ forms a Markov chain. Assume that start-slot l corresponds with slot k. Relating start-slots l and $l+1$ establishes the following set of system equations:

$$\begin{aligned} n_{1,l+1} &= \begin{cases} \min(N, a_{1,k}) & \text{if } n_{1,l} = 0 \\ \min(N, n_{1,l} - 1 + e_{1,k}) & \text{if } n_{1,l} > 0 \end{cases} , \\ n_{2,l+1} &= \begin{cases} (n_{2,l} - 1)^+ + a_{2,k} & \text{if } n_{1,l} = 0 \\ n_{2,l} + e_{2,k} & \text{if } n_{1,l} > 0 \end{cases} . \end{aligned} \tag{13}$$

Here $(.)^+$ is shorthand for $\max(0, .)$. The system equations can be explained as follows: if $n_{1,l} > 0$, a class-1 packet starts service at the beginning of start-slot l and it leaves the system immediately before start-slot $l+1$. For each class, admitted arrivals during this service contribute to the system contents at the beginning of start-slot $l+1$. On the other hand, if $n_{1,l} = 0$, a class-2 packet starts service at the beginning of start-slot l if there are class-2 packets present in the system. As this service only takes a single slot, start-slot $l+1$ is the next slot. If the system is empty, the server is idle and start-slot $l+1$ is the next slot. Note that the class-1 system contents at the beginning of start-slots cannot exceed N.

We now define the $(N+1) \times (N+1)$ matrix

$$\mathbf{X}(z) = \begin{bmatrix} A_0(z) & A_1(z) & \cdots & A_{N-1}(z) & A_N^*(z) \\ E_0(z) & E_1(z) & \cdots & E_{N-1}(z) & E_N^*(z) \\ 0 & E_0(z) & \cdots & E_{N-2}(z) & E_{N-1}^*(z) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & E_0(z) & E_1^*(z) \end{bmatrix} , \tag{14}$$

and the row vector of $N+1$ elements

$$\mathbf{n}_l(z) = \left[N_{0,l}(z) \; N_{1,l}(z) \cdots \; N_{N,l}(z) \right] , \tag{15}$$

which corresponds with the system contents at the lth start-slot and we will use this phrase to determine vectors like (15) throughout this paper. Using standard z-transform techniques, a relation between $\mathbf{n}_l(z)$ and $\mathbf{n}_{l+1}(z)$ is derived from the system equations (13). We have

$$\mathbf{n}_{l+1}(z) = \mathbf{n}_l(z) \begin{bmatrix} \frac{1}{z} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \mathbf{X}(z) + \mathbf{n}_l(0) \begin{bmatrix} \frac{z-1}{z} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \mathbf{X}(z) \ . \tag{16}$$

Assume that the system has reached steady-state and define following steady-state values

$$\mathbf{n}(z) = \lim_{l\to\infty} \mathbf{n}_l(z) = \lim_{l\to\infty} \mathbf{n}_{l+1}(z) = \begin{bmatrix} N_0(z) \ N_1(z) \cdots \ N_N(z) \end{bmatrix} \ . \tag{17}$$

Taking the limit of (16) for $l \to \infty$ induces

$$\mathbf{n}(z)\left(z\mathbf{I} - \begin{bmatrix} 1 & & & \\ & z & & \\ & & \ddots & \\ & & & z \end{bmatrix} \mathbf{X}(z)\right) = \Big((z-1)N_0(0) \begin{bmatrix} 1 \ 0 \cdots \ 0 \end{bmatrix} \mathbf{X}(z)\Big) \ . \tag{18}$$

The constant $N_0(0)$ is still unknown. Note that $\mathbf{X}(1)$ is a right-stochastic matrix by construction. Therefore, observe that

$$\big(\mathbf{I} - \mathbf{X}(1)\big)\mathbf{e} = \mathbf{0} \ , \begin{bmatrix} 1 \ 0 \cdots \ 0 \end{bmatrix} \mathbf{X}(1)\mathbf{e} = 1 \ . \tag{19}$$

Keeping these identities in mind, derivation of (18) with respect to z, evaluation in $z = 1$ and multiplication of both sides of the resulting equation by $\mathbf{e}$ yields

$$N_0(0) = \mathbf{n}(1)\left(\mathbf{I} - \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \mathbf{X}(1) - \mathbf{X}'(1)\right)\mathbf{e} \ . \tag{20}$$

The vector $\mathbf{n}(1)$ is yet to be obtained. Evaluating (18) in $z = 1$ produces

$$\mathbf{n}(1)\Big(\mathbf{I} - \mathbf{X}(1)\Big) = \begin{bmatrix} 0 \cdots \ 0 \end{bmatrix} \ . \tag{21}$$

As $\mathbf{X}(1)$ is right-stochastic, each row of matrix $[\mathbf{I} - \mathbf{X}(1)]$ sums to 0 and it hence has rank N and is not invertible. We thus require an additional relation in order to obtain the vector $\mathbf{n}(1)$. Observe that $N_i(1)$ represents the probability that the class-1 system contents at the beginning of a start-slot in steady state equal i and thus

$$N_i(1) = \lim_{l\to\infty} \Pr[n_{1,l} = i] \ . \tag{22}$$

The normalization condition provides $\mathbf{n}(1)\mathbf{e} = 1$. Combining this observation with (21) yields

$$\mathbf{n}(1) = \begin{bmatrix} 0 \cdots \ 0 \ 1 \end{bmatrix} \Big(\big[\mathbf{I} - \mathbf{X}(1) \big\| \mathbf{e}\big]\Big)^{-1} \ . \tag{23}$$

By $[\mathbf{A}\|\mathbf{b}]$ we denote the matrix $\mathbf{A}$ with the last column replaced by the column vector $\mathbf{b}$.

The probability mass function (pmf) of the class-1 system contents at the beginning of a start-slot in steady state has been obtained in (23). Substituting it in (20) produces $N_0(0)$, the only unknown in (18). The latter yields the pgf of the class-2 system contents at the beginning of a start-slot in steady state as

$$\lim_{l\to\infty} \mathrm{E}[z^{n_{2,l}}] = \mathbf{n}(z)\mathbf{e} \ . \tag{24}$$

3.4 Queue Contents at the Beginning of Non Start-Slot Slots

If a random slot k is not a start-slot, a class-1 packet started service in the start-slot preceding the random slot (start-slot l). We know that no packets leave the server between these two slots. Hence, we study the queue contents, instead of the system contents, at the beginning of slots that are not start-slots. The system certainly contains class-1 packets at the beginning of start-slot l, one of which enters the server (leaves the queue) at the beginning of start-slot l. Therefore, the steady-state queue contents of both classes, at the beginning of a start-slot in steady state where a class-1 packet starts service, are characterized by the vector of $N+1$ elements

$$\mathbf{m}(z) = \frac{1}{1-N_0(1)}\left[N_1(z)\cdots\ N_N(z)\ 0\right] \ . \tag{25}$$

Slot k lies in the time epoch between start-slots l and $l+1$. No packets leave the system (and hence the queue) during this epoch. In start-slot l and in the slots up to start-slot $l+1$, packets (of both classes) arrive at the queue according to the matrix $\mathbf{Y}(z)$ given in (9). Slot k is one of the s_1-1 slots between start-slot l and $l+1$ with s_1 the service time of the class-1 packet in service. Standard renewal theory [6] yields that $\mathbf{q}(z)$, the vector of $N+1$ elements representing the queue contents of both classes at the beginning of a non start-slot in steady state is given by

$$\mathbf{q}(z) = \mathbf{m}(z)\frac{E\left[\sum_{i=1}^{s_1-1}\mathbf{Y}(z)^i\right]}{\mu-1} \ . \tag{26}$$

Define the function $S^n(x)$ as

$$S^n(x) = E\left[\sum_{i=1}^{s_1-1} x^i\right] = \frac{S(x)-x}{x-1}. \tag{27}$$

By combining (26) and (27) and keeping in mind that the spectral decomposition theorem (3) enables evaluation of $S^n\big(\mathbf{Y}(z)\big)$, we have

$$\mathbf{q}(z) = \mathbf{m}(z)\frac{S^n\big(\mathbf{Y}(z)\big)}{\mu-1} \ . \tag{28}$$

3.5 System Contents at the Beginning of a Random Slot

On average, a start-slot corresponds with μ slots if a class-1 packet starts service and with one slot if this is not the case (the system is void of class-1 packets). Therefore, γ, the (long-run) probability that a random slot is a start-slot, is defined as

$$\gamma = \lim_{k\to\infty} \Pr[\text{slot k is a start-slot}] = \frac{1}{N_0(1) + \big(1 - N_0(1)\big)\mu} \ . \tag{29}$$

The class-i system contents at the beginning of a random slot are denoted by $u_{i,k}$. Note that $0 \leq u_{1,k} \leq N+1$. The system contents (of both classes) at the beginning of a random slot in steady state are determined by $\mathbf{u}(z)$, a vector of $N+2$ elements. The class-1 system contents at the beginning of a start-slot never exceed N and the server contains a class-1 packet during non start-slots, yielding

$$\mathbf{u}(z) = \big[U_0(z) \cdots U_{N+1}(z)\big] = \gamma\left[\mathbf{n}(z)\ 0\right] + (1-\gamma)\left[0\ \mathbf{q}(z)\right] \ . \tag{30}$$

The pmf of the class-1 and the pgf of the class-2 system contents at the beginning of a slot are respectively determined by $\mathbf{u}(1)$ and $\mathbf{u}(z)\mathbf{e}$.

The number of class-1 packets effectively entering the system and leaving the system in steady-state must be equal. This allows us to determine the effective class-1 load ρ_1^e, the mean number of effective class-1 arrivals $\bar{a}_1^e$ and the class-1 packet loss ratio plr_1, the fraction of class-1 packets rejected by the system. We have

$$\rho_1^e = 1 - U_0(1) \ , \ \bar{a}_1^e = \frac{\rho_1^e}{\mu} \ , \ plr_1 = \frac{\bar{a}_1 - \bar{a}_1^e}{\bar{a}_1} \ . \tag{31}$$

3.6 Class-1 Delay

Tag an arbitrary class-1 packet that effectively arrives at the system in a slot in steady-state. The arrival slot of the packet is assumed to be slot k. Let the delay of the packet be denoted by d_1. Recall that class-1 packets are not affected by class-2 packets. We obtain the amount of class-1 packets in the system at the moment the tagged packet arrives. As the service times are i.i.d., each of these packets (except the class-1 packet in service during slot k) will contribute a random number of s_1 slots to the delay, as will the tagged packet itself. Therefore, once a class-1 packet arrives at the system, its delay is known.

Let $f_{1,k}$ denote the amount of class-1 packets arriving in slot k but before the tagged packet. Renewal theory states that a random packet is more likely to arrive in a slot with a lot of arrivals. This yields, considering that the tagged packet has to be an effective arrival,

$$\begin{aligned} &\Pr[f_{1,k} = m \mid (u_{1,k}-1)^+ = i] = \frac{A^*_{m+1}(1)}{\bar{a}_1^e} \ , \quad m = 0 \ldots N-i-1 \ , \\ &\Pr[f_{1,k} = m \mid (u_{1,k}-1)^+ = i] = 0 \ , \quad m > N-i-1 \ . \end{aligned} \tag{32}$$

Define the matrix $\mathbf{F}_1^e$ such that the element on row i, column j ($1 \le i \le N+1,\ 1 \le j \le N$) corresponds with $\Pr[f_{1,k} = j - i \mid (u_{1,k}-1)^+ = i-1]$. We have

$$\mathbf{F}_1^e = \frac{1}{\bar{a}_1^e} \begin{bmatrix} A_1^*(1) & A_2^*(1) & \cdots & A_N^*(1) \\ 0 & A_1^*(1) & \cdots & A_{N-1}^*(1) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & A_1^*(1) \\ 0 & \cdots & \cdots & 0 \end{bmatrix} . \tag{33}$$

Note that the queue cannot be entirely full upon arrival of the tagged packet as the latter must be able to enter the system.

If the system does not contain class-1 packets at the beginning of slot k the delay is rather straightforward as only the tagged packet and the packets arriving before it in slot k contribute to the delay. On the other hand, if $u_{1,k} > 0$ a class-1 packet is in service and additional random variables are involved. Let s_1^- denote the elapsed service time and let s_1^+ denote the remaining service time (slot k excluded). The packet in service only contributes s_1^+ slots to the tagged packet's delay. The class-1 packets in the queue at the moment the tagged packet arrives each contribute s_1 slots to the delay. They are constituted by $\mathbf{m}(1)$, the queue content at the start-slot preceding slot k, obtained in (25), the number of arriving class-1 packets during s_1^- and $f_{1,k}$, the number of class-1 packets arriving before the tagged packet in slot k.

Define the function

$$S^b(x,y,z) \triangleq \mathrm{E}[x^{s^-}\, y\, z^{s^+}] , \tag{34}$$

where the arguments can be matrices and the order in which the arguments appear is hence important as matrix multiplication does not commute. From the discussion above follows that we need to calculate $S^b\big(\mathbf{Y}(1), \mathbf{F}_1^e, z\big)$ in order to obtain the class-1 delay. Considering that $\mathbf{F}_1^e$ does not contain stochastic variables and that scalar multiplication of a matrix commutes, we have $S^b\big(\mathbf{Y}(1), \mathbf{F}_1^e, z\big) = S^b\big(\mathbf{Y}(1), 1, z\big)\mathbf{F}_1^e$.

The random variables s_1^- and s_1^+ are generally dependent. Slot k may be any slot in s_1 with equal probability [6]. For scalar arguments x, y, z this yields

$$S^b(x,y,z) = \mathrm{E}[x^{s^-}\, y\, z^{s^+}] = \frac{S(x) - S(z)}{\mu(x-z)} y . \tag{35}$$

Using the spectral decomposition theorem (3), we can express the image of a matrix under the function S^b, as it can be seen as a scalar function in a single variable by considering the other two variables to be constant. This allows us to obtain $S^b\big(\mathbf{Y}(1), 1, z\big)$ from (35). Bringing everything together, the pgf $D_1(z)$ of the steady-state class-1 delay is given by

$$D_1(z) = \Big(\big[U_0(1)\ 0 \cdots\ 0\big] + \big(1 - U_0(1)\big)\mathbf{m}(1) S^b\big(\mathbf{Y}(1), 1, z\big) \Big) \mathbf{F}_1^e \begin{bmatrix} S(z) \\ S(z)^2 \\ \vdots \\ S(z)^N \end{bmatrix} . \tag{36}$$

3.7 Class-2 Delay

The delay of class-2 packets is more intricate as it is influenced by class-1 packets arriving at the system until the class-2 packet enters the server. In order to capture this influence we first study the (remaining) class-1 busy period.

The remaining class-1 busy period in start-slot l, denoted by r_l, is the number of slots until the system is void of class-1 packets (for the first time). Obviously, it depends on the number of class-1 packets in the system at start-slot l. The conditional pgf of the remaining class-1 busy period in start-slot l, if the class-1 system contents at the beginning of start-slot l equal j is denoted by

$$R_l(z|j) = \mathrm{E}[z^{r_l}|n_{1,l} = j],\ j = 0 \ldots N \ . \tag{37}$$

Define the vector $\mathbf{R}_l(z) = \big[R_l(z|0) \cdots R_l(z|N)\big]^T$. Relating start-slot l and $l+1$ yields

$$\mathbf{R}_l(z) = \big[1\ 0 \cdots\ 0\big]^T + \begin{bmatrix} \mathbf{0}^T & 0 \\ \mathbf{I} & \mathbf{0} \end{bmatrix} S\big(\mathbf{Y}(1)z\big)\mathbf{R}_{l+1}(z) \ . \tag{38}$$

The first term results from $R_l(z|0)$ marking the end of the (remaining) busy period as the system is empty. The second term expresses that for $R_l(z|j),\ j > 0$ the packet in service leaves the system by the next start-slot and that we keep track of the number of slots during the epoch s_1 between start-slots l and $l+1$ and the arrivals during this epoch. In each slot of this epoch class-1 packets arrive according to $\mathbf{Y}(1)$. Spectral decomposition (3) again yields evaluation of $S\big(\mathbf{Y}(1)z\big)$. In steady-state, taking the limit for l of (38) results in a simple expression for $\mathbf{R}(z) = \lim_{l\to\infty} \mathbf{R}_l(z) = \lim_{l\to\infty} \mathbf{R}_{l+1}(z)$.

A class-1 busy period b is the number of consecutive slots with class-1 system contents greater than zero. Notice that a class-1 busy period is simply the remaining class-1 busy period in a random start-slot preceded by a start-slot with empty class-1 system contents at the beginning of the slot and a number of class-1 arrivals larger than 0. Thus we obtain the pgf of the steady-state class-1 busy period as

$$B(z) = \frac{\sum_{m=1}^{N-1} R(z|m)A_m(1) + R(z|N)A_N^*(1)}{1 - A_0(1)} \ . \tag{39}$$

The extended service completion time of a class-2 packet, denoted by t_2, starts at the slot where the packet starts service and lasts until the next slot wherein a class-2 packet can be serviced [7]. If no class-1 packets arrive during the service-slot of the packet, the server can handle another class-2 packet in the next slot. If there are class-1 arrivals, we have to wait for a class-1 busy period after the service-slot until the service of another class-2 packet can start. We can thus express the pgf of the extended service completion time in steady state as $T_2(z) = A_0(1)z + \big(1 - A_0(1)\big)B(z)z$.

Now, we can finally tackle the class-2 delay. Tag an arbitrary class-2 packet arriving at the system in a slot in steady-state. The arrival slot of the packet is assumed to be slot k. Let the delay of the packet be denoted by d_2. It resembles the class-1 delay but here we need to keep track of packets of both classes. Consider the first start-slot succeeding slot k. The remainder of the delay of the tagged packet is simply the remaining class-1 busy period in this start-slot followed by an extended service completion time for each class-2 packet to be served before the tagged packet and a single slot to serve the tagged packet itself.

Let $f_{2,k}$ denote the amount of class-2 packets arriving in slot k but before the tagged packet. We determine the number of class-2 arrivals before the tagged packet. It is clear that $a_{1,k}$ and $f_{2,k}$ are correlated. The corresponding matrix can be found using renewal arguments [6]. We have

$$\hat{\mathbf{A}}(z) = \frac{\mathbf{Y}(z) - \mathbf{Y}(1)}{\bar{a}_2(z-1)} \ . \tag{40}$$

Given that the class-1 queue contents are $i-1$ at the beginning of the slot, $\hat{\mathbf{A}}(z)_{ij}$ is the partial pgf of the class-2 packets arriving before the tagged packet while $j-i$ class-1 packets are effectively allowed into the system in this slot.

We obtain the system state in the first start-slot succeeding slot k as follows. If the system does not contain class-1 packets at the beginning of slot k the next start-slot is simply the next slot and the class-2 system contents at the beginning of slot k (if any) contribute to the delay. On the other hand, if $u_{1,k} > 0$ a class-1 packet is in service and additional random variables are involved. Let s_1^- denote the elapsed service time and let s_1^+ denote the remaining service time (slot k excluded). The packets contributing to the delay are $\mathbf{m}(z)$, the queue contents at the start-slot preceding slot k, obtained in (25), the number of arriving packets of both classes during s_1^- and the number of arriving class-1 packets during s_1^+. Note that s_1^+ contributes to the delay as well.

This discussion leads to the following pgf $D_2(z)$ of the steady-state class-2 delay as

$$\begin{aligned} D_2(z) = \Bigg(& \left[\tfrac{U_0\left(T_2(z)\right) + \left(T_2(z)-1\right) U_0(0)}{T_2(z)} \ 0 \cdots 0 \right] \hat{\mathbf{A}}(T_2(z)) \\ & + \big(1 - U_0(1)\big) \mathbf{m}(T_2(z)) S^b\big(\mathbf{Y}(T_2(z)), \hat{\mathbf{A}}(T_2(z)), \mathbf{Y}(1)z\big) \Bigg) \mathbf{R}(z) z \ . \end{aligned} \tag{41}$$

Finally we calculate $S^b\big(\mathbf{Y}(T_2(z)), \hat{\mathbf{A}}(T_2(z)), \mathbf{Y}(1)z\big)$. As matrices generally do not commute, there is no multivariate version of the spectral decomposition theorem. However, if we specify the function S^b by its power series expansion we can apply the spectral decomposition theorem on the arguments separately. Power series expansion produces

$$\begin{aligned} S^b\big(\mathbf{Y}\big(T_2(z)\big), \hat{\mathbf{A}}(T_2(z)), \mathbf{Y}(1)z\big) &= E\left[\mathbf{Y}\big(T_2(z)\big)^{s_1^-} \hat{\mathbf{A}}(T_2(z)) \big(\mathbf{Y}(1)z\big)^{s_1^+} \right] \\ &= \frac{1}{\mu} \sum_{n=0}^{\infty} \mathrm{Prob}[s_1 = n+1] \sum_{i=0}^{n} \mathbf{Y}\big(T_2(z)\big)^i \hat{\mathbf{A}}(T_2(z)) \big(\mathbf{Y}(1)z\big)^{n-i} \end{aligned} \tag{42}$$

Spectral decomposition (3) enables evaluation of $\mathbf{Y}\big(T_2(z)\big)^i$ and $\big(\mathbf{Y}(1)z\big)^{n-i}$. Note that both decompositions share the same spectral projectors $\mathbf{G}_1$ and $\mathbf{G}_2$. The eigenvalues and their index are respectively denoted by

$$\begin{aligned}\lambda_1 &= A_0^*(T_2(z)) \text{ with } k_1 = 1 \ , \ \lambda_2 = A_0(T_2(z)) \text{ with } k_2 = N \ , \\ \lambda_1' &= A_0^*(1)z \text{ with } k_1' = 1 \ , \ \lambda_2' = A_0(1)z \text{ with } k_2' = N \ .\end{aligned} \tag{43}$$

After the spectral decomposition we can reconstruct the power series yielding

$$\begin{aligned}&S^b\big(\mathbf{Y}\big(T_2(z)\big), \hat{\mathbf{A}}(T_2(z)), \mathbf{Y}(1)z\big) \\ &\quad = \sum_{j=1}^{2}\sum_{i=0}^{k_j-1}\sum_{j'=1}^{2}\sum_{i'=0}^{k'_{j'}-1} Q_{ii'}(\lambda_j, \lambda'_{j'})(\mathbf{Y}\big(T_2(z)\big) - \lambda_j \mathbf{I})^i \mathbf{G}_j \\ &\qquad\qquad \times \hat{\mathbf{A}}(T_2(z))\,(\mathbf{Y}(1)z - \lambda'_{j'}\mathbf{I})^{i'}\,\mathbf{G}_{j'} \ .\end{aligned} \tag{44}$$

$$\text{with } Q_{ii'}(\lambda_j, \lambda'_{j'}) \triangleq \frac{1}{i!}\frac{1}{i'!}\frac{\partial^{i+i'}}{\partial x^i y^{i'}} S^b\big(x, 1, y\big)\bigg|_{\substack{x=\lambda_j \\ y=\lambda'_{j'}}} \ .$$

By taking proper derivatives of the pgfs obtained in this paper, all moments of the corresponding random variables can be calculated.

4 Numerical Examples

With the formulas at hand, we study an output-queueing switch with L inlets and L outlets and two types of traffic as in [2]. On each inlet a batch arrives according to a Bernoulli process with parameter ν_T. A batch contains b (fixed) packets of class 1 with probability ν_1/ν_T or b packets of class 2 with probability ν_2/ν_T (with $\nu_1 + \nu_2 = \nu_T$). Incoming packets are routed uniformly to the outlets where they arrive at a queueing system as described in this paper. Therefore, all outlets can be considered identical and analysis of one of them is sufficient. The arrival process at the queueing system can consequently be described by the pmf

$$a(bn, bm) = \frac{L!\big(\frac{\nu_1}{L}\big)^n \big(\frac{\nu_2}{L}\big)^m \big(1 - \frac{\nu_T}{L}\big)^{L-n-m}}{n!m!(L-n-m)!}, \quad n + m \leq L \ , \tag{45}$$

and by $a(p, q) = 0$, for all other values of p and q. Obviously the number of arrivals of class-1 and class-2 are negatively correlated as there can be no more than $Lb - i$ class-2 arrivals in a slot with i class-1 arrivals. For increasing values of L the correlation increases and for L going to infinity the numbers of arrivals of both types become uncorrelated. We now study a 4×4 output-queueing switch.

For Fig. 1, let $\nu_1 = \nu_2$. On average the system thus receives the same amount of packets of both classes. On the left, the batch size is $b = 10$, $\nu_1 = \nu_2 = 0.02$ and service of a class-1 packet takes the distribution $S(z) = 0.25z + 0.75z^4$ yielding a mean class-1 of service time $\mu = 3.25$ slots and hence $\rho_T = 0.85$. The mean and the standard deviation of the system contents at the beginning of random slots of both classes are plotted versus the class-1 queue capacity N. The values increase for increasing N, as the number of dropped class-1 packets decreases. For larger N the values clearly converge to the values corresponding with the

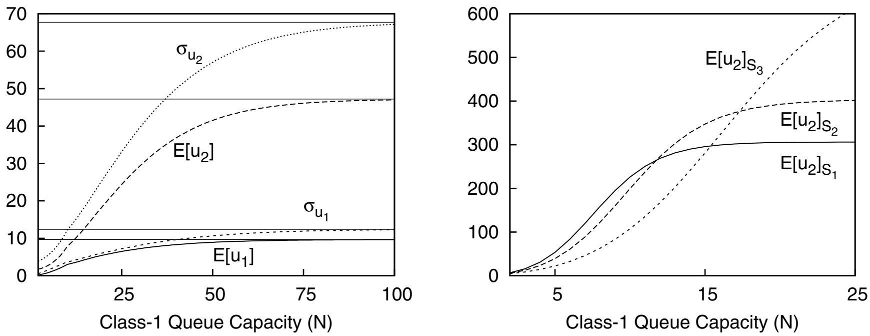

Fig. 1. System contents versus class-1 queue capacity

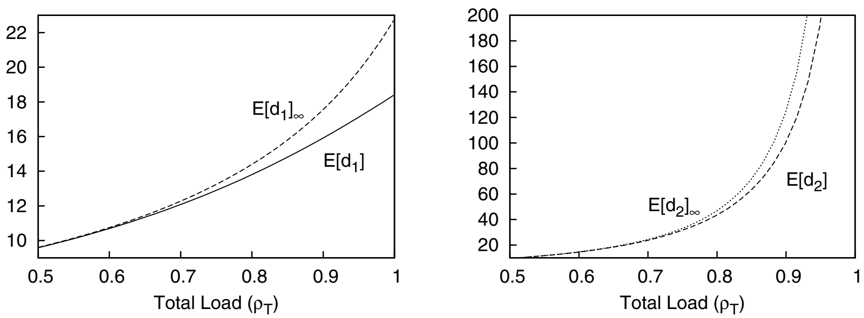

Fig. 2. Mean delays versus total load

infinite system [2], represented by the horizontal lines. However, the convergence is rather slow, especially for class-2. On the right, $b = 1$, $\nu_1 = \nu_2 = 0.199$ and we have plotted the mean class-2 system contents versus the class-1 queue capacity for three distributions with $\mu = 4$ slots yielding $\rho_T = 0.995$. These distributions have different variances. In order of increasing variance, we have

$$S_1(z) = z^4 \ , \ S_2(z) = 0.25(z + z^3 + z^5 + z^7) \ , \ S_3(z) = 0.7z + 0.3z^{11} \ . \qquad (46)$$

For large values of N the expected (from the infinite model) behaviour arises as increased variance normally yields increased system content. However, for small values of N the inverse effect occurs as the class-1 queue is likely to get full during a large service time causing arriving packets to be dropped. Therefore, the effective class-1 load will be lower when the variance of the class-1 service times is larger, increasing class-2 performance. As the queue capacity gets bigger less packets are lost and the normal behaviour is exemplified. Evidently, this effect cannot be predicted by infinite capacity queueing models.

For Fig. 2, we assume that $b = 3$, $\nu_1 = \nu_2$ and that the service of a class-1 packet takes the distribution $S(z) = 0.25z + 0.75z^4$. We keep $N = 15$ constant and vary ν_1 and ν_2 and hence the total load. The class-1 delay (on the left) and the class-2 delay (on the right) are plotted versus the total load and are compared to results for the infinite model. We clearly see the effect of the priority scheduling as the low mean for the class-1 delay delivers the performance required for real-time traffic at the cost of the class-2 delay. Note that the starvation effect is alleviated (compared to the infinite model) when the load gets high, as an increasing amount of class-1 packets are dropped, in turn improving the delay performance of packets (of both classes) allowed into the system.

5 Conclusions

A two-class priority queue with finite capacity for high-priority packets has been studied in order to model a DiffServ router with Expedited Forwarding Per-Hop Behaviour for high-priority traffic. The service times of class-1 packets are generally distributed, which considerably complicates the analysis. Analytical formulas for system content and packet delay of all traffic classes were determined making extensive use of the spectral decomposition theorem. In a DiffServ router, the capacity for high-priority packets is often small to prevent this traffic monopolizing the system. Opposed to existing models, the presented model takes the exact (finite) high-priority queue capacity into account. The resulting impact on system performance is clearly indicated by numerical examples.

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