# Equilibrium in Size-Based Scheduling Systems

Sebastien Soudan[1,⋆], Dinil Mon Divakaran[1], Eitan Altman[2], and Pascale Vicat-Blanc Primet[1,⋆⋆]

[1] INRIA / Université de Lyon / ENS Lyon
Tel.: +33-472728037; Fax: +33-472728080
{Sebastien.Soudan,Dinil.Mon.Divakaran,Pascale.Primet}@ens-lyon.fr
[2] INRIA
Tel.: +33-492387786; Fax: +33-492387858
Eitan.Altman@sophia.inria.fr

**Abstract.** Size-based scheduling is advocated to improve response times of small flows. While researchers continue to explore different ways of giving preferential treatment to small flows without causing starvation to other flows, little focus has been paid to the study of stability of systems that deploy size-based scheduling mechanisms. The question on stability arises from the fact that, users of such a system can exploit the scheduling mechanism to their advantage and split large flows into multiple small flows. Consequently, a large flow in the disguise of small flows, may get the advantage aimed for small flows. As the number of misbehaving users can grow to a large number, an operator would like to learn about the system stability before deploying size-based scheduling mechanism, to ensure that it won't lead to an unstable system. In this paper, we analyse the criteria for the existence of equilibria and reveal the constraints that must be satisfied for the stability of equilibrium points. Our study exposes that, in a two-player game, where the operator strives for a stable system, and users of large flows behave to improve delay, size-based scheduling doesn't achieve the goal of improving response time of small flows.

## 1 Introduction

Scheduling based on flow size (or flow age) has been gaining importance in the recent times. Researchers have proposed different ways of scheduling based on size, ranging from SRPT (Shortest Remaining Processing Time) to LAS (Least Attained Service) to MLPS (Multi-level Processor Sharing) scheduling mechanisms [1,2,3]. These scheduling strategies differ from the general model for flow scheduling in the Internet. The queues in the Internet nodes, though are served in an FCFS order at packet level, can be modeled using an M/G/1-PS (processor sharing) queue at flow level. The motivation to deviate from this norm,

⋆ Corresponding author.
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and schedule flows based on size, is to give better completion time to small flows. Strictly speaking, the aim has been to improve the conditional mean response time of small flows, at negligible cost to large flows. LAS, for example, always gives highest priority to the flow that has attained the least service. More details on size-based scheduling policies and the advantages they bring, can be found in [4] and [2]. Note that, researchers use *age-based scheduling* to refer to the scheduling schemes that are *blind*, in the sense that, they do not have information about the size of the flow when it arrives, and hence uses its age (the number of bytes/packets already scheduled) to make scheduling decision. Whereas, in this paper, we use the broader phrase *size-based scheduling* to include all the policies that use *age* or *size* to make scheduling decisions.

A user (an end-user or an application) sends a file as a single flow across the Internet. We take this as a normal behaviour. If size-based scheduling is deployed by an operator, there is a clear motivation for one or more users to deviate from the normal behaviour. Indeed, there is an incentive in splitting a flow (possibly large, but more precisely, one that is not small) into multiple small flows to exploit the advantage (say, priority in scheduling) given to small flows to improve the response time. If a considerable number of users deviate from the normal behaviour, then the operator's aim of giving shorter response time to small flows might well be deceived. More importantly, an operator would like to know if such user manipulations would lead to an unstable system behaviour. This poses an important problem in the context of size-based scheduling systems which, to the best of our knowledge, has not been addressed yet. This is the problem we address in this work. In the scenario where users do not misbehave, the stability issue (for network of queues) has been addressed in [5] recently.

The focus of this work is to study the equilibria in size-based scheduling system where users misbehave. We believe this would lead to better understanding of the implication of deploying a size-based scheduling mechanism. More description of the problem is given in Section 2. The model is elaborated in Section 3. The existence of equilibria are studied for two kinds of system behaviours: one in which the service rates are fixed, is studied in Section 4; and the other in which the service rates are varying, is studied in Sections 5 and 6. We summarize our analysis as a game between the operator and users, in Section 7.

## 2 Problem Statement and Assumptions

We study the problem that arises when an operator deploys a size-based scheduling mechanism. Though there are different ways of scheduling based on size, our focus is on size-based scheduling using two queues. Here, flows are classified based on their sizes. Small flows are sent to one queue, and large flows to another[1]. Each queue is assigned a specific service rate, such that the total service

[1] A flow is called small if its size is less than a threshold, $\theta$. In practice, $\theta$ bytes of every large flow also go to the small queue. But, we ignore this to keep the model simple. Besides, this affects neither the analysis nor the results given here.

rate equals the line capacity. The aim of operator in setting such a mechanism is to give to reduce the average response times of small flows.

To formulate the objective of the operator, we assume Poisson flow arrivals. Arrivals and service rates are in units of small flow. $\lambda_x$ and $\lambda_y$ are the arrival rates for small and large flows respectively. Each large is $F$ times a small flow. The service rates at small and large queues are $\phi_x$ and $\phi_y$ respectively, such that if $C$ denotes the line capacity, $\phi_x + \phi_y = C$. Each queue is served using the PS discipline; hence it is an $M/G/1 - PS$ queue.

We study the existence of equilibria under the scenario where users *cheat* by splitting a large flow into multiple small flows to improve their delay. This is explored in two cases: (i) where the service rates assigned are static, (ii) where the operators exhibits control by dynamically changing the service rates. In the latter case, we explore the existence of interesting equilibria, and state the conditions required for stability, under the assumption that the incentive for players to migrate is to minimize the delay the flow will incur. Note that, by 'players', we consider only the users who migrate.

## 3 Model Description

The fluid model used in this work is inspired by the one used in [6], where the authors analyse dynamic bandwidth resource allocation and migration between *guaranteed performance* and *best effort* traffic classes.

The two-queues model is depicted in Fig. 1. The queue for small flows is called *small queue* and is referred to as $Q_x$. The other queue is called the *large queue* which is denoted by $Q_y$. The number of flows at $Q_x$ is represented by $x$. At the large queue, this number (in number of small flows) is denoted by $y$. We assume infinite queues. The service rates, $\phi_x$ and $\phi_y$, are also in number of small flows. They are both assumed to take non-zero values.

The system parameters $\phi_x$ and $\phi_y$ are set by the operator. System state is modeled using averaged queue sizes: $x$ and $y$. Depending on the measured delay values, a user might decide to split a large flow into multiple small flows. Therefore, a fraction of the flows arriving at the large queue might be *migrated* to the small queue. This migration function, which is a result of aggregate user behaviour, is represented as $m(x, y)$. It is linear in $\lambda_y F$ as a result of the integration of individual user that send $\mathrm{d}\lambda_y$ each: $\int m \mathrm{d}\lambda_y = \lambda_y m$. We take $m$ to be a non-negative and continuous function of $x$ and $y$. $m$ represents the fraction of $\lambda_y$ which goes to $Q_x$.

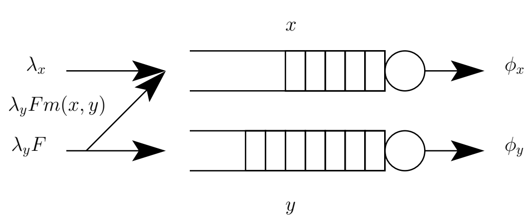


**Fig. 1.** Two-queues model

$$0 \leq m(x, y) \leq 1 \tag{1}$$

For every large flow that migrates, it adds an overhead of $\eta$ (e.g. connection establishment cost, slow-start cost). The rate equations can now be written as:

$$\frac{dx}{dt} = \lambda_x - \phi_x + \lambda_y F m(x, y)(1 + \eta), \quad x > 0 \tag{2}$$

$$\frac{dy}{dt} = \lambda_y F - \phi_y - \lambda_y F m(x, y), \quad y > 0 \tag{3}$$

The rate equations are different at the borders. For $x = 0$,

$$\left.\frac{dx}{dt}\right|_{x=0} = [\lambda_x - \phi_x + \lambda_y F m(x, y)(1 + \eta)]^+ \tag{4}$$

and for $y = 0$,

$$\left.\frac{dy}{dt}\right|_{y=0} = [\lambda_y F - \phi_y - \lambda_y F m(x, y)]^+. \tag{5}$$

## 4 System Analysis for Static Service Rates

This section details the analysis of a system where the service rates at both the queues are fixed.

**Proposition 4.1.** *An interior point* $(x, y)$ *is an equilibrium iff* $\phi_x - \lambda_x = \lambda_y F - \phi_y$ *and* $m$ *is such that* $m(x, y) = \frac{\phi_x - \lambda_x}{\lambda_y F}$ *and* $0 \leq m(x, y) \leq 1$.

*Proof (Proof of Prop. 4.1)*
Let $(x, y)$ be an interior point. It is an equilibrium if and only if:

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \\ 0 \leq m(x, y) \leq 1 \end{cases} \iff \begin{cases} m(x, y) = \frac{\phi_x - \lambda_x}{\lambda_y F(1+\eta)} \\ m(x, y) = \frac{\lambda_y F - \phi_y}{\lambda_y F} \\ 0 \leq m(x, y) \leq 1 \end{cases} \qquad \square$$

**Remark 4.2.** *Existence of interior equilibrium does not only depend on m function but also on the arrival rates and service rates. Meaning that they can only exist in very specific cases.*

**Proposition 4.3.** $(0, 0)$ *is an equilibrium point if and only if:*

$$\begin{cases} m(0, 0) \leq \frac{\phi_x - \lambda_x}{\lambda_y F(1+\eta)} \\ \frac{\lambda_y F - \phi_y}{\lambda_y F} \leq m(0, 0) \\ 0 \leq m(0, 0) \leq 1 \end{cases}$$

*Proof (Proof of Prop. 4.3).* Using equations (4) and (5), we obtain that $(0,0)$ is an equilibrium point if and only if:

$$\begin{cases} \frac{dx}{dt}\Big|_{x=0} &= 0 \\ \frac{dy}{dt}\Big|_{y=0} &= 0 \\ 0 \leq m(0,0) \leq 1 \end{cases} \iff \begin{cases} \frac{\lambda_x - \phi_x}{1+\eta} + \lambda_y F m(0,0) \leq 0 \\ \lambda_y F - \phi_y - \lambda_y F m(0,0) \leq 0 \\ 0 \leq m(0,0) \leq 1 \end{cases}$$

$$\iff \begin{cases} m(0,0) \leq \frac{\phi_x - \lambda_x}{\lambda_y F(1+\eta)} \\ \frac{\lambda_y F - \phi_y}{\lambda_y F} \leq m(0,0) \\ 0 \leq m(0,0) \leq 1 \end{cases}$$ □

**Proposition 4.4.** $(0,y)$ *with* $y > 0$ *is an equilibrium point if and only if:*

$$\begin{cases} m(0,y) \leq \frac{\phi_x - \lambda_x}{\lambda_y F(1+\eta)} \\ m(0,y) = \frac{\lambda_y F - \phi_y}{\lambda_y F} \\ 0 \leq m(0,y) \leq 1 \end{cases}$$

*Proof (Proof of Prop. 4.4).* Using equations (4) and (3), we obtain that $(0,y)$ is an equilibrium point if and only if:

$$\begin{cases} \frac{dx}{dt}\Big|_{x=0} &= 0 \\ \frac{dy}{dt} &= 0 \\ 0 \leq m(0,y) \leq 1 \end{cases} \iff \begin{cases} \frac{\lambda_x - \phi_x}{1+\eta} + \lambda_y F m(0,y) \leq 0 \\ \lambda_y F - \phi_y - \lambda_y F m(0,y) = 0 \\ 0 \leq m(0,y) \leq 1 \end{cases}$$

$$\iff \begin{cases} m(0,y) \leq \frac{\phi_x - \lambda_x}{\lambda_y F(1+\eta)} \\ m(0,y) = \frac{\lambda_y F - \phi_y}{\lambda_y F} \\ 0 \leq m(0,y) \leq 1 \end{cases}$$

□

**Proposition 4.5.** $(x,0)$ *with* $x > 0$ *is an equilibrium point if and only if:*

$$\begin{cases} m(x,0) = \frac{\phi_x - \lambda_x}{\lambda_y F(1+\eta)} \\ \frac{\lambda_y F - \phi_y}{\lambda_y F} \leq m(x,0) \\ 0 \leq m(x,0) \leq 1 \end{cases}$$

*Proof (Proof of Prop. 4.5).* Using equations (2) and (5), we obtain that $(x,0)$ is an equilibrium point if and only if:

$$\begin{cases} \frac{dx}{dt} &= 0 \\ \frac{dy}{dt}\Big|_{y=0} &= 0 \\ 0 \leq m(0,y) \leq 1 \end{cases} \iff \begin{cases} \frac{\lambda_x - \phi_x}{1+\eta} + \lambda_y F m(x,0) = 0 \\ \lambda_y F - \phi_y - \lambda_y F m(x,0) \leq 0 \\ 0 \leq m(x,0) \leq 1 \end{cases}$$

$$\iff \begin{cases} m(x,0) = \frac{\phi_x - \lambda_x}{\lambda_y F(1+\eta)} \\ \frac{\lambda_y F - \phi_y}{\lambda_y F} \leq m(x,0) \\ 0 \leq m(x,0) \leq 1 \end{cases}$$

□

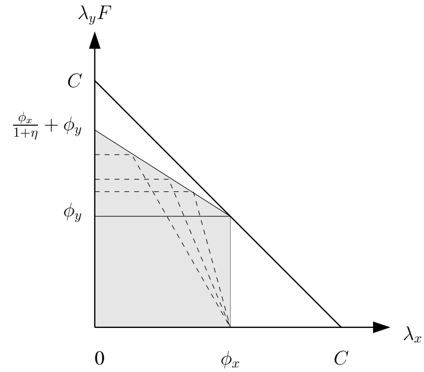


**Fig. 2.** Existence region of equilibrium $(0, 0)$ under static service rate

### 4.1 Discussion

The aim of a network operator in deploying such a scheduling mechanism is to give shorter delays to small flows, at negligible cost to large flows. With this in mind, we can now evaluate which among the equilibrium points are interesting and useful (from the perspective of a network operator).

To start with, let us consider the equilibrium point $(0, 0)$. The inequalities of Prop. 4.3 give the shaded region of Fig. 2, where one $m$ can exist to make $(0, 0)$ an equilibrium. This region is dominated by the line $\lambda_x + \lambda_y F = C$, which defines the region where a single queue system would have empty queue equilibrium. Thus, this equilibrium (in the two queue system) is not of great interest for the network operator.

The lines $(x, 0)$ and $(0, y)$ constitute the remaining border point equilibria. $(x, 0)$ is the set of those points where there is queueing in the small queue, but not at the large queue. For this reason, these are not desirable equilibria from operator's point of view. Similarly existence of $(0, y)$ means, there is nothing queueing at $Q_x$. So, there is incentive for users to migrate to $Q_x$. Hence $(0, y)$ will not be stable.

As seen in previous section, interior point equilibrium are only possible in limiting cases where the surplus rate at the large queue is exactly equal to the surplus of service of $x$, with the additional constraint that $m$ transfers exactly this. This situation is too constrained to happen in a real scenario. To introduce more flexibility, the operator can control the service rate. But this requires the use of some observable parameters of the system. In this system, the only observable parameters are $x$ and $y$ as arrival rates $\lambda_x$ and $\lambda_y$ are not separable at the queues.

## 5 Control on $\phi_x$ Using Parameter $x$

In this section we study the system when operator controls the service rates using a single parameter. Let $f$ be the control function, and $x$ be the control parameter. In the remaining of this section we use the following definition for $\phi_x(x)$ and $\phi_y(x)$.

**Definition 5.1.** *$\phi_x(x)$ and $\phi_y(x)$*
$\phi_x(x) = f(x)$
$\phi_y(x) = C - f(x)$

$C$ being the maximum link capacity (or service rate), let:

$$0 < f(x) < C \tag{6}$$

so that the service rate at any queue doesn't vanish.

### 5.1 Delay Condition

We introduce the delay condition which is satisfied at equilibrium, as the users have no incentive to migrate once the delays at both queues are equal. Let us look the delay a large flow will incur $Q_x$, if it is split into $F$ small flows. For a service rate of $\phi_x$ at $Q_x$, each small flow gets $\frac{\phi_x}{x+F}$ of service. Hence the time to transfer a large flow through $Q_x$ is $T_x = \frac{x+F}{F\phi_x}(1+\eta)$. On the other hand, if the arriving large flow decides to queue at $Q_y$, the delay experienced will be $T_y = \frac{y+1}{\phi_y}$.

At equilibrium, $T_x = T_y$; thus,

$$\frac{(x+F)(1+\eta)}{F\phi_x} = \frac{y+1}{\phi_y} \tag{7}$$

### 5.2 Analysis of Equilibrium

For equilibrium to exist, the equations (2) and (3) should be equated to zero.

**Proposition 5.2.** *If $\eta$ is zero, no equilibrium will exist unless $C = \lambda_x + \lambda_y F$.*

*Proof (Proof of Prop. 5.2).*

From the combination (2) + (3) at equilibrium, when $\eta$ is 0, we get $C = \lambda_x + \lambda_y F$. □

In the remaining, $\eta$ is taken to be strictly positive.

Using equations (2) and (3) at equilibrium, gives the constraints (8) on $f$ for the existence of such an equilibrium point.

$$f(x_e) = \frac{(1+\eta)(C - \lambda_y F) - \lambda_x}{\eta} \tag{8}$$

There can be multiple such points $x_e$ or no depending on $f$.

**Proposition 5.3.** *For a given set of parameters $(\lambda_x, \lambda_y, C, \eta, f, m)$ with $\eta > 0$, the system has inner equilibrium points $(x_e, y_e)$ where:*

$$x_e \in f^{-1}\left(\frac{(1+\eta)(C - \lambda_y F) - \lambda_x}{\eta}\right) \tag{9}$$

*and:*

$$y_e = \frac{C - f(x_e)}{Ff(x_e)}(x_e + F)(1 + \eta) - 1$$

*iff:*

$$\begin{cases} \lambda_x + \lambda_y F \leq C \\ \lambda_x + \lambda_y F(1 + \eta) > C \\ f^{-1}(\frac{(1+\eta)(C-\lambda_y F)-\lambda_x}{\eta}) \neq \emptyset \\ m(x_e, y_e) = \frac{C-(\lambda_x+\lambda_y F)}{\eta\lambda_y F} \end{cases} \tag{10}$$

*Proof (Proof of Prop. 5.3)*

From the combination, (2) + $(1 + \eta)$(3), at equilibrium, we obtain Eq. (8). The system has equilibriums iff there is point $x_e$ satisfying this equation, meaning $f^{-1}(\frac{(1+\eta)(C-\lambda_y F)-\lambda_x}{\eta})$ is not empty ($\eta \neq 0$). From Eq. (7), we get corresponding $y_e$. Then from Eq. (2) at equilibrium, we have $m$ as defined in Eq. (10).

Due to constraint (1) on $m$, and constraint (6) on $f$, we have the existence of this equilibrium iff:

$$\begin{cases} \lambda_x + \lambda_y F \leq C \\ \qquad\qquad C < \lambda_x + \lambda_y F(1 + \eta) \end{cases} \tag{11}$$

Second inequality is strict because of Eq. (6). □

Fig. 3 shows the region of arrival rates where equilibrium can exist, dashed-line is excluded from this.

**Corollary 5.4.** *If $f$ is strictly monotonic. For every 2-tuple of $(\lambda_x, \lambda_y)$ satisfying the line equation: $(1 + \eta)(C - \lambda_y F) - \lambda_x = k$ (for a constant $k$), there is maximum of one equilibrium point.*

*Proof.* Corollary 5.4

If $f$ is strictly monotonic, there is utmost one pre-image by $f^{-1}$. As potential equilibria are determined by Eq. (9) (and $y_e$ which only depends on $x_e$), all points of the line of arrival rates: $(1 + \eta)(C - \lambda_y F) - \lambda_x = k$ have the same potential equilibrium. Since $m(x_e, y_e)$ has to satisfy Eq. (10), which gives a different line in $\lambda_x$ and $\lambda_y$, there is at most one equilibrium point (the intersection). □

From the above, it can be observed that, for a monotonic $f$, there exists utmost one equilibrium point for the whole line of arrival rates. This gives only a few equilibrium points for a wide range of arrival rates. A non-monotonic $f$ will give more equilibrium points. But still, it is not feasible to obtain equilibrium points for all values of $(\lambda_x, \lambda_y)$ satisfying the line of arrival rates, as it would require an infinite queue or an infinite variability of $f$.

Hence, we conclude that control using a function of $x$ alone, is not of any use to the operator.

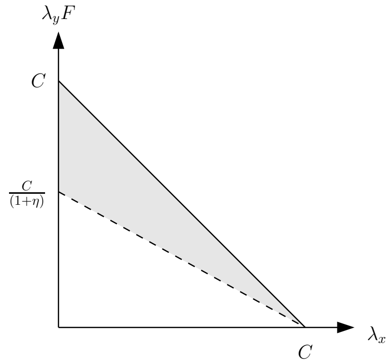


**Fig. 3.** Interior equilibrium existence region under $\phi_x(x) = f(x)$

# 6 Control on $\phi_x$ Using Parameters $x$ and $y$

As seen in previous section, using only one parameter is not enough to stabilize the system as the control space is too small. We thus use a control function with two parameters: $x$ and $y$.

**Definition 6.1.** *$\phi_x(x, y)$ and $\phi_y(x, y)$*

$\phi_x(x, y) = g(x, y)$
$\phi_y(x, y) = C - g(x, y)$

Similar to what have been done with $f$, if $C$ is the maximum link capacity (or service rate), let:

$$0 < g(x, y) < C \tag{12}$$

Note that, definition of delay equation at equilibrium as given in (7) remains the same and so we directly proceed to the analysis of potential equilibria.

## 6.1 Analysis of Equilibrium

For equilibrium to exist, the equations (2) and (3) should be equated to zero. Prop. 5.2 still holds in this case as $\phi_x$ and $\phi_y$ also sum to $C$; therefore from equations (2) and (3) we can prove the same. Hence $\eta$ is also taken strictly positive here.

Similarly to what have been done for Prop.5.3, at equilibrium, using Eq. (2) and (3), we obtain the following constraint on $g$:

$$g(x_e, y_e) = \frac{(1 + \eta)(C - \lambda_y F) - \lambda_x}{\eta} \tag{13}$$

**Proposition 6.2.** *For a given set of parameters* $(\lambda_x, \lambda_y, C, \eta, g, m)$ *with* $\eta > 0$*, the system has inner equilibrium points* $(x_e, y_e)$ *iff:*

$$\begin{cases} g(x_e, y_e) = \frac{(1+\eta)(C-\lambda_y F)-\lambda_x}{\eta} \\ m(x_e, y_e) = \frac{C-(\lambda_x+\lambda_y F)}{\eta\lambda_y F} \\ \frac{(x+F)(1+\eta)}{F\phi_x} = \frac{y+1}{\phi_y} \\ \lambda_x + \lambda_y F \leq C \\ \lambda_x + \lambda_y F(1+\eta) > C \end{cases} \tag{14}$$

*Proof (Proof of Prop. 6.2).* Same as Prop. 5.3 except that (8) has been replaced by (13). □

Note that the region of arrival rates where equilibrium points can exist is the same.

We define an *equivalent load* $\Gamma$:

**Definition 6.3.** $\Gamma(\lambda_x, \lambda_y) = \frac{\lambda_x}{1+\eta} + \lambda_y F$.

**Definition 6.4.** $D(\Gamma)$ *is the set of* $(x, y)$ *satisfying:*

$$y = a(\Gamma)x + b(\Gamma) \tag{15}$$

*where*

$$a(\Gamma) = \frac{(1+\eta)\Gamma - C}{F(C-\Gamma)}$$

*and*

$$b(\Gamma) = \frac{(2+\eta)\Gamma - 2C}{C-\Gamma}$$

**Proposition 6.5.** *For a given setting of arrival rates* $(\lambda_x, \lambda_y)$ *satisfying*

$$\Gamma(\lambda_x, \lambda_y) = k \tag{16}$$

*and the two inequalities of Prop. 6.2, equilibria* $(x_e, y_e)$ *under this load are on* $D(k)$*. Besides, for all the equilibrium points in* $D(k)$*,* $g$ *satisfies* (13) *and is constant:*

$$g(x_e, y_e) = \frac{1+\eta}{\eta}(C - \Gamma) \tag{17}$$

*Proof (Proof of Prop. 6.5).* Let $(\lambda_x, \lambda_y)$ be a setting of arrival rates satisfying Eq. (16) and the two inequalities of Prop. 6.2.

We first show that $D(k)$ contains all the potential equilibrium points. By replacing $g$ using Eq. (13) in the delay equation (7), we obtain Eq. (15).

All equilibrium points of arrival settings satisfying (16) have the same value of $g$ as Eq. (13) holds and gives Eq. (17) which is constant in $\Gamma(\lambda_x, \lambda_y)$. □

Fig. 4 shows $\Gamma$-lines in the $\lambda_x\lambda_y$-plane and their corresponding $D(\Gamma)$-lines in the $xy$-plane. On each such line in the $xy$-plane, $g$ is constant and thus gradient is orthogonal. From Eq. (17), we also know $\frac{\partial g}{\partial \Gamma}$ is negative which justifies the orientation of gradient on the figure.

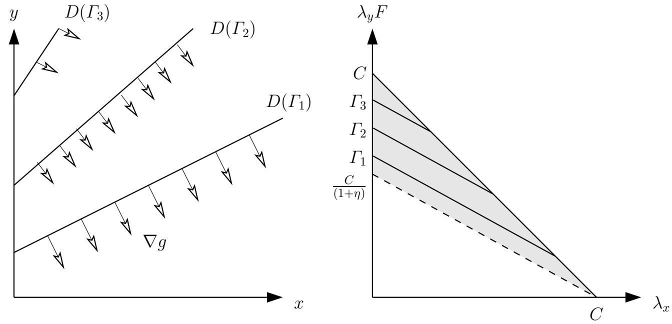


**Fig. 4.** Interior equilibrium existence region and mapping of $\Gamma(\lambda_x, \lambda_y)$ lines to $D(\Gamma)$, level sets and gradient field of $g(x, y)$

**Proposition 6.6.** *For any* $(\lambda_x, \lambda_y)$ *verifying the two inequalities of Prop. 6.2,* $D(\Gamma(\lambda_x, \lambda_y))$ *for does not intersect in first quadrant.*

*Proof (Proof of Prop. 6.6).* For $\Gamma(\lambda_x, \lambda_y) = \Gamma$ satisfying the two inequalities of Prop. 6.2, satisfy: $(1 + \eta)\Gamma > C$ and $C \geq \lambda_x + \lambda_y F > \Gamma$.

Under this, $\frac{da}{d\Gamma}$ and $\frac{db}{d\Gamma}$ are strictly positive and $a$ is strictly positive. Thus, $D(\Gamma(\lambda_x, \lambda_y))$ do not intersect in the first quadrant. □

This basically means $g$ is 'feasible'. As a corollary of Prop. 6.6, we give:

**Corollary 6.7.** *$g$ can exist in the sense that there are no incompatible constraints resulting from Prop. 6.2.*

*Proof (Proof of Corollary 6.7).* Prop. 6.5 gives the value $g$ must have on $D(\Gamma)$ in order to have equilibria on it and according to Prop. 6.6 these lines do not intersect in the first quadrant where $g$ can be defined. It proves there is no incompatibility in the definition of $g$. □

**Proposition 6.8**

$$\lim_{\Gamma \to C^-} a(\Gamma) = +\infty \ ; \qquad \lim_{\Gamma \to \frac{C}{1+\eta}^+} a(\Gamma) = 0$$
$$\lim_{\Gamma \to C^-} b(\Gamma) = +\infty \ ; \qquad \lim_{\Gamma \to \frac{C}{1+\eta}^+} b(\Gamma) = -1$$

*Proof (Proof of Prop. 6.8).* Trivial. □

In particular, this last proposition implies that $g$ can be defined in the whole first quadrant using Eq. (13) and lines $D(\Gamma)$.

As of now, we demonstrated that it is feasible to define $g$ so that point of $D(\Gamma)$ can be equilibria for $(\lambda_x, \lambda_y)$ on $\Gamma$ line. Next, we study the stability of the potential equilibria in order to define the additional constraints on $m$. The only constraint on $m$ coming from existence of equilibrium (Prop. 6.2) is that $m(x_e, y_e) = \frac{C - (\lambda_x + \lambda_y F)}{\eta \lambda_y F}$. The point where this will hold is not specified and depends on $m$. Defining $m$ will thus define a mapping of arrival rates $(\lambda_x, \lambda_y)$ to the actual equilibrium point.

### 6.2 Stability of the Equilibria and Definition of $m$

As demonstrated in the previous section, $g$ that doesn't prevent existence of equilibrium is feasible. We now like to have the constraints that $m$ has to satisfy. We already know from the previous section the range $m$ must cover, but we don't know where they have to be located in $xy$-plane. In order to get more constraints on $m$, we study the conditions for stable equilibriums. To do so we rely on Hartman Grobman theorem and the study of the stability of the linearized system.

**Proposition 6.9.** *For an equilibrium point $(x_e, y_e)$ as defined by Prop. 6.2 if the following equations hold:*

$$\begin{cases} \dfrac{\partial m}{\partial x} < \frac{(1+y_e)C}{\lambda_y(F(2+\eta+y_e)+x_e(1+\eta))^2} \\ \dfrac{\partial m}{\partial x} + \left(\dfrac{y_e+1}{x_e+F}\right)\dfrac{\partial m}{\partial y} < 0 \end{cases} \tag{18}$$

*then $(x_e, y_e)$ is asymptotically stable.*

*Proof (Proof of Prop. 6.9)*

To analyse of the equilibrium point $(x_e, y_e)$, we take the Jacobian $J$ of the rate equations (2) and (3) at this point. The partial derivatives $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(x_e, y_e)$ are obtained from the delay equation, Eq. (7).

$$\frac{\partial g}{\partial x}(x_e, y_e) = \frac{(1+\eta)(y_e+1)FC}{(Fy_e + x_e(1+\eta) + F(2+\eta))^2} \tag{19}$$

$$\frac{\partial g}{\partial y}(x_e, y_e) = -\frac{(1+\eta)(x_e+F)FC}{(Fy_e + x_e(1+\eta) + F(2+\eta))^2} \tag{20}$$

The equilibrium point $(x_e, y_e)$ is asymptotically stable if the eigenvalues of the $J$ at $(x_e, y_e)$ have strictly negative real parts [7, Ch. 2 & 5]. Characteristic polynomial of $J$ is:

$$\begin{array}{r} \lambda^2 + \\ (\lambda_y F(\frac{\partial m}{\partial y} - (1+\eta)\frac{\partial m}{\partial x}) + \frac{\partial g}{\partial x} - \frac{\partial g}{\partial y})\ \lambda + \\ \eta(\lambda_y F(\frac{\partial m}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial m}{\partial y}\frac{\partial g}{\partial x})) \end{array}$$

From this and equations (19) and (20), real parts of the roots are strictly negative iff:

$$(1+\eta)\frac{\partial m}{\partial x} - \frac{\partial m}{\partial y} < \frac{(1+\eta)C(1+x_e+y_e+F)}{\lambda_y(F(2+y_e+\eta)+x_e(1+\eta))^2} \tag{21}$$

and

$$\frac{\partial m}{\partial x} + \left(\frac{y_e+1}{x_e+F}\right)\frac{\partial m}{\partial y} < 0 \tag{22}$$

Inequalities of the proposition are obtained using combination of equations (21) and (22). □

Proposition 6.2 and 6.9 give sufficient conditions on $m$ to define stable equilibria. Next, we prove that there exists $m$ which stabilizes the system for any arrival setting.

**Proposition 6.10.** *There exists an $m$ satisfying the constraints of propositions 6.2 and 6.9 which stabilizes the system for any arrival rates in the shaded region of the Fig. 4.*

*Proof (Proof of Prop. 6.10)*

We prove this by exhibiting one such $m$. Let $m$ be such that

$$m(x, y) = e^{-xy}$$

The $m$ satisfies the constraints of Eq. (18) for any $\lambda_y$ in $(0, C)$ as $\frac{\partial m}{\partial x} = -ye^{-xy}$ and $\frac{\partial m}{\partial y} = -xe^{-xy}$, are both strictly negative on the interior. Besides, as $m$ ranges from 1 to 0, from the borders ($y = 0$ and $x = 0$) to infinity, thus by continuity, there exists an equilibrium point $(x_e, y_e)$ where $m(x_e, y_e) = \frac{C-(\lambda_x+\lambda_y F)}{\eta\lambda_y F}$ for any arrival rates as all $D(\Gamma)$-lines enter the first quadrant by one its borders. □

Note that if $m$ is strictly monotonic, there is only one equilibrium point for any arrival rate setting in the equilibrium existence region (refer Fig. 3) located at the intersection of the level set of $g$ and $m$. In addition, it is not possible to apply this for all setting of arrival rates in order to get equilibria for all of them, unless queue are infinite.

**Proposition 6.11.** *If queues are finite, some setting of arrival rates can't have equilibrium.*

*Proof (Proof of Prop. 6.11).* As $a(\Gamma)$ tends to 0 when $\Gamma$ tends to $C/1 + \eta^+$, and $b(\Gamma)$ tends to -1, intersection of $y = 0$ and $D(\Gamma)$ tends to infinity. Hence, for any $x_{max}$, it is possible to find $\Gamma$ close enough to $C/1 + \eta$ so that equilibrium which have to be on $D(\Gamma)$ (due to Prop. 6.5) would have to be after $x_{max}$.

Using limits of $a(\Gamma)$ and $b(\Gamma)$ when $\Gamma$ tends to $C$, it is possible to pursue the same reasoning and prove that for some settings of arrival rates, there can't be equilibrium under finite queue for large flows. □

Thus we see that, the system can attain stability depending on the decision of users, and the control function used by operator.

## 7 Game

We summarize our results in the form of a game with two players: operator and user (with a large flow to send). Here, we make the fair assumption that $T_x < T_y$. The operator can take one of the two actions:

– AFP: Assume fair play, and not use a $g$.
– AUP: Assume unfair play, and use a $g$.

From the users, we consider a collective behaviour.

- UC: Users cheat,
- UR: Users rightful

Under AUP, $T_x = T_y$. We use preferential ordering of payoffs for both players. That is $(a_o, a_u) \prec_p (a'_o, a'_u)$, if player $p$ prefers second strategy over the first. The letter $o$ is used to refer to operator, and $u$ to refer to users.

- $(\mathrm{AFP}, \mathrm{UR}) \prec_u (\mathrm{AFP}, \mathrm{UC})$: Users prefer to cheat when the operator does nothing to stop them from cheating, as this would give them shorter response time in the small queue (when $T_x < T_y$).
- $(\mathrm{AUP}, \mathrm{UR}) \prec_u (\mathrm{AUP}, \mathrm{UC})$: Users also prefer to cheat when the operators are aware and are setting service rates dynamically to achieve stability, as this would ensure a finite queue; hence a finite delay. Observe that, if the don't cheat (and stay in $Q_y$), there is no equilibrium (from Prop. 6.2); hence the queue will build up without bound.

Therefore, it can be drawn that UC strictly dominates UR under any action of the operator (AFP or AUP). Hence, the action UR can be eliminated [8]. So, what lefts to be analysed is the preference of operator under this user action (UR). We see, $(\mathrm{AFP}, \mathrm{UC}) \prec_o (\mathrm{AUP}, \mathrm{UC})$, as there is no equilibrium for general arrival rates (from Prop. 6.2, and if $T_x$ remains less than $T_y$, migration will create additional load due to $\eta$) leading to overflow.

From the above analysis, $(\mathrm{AUP}, \mathrm{UC})$ is a Nash equilibrium in the two-players game. That is, assuming operators and users are rational, users will tend to cheat, and operators will look to stabilize the system to maintain finite queues (when the system is operating near to saturation, depending on $\eta$).

Note that if the operator's setting of service rates is such that $T_x > T_y$, then migrating to small queue is no more an incentive for large flows. This doesn't preclude operator from favoring small flows as $\frac{x}{\phi_x} < \frac{y}{\phi_y}$ can still hold. In such a scenario, it can be seen that $(\mathrm{AFP}, \mathrm{UR})$ will be a Nash equilibrium. This situation can happen if $\eta$ is large enough and $\frac{\phi_y}{\phi_x}$ can be maintained such that:

$$\begin{cases} \frac{\phi_y}{\phi_x} < \frac{y}{x} \\ \frac{y+1}{x+f} \frac{F}{1+\eta} < \frac{\phi_y}{\phi_x} \end{cases}$$

Second constraint will not be satisfied if operator want to favor small flows too much, say, as in the priority based scheduling proposed in [2]; meaning that it will be of interest for users to cheat.

## 8 Conclusions

Starting from the setting of static service rates, and moving to dynamic service rate settings, we analysed the existence of equilibria. For the existence of equilibria that is of interest to the operator, it is necessary to have control over the service rate as a function of the queue lengths. Even then, not all the stable equilibrium points are of interest to the operator, as they give the same delay to small and large flows. Therefore, if a large number of users cheat, the operator has no visible incentive in deploying a size-based scheduling system.

The focus of our study revolved around saturation (of the line capacity) as we assumed that there is some cost $\eta$ incurred due to migration. In the future, we plan to analyse the system in overload. Similarly, it would be interesting to understand what happens if the operator deploys a mechanism to detect and shift some of the disguised large flows from the small queue to the large queue.

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