Chapter 6
Overview and Directions for Future Research

In Chap. 1 we introduced the classical Cournot model and after setting up the general framework we focused on a number of specific examples involving combinations of linear and hyperbolic price functions and linear and quadratic cost functions, also taking careful account of capacity constraints. These examples illustrated the variety of reaction functions that can occur and the various types of equilibria (possibly multiple) both in the interior of the domain of interest and on its boundaries. We then went on to introduce the various types of adjustment processes that underpin the dynamic processes, the study of the local and global dynamics of which has occupied much of the space in this book. In particular we considered discrete time and continuous time versions of partial adjustment towards the best response with naive expectations and adaptive expectations as well as the gradient adjustment process. We then introduced some of the basic tools for the analysis of global dynamics via some examples involving duopoly or symmetric and semi-symmetric oligopolies. We introduced the important concept of basins of attraction of different equilibria and the important tool of the critical curve and the concept of border collision bifurcations. Already with the simple examples considered we see the types of complexity that can arise in oligopoly models under the type of dynamic adjustment processes we consider here.

In the second chapter we considered the widely studied class of concave oligopolies. We first obtained the properties of the reaction function both with and without cost externalities, and then used these to study the local and global dynamics of discrete time and continuous time concave oligopolies under the various best response processes of Chap. 1. We made use of the determinantal relation in Appendix E to obtain results on local stability. The full array of the tools for the analysis of the global dynamics were brought to bear to obtain interesting results in a number of special cases of price and cost functions, as well as on the nature of the oligopoly, such as whether it is a duopoly or semi-symmetric. We saw in particular the important role of border collision bifurcations in determining the global dynamics and how the number of firms in the oligopoly, the capacity constraints of firms and their speeds of adjustment are all important bifurcation parameters. The chapter concluded with a study of the local dynamics of continuous time oligopolies with continuously distributed informational time lags, and we saw how such time lags can have a strong influence of local bifurcation behavior.

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Chapter 3 considered general oligopolies and started with an analysis of the case of isoelastic price functions under both continuous time and discrete time adjustment processes. Again the results of Appendix E were invoked to analyze the local stability and we saw that for semi-symmetric oligopolies in both the discrete time and continuous time cases this is determined qualitatively by the same set of graphs, though of course quantitatively the two cases differ. With regard to the global analysis of the discrete time model we saw the richness of the local bifurcations with respect to the number of firms, the cost ratio and speeds of adjustment in the semi-symmetric case. We found also that the speeds of adjustment played a role in the generation of border collisions and hence global bifurcations. The remainder of the chapter considered the role of cost externalities that are captured by the assumption of a certain type of non-monotonic reaction function. Here we focused on the duopoly case and saw that such models can generate situations of several coexisting equilibria that are locally stable, each having its own basin of attraction. In the case of identical speeds of adjustment we were able to analyze and understand in some detail the way in which the different equilibria can be born and the way in which the structure of their basins of attraction change with key parameters, due mainly to the occurrence of contact bifurcations. Some numerical examples of the non-identical speed of adjustment situation illustrate how the basins can become even more complex in this case. The disconnected nature of the basins of attraction means that the outcome (in the sense of to which equilibrium the game converges) of oligopolies with cost externalities is highly path dependent. These examples convey in a very clear way the important distinction between local bifurcations and global bifurcations.

In Chap. 4 we apply the analysis of the first three chapters to a number of models that are an extension of the basic oligopoly set-up or are dynamic economic games that essentially reduce to classical oligopolies. These are market share attraction games, labor-managed oligopolies, oligopolies with intertemporal demand interaction, oligopolies with production adjustment costs and oligopolies with partial cooperation amongst the firms. Such extensions of the basic oligopoly model and dynamic economic games exhibit the range of behaviors observed in the basic oligopolies of the previous chapters.

Finally in Chap. 5 we considered learning behavior under incomplete knowledge of the demand relationship. We started by considering oligopolies in which firms have misspecified price functions but otherwise we still use the adjustment processes of Chap. 1. Now the possibility of subjective equilibria arises, the local and global dynamics of which are studied through some examples that illustrate how such subjective equilibria are born, their local stability properties and the (sometimes complicated) nature of their basins of attraction. Next we assume that firms use some kind of approximate learning procedure to resolve their incomplete knowledge of the price function. Using a number of specific examples we study the local stability of the equilibria and how its loss can give rise to fluctuating attractors as various parameters change. Global analysis indicates how the learning scheme can affect the basin of attraction of a stable equilibrium. Next we study other types of learning schemes by firms as they try to determine the true shape of the price function. We focus on the case of linear price and cost functions and consider three scenarios in which firms have different types of partial information about some parameters of the price function and seek to learn about the remaining parameters by some adjustment process. Again via specific examples we see that these learning schemes can generate the type of local and global bifurcations seen in the previous chapters. Finally we conclude this chapter with a brief discussion of uncertain price functions which brings us to the edge of the field of statistical learning, which presents a whole different field of research.

From the point of view of nonlinear dynamical systems the book has introduced the still relatively new (at least for economists) concept of border collisions and illustrated its use in a number of examples. The examples have emphasized how there are in fact two types of complexity of importance in dynamic economic models. The first is the familiar one arising as a result of local bifurcations, which frequently occur when equilibria lose local stability via Hopf or flip bifurcations and local stability of an equilibrium gives way to some sort of fluctuation around it. The other, less familiar one, arises when a border collision occurs, and basins of attraction of different equilibria undergo a change in their structure. Also there may be co-existing attractors within the same basin of attraction. A typical result of such bifurcations is that the outcome of the economic adjustment process under consideration may be highly sensitive to initial conditions. Future research in economic applications in this area will probably focus on the systematic description of the different sources of such bifurcations, the elaboration of the types of examples where such border collision bifurcations can occur and the typical sorts of behavior that can emerge from them. It would also be useful to try to understand the economic origins of the different types of such bifurcations.

With regard to the specific models we have studied in this book, a number of issues are likely to occupy the attention of researchers in the years ahead.

Considering first the case of concave oligopolies, we have derived most of our results under the assumptions (A)–(C) in Sect. 2.1 (or their modifications in different model types) which we recall placed restraints on the inverse demand function and cost functions so as to guarantee the concavity of the profit function and in the concave case the monotonicity of the best response functions. An important task for future research will be to study the implications of relaxing any one of these assumptions. We have seen in Example 1.2 that just by relaxing the condition (C) how more complicated equilibrium situations can arise.

A number of our examples involved the isoelastic price function, which is widely used in the literature on oligopoly because it affords a lot of analytical tractability. However this price function has the disadvantage that it has no reservation price (or rather the reservation price is infinite) and this is rather unrealistic. Future research should try to introduce a reservation price into this price function, either by using a translated hyperbola, or by truncating the function at some (presumably high) price. The isoelastic price function has also been used frequently in conjunction with a convex cost function, but what would happen if we were to allow a certain amount of concavity into the cost function? This could arise for instance if there were an increasing return to scales effect at low outputs and a decreasing returns to scale effect at higher levels of output. Uniqueness of the equilibrium could be lost in such situations or there may be no equilibrium at all. Also many of the local and global stability results we have derived rely heavily on the special analytical properties of the cost functions, so we might expect to see a much richer set of dynamic outcomes.

With regard to the modified and extended oligopolies of Chap. 4 a number of extensions can be envisaged. The market share attraction games are equivalent to oligopolies with isoelastic price functions, so all the remarks of the previous paragraph apply to this class of model as well. In our analysis of labor-managed oligopolies we assumed very special forms for the labor demand functions, but both equilibrium results as well as the dynamic analysis will change if we consider more general forms for these demand functions. The models with intertemporal demand interaction were analyzed under the assumptions of the concave oligopolies so again the relaxation of the assumptions on the price and cost functions will lead to a richer set of outcomes for the equilibria and the dynamics. In the models with production adjustment costs we have assumed that this additional cost component depends on the output change from the previous period. A more realistic assumption might be to make this cost depend on a state variable related to the capacity limit that adjusts dynamically in such a way that the firm increases it if output needs to go beyond it. The analysis of oligopolies under partial cooperation also relies very much on the concavity assumptions being satisfied by the profit functions. Here also it would be of interest to study the situations in which these concavity conditions are relaxed. It would also be interesting to include partial cooperation into some of the extensions described earlier in this chapter.

In the learning schemes in the models with misspecified and uncertain cost functions we have adopted various assumptions on the learning behavior of the firms, from remaining statically with the same misspecification over every time period, to updating their estimate of it based on the most recently observed price. There is now a vast literature on learning in dynamic economic models, see for example Fudenberg and Levine (1998), and many of these ideas could be brought into the problems considered in Chap. 5. Many of these schemes are probabilistic in nature so this strand of research will involve the analysis of economic models evolving dynamically under random influences, this is an area into which research has barely begun as it involves bringing together the theory of dynamical systems and the theory of stochastic processes.