# Chapter 1
# The Classical Cournot Model

In this chapter we will introduce the classical Cournot model, which is also known as the single-product quantity setting oligopoly model without product differentiation. In the first section of the chapter the Cournot model will be discussed as an $N$-firm static game and the best responses of the firms and the equilibria will be determined in a series of examples, many of which will be built upon in developing the ideas in subsequent chapters. Section 1.2 introduces the dynamic adjustment processes via which we shall assume that firms adjust output over time. We will in particular discuss expectation formation processes and adaptive adjustments and gradient adjustments. The final section will illustrate by simple examples the complexity of the dynamics that can arise in these models due to certain nonlinear features to be described below. The fundamental techniques for the global analysis of the dynamics of such models will be explained in Sect. 1.3.

## 1.1 Introduction

The basic model can be described as follows. Consider an industry of $N$ firms producing a homogeneous product. Let $k = 1, 2, \ldots, N$ denote the firms and let $x_k$ be the output quantity of firm $k$. We assume that the inverse demand (or price) function depends on the total output level of the industry, so the market price may be written $p = f\left(\sum_{k=1}^{N} x_k\right)$. The particular form of the function $f$ can be derived from microeconomic principles (see for example, Vives (1999)), and several function types are discussed in the literature.

An important example of an inverse demand function which is linear is obtained by assuming that the utility function of a typical consumer is quadratic,

$$U(q) = aq - \frac{1}{2}bq^2, \qquad (a, b > 0),$$

where $q$ is the quantity of the good purchased by the consumer. If we denote the market price of the good by $p$, then for a sufficiently large income the consumer

G.I. Bischi et al., *Nonlinear Oligopolies*, DOI 10.1007/978-3-642-02106-0_1, 

solves the optimization problem

$$\max(U(q) - pq).$$

Assuming an interior optimum, the first order condition implies that

$$0 = U'(q) - p = a - bq - p,$$

so that the individual demand at the price $p$ is therefore

$$q(p) = \frac{a}{b} - \frac{1}{b}p.$$

Consider now $n$ heterogenous consumers with quadratic utility and preference parameters $a_i$ and $b_i$. From the previous description we know that for any fixed price consumer $i$ will buy the amount $q_i = (a_i - p)/b_i$, so the total demand becomes

$$D = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \frac{a_i}{b_i} - \sum_{i=1}^{n} \frac{1}{b_i} p,$$

and hence the relationship between total demand and market price is linear. Notice that if price increases, demand decreases and that there is a maximum price, usually referred to as the *reservation price*, above which demand reduces to zero. If we denote by $Q = \sum_{k=1}^{N} x_k$ the quantity supplied by the $N$ firms in the industry and we assume that at the price $p$ the market clears, that is $D = Q$, then it also follows that the relation between industry output and price is linear. Hence, by inverting this relationship we finally obtain

$$p = f(Q) = A - BQ,$$

where

$$A = \sum_{i=1}^{n} \frac{a_i}{b_i} \Big/ \sum_{i=1}^{n} \frac{1}{b_i}, \quad B = 1 \Big/ \sum_{i=1}^{n} \frac{1}{b_i}.$$

Obviously, this representation is only valid for $Q \leq A/B$, that is as long as the industry output is below the *market saturation* point. Otherwise, we have $p = 0$.

In the case of a general inverse demand function the profit of firm $k$ $(1 \leq k \leq N)$ is the difference between its revenue and its cost and so is given by

$$\varphi_k(x_1, \ldots, x_N) = x_k f\left(\sum_{l=1}^{N} x_l\right) - C_k(x_1, \ldots, x_N), \tag{1.1}$$

where $C_k$ is the cost function of firm $k$.[1] Our formulation takes into account the fact that the cost of each firm depends not only on its own output but also on the outputs

[1] In the game theory context the profit functions are usually called the payoff functions, and the firms are called the players. We will occasionally make use of these terms throughout this book.

of the competitors. The firms have to compete in the secondary market to ensure capital, manpower, energy, material, etc. for their production processes. The technological and intellectual spillover between companies is another cost externality which adds to the interdependence of the firms. In the literature on oligopoly theory the interdependence of the firms through their cost functions is either ignored by assuming that the cost of firm $k$ is $C_k(x_k)$, or it is assumed that the cost of firm $k$ depends on its own production level $x_k$ and also on the total production level of the rest of the industry, which we will denote by $Q_k = \sum_{l \neq k} x_l$ so that the cost function of firm $k$ may be written more generally as $C_k(x_k, Q_k)$. In the rest of the book we will consider various cases where cost externalities arise. Note that under this assumption the profit of any firm $k$ just depends on its own output and the output of the rest of the industry, it does not depend on the individual output level of any competitor. For this reason it is convenient to rewrite the profit function of firm $k$ as

$$\varphi_k(x_1, \ldots, x_N) = x_k f(x_k + Q_k) - C_k(x_k, Q_k). \tag{1.2}$$

Taken together, the above set-up yields a static $N$-person game, where the players are the firms, the strategy set of firm $k$ is the interval $[0, L_k]$, where $L_k$ is the capacity limit of firm $k$ and its payoff function is given by (1.2). If we assume that all firms are rational in the sense that they want to maximize their own profits, then we can derive the firms' best responses. That is, if firm $k$ knows the total production $Q_k$ of the rest of the industry, then it will select a production level $x_k$ that maximizes its profit (1.2). For each value of $Q_k$ let $R_k(Q_k)$ denote the set of all optimal solutions, that is

$$R_k(Q_k) = \left\{ x_k \mid x_k = \arg \max_{0 \leq x_k \leq L_k} \{x_k f(x_k + Q_k) - C_k(x_k, Q_k)\} \right\}, \tag{1.3}$$

which is called the *best response* or *best reply mapping* of firm $k$. In the general case this is a point-to-set mapping, and in this case it is usually called the *best reply correspondence*. In the case of a unique optimal solution, $R_k(Q_k)$ is called the *best reply* or *reaction function* of firm $k$. The *Nash equilibrium* of the game is a simultaneous production vector $(\bar{x}_1, \ldots, \bar{x}_N)$ which is a best response for each firm, under the assumption that all others maintain their corresponding equilibrium production levels. This concept can be mathematically expressed for all $k$ as,

$$\bar{x}_k \in R_k(\bar{Q}_k) \text{ with } \bar{Q}_k = \sum_{l \neq k} \bar{x}_l. \tag{1.4}$$

At the equilibrium all firms simultaneously select their best responses to the corresponding equilibrium choices of the competitors. In other words, no firm has any interest to deviate unilaterally from its equilibrium level.

In the following examples we will show that best responses might have a large variety of forms, and also, that oligopolies may have no equilibrium at all. Furthermore, in the case of existence there may be multiple equilibria, and the number of equilibria may be finite or infinite. In the case of multiple equilibria, the problem of equilibrium selection arises. In such situations, the non-negativity of the profits and the dynamic evolution of the oligopoly game, determined by the adjustment processes and the degree of bounded rationality of the players, can be used to determine which equilibria are realistic and which are not. We will return to this problem in later chapters.

*Example 1.1.* Consider the case of a linear oligopoly where the price function has the form $f(Q) = \max\{0, A - BQ\}$ with $Q = \sum_{k=1}^{N} x_k$ and $C_k(x_k) = d_k + c_k x_k$ $(1 \le k \le N)$ with $A$, $B$, $c_k$, $d_k$ being all positive. Note that the max operation ensures that the price is zero for total output above the market saturation point $A/B$. In this case $\varphi_k$ is strictly concave in $x_k$ with derivative

$$\frac{\partial \varphi_k}{\partial x_k} = \begin{cases} A - BQ_k - 2Bx_k - c_k & \text{if} \quad Q_k + x_k < \frac{A}{B}, \\ -c_k & \text{if} \quad Q_k + x_k > \frac{A}{B}, \end{cases}$$

and this derivative does not exist if $Q_k + x_k = A/B$.

If for any firm $k$ it is the case that $A - c_k \le 0$, then $\partial\varphi_k/\partial x_k$ is always negative, so the best response of this firm is always zero, and hence entry for this firm is blocked. Hence such firms do not participate in production, and therefore we can ignore them in all further discussions. If for firm $k$, the capacity limit $L_k$ is sufficiently large, then with $A > c_k$, its monopoly quantity is $x_k^M = (A - c_k)/(2B)$, which can be obtained from the first order condition with $Q_k = 0$.

In order to determine the best response of the firms, consider firm $k$ and assume that the total production level $Q_k$ of the rest of the industry is fixed. Notice first that the best response of this firm cannot exceed $A/B - Q_k$, that is, the total industry output cannot be larger than the market saturation point. In contrast, assume that $x_k > A/B - Q_k$, then the price is zero, and by decreasing the value of $x_k$ by a small amount, the price will be still zero and the cost decreases. So the payoff of this firm would increase contradicting the assumption that $x_k$ is the firm's best response. Therefore with fixed values of $Q_k$ the best response of firm $k$ is selected in the interval $[0, \bar{L}_k]$ with $\bar{L}_k = \min\{L_k, A/B - Q_k\}$. If the capacity limits of the firms are sufficiently small, that is, when $\sum_{k=1}^{N} L_k \le A/B$, then the zero segment of the price function cannot occur, so $\bar{L}_k = L_k$ for all $k$ and $Q_k$. For the sake of simplicity in the following discussion we will assume that this is the case. Since $\varphi_k$ is strictly concave in $x_k$, the best response of firm $k$ is unique and is given as

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad \frac{\partial \varphi_k}{\partial x_k} \,|_{x_k=0} \le 0, \\ L_k & \text{if} \quad \frac{\partial \varphi_k}{\partial x_k} \,|_{x_k=L_k} \ge 0, \\ z_k^* & \text{otherwise}, \end{cases}$$

where $z_k^*$ is the solution of

$$\frac{\partial \varphi_k}{\partial x_k} = 0,$$

implying in the present case that

$$z_k^* = -\frac{1}{2}Q_k + \frac{A - c_k}{2B}. \tag{1.5}$$

Straightforward calculations reveal that

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad Q_k \geq (A - c_k)/B, \\ L_k & \text{if} \quad Q_k \leq (A - c_k - 2BL_k)/B, \\ -\frac{1}{2}Q_k + (A - c_k)/(2B) & \text{otherwise.} \end{cases} \tag{1.6}$$

In the case of two firms, when $Q_1 = x_2$ and $Q_2 = x_1$, we can illustrate graphically the existence of a unique equilibrium. Figure 1.1 shows the best response functions of the two firms in the situation where $L_1 < x_1^M$ and $L_2 < x_2^M$. If $L_1 \geq x_1^M$, then the vertical segment of $R_1(x_2)$ disappears and we simply have $R_1(0) = x_1^M$. A similar situation occurs when $L_2 \geq x_2^M$. The best replies intersect at a unique point, which is the Nash equilibrium. It can also be proved that with an arbitrary value of $N$, the oligopoly always has a unique equilibrium (see for example Sect. 2.1, and Okuguchi and Szidarovszky (1999)). If the market saturation point and the capacity limits are sufficiently large, then we can even compute the unique equilibrium.

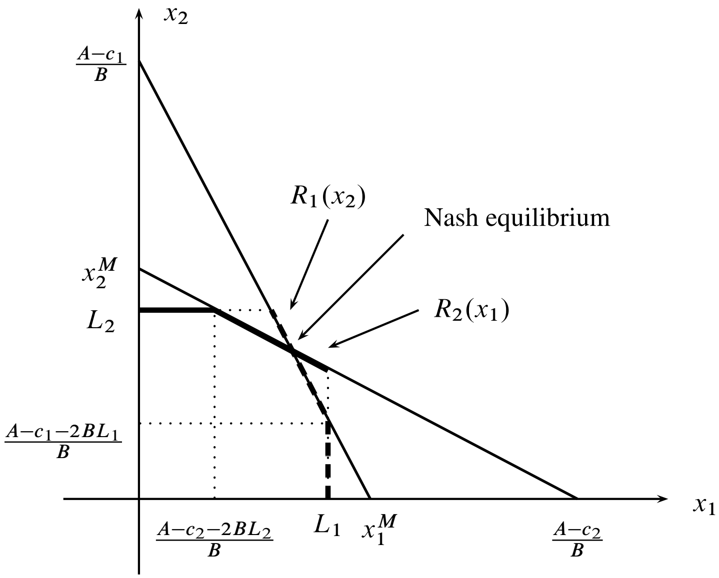


**Fig. 1.1** Example 1.1; the Cournot model in the case of duopoly ($N = 2$) with linear price and cost functions. The figure shows the reaction functions $R_1(x_2)$ (*dashed line*), $R_2(x_1)$ (*solid line*) and the unique equilibrium

Assume that all equilibrium outputs are positive, the other case can be examined similarly. The first order conditions imply that

$$\begin{aligned}\frac{\partial \varphi_k}{\partial x_k} &= \frac{\partial}{\partial x_k}[x_k(A - Bx_k - BQ_k) - (d_k + c_k x_k)] \\ &= A - 2Bx_k - BQ_k - c_k \\ &= A - Bx_k - BQ - c_k = 0, \end{aligned} \tag{1.7}$$

where $Q$ is the total output of the industry. So

$$x_k = \frac{A - BQ - c_k}{B} = \frac{A - c_k}{B} - Q. \tag{1.8}$$

By summing this last equation over all firms we obtain for $Q$ the single equation

$$Q = \frac{NA - \sum_{i=1}^{N} c_i}{B} - NQ, \tag{1.9}$$

implying that at the equilibrium

$$\bar{Q} = \frac{NA - \sum_{i=1}^{N} c_i}{(N+1)B}. \tag{1.10}$$

Notice that $\bar{Q} < A/B$, so the price is always positive. From (1.8) and (1.10) we can compute the equilibrium output levels of the firms as

$$\bar{x}_k = \frac{A - c_k}{B} - \frac{NA - \sum_{i=1}^{N} c_i}{(N+1)B} = \frac{A - (N+1)c_k + \sum_{i=1}^{N} c_i}{(N+1)B}. \tag{1.11}$$

The output levels in (1.11) can be an equilibrium only if they are all non-negative and below the corresponding capacity limits. The equilibrium price is then

$$\bar{p} = A - B\bar{Q} = \frac{A + \sum_{i=1}^{N} c_i}{N+1}.$$

At the equilibrium, the profit of firm $k$ is given by

$$\begin{aligned}\bar{\varphi}_k &= \bar{x}_k \bar{p} - (d_k + c_k \bar{x}_k) = \bar{x}_k \left( \frac{A + \sum_{i=1}^{N} c_i}{N+1} - c_k \right) - d_k \\ &= \frac{1}{(N+1)^2 B} \left( A - (N+1)c_k + \sum_{i=1}^{N} c_i \right)^2 - d_k. \end{aligned}$$

Notice that with zero fixed cost the equilibrium profit of firm $k$ is non-negative, and if $\bar{x}_k > 0$ and $d_k$ is sufficiently small, then $\bar{\varphi}_k$ is necessarily positive. If capacity limits are present and this "unconditional" equilibrium becomes infeasible, then the "conditional" equilibrium can still be computed, but cannot be represented by simple equations. Okuguchi and Szidarovszky (1999) discuss algorithms to compute such equilibria. ▼

If nonlinearity (which was in the form of capacity constraints in the above example) is introduced into the models, then usually numerical methods are required to compute the equilibrium in the general case. Analytical methods are available in only very special cases, for example by assuming symmetric or semi-symmetric firms. If all firms have identical capacity limits and cost functions, and their initial outputs are also the same, then the oligopoly is called *symmetric*. If $(N-1)$ firms are identical in this sense and one firm is different, then we have a *semi-symmetric* case. We will frequently make use of such special cases in later chapters.

*Example 1.2.* Assume again a linear price function $f(Q) = \max\{0, A - BQ\}$ but quadratic cost functions $C_k(x_k) = c_k x_k + e_k x_k^2$. The profit of firm $k$ now has the form

$$\varphi_k(x_1, \ldots, x_N) = \begin{cases} x_k(A - Bx_k - BQ_k) - (c_k x_k + e_k x_k^2) & \text{if} \quad x_k + Q_k \le \frac{A}{B}, \\ -(c_k x_k + e_k x_k^2) & \text{otherwise.} \end{cases}$$

For the sake of simplicity we assume again that $\sum_{k=1}^{N} L_k \le A/B$, that is, the zero segment of the price function cannot occur.

(i) Assume first that for all $k$, $0 < e_k$. Then the cost function is convex, so that marginal costs are increasing in $x_k$, and the profit is concave in $x_k$. Since

$$\frac{\partial \varphi_k}{\partial x_k} = A - 2Bx_k - BQ_k - c_k - 2e_k x_k,$$

the best response is unique and has the form

$$R_k(Q_k) = \begin{cases} 0 \quad \text{if} \quad A - BQ_k - c_k \le 0, \\ L_k \;\; \text{if} \quad A - 2BL_k - BQ_k - c_k - 2e_k L_k \ge 0, \\ (A - BQ_k - c_k)/(2(B + e_k)) \quad \text{otherwise,} \end{cases}$$

which is piece-wise linear, similar to the case of the previous example where both demand and cost were linear. Notice that if $A \le c_k$, then $R_k(Q_k) = 0$ regardless of the value of $Q_k$, so we assume that $A > c_k$ for all firms. In the case of duopoly the $x_1$ intercept of $R_1(x_2)$ is the monopoly output $x_1^M$ of firm 1, and the $x_2$ intercept of $R_2(x_1)$ is the monopoly output $x_2^M$ of firm 2. It can be proved (see Chap. 2) that there is always a unique Nash equilibrium in this case.

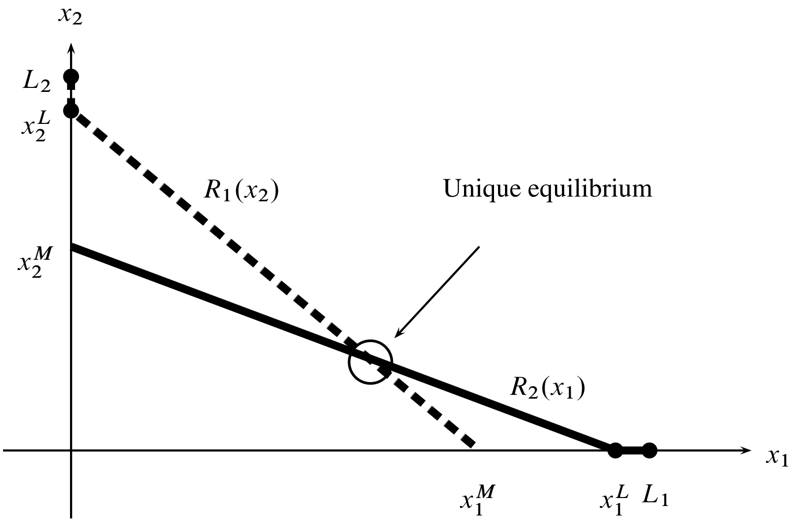


**Fig. 1.2** Example 1.2; the Cournot model with linear price function and quadratic cost function in the case of duopoly ($N = 2$). The reaction functions $R_1(x_2)$, $R_2(x_1)$ and the unique equilibrium. The figure illustrates case (ii) when $B^2 < 4(B + e_1)(B + e_2)$ and $x_k^L > x_k^M$, $k = 1, 2$

(ii) Assume next that for all $k$, $-B < e_k < 0$, then the cost function is concave, however $\varphi_k$ remains concave in $x_k$, so the best response remains the same as above. However, this case raises the possibility of multiple equilibria. Consider a duopoly ($N = 2$). Figure 1.2 depicts the reaction functions in the case where

$$B^2 < 4(B + e_1)(B + e_2),$$

that is when marginal costs are decreasing but not too strongly.[2] Furthermore, the "limit quantities" $x_k^L = (A-c_k)/B$, that is the corresponding quantity levels which guarantee that the other firm is kept out of the market, are larger than the monopoly quantities $x_k^M = (A-c_k)/(2(B+e_k))$. Under these conditions there is still a unique interior equilibrium given by

$$\begin{aligned} E &= (\bar{x}_1, \bar{x}_2) \\ &= \left( \frac{2(B + e_2)\,(A - c_1) - B(A - c_2)}{4(B + e_1)(B + e_2) - B^2}, \right. \\ &\qquad \left. \frac{2(B + e_1)\,(A - c_2) - B(A - c_1)}{4(B + e_1)(B + e_2) - B^2} \right) \end{aligned} \tag{1.12}$$

[2] This interpretation is based on the fact that the condition is satisfied if $-e_k$ ($k = 1, 2$) does not get too close to $B$.

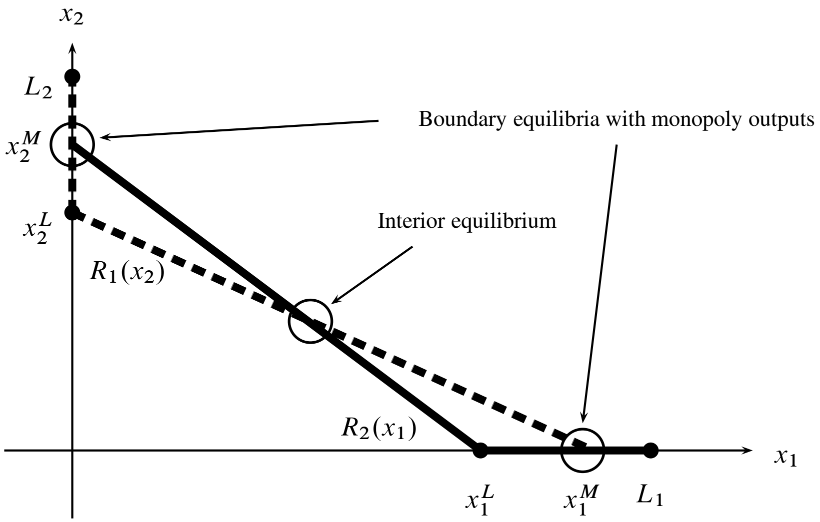


**Fig. 1.3** Example 1.2; the Cournot model with linear price function and quadratic cost function in the case of duopoly ($N = 2$). The figure shows case (ii) when $B^2 > 4(B + e_1)(B + e_2)$ and $x_k^L < x_k^M$, $k = 1, 2$. Three equilibria occur in this case

and the equilibrium profits are

$$\bar{\varphi}_k = (B + e_k)(\bar{x}_k)^2, \qquad k = 1, 2.$$

If in contrast

$$B^2 > 4(B + e_1)(B + e_2),$$

so that marginal costs are decreasing strongly, then the uniqueness of the equilibrium is no longer guaranteed. For example, Fig. 1.3 shows a case where

$$x_k^L = \frac{A - c_k}{B} < \frac{A - c_k}{2(B + e_k)} = x_k^M,$$

so that there is an interior equilibrium and there are also two boundary equilibria given by

$$E_1 = \left(\frac{A - c_1}{2(B + e_1)}, 0\right) \quad \text{and} \quad E_2 = \left(0, \frac{A - c_2}{2(B + e_2)}\right),$$

where we assume again that $A > c_k$ for both firms. Observe in addition, that $E_k$ includes the monopoly output for firm $k$ ($k = 1, 2$). At the boundary equilibrium $E_k$, the profit of firm $k$ is

$$(A - c_k)^2 / (4(B + e_k)) > 0.$$

In the borderline case, when

$$B^2 = 4(B + e_1)(B + e_2),$$

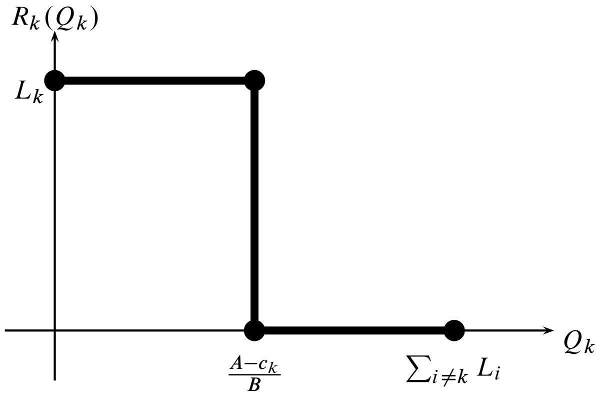


**Fig. 1.4** Example 1.2; the Cournot model with linear price function and quadratic cost function in the case of duopoly ($N = 2$). The figure shows the reaction function of a typical firm in case (iii) when $e_k = -B$. The number of equilibria may be $1, 3$ or infinite

the two straight lines either coincide or are parallel. Therefore there are either infinitely many equilibria, or a unique boundary equilibrium.

(iii) In the case where $e_k = -B$ for all $k$, the profit function assumes the linear form

$$\varphi_k = x_k(A - BQ_k - c_k),$$

therefore

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad A - BQ_k - c_k < 0, \\ L_k & \text{if} \quad A - BQ_k - c_k > 0, \\ \text{arbitrary} \quad x_k & \text{if} \quad A - BQ_k - c_k = 0. \end{cases}$$

We can assume again that $c_k < A$, otherwise $R_k(Q_k) = 0$ for all $Q_k$. This best response function is illustrated in Fig. 1.4 in the case when

$$\frac{A - c_k}{B} < \sum_{i \neq k} L_i.$$

In the case when the last inequality becomes an equality, the vertical segment moves to $Q_k = \sum_{i\neq k} L_i$. If however the above relation is violated with strict inequality, then $R_k(Q_k) = L_k$ for all $Q_k$. Depending on the values of $(A - c_k)/B$ and $L_k$, in the duopoly case the number of equilibria can be 1, 3 or infinite; Fig. 1.5 shows a case where three equilibria exist.

(iv) Assume finally that for all $k$, $e_k < -B$. In this case $\varphi_k$ is convex in $x_k$, so the best response is located at an endpoint of the feasible interval $[0, L_k]$ and is of the form

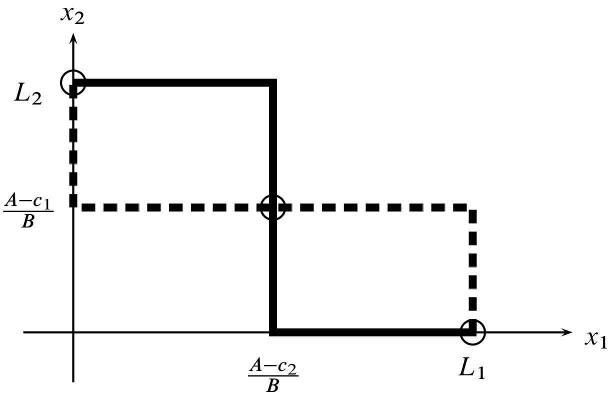


**Fig. 1.5** Example 1.2; the Cournot model with linear price function and quadratic cost function in the case of duopoly ($N = 2$). The figure shows case (iii) when $e_k = -B$, and there exist three equilibria

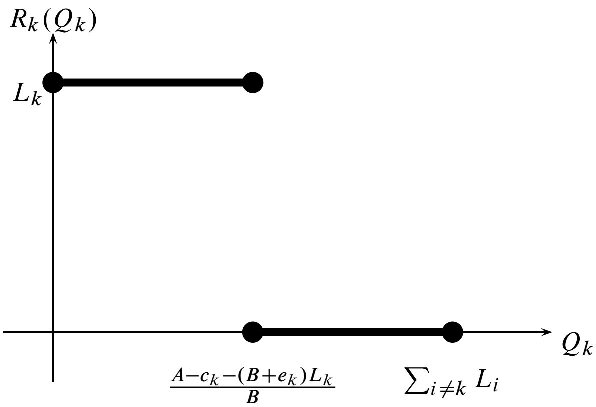


**Fig. 1.6** Example 1.2; the Cournot model with linear price function and quadratic cost function. The figure shows case (iv) when $e_k < -B$. The best response of the typical firm is determined by the fact that the profit function is linear in this case

$$R_k(Q_k) = \begin{cases} L_k & \text{if} \quad L_k(A - BL_k - BQ_k) - (c_k L_k + e_k L_k^2) > 0, \\ 0 & \text{if} \quad L_k(A - BL_k - BQ_k) - (c_k L_k + e_k L_k^2) < 0, \\ \{0; L_k\} & \text{if} \quad L_k(A - BL_k - BQ_k) - (c_k L_k + e_k L_k^2) = 0. \end{cases}$$

This function is illustrated in Fig. 1.6 in the case when

$$0 < (A - c_k - (B + e_k)L_k)/B < \sum_{i \neq k} L_i.$$

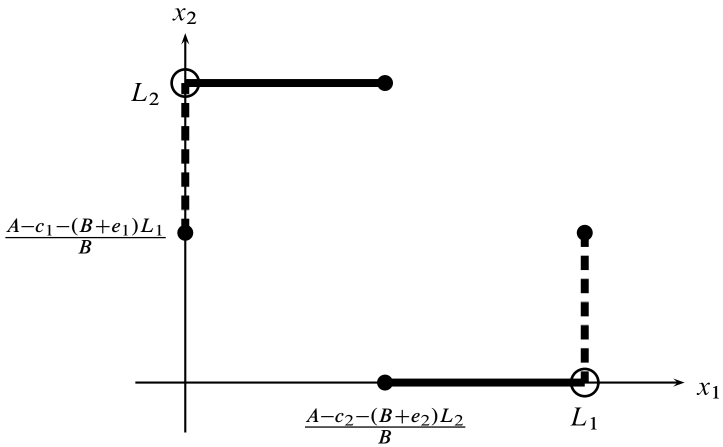


**Fig. 1.7** Example 1.2; the Cournot model with linear price function and quadratic cost function in the case of duopoly ($N = 2$). The figure shows case (iv) when $e_k < -B$ and the existence of two equilibria with convex profit functions

In the duopoly case ($N = 2$) the Nash equilibrium is at the intersection of the two best response functions. The number of equilibria can be 1, 2 or 3 depending on the relative order of magnitude of the values $(A - c_k - (B + e_k)L_k)/B$ and $L_l$ $(l \neq k)$. In Fig. 1.7 we show the case of two equilibria $(L_1, 0)$ and $(0, L_2)$.

Notice that in all cases at $x_k = 0$ the profit of firm $k$ is zero, therefore at the best response it has to be non-negative. Hence, at any equilibrium the profit of each firm is also non-negative. ▼

*Example 1.3.* Consider again the duopoly in which $N = 2$, furthermore take $L_1 = L_2 = 1.5$, $C_1(x_1) = 0.5x_1$, $C_2(x_2) = 0.5x_2$ and assume that the price function is given by

$$f(Q) = \begin{cases} 1.75 - 0.5Q & \text{if} \quad 0 \leq Q \leq 1.5, \\ 2.5 - Q & \text{if} \quad 1.5 \leq Q \leq 2.5, \\ 0 & \text{if} \quad Q \geq 2.5. \end{cases} \tag{1.13}$$

Notice that the cost functions are linear but that the price function is piece-wise linear. Because of the kink in the price function the profit functions are not differentiable at $Q = 1.5$. By calculating and comparing the left and right hand derivatives of the profit function, it is easy to show that there are infinitely many equilibria and they form the set

$$\bar{X} = \{(\bar{x}_1, \bar{x}_2) | 0.5 \leq \bar{x}_1 \leq 1, \quad 0.5 \leq \bar{x}_2 \leq 1, \quad \bar{x}_1 + \bar{x}_2 = 1.5\}.$$

Notice that the total output of the two firms is unique, satisfying $x_1 + x_2 = 1.5$, but this total output can be divided between the two firms in infinitely many different ways. At any equilibrium, $\bar{Q} = 1.5$, so the equilibrium price is $f(\bar{Q}) = 1$, and therefore the profit of firm $k$ is always positive, being given by

$$\varphi_k(\bar{x}_1, \bar{x}_2) = \bar{x}_k \cdot 1 - 0.5\bar{x}_k = 0.5\bar{x}_k. \qquad \blacktriangledown$$

*Example 1.4.* In this example we assume linear cost functions, $C_k(x_k) = c_k x_k$ with some positive constant $c_k$, and a quadratic price function where

$$f(Q) = \begin{cases} A - Q^2 & \text{if} \quad 0 \le Q \le \sqrt{A}, \\ 0 & \text{if} \quad Q > \sqrt{A}. \end{cases}$$

It is also assumed that $A > c_k$ for all $k$. Notice that at the best response of firm $k$ it is the case that $Q_k + x_k \le \sqrt{A}$, otherwise the value of $x_k$ can be decreased by a small amount, when the price is still zero and the cost would decrease. Therefore at the best response of all firms the total output has to be less than or equal to $\sqrt{A}$. For the sake of simplicity assume that $\sum_{k=1}^{N} L_k \le \sqrt{A}$, the other case can be discussed in a similar way. By assuming an interior optimum, the first order condition implies that

$$\frac{\partial}{\partial x_k}[x_k(A - (x_k + Q_k)^2) - c_k x_k] = A - 3x_k^2 - 4x_k Q_k - Q_k^2 - c_k = 0.$$

If $c_k \ge A$, then $\varphi_k$ is strictly decreasing in $Q_k$, so the best response of firm $k$ is always zero. Therefore we may assume that $c_k < A$ for all $k$. The solution of the above quadratic equation is

$$z_k^* = \frac{1}{3}\left(\sqrt{Q_k^2 + 3(A - c_k)} - 2Q_k\right).$$

Since the payoff function of firm $k$ is strictly concave in $x_k$, the best response assumes the form

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad z_k^* < 0, \\ L_k & \text{if} \quad z_k^* > L_k, \\ z_k^* & \text{otherwise.} \end{cases}$$

This function is illustrated in Fig. 1.8. Simple differentiation shows that $z_k^*$ is strictly decreasing and convex in $Q_k$. It can be proved that there is always a unique equilibrium. Since at $x_k = 0$ the profit of firm $k$ is zero, the profits at the best responses and therefore the equilibrium profits must be non-negative for all firms. In the case of an interior equilibrium the equilibrium quantities can be derived in closed-form. The first order condition may be rewritten as

$$A - Q^2 + x_k(-2Q) - c_k = 0,$$

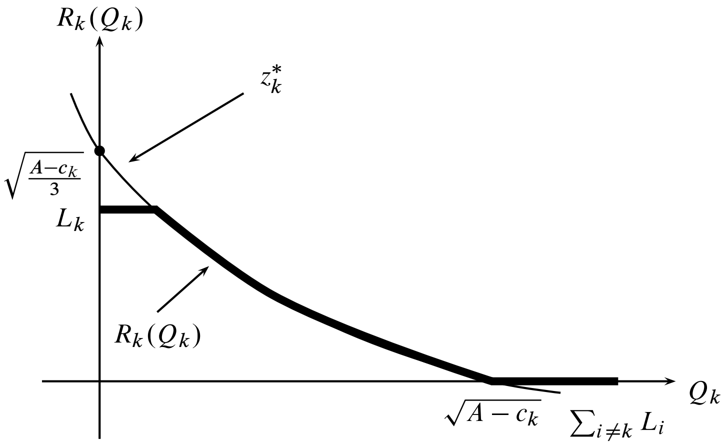


**Fig. 1.8** Example 1.4; the best response function (*thick line*) of a typical firm $k$ with a linear cost function and quadratic price function

implying that at the interior equilibrium

$$\bar{x}_k = \frac{A - \bar{Q}^2 - c_k}{2\bar{Q}}.$$

Summation over all $N$ firms yields

$$\bar{Q} = \frac{NA - N\bar{Q}^2 - \sum_{l=1}^{N} c_l}{2\bar{Q}},$$

and therefore

$$\bar{Q}^2 = \frac{NA - \sum_{l=1}^{N} c_l}{N+2}.$$

The individual quantities in equilibrium are then obtained as

$$\begin{aligned} \bar{x}_k &= \frac{1}{2\sqrt{(NA - \sum_{l=1}^{N} c_l)/(N+2)}} \left( A - \frac{NA - \sum_{l=1}^{N} c_l}{N+2} - c_k \right) \qquad (1.14) \\ &= \frac{2A + \sum_{l=1}^{N} c_l - (N+2)c_k}{2\sqrt{(N+2)(NA - \sum_{l=1}^{N} c_l)}}. \end{aligned}$$

For positivity of all equilibrium quantities, additional conditions are required, namely that

$$c_k < \frac{2A + \sum_{l\neq k} c_l}{N+1} \quad \text{for all } k.$$

Obviously, if firm $k$'s unit costs $c_k$ are too high (for a given number of firms $N$), production might not be feasible (so that firm $k$ offers $x_k = 0$). Furthermore, for increasing $N$ (and given unit costs) some (high-cost) firms might drop out of the market. The equilibrium price is given by

$$\bar{p} = \frac{2A + \sum_{l=1}^{N} c_l}{N+2} > 0$$

and the equilibrium profit of firm $k$ is

$$\bar{\varphi}_k = \frac{(2A + \sum_{l=1}^{N} c_l - (N+2)c_k)^2}{2(N+2)\sqrt{(N+2)(NA - \sum_{l=1}^{N} c_l)}}. \qquad \blacktriangledown$$

*Example 1.5.* Assume again linear cost functions, $C_k(x_k) = d_k + c_k x_k$, but isoelastic (hyperbolic) price function, $f(Q) = A/Q$. The form of the profit of firm $k$ depends on whether $Q_k$ is positive or zero. If $Q_k > 0$, then

$$\varphi_k(x_1, \ldots, x_N) = \frac{Ax_k}{x_k + Q_k} - (d_k + c_k x_k),$$

and if $Q_k = 0$, then

$$\varphi_k(x_1, \ldots, x_N) = \begin{cases} A - (d_k + c_k x_k) & \text{if} \quad x_k > 0, \\ -d_k & \text{if} \quad x_k = 0, \end{cases}$$

where we assume that firm $k$ cannot exit the market, so with zero production level it must face fixed costs. Notice that if $Q_k = 0$, then with any $x_k > 0$, the revenue of firm $k$ is always $A$. In this case firm $k$ has no best response and its interest is to select a very small output level, since the supremum of its profit occurs at $x_k = 0$. Assume next that $Q_k > 0$. In maximizing $\varphi_k$, the first order condition is

$$\frac{AQ_k}{(x_k + Q_k)^2} - c_k = 0.$$

Since $\varphi_k$ is strictly concave in $Q_k$, the best response of firm $k$ is

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad \sqrt{\frac{AQ_k}{c_k}} - Q_k \le 0, \\ L_k & \text{if} \quad \sqrt{\frac{AQ_k}{c_k}} - Q_k \ge L_k, \\ \sqrt{AQ_k/c_k} - Q_k & \text{otherwise.} \end{cases}$$

This function is illustrated in Fig. 1.9. We note that the best response is first increasing and then decreasing. This is in contrast to the examples considered previously, where the best responses were decreasing everywhere. Some authors consider

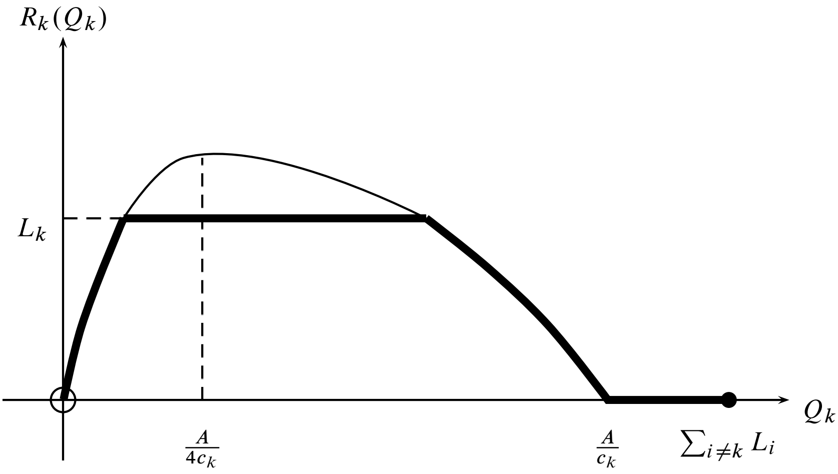


**Fig. 1.9** Example 1.5; the best response function (*thick line*) of a typical firm $k$ with a linear cost function and hyperbolic price function

$\bar{x}_1 = \cdots = \bar{x}_N = 0$ as a trivial equilibrium in a limiting sense.[3] In a non-trivial equilibrium, when $\bar{Q} > 0$, still some equilibrium outputs might be zero, when the marginal costs, $c_k$, for some firms are very large. By assuming that the value of $L_k$ is sufficiently large for all firms, the positive equilibrium can be computed as follows. Since for all $k$,

$$x_k = \sqrt{\frac{A(Q - x_k)}{c_k}} - (Q - x_k),$$

we have

$$c_k Q^2 = A(Q - x_k),$$

implying that

$$x_k = \frac{AQ - c_k Q^2}{A}.$$

Summing this equation over all $N$ firms, we obtain

$$Q = \frac{NAQ - Q^2 \sum_{k=1}^{N} c_k}{A}.$$

So the total output of all firms is

$$\bar{Q} = \frac{(N-1)A}{\sum_{k=1}^{N} c_k},$$

[3] See Agliari et al. (2005, 2006), Agliari (2006) and Matsumoto and Serizawa (2007).

and by substituting it into the above expression for $x_k$, the equilibrium output of firm $k$ is given by

$$\bar{x}_k = \frac{(N-1)A}{\sum_{l=1}^N c_l} - \frac{A(N-1)^2 c_k}{\left(\sum_{l=1}^N c_l\right)^2},$$

and the equilibrium profit of firm $k$ is given by

$$\bar{\varphi}_k = \frac{A\bar{x}_k}{\bar{Q}} - c_k\bar{x}_k - d_k = \left(\frac{\sum_{l=1}^N c_l}{N-1} - c_k\right)\bar{x}_k - d_k = A\left(1 - \frac{(N-1)c_k}{\sum_{l=1}^N c_l}\right)^2 - d_k.$$

In order to guarantee that all equilibrium outputs of the firms are positive, we have to assume that

$$c_k < \frac{\sum_{l \neq k} c_l}{N-2},$$

that is, the marginal costs cannot be too high. ▼

The examples above considered the case in which the cost function of a firm depends only on its own output. We will next present two particular examples including cost externalities, with linear price and cost functions, where the fixed costs are equal to zero and the marginal cost of each firm depends on the output of the rest of the industry.

*Example 1.6.* In the case of $N$ firms assume a linear price function $f(Q) = A - BQ$, and furthermore assume that the marginal cost of each firm is a function of the output of the rest of the industry, $M_k(Q_k)$. If zero fixed cost is assumed, then the cost function of firm $k$ is given as (see Howroyd and Russell (1984), Russell et al. (1986) and Furth (2009))

$$C_k(x_k, Q_k) = x_k M_k(Q_k),$$

so the profit of firm $k$ is

$$x_k(A - Bx_k - BQ_k) - x_k M_k(Q_k),$$

by assuming that $x_k + Q_k \leq A/B$. Notice that this function is strictly concave in $x_k$, so in the case of sufficiently small capacity limits there is a unique best response function given by

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad A - BQ_k - M_k(Q_k) \leq 0, \\ L_k & \text{if} \quad A - 2BL_k - BQ_k - M_k(Q_k) \geq 0, \\ z_k^* & \text{otherwise,} \end{cases}$$

where $z_k^*$ is the solution of the equation

$$A - 2Bz_k - BQ_k - M_k(Q_k) = 0$$

inside the interval $(0, L_k)$. That is,

$$z_k^* = \frac{A - BQ_k - M_k(Q_k)}{2B}.$$

We mention here that for an arbitrary value of $Q_k$, the profit of each firm $k$ is zero with $x_k = 0$, so the payoff at the best response also must be non-negative. Hence at any equilibrium the firms have non-negative profit values.

If $M_k(Q_k)$ is a linear function, then $z_k^*$ is also linear in $Q_k$, so $R_k(Q_k)$ is a piece-wise linear function similar to Example 1.1. If we assume that $M_k(Q_k)$ is a quadratic function, then $z_k^*$ is also quadratic in $Q_k$. Thus if we write

$$M_k(Q_k) = \alpha_k + \beta_k Q_k + \gamma_k Q_k^2,$$

then

$$z_k^* = \frac{(A - \alpha_k) + (-B - \beta_k)Q_k - \gamma_k Q_k^2}{2B}.$$

Let $\mu_k > 1$ be a given constant and select

$$\alpha_k = A, \quad \beta_k = -B(1 + 2\mu_k) \quad \text{and} \quad \gamma_k = 2B\mu_k,$$

then we have the relatively simple form

$$z_k^* = \mu_k Q_k(1 - Q_k).$$ ▼

*Example 1.7.* Consider again the oligopoly of the previous example with the only difference being that the marginal cost of each firm $k$ is a hyperbola of the form

$$M_k(Q_k) = \frac{c_k}{1 + \gamma_k Q_k}.$$

In this case $R_k(Q_k)$ has the same structure as in the previous example with

$$z_k^* = \frac{A - BQ_k - M_k(Q_k)}{2B} = \frac{1}{2B}\left(A - BQ_k - \frac{c_k}{1 + \gamma_k Q_k}\right).$$

In Chap. 3 we will give a detailed analysis of this example. ▼

In our last example we show an oligopoly for which no equilibrium exists.

*Example 1.8.* Consider the case of two firms, $N = 2$, with capacity limits $L_1 = L_2 = 0.5$, linear price function $f(Q) = 1 - Q$ with $Q = \sum_{k=1}^{2} x_k$, and discontinuous cost functions

$$C_k(x_k) = \begin{cases} 10 & \text{if} \quad x_k = 0, \\ 10x_k + 5 & \text{if} \quad 0 < x_k \le \frac{1}{2}. \end{cases} \tag{1.15}$$

The higher costs at zero reflect exit barriers, which do not occur when the firms start producing. We will show that this oligopoly has no equilibrium. On the contrary, assume that $(\bar{x}_1, \bar{x}_2)$ is an equilibrium. Assume first that $x_1 > 0$, then

$$\varphi_1(x_1, \bar{x}_2) = x_1(1 - x_1 - \bar{x}_2) - (10x_1 + 5) = -x_1^2 - (9x_1 + x_1\bar{x}_2 + 5)$$

with derivative

$$\frac{\partial \varphi_1}{\partial x_1}(x_1, \bar{x}_2) = -2x_1 - 9 - \bar{x}_2 < 0.$$

Therefore $\varphi_1$ is strictly decreasing in $x_1$. Assume next that $x_1 = 0$. Then $\varphi_1(0, \bar{x}_2) = -10$ with $\lim_{x_1 \to 0+} \varphi_1(x_1, \bar{x}_2) = 0 \cdot f(Q) - 5 = -5 > \varphi_1(0, \bar{x}_2)$ showing that at $\bar{x}_2$, firm 1 has no best response. Hence no equilibrium exists. ▼

## 1.2 Dynamic Adjustment Processes

In this section dynamic adjustment processes in the Cournot model will be introduced. If all firms simultaneously select the corresponding output levels of an equilibrium, then none of the firms can change unilaterally its output level and increase profit. So without coordination and cooperation between the firms, the output level of all firms will remain steady at the equilibrium levels. If the selected output levels do not form an equilibrium, then at least one firm is able to increase its profit by changing its output level unilaterally. Since the firms are rational, all firms will do the same. Since the firms change their output levels simultaneously, they cannot reach their best response levels, because the competitors simultaneously move away from their previously assumed output levels at the same time. In this way the firms usually would not reach an equilibrium, so output changes are again undertaken, and a dynamic process develops. The model of the resulting process depends on the assumed nature of the time scales and on the way the firms adjust output levels, which in turn depends on their expectation formation.

In the *discrete time* case let $t = 0, 1, 2 \cdots$ denote the time periods, then here we shall assume that in each time period each firm changes its output level to the best response based on its latest belief of the total production level of the rest of the industry. This process can be written as

$$x_k(t+1) = R_k\left(Q_k^E(t+1)\right), \tag{1.16}$$

where $Q_k^E(t+1)$ is the total output of the rest of the industry expected by firm $k$ for the next time period $t+1$. We emphasize here the fact that expectation is not meant in its probabilistic sense, rather it is a deterministic predicted value. The most simple expectation scheme is the one in which the firms use the latest available information,

$$Q_k^E(t+1) = \sum_{l \neq k} x_l(t), \tag{1.17}$$

which is sometimes called the *static*, or *naive*, or *Cournot expectation*.

The firms are also able to develop certain learning procedures based on earlier data. The most popular such learning scheme is obtained when the firms adjust their expectations *adaptively* according to

$$Q_k^E(t+1) = Q_k^E(t) + a_k \left( \sum_{l \neq k} x_l(t) - Q_k^E(t) \right), \tag{1.18}$$

with $a_k$ being a positive constant known as the *speed of adjustment* of firm $k$. It is usually assumed that $0 < a_k \leq 1$ for all $k$. The interpretation of this dynamic learning scheme is that, if firm $k$ underestimated (overestimated) the output of the rest of the industry in the previous time period, then in the next time period this firm wants to increase (decrease) its estimate. This increase (decrease) is represented by the second term, and the coefficient $a_k$ determines the speed (or rate) of adjustment. If the expectation of a firm were correct in the previous time period, then there would be no need to change the expectation, in this case the second term would be zero. Notice that the special case of $a_k = 1$ reduces to the static or Cournot expectation.

Mathematically, the dynamic process (1.16), together with naive expectations (1.17) form the $N$-dimensional dynamical system

$$x_k(t+1) = R_k \left( \sum_{l \neq k} x_l(t) \right) \qquad (k = 1, 2, \ldots, N), \tag{1.19}$$

to which we will refer as *best response dynamics with naive expectations*.

Under the adaptive expectations scheme (1.18), the dynamic process (1.16) becomes the $2N$-dimensional dynamical system

$$x_k(t+1) = R_k \left( a_k \sum_{l \neq k} x_l(t) + (1 - a_k) Q_k^E(t) \right), \tag{1.20}$$

$$Q_k^E(t+1) = a_k \sum_{l \neq k} x_l(t) + (1 - a_k) Q_k^E(t), \tag{1.21}$$

for $k = 1, 2, \ldots, N$. We will refer to this process as the *best response dynamics with adaptive expectations*.

In the latter formulation we have formally $2N$ state variables, however it is easy to show that the best response dynamics with adaptive expectations are actually driven by the $N$ expectation variables and the production outputs can be computed directly from them. In fact, for all $k$, (1.18) can be written as

$$Q_k^E(t+1) = a_k \sum_{l \neq k} x_l(t) + (1-a_k) Q_k^E(t) = a_k \sum_{l \neq k} R_l \left( Q_l^E(t) \right) + (1-a_k) Q_k^E(t). \tag{1.22}$$

The dynamic process now reduces to an $N$-dimensional dynamical system in the expected variables $Q_1^E(t), \ldots, Q_N^E(t)$, and at each time period $t$ the output of firm $k$ is given as

$$x_k(t) = R_k \left( Q_k^E(t) \right),$$

which is a static mapping from beliefs to realizations in the sense that both sides of the mapping are computed at the same time $t$.

In most industries any increase of the output level of any firm requires time, new hirings, purchase of new machinery, or sometimes even the opening up of a new plant. Therefore output changes are made gradually. For example, in the case of the dynamic process (1.19) instead of selecting the best response directly, the new output level of firm $k$ is selected somewhere in between the current level and the best response to ensure that the output level change occurs in the right direction. This concept of *partial adjustment towards the best response with naive expectations* can be described by the modified $N$-dimensional dynamical system

$$x_k(t+1) = a_k R_k \left( \sum_{l \neq k} x_l(t) \right) + (1-a_k) x_k(t), \tag{1.23}$$

for some $a_k \in (0, 1]$. In the case of $a_k = 0$ the output level would never change, therefore this value is excluded. Notice that in the case of $a_k = 1$, the partial adjustment towards the best response with naive expectations (1.23) reduces to best response dynamics with naive expectations (1.19).

In the special case of two firms ($N = 2$) both dynamical systems (1.22) and (1.23) have the common form

$$\begin{aligned} y_1(t+1) &= a_1 R_1 \left( y_2(t) \right) + (1-a_1) y_1(t), \\ y_2(t+1) &= a_2 R_2 \left( y_1(t) \right) + (1-a_2) y_2(t) \end{aligned}$$

with $y_1 = x_1$ and $y_2 = x_2$ in (1.23), and $y_1 = Q_2^E$, $y_2 = Q_1^E$ and $a_1$ and $a_2$ being interchanged in (1.22). If $N > 2$, then systems (1.22) and (1.23) are equivalent if $R_k \left( \sum_{l \neq k} y_l(t) \right) = \sum_{l \neq k} R_l \left( y_l(t) \right)$ holds for all $k$. In the symmetric case (when $R_k \equiv R$), this condition holds if $R(Q_k) = r Q_k$ with some constant $r$.

It is important to realize that dynamic adjustment processes of the kind considered above are defined on the action space $\Pi_{k=1}^N [0, L_k]$ and incorporate only the firms' quantity decision. In order to obtain economically feasible trajectories, we need to keep in mind the fact that prices (and profits) have to be non-negative in the long run, though it is possible (as we shall indeed find) that over some periods negative profits may occur. In some of the models we study it will be possible to ensure non-negative prices simply by selecting suitable parameter values. For example, for the $N$-firm oligopoly model with linear inverse demand function a sufficient condition for non-negative prices is $\sum_{k=1}^{N} L_k \leq A/B$ (see Example 1.1) and for the model with quadratic price function and linear costs, we can simply select $\sum_{k=1}^{N} L_k \leq \sqrt{A}$ (see Example 1.4).

If the time scales are *continuous*, then output changes are made continuously, without direct jumps to the best response levels. It is always assumed that in each time period the output level moves in a direction towards the best response. This concept is modeled by an $N$-dimensional system of ordinary differential equations of the form

$$\dot{x}_k(t) = a_k \left( R_k(\sum_{l \neq k} x_l(t)) - x_k(t) \right) \qquad (k = 1, 2, \ldots, N). \tag{1.24}$$

Here $a_k > 0$ is a given constant and also called the *speed of adjustment* of firm $k$. This is the continuous time counterpart of the discrete system (1.23), which is also called the partial adjustment dynamics.

*Example 1.9.* Consider again the case of linear oligopolies with linear inverse demand and linear cost functions, which was discussed earlier in Example 1.1. By ignoring the non-negativity condition of the outputs and assuming that $L_k = \infty$ for all $k$, the best reply of firm $k$ is given as (see (1.6))

$$R_k(Q_k) = -\frac{1}{2} Q_k + \frac{A - c_k}{2B}.$$

Since for all $k$, $R_k(Q_k)$ is linear with identical derivative, the dynamical systems (1.22) and (1.23) have the same coefficient matrix, so the asymptotic behavior of the discrete dynamics with adaptive expectations and with adaptive adjustments are equivalent. The dynamical system (1.23) for partial adjustment towards the best response can be written as

$$x_k(t+1) = a_k \left( -\frac{1}{2} \sum_{l \neq k} x_l(t) + \frac{A - c_k}{2B} \right) + (1 - a_k) x_k(t), \tag{1.25}$$

which is a linear system with coefficient matrix

$$\begin{pmatrix} 1 - a_1 & -\frac{a_1}{2} & \ldots & -\frac{a_1}{2} \\ -\frac{a_2}{2} & 1 - a_2 & \ldots & -\frac{a_2}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_N}{2} & -\frac{a_N}{2} & \ldots & 1 - a_N \end{pmatrix}.$$

In Chap. 2 (Theorem 2.1) we will see that the eigenvalues of this matrix lie inside the unit circle if and only if $a_k < 4$ for all $k$, and

$$\sum_{k=1}^{N} \frac{a_k}{4 - a_k} < 1.$$

In the case of linear systems local and global asymptotic stability are the same, so the equilibrium is globally asymptotically stable if and only if the above conditions are satisfied.

In the case of continuous time scales the dynamical system for partial adjustment (1.24) can be written as

$$\dot{x}_k(t) = a_k \left( -\frac{1}{2} \sum_{l \neq k} x_l(t) + \frac{A - c_k}{2B} - x_k(t) \right), \tag{1.26}$$

which is again a linear system with coefficient matrix

$$\begin{pmatrix} -a_1 & -\frac{a_1}{2} & \dots & -\frac{a_1}{2} \\ -\frac{a_2}{2} & -a_2 & \dots & -\frac{a_2}{2} \\ \vdots & \vdots & & \vdots \\ -\frac{a_N}{2} & -\frac{a_N}{2} & \dots & -a_N \end{pmatrix}.$$

In Chap. 2 (Theorem 2.2) we will see that all eigenvalues of this matrix always have negative real parts so the equilibrium is locally asymptotically stable. The linearity of the system implies that the Nash equilibrium is also globally asymptotically stable. ▼

Introducing the non-negativity conditions and the capacity limits into the model makes the best reply functions nonlinear. Nonlinearity can also occur by assuming nonlinear cost or price functions. Then the corresponding dynamical systems become nonlinear, and local asymptotic stability does not imply global asymptotic stability. This observation points to the need to perform detailed global analysis of the dynamical behavior. The next section will present the foundation of the relevant methodology.

In models (1.20)–(1.21), for the best response dynamics with adaptive expectations, and (1.23) and (1.24) for the dynamics of partial adjustment towards the best response with naive expectations, we have used simple linear adjustment rules. However these can be easily extended to the nonlinear case by introducing sign-preserving adjustment functions. A real-variable, real-valued function $\alpha : \mathbb{R} \to \mathbb{R}$ is called sign-preserving, if $\alpha(x)$ has the same sign as $x$, that is,

$$\alpha(x) \begin{cases} > 0 & \text{if } x > 0, \\ = 0 & \text{if } x = 0, \\ < 0 & \text{if } x < 0. \end{cases} \tag{1.27}$$

Assume now that for all $k$, $\alpha_k$ is a sign-preserving function, then the dynamical system (1.20)–(1.21) for the best response with adaptive expectations can be extended to

$$x_k(t+1) = R_k\left(Q_k^E(t) + \alpha_k\left(\sum_{l \neq k} x_l(t) - Q_k^E(t)\right)\right), \tag{1.28}$$

$$Q_k^E(t+1) = Q_k^E(t) + \alpha_k\left(\sum_{l \neq k} x_l(t) - Q_k^E(t)\right). \tag{1.29}$$

Similarly the discrete time dynamical system (1.23) for the dynamics of partial adjustment towards the best response with naive expectations becomes

$$x_k(t+1) = x_k(t) + \alpha_k\left(R_k\left(\sum_{l \neq k} x_l(t)\right) - x_k(t)\right), \tag{1.30}$$

whilst the continuous time dynamical system (1.24) for the same process becomes

$$\dot{x}_k(t) = \alpha_k\left(R_k\left(\sum_{l \neq k} x_l(t)\right) - x_k(t)\right). \tag{1.31}$$

Another important class of adjustment processes that has been investigated in the literature on dynamic oligopolies by many authors is that of the gradient adjustment process. This adjustment process is based on the observation that if for firm $k$ at a certain time period, $\partial\varphi_k/\partial x_k$ is positive, then it is in firm $k$'s interest to increase the output level, if $\partial\varphi_k/\partial x_k$ is negative, then the firm wants to decrease it, and if $\partial\varphi_k/\partial x_k = 0$, then firm $k$ believes that it is already at its maximum level, so it wants to maintain the same output level. This idea can be mathematically realized in the gradient adjustment processes

$$x_k(t+1) = x_k(t) + \alpha_k\left(\frac{\partial\varphi_k(x_1(t), \ldots, x_N(t))}{\partial x_k}\right) \quad (1 \leq k \leq N), \tag{1.32}$$

in discrete time and

$$\dot{x}_k(t) = \alpha_k\left(\frac{\partial\varphi_k(x_1(t), \ldots, x_N(t))}{\partial x_k}\right) \quad (1 \leq k \leq N), \tag{1.33}$$

in continuous time, where $\alpha_k$ is a sign-preserving function. Notice that dynamic processes based on best response functions require the solution of optimization problems in order to determine the best responses. In contrast gradient adjustment processes do not need the computation of best responses, rather they need only local information about the profit functions. Therefore the uniqueness of best responses is not an issue with gradient adjustment processes. Observe, however, that in the case of gradient adjustment, we need to check whether the obtained quantity is non-negative and also whether it is below the capacity limit.

Clearly the steady states of the dynamic processes (1.28)–(1.29), for the generalised best response with adaptive expectations, and (1.30)–(1.31) for the generalised partial adjustment towards the best response with naive expectations, are the *Nash equilibria*. However only interior equilibria can be the steady states of the gradient adjustment processes (1.32)–(1.33). Therefore boundary equilibria can be obtained as the limits of the trajectories as $t \to \infty$ only in special cases. The foregoing reasoning is based on the fact that a point is a steady state of best response based adjustment if and only if the output levels equal the best responses for all firms, that is, when they are at an equilibrium. However in the case of gradient adjustment a point is a steady state if and only if all partial derivatives are zero, which is not the case if the equilibrium lies on the boundary. Therefore even in the case of asymptotic stability the trajectory does not need to converge to the equilibrium, since the solutions of the first order conditions may lie outside the feasible region, so they are not necessarily steady states. This behavior may be regarded as a drawback of gradient adjustment processes.

*Example 1.10.* In the case of linear oligopoly, discussed in Example 1.9, we can calculate

$$\begin{aligned}\frac{\partial \varphi_k}{\partial x_k} &= \frac{\partial}{\partial x_k}\left\{x_k\left(A - Bx_k - B\sum_{l\neq k} x_l\right) - (c_k x_k + d_k)\right\}\\ &= A - 2Bx_k - B\sum_{l\neq k} x_l - c_k,\end{aligned}$$

so the gradient adjustment dynamical system (1.32) in discrete time with linear sign-preserving functions ($\alpha_k(x) = a_k x$ with $a_k > 0$) can be written as

$$\begin{aligned}x_k(t+1) &= x_k(t) + a_k\left(-2Bx_k(t) - B\sum_{l\neq k} x_l(t) + A - c_k\right)\\ &= 2Ba_k\left(-\frac{1}{2}\sum_{l\neq k} x_l(t) + \frac{A - c_k}{2B}\right) + (1 - 2Ba_k)x_k(t),\end{aligned}$$

which is the same as the dynamical system (1.25) for partial adjustment towards the best response, with $a_k$ replaced by $2Ba_k$. The continuous time system (1.33) with linear sign-preserving functions now assumes the form

$$\begin{aligned}\dot{x}_k(t) &= a_k\Big(-B\sum_{l\neq k} x_l(t) - 2Bx_k(t) + A - c_k\Big)\\ &= 2Ba_k\Big(-\frac{1}{2}\sum_{l\neq k} x_l(t) - x_k(t) + \frac{A-c_k}{2B}\Big),\end{aligned}$$

which is the same as system (1.26) with $a_k$ replaced by $2Ba_k$. ▼

The dynamical behavior of these adjustment process systems largely depends on the type and the parameters of the adjustment schemes as well as on the analytical properties of the best response functions, which in turn depend on the shapes of the price and cost functions.

There has been some criticism of the modeling of boundedly rational firms in dynamic oligopoly models using the previously discussed adjustment processes (see for example, Friedman (1977, 1982)). The essence of the criticism is that the firms ignore the fact that their current actions will have an impact on the future actions of the competitors (that is the limit of the adjustment process itself may not be an equilibrium of the repeated game). Therefore, it has been suggested that it would be more reasonable to assume that firms operating in markets over many time periods would seek to maximize a discounted stream of profits over a finite or infinite time horizon taking the strategic behavior of their competitors into account. Beside the fact that such an approach necessarily assumes a high degree of information and rationality on the part of the firms, one justification for the interest in models of the type studied in this book is given by more recent results demonstrating that myopic play is (approximately) optimal if the discount factor is very small (see Dana and Montrucchio (1986, 1987)). Moreover, non-equilibrium adjustment processes like the adjustment processes presented above can be shown to implicitly rely on a combination of "lock-in" and impatience, and this may serve as a further explanation for the players' myopia (see Fudenberg and Levine (1998), and Tirole (1988)). In any case, in this book we follow the argument that the kind of adjustment processes introduced above can "... be interpreted as a crude way of expressing the bounded rationality of agents" (Vives (1999), p. 49). Readers interested in dynamic games where players are more rational and forward-looking might want to consult the book by Dockner et al. (2000) who present a variety of models and summarize many interesting results. In this book we will mainly concentrate on best response based dynamic processes.

## 1.3 An Introduction to the Analysis of Global Dynamics

The purpose of this section is to introduce the main concepts and tools for the analysis of the global properties of a discrete time dynamical system. In order to do so we will use the example of a simple Cournot oligopoly with linear inverse demand and quadratic costs. This example has already been introduced in Sect. 1.1 (see Example 1.2), where we denoted the linear price function as $p = f(Q) = A - BQ$ and the quadratic production cost functions as $C_k(x_k) = c_k x_k + e_k x_k^2$. In order to avoid trivial best responses we assume again that $A > c_k$ for $k = 1, 2$.

### 1.3.1 A Cournot Duopoly Game

We first consider a duopoly game ($N = 2$), where the firms use partial adjustment towards the best response. The reaction functions in this case become

$$R_1(x_2) = \begin{cases} 0 & \text{if} \quad z_1^* < 0, \\ L_1 & \text{if} \quad z_1^* > L_1, \\ z_1^* & \text{otherwise}, \end{cases} \tag{1.34}$$

and

$$R_2(x_1) = \begin{cases} 0 & \text{if} \quad z_2^* < 0, \\ L_2 & \text{if} \quad z_2^* > L_2, \\ z_2^* & \text{otherwise}, \end{cases} \tag{1.35}$$

where $z_k^* = \frac{A - c_k - BQ_k}{2(B+e_k)}$ ($k = 1, 2$) with $Q_1 = x_2$ and $Q_2 = x_1$. If the duopolists partially adjust their quantities towards the best replies (based on naive expectations) and if the speeds of adjustment are constant, the dynamical system is generated by the iteration of the map $T_a : [0, L_1] \times [0, L_2] \to [0, L_1] \times [0, L_2]$, where

$$T_a : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1 R_1\left(x_2(t)\right) \\ x_2(t+1) = (1-a_2)x_2(t) + a_2 R_2\left(x_1(t)\right) \end{cases}, \tag{1.36}$$

with $0 < a_k \leqslant 1$. Recall from Sect. 1.2 that the best reply dynamics with naive expectations is obtained as a special case with $a_k = 1$ for $k = 1, 2$. We have also shown in Sect. 1.2 that in a duopoly partial adjustment towards the best response and the best reply dynamics with adaptive expectations are equivalent. Hence, the results obtained in this section also describe what happens if best reply dynamics with adaptive expectations are considered. Using (1.36) together with the steady state conditions $x_k(t+1) = x_k(t)$, $k = 1, 2$, leads to the equations $x_1 = R_1(x_2)$, $x_2 = R_2(x_1)$, which shows that the steady states of this dynamical system coincide with the Cournot–Nash equilibria of the underlying game and that they are located at the intersections of the reaction curves. Clearly, the steady states do not depend on the adjustment speeds $a_k$. As demonstrated in Sect. 1.1, the number of equilibria depends on the marginal costs. If marginal costs are increasing or even decreasing but not too strongly such that $B + e_k > 0$ and

$$B^2 < 4(B + e_1)(B + e_2), \tag{1.37}$$

then for $x_k^L > x_k^M$ $(k = 1,2)$ we have a unique interior equilibrium. The quantities at this interior equilibrium are given by

$$\begin{aligned} E &= (\bar{x}_1, \bar{x}_2) \\ &= \left( \frac{2(B+e_2)\,(A-c_1) - B(A-c_2)}{4(B+e_1)(B+e_2) - B^2}, \frac{2(B+e_1)\,(A-c_2) - B(A-c_1)}{4(B+e_1)(B+e_2) - B^2} \right). \end{aligned}$$

On the other hand, if $-B < e_k < 0$, $x_k^L < x_k^M$ $(k = 1,2)$ as before, but

$$B^2 > 4(B+e_1)(B+e_2), \tag{1.38}$$

then a situation of multiple equilibria might be obtained. This is the situation depicted in Fig. 1.3, where in addition to the interior equilibrium there also appear two boundary equilibria. The two coexisting boundary equilibria are given by

$$E_1 = (x_1^M, 0); \quad E_2 = (0, x_2^M),$$

where

$$x_1^M = \frac{A - c_1}{2(B+e_1)}; \quad x_2^M = \frac{A - c_2}{2(B+e_2)},$$

are the monopoly quantities.

Let us first try to give conditions for the global asymptotic stability of an equilibrium, which would also imply its uniqueness. We recall that an equilibrium is globally asymptotically stable if any trajectory starting from an initial condition in the strategy space converges to the equilibrium as $t \to \infty$. In the case of the model (1.36) the strategy space is given by the trapping region $\mathbb{D} = [0, L_1] \times [0, L_2]$. However the map (1.36), whose iteration gives the time evolution of the duopoly game, is not differentiable in the whole strategy space $\mathbb{D}$ because the reaction functions are piecewise differentiable functions defined by

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad Q_k \geq \frac{A-c_k}{B}, \\ L_k & \text{if} \quad Q_k \leq \frac{A-c_k-2(B+e_k)L_k}{B}, \\ (A - c_k - BQ_k)/\big(2(B+e_k)\big) & \text{otherwise.} \end{cases}$$

Accordingly, the phase space $\mathbb{D}$ can be subdivided into nine regions defined by the break points of the reaction functions (see Fig. 1.10), such that the map $T_a$ is differentiable (indeed linear in this case) inside each of them, it is defined differently in each region and it is not differentiable on the boundaries between the regions. Depending on the possible combination of the reaction functions the different components of the map are given by

$$T_a|_{\mathbb{D}^{(1)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1(A - c_1 - Bx_2)/\big(2(B+e_1)\big), \\ x_2(t+1) = (1-a_2)x_2(t) + a_2(A - c_2 - Bx_1)/\big(2(B+e_2)\big), \end{cases}$$

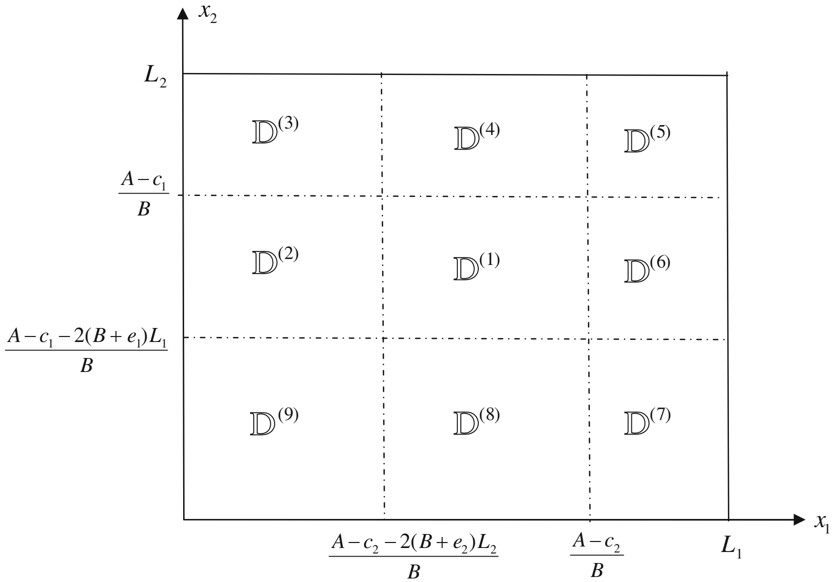


**Fig. 1.10** Phase space regions for the Cournot duopoly game where firms use partial adjustment towards the best response

$$T_a|_{\mathbb{D}^{(2)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1(A - c_1 - Bx_2)/\big(2(B+e_1)\big), \\ x_2(t+1) = (1-a_2)x_2(t) + a_2 \cdot L_2, \end{cases}$$

$$T_a|_{\mathbb{D}^{(3)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1 \cdot 0, \\ x_2(t+1) = (1-a_2)x_2(t) + a_2 \cdot L_2, \end{cases}$$

$$T_a|_{\mathbb{D}^{(4)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1 \cdot 0, \\ x_2(t+1) = (1-a_2)x_2(t) + a_2(A - c_2 - Bx_1)/(2(B+e_2)), \end{cases}$$

$$T_a|_{\mathbb{D}^{(5)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1 \cdot 0, \\ x_2(t+1) = (1-a_2)x_2(t) + a_2 \cdot 0, \end{cases}$$

$$T_a|_{\mathbb{D}^{(6)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1(A - c_1 - Bx_2)/\big(2(B+e_1)\big), \\ x_2(t+1) = (1-a_2)x_2(t) + a_2 \cdot 0, \end{cases}$$

$$T_a|_{\mathbb{D}^{(7)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_2 \cdot L_1, \\ x_2(t+1) = (1-a_2)x_2(t) + a_2 \cdot 0, \end{cases}$$

$$T_a|_{\mathbb{D}^{(8)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1 \cdot L_1, \\ x_2(t+1) = (1-a_2)x_2(t) + a_2(A - c_2 - Bx_1)/\big(2(B+e_2)\big), \end{cases}$$

$$T_a|_{\mathbb{D}^{(9)}} : \begin{cases} x_1(t+1) = (1-a_1)x_1(t) + a_1 \cdot L_1, \\ x_2(t+1) = (1-a_2)x_2(t) + a_2 \cdot L_2. \end{cases}$$

The derivative of the best response function of firm $k$ is either zero or $-B/(2(B+e_k))$, or does not exist in the cases when $Q_k = (A - c_k)/B$ and $Q_k = (A - c_k - 2(B + e_k)L_k)/B$.

Hence we need to consider the four different Jacobian matrices given by

$$\boldsymbol{J}^{(1)} = \begin{pmatrix} 1-a_1 & -\frac{a_1 B}{2(B+e_1)} \\ -\frac{a_2 B}{2(B+e_2)} & 1-a_2 \end{pmatrix}; \quad \boldsymbol{J}^{(2)} = \boldsymbol{J}^{(6)} = \begin{pmatrix} 1-a_1 & -\frac{a_1 B}{2(B+e_1)} \\ 0 & 1-a_2 \end{pmatrix};$$

$$\boldsymbol{J}^{(4)} = \boldsymbol{J}^{(8)} = \begin{pmatrix} 1-a_1 & 0 \\ -\frac{a_2 B}{2(B+e_2)} & 1-a_2 \end{pmatrix};$$

$$\boldsymbol{J}^{(3)} = \boldsymbol{J}^{(5)} = \boldsymbol{J}^{(7)} = \boldsymbol{J}^{(9)} = \begin{pmatrix} 1-a_1 & 0 \\ 0 & 1-a_2 \end{pmatrix}.$$

Select a diagonal matrix $\boldsymbol{P} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ with $x > 0$, then the row norms of these Jacobians generated by the matrix $\boldsymbol{P}$ are bounded by the row norm of the matrix

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-a_1 & \frac{a_1 B}{2(B+e_1)} \\ \frac{a_2 B}{2(B+e_2)} & 1-a_2 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-a_1 & \frac{a_1 B x}{2(B+e_1)} \\ \frac{a_2 B}{2(B+e_2)x} & 1-a_2 \end{pmatrix}, \tag{1.39}$$

which is below one if and only if

$$1 - a_1 + \frac{a_1 B x}{2(B + e_1)} < 1,$$

and

$$1 - a_2 + \frac{a_2 B}{2(B + e_2)x} < 1.$$

Since we assume that $0 < a_k \leq 1$ $(k = 1, 2)$, these relations can be rewritten as

$$\frac{B}{2(B + e_2)} < x < \frac{2(B + e_1)}{B},$$

and a feasible $x$ exists if and only if $B^2 < 4(B + e_1)(B + e_2)$.

Hence under this condition the equilibrium is unique and is globally asymptotically stable regardless of whether it is interior or not. (See Appendix B, Theorem B.3 for the relevant theoretical background.)

Next we will examine the local asymptotic stability of an interior steady state $E$. Let us consider the Jacobian matrix evaluated at the steady state,

$$\boldsymbol{J} = \begin{pmatrix} 1-a_1 & -a_1 \frac{B}{2(B+e_1)} \\ -a_2 \frac{B}{2(B+e_2)} & 1-a_2 \end{pmatrix}.$$

The characteristic equation of this Jacobian is given by $\lambda^2 + p\lambda + q = 0$, where $p = -2 + a_1 + a_2$ and $q = (1 - a_1)(1 - a_2) - a_1 a_2 B^2 / (4(B + e_1)(B + e_2))$. The necessary and sufficient conditions for the eigenvalues to be located inside the unit circle, which are conditions for the local asymptotic stability of the interior Nash equilibrium $E$, are given by the inequalities (see Appendix F, Lemma F.1)

$$1 + p + q > 0 \quad , \quad 1 - p + q > 0 \quad , \quad q < 1. \tag{1.40}$$

These inequalities, respectively, reduce to

$$\begin{aligned}
\frac{B^2}{4(B + e_1)(B + e_2)} &< 1, \\
\frac{B^2}{4(B + e_1)(B + e_2)} &< 1 + 2\frac{2 - a_1 - a_2}{a_1 a_2}, \\
\frac{B^2}{4(B + e_1)(B + e_2)} &> 1 - \frac{a_1 + a_2}{a_1 a_2}.
\end{aligned}$$

Observe that the first stability condition coincides with condition (1.37) under which this is the only equilibrium and so is globally asymptotically stable. The other conditions do not affect the stability properties, because the second condition is implied by the first one (since $0 < a_k \leq 1$) and the last condition is always satisfied (since the left hand side is positive, whereas the right hand side is negative). If $B^2 > 4(B + e_1)(B + e_2)$, then the interior equilibrium is unstable. This is the situation in case (ii) of Example 1.2, where we might have three equilibria with an unstable interior equilibrium.

Consider now the case shown in Fig. 1.3 and the monopoly equilibrium $(0, x_2^M)$. In the neighborhood of this equilibrium $x_2^L < x_2 < L_2$, so $R_1(x_2) = 0$. Furthermore $x_1 = 0$ or a small positive value. Notice that the segments where $R_1(x_2) = L_1$, or $R_2(x_1) = L_2$ are empty, which implies that the sets $\mathbb{D}^{(k)}$ for $k = 3, 2, 9, 8, 7$ are also empty. Therefore any point in a small neighborhood of the equilibrium $(0, x_2^M)$ is in the region $\mathbb{D}^{(4)}$ where the Jacobian matrix is

$$\begin{pmatrix} 1 - a_1 & 0 \\ -\frac{a_2 B}{2(B+e_2)} & 1 - a_2 \end{pmatrix}. \tag{1.41}$$

Let

$$\boldsymbol{P} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \tag{1.42}$$

be a diagonal matrix with $x > 0$. Then the row norm generated by this matrix is bounded by the row norm of the matrix

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - a_1 & 0 \\ \frac{a_2 B}{2(B+e_2)} & 1 - a_2 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.43}$$

which is below one if

$$1 - a_2 + \frac{a_2 B}{2(B + e_2)x} < 1,$$

since $0 < a_k \leq 1$ for $k = 1, 2$. This relation can be rewritten as

$$x > \frac{B}{2(B + e_2)},$$

so a feasible positive $x$ exists. From the local stability result of Appendix B we conclude that the monopoly equilibrium $(0, x_2^M)$ is locally asymptotically stable. The stability of the other monopoly equilibrium $(x_1^M, 0)$ can be proved similarly.

This provides a first conclusion with regard to the equilibrium selection problem, because even if we obtain three Nash equilibria, from an evolutionary perspective a stability argument suggests that the interior equilibrium will not be selected. It remains an open question, however, as to which one of the two monopoly equilibria is more likely to be observed in the long run. The situation is even more intricate, since in addition to the two asymptotically stable boundary equilibria, in the strategy space another attracting set might coexist. This can be demonstrated by considering the best reply dynamics obtained for $a_k = 1, k = 1, 2$. In the case when $x_2^M > (A - c_1)/B$ and $x_1^M > (A - c_2)/B$ we have $(R_1(0), R_2(0)) = (x_1^M, x_2^M)$ and $\left(R_1(x_2^M), R_2(x_1^M)\right) = (0, 0)$. Therefore, under best reply dynamics the periodic cycle $C_2 = \left\{(0, 0) ; \left(x_1^M, x_2^M\right)\right\}$ coexists with the two stable monopoly equilibria. It is also easy to see that $C_2$ is stable, so it may even occur that an adjustment process fails to converge towards any Nash equilibrium in the long run. In such a situation, where several attractors coexist, the question of which attractor will be reached in the long run crucially depends on the initial conditions and the observed outcome becomes path dependent. Each of these long run outcomes has its own basin of attraction (see Appendix C for definitions of these concepts from the qualitative theory of dynamical systems) and any external random factor (a so-called "historical accident") that causes a displacement of some of the initial outputs may cause the trajectory to move across a basin boundary and, consequently, it will converge to a different attractor.

We can shed some light on this issue by using a mixture of analytical, geometrical and numerical methods, an approach which is typically used in the study of the global dynamical properties of nonlinear systems of dimension greater than one (see for example Mira et al. (1996), Brock and Hommes (1997) and Puu (2003)).

To get a better feeling for the global dynamics of our duopoly game where firms use partial adjustment towards the best response, we numerically compute the basins of attraction for the coexisting attractors. Let the reservation price be $A = 450$ and the slope of the linear inverse demand function be $B = 30$. For the sake of simplicity, we consider identical firms with cost parameters $c_1 = c_2 = c = 275$ and $e_1 = e_2 = e = -17$, so that production costs are increasing, but marginal costs are decreasing. (Similar values were chosen by Cox and Walker (1998) in an experimental setup). In order to guarantee non-negative prices, we select $L_1 = L_2 = 7.5$, which ensures that $L_1 + L_2 \leq A/B$. For these parameter values condition (1.38) is fulfilled and the interior equilibrium is unstable. In addition, $A - c_k - 2(B_k + e_k)L_k < 0$ implying that the output space $[0, L_1] \times [0, L_2]$ is divided into only four regions rather than the nine shown in Fig. 1.10).

- In Fig. 1.11a the basins of attraction of $E_1$, $E_2$, and the coexisting 2-cycle $C_2$ are shown for the best reply dynamics, namely for $a_1 = a_2 = 1$. The basin of attraction of $E_1$ is represented by the light-grey region, the basin of $E_2$ by the dark-grey region, and the basin of the cycle $C_2$ by the white region. The peculiar rectangular-shaped structure of the basins is related to the particular structure of the best reply process, $x_1(t+1) = R_1(x_2(t))$, $x_2(t+1) = R_2(x_1(t))$, where next period's output of firm $i$ only depends on the current output of the other firm. This implies that the eigenvectors associated with the unstable equilibrium $E$ (that belongs to the basin boundaries) are parallel to the coordinate axes. Moreover, the map which generates the dynamics transforms vertical lines into horizontal lines and vice versa. Hence, the invariant sets associated with the unstable node $E$, that form the boundaries of the basins, are formed by vertical and horizontal lines (on this point see also Bischi et al. (2000b)).
- If the speeds of adjustment are smaller than 1, important differences can be observed in the global dynamics. For example, Fig. 1.11b has been obtained with $a_1 = 0.97$, $a_2 = 0.98$, leaving all the other parameters unchanged. Now the stable 2-cycle has both periodic points characterized by positive coordinates, namely $C_2 = \{(0.19, 0.13); (6.39, 6.38)\}$, and the structure of the basins is different, in particular the basin of the cycle $C_2$ is smaller. The rectangular shape of the basins is lost since in the case of partial adjustment the eigenvectors associated with $E$ are no longer parallel to the coordinate axes.
- If the speeds of adjustment are even further decreased, the basin of the cycle $C_2$ shrinks; see Fig. 1.11c obtained with $a_1 = 0.93$, $a_2 = 0.95$. The periodic points of $C_2$ approach the boundary of its basin and after a contact with such a boundary, the cycle $C_2$ becomes unstable. As a consequence, the whole strategy space is shared by the basins of the two asymptotically stable boundary Nash equilibria $E_1$ and $E_2$, as depicted in Fig. 1.11d obtained with $a_1 = 0.9$, $a_2 = 0.92$.

Our analysis suggests the following insights. First, the basins of the Nash equilibria $E_1$ and $E_2$ are always simply connected. We emphasize this fact since later on we will encounter examples where the basins will not have such a simple structure. Second, whereas the local asymptotic stability of the boundary Nash equilibria does not depend on the adjustment speeds, the shape of the basins changes significantly when adjustment speeds become smaller. If the players' speeds of adjustment are lower, then the size of the basins of the equilibria is larger. As far as local asymptotic stability is concerned, it is well-known in the literature that decreasing the speeds of adjustments usually stabilizes the system (see for instance Fisher (1961), McManus and Quandt (1961) and some results to be presented in Chap. 2). Here, however, we emphasize that (in the present example) this also holds for the global dynamics. Finally, since the firm with the smaller adjustment speed has the larger basin, this firm is more likely to achieve the role of the monopolist, if initial production quantities are selected randomly from a close to uniform distribution.

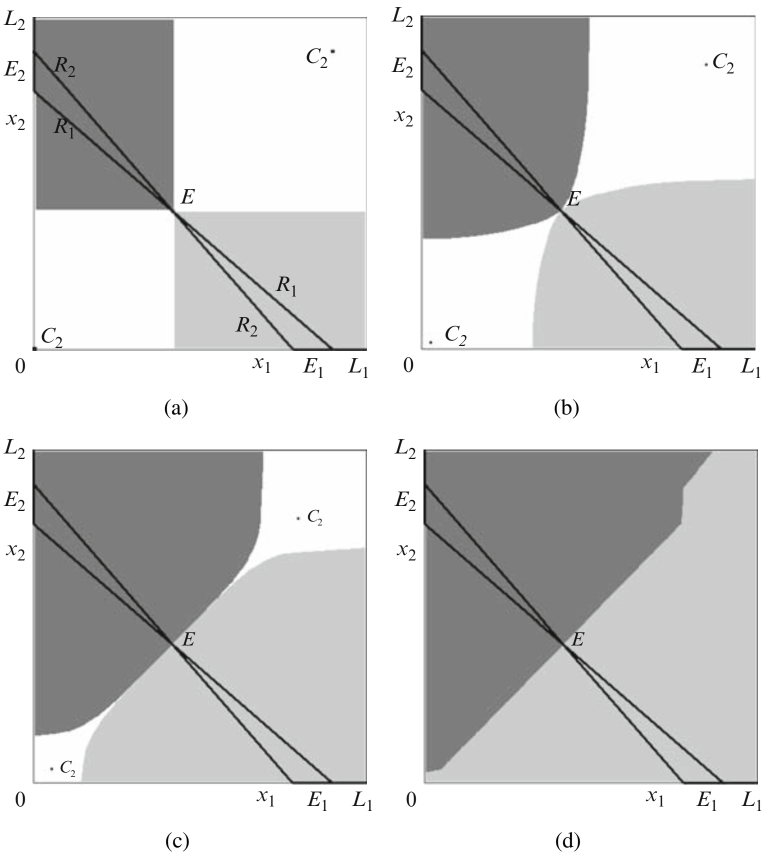


**Fig. 1.11** Basins of attraction for the Cournot duopoly when firms use partial adjustment towards the best response with linear demand/quadratic cost. *Light grey* basin of $E_1$; *dark grey* basin of $E_2$; *white* basin of the 2-cycle $C_2$. (**a**) Full adjustment, $a_1 = a_2 = 1$. The basins are rectangular. (**b**) Partial adjustment, $a_1 = 0.97$, $a_2 = 0.98$. The basins lose their rectangular shape. (**c**) Partial adjustment, $a_1 = 0.93$, $a_2 = 0.95$. The basin of $C_2$ shrinks. (**d**) Partial adjustment, $a_1 = 0.9$, $a_2 = 0.92$. The 2-cycle $C_2$ has become unstable, and its basin has disappeared

As a final remark we note that although the cyclic outcome $C_2$ is an attractor from a mathematical point of view, it has several shortcomings as a potential description of real-world economic behavior. First, whereas convergence to a steady state implies that the players' naive expectations are fulfilled at least in the long run, a sustained low-periodic oscillation implies that the players' expectations are permanently wrong. It seems plausible that in such a situation the players would learn how to improve their forecasts. Second, although profits are always positive in all Nash equilibria, this is not necessarily true in general for the cycle $C_2$. As an example consider again the best reply dynamics, where $C_2 = \{(0,0)\,;\,(x_1^M, x_2^M)\}$. The corresponding profits along the 2-cycle are $\varphi_k(0,0) = 0$ for firm $k$, with

$\varphi_1\left(x_1^M, x_2^M\right) = (A - c_1)\left[e_2(A - c_1) + B(c_2 - c_1)\right]/(4(B + e_1)(B + e_2))$, and $\varphi_2\left(x_1^M, x_2^M\right) = (A - c_2)\left[e_1(A - c_2) + B(c_1 - c_2)\right]/ \ (4(B + e_1)(B + e_2))$. This shows that for at least one of the firms, profits are negative along the cycle. Moreover, if $-B < e_k < 0$ and $c_1 = c_2$, then we have negative profits for both firms, a situation which is not sustainable for any firm. As a consequence of these considerations, what our analysis of the global dynamics reveals is that for some initial production choices an economically infeasible situation will emerge for the firms. Notice that this important result can only be obtained through a global study of the structure of the basins of attraction.

We also would like to draw the reader's attention to a global bifurcation which is responsible for the drastic change in the dynamics obtained in this simple duopoly model. In a situation where marginal costs are decreasing strongly and $x_k^M < x_k^L$, we obtain three coexisting attractors: two boundary equilibria and a 2-cycle. Notice that the limiting quantities $x_k^L$ are located on a line where the map is not differentiable. Consider now what happens if marginal costs increase. At a certain point, a boundary equilibrium $x_k^M$ will collide with $x_k^L$, and if marginal costs are increased even further, then the interior equilibrium becomes globally stable. This is actually a first example of a border collision bifurcation, a global bifurcation occurring whenever a qualitative change in the phase diagram (that is, creation/destruction of invariant sets and/or stability change of existing ones) is due to a contact (and crossing) of an invariant set with a border where the map is not differentiable separating regions where it is differentiable. In this case the boundary that separates regions $\mathbb{D}^{(5)}$ and $\mathbb{D}^{(1)}$ is the one involved in the contact, and such a border is due to the presence of non-negativity constraint. This kind of global (or contact) bifurcations, specific to piece-wise differentiable dynamical systems, will be examined in more detail in Chap. 2, in particular in Examples 2.3 and 2.4.

### *1.3.2 A Cournot Oligopoly Game*

In his seminal paper, Theocharis (1960) studied the asymptotic stability of the Cournot–Nash equilibrium under discrete-time best reply dynamics with naive expectations. For this quantity-setting model with linear demand and linear costs, he found that the (unique) equilibrium is asymptotically stable only in the case of two competitors. It is marginally stable (see definition (A.1) in Appendix A) for three firms and unstable for more than three firms. Among others, McManus and Quandt (1961) and Fisher (1961) demonstrated that this result depends on the type of adjustment process the firms use to determine their production quantities. They showed that for certain adjustment processes in continuous-time the equilibrium is stable no matter what the number of firms is. These facts will be later discussed in Chap. 2. Despite this result Fisher (1961, p.125) notes that "... the tendency to instability does rise with the number of sellers for most of the processes considered". These early papers gave rise to a lively discussion that has endured until the present day. One of the main topics in this body of literature is the relation between the following issues: the quasi-competitiveness of the economy, that is the question as to whether output increases and the market price decreases with an increasing number of firms in the industry; the asymptotic stability of the equilibrium if entry occurs; the question as to whether perfect competition is obtained in the limit as the number of competitors is increased. The interested reader should consult for example Frank (1965), Ruffin (1971), Howrey and Quandt (1968), Okuguchi (1976), or more recently, Seade (1980) and Amir and Lambson (2000) to get an impression of the variety of interesting results obtained concerning this issue. In this section we focus on asymptotic stability issues and we try to answer the question: is local asymptotic stability obtained when the number of firms increases? Furthermore, we also address the topic of global dynamics, that is we look at the changes in the basins of attraction of the stable equilibria. Clearly, a discussion of these issues becomes more complicated when the model is nonlinear, since increasing the number of players means increasing the dimension of the dynamical system. This is so since such increases lead to greater complexity in the dynamics of nonlinear systems, whereas in the case of linear systems no new dynamic phenomena arise.

In order to keep the mathematical analysis tractable, but at the same time to also shed some light on the relation between asymptotic stability and the number of firms, in what follows we will consider both the symmetric and semi-symmetric models. Recall that in the *symmetric case* it is assumed that all firms are identical, so that they have identical cost functions and all firms start from the same initial production quantities. Since the cost and demand parameters are identical for all firms, the reaction functions $R_k$ will be identical, say $R_k = R$ for each $k$. Consequently, the quantities will be identical for all periods, and the dynamics are governed by a 1-dimensional system. If we let $x(t)$ denote the common output of the representative firm, then the one-dimensional model in the symmetric case is obtained by setting $Q_k = (N-1)x$ for each $k$. It is worth noting that the symmetric case may be structurally unstable, that is the outcome obtained for the representative firm in the symmetric case may be completely different from the outcome of the model with almost identical, but nevertheless heterogeneous firms (the firms might differ in their production costs or might select slightly different initial quantities). Therefore, the insights obtained from the symmetric model need to be accepted with some caution. In order to derive some results which can be compared with the existing literature, we reconsider the partial adjustment towards the best response process given by (1.23).

The symmetric case is obtained if we assume $N$ players with identical quadratic cost functions (as in Example 1.2), that is $c_1 = c_2 = \cdots = c_N = c$ and $e_1 = e_2 = \cdots = e_N = e$, identical adjustment speeds, that is $a_1 = a_2 = \ldots, a_N = a$, and identical capacity limits $L_1 = L_2 = \cdots = L_N = L$. It is also assumed that $B + e > 0$, so the payoff functions of the firms are strictly concave in their strategies. Then from (1.23) the 1-dimensional model which summarizes the common behavior of all identical firms starting from identical initial condition $x_1(0) = x_2(0) = \cdots = x_N(0) = x(0)$ is

$$x(t+1) = T(x(t)) \equiv (1-a)\, x(t) + aR\left((N-1)\, x(t)\right),$$

where (see the reaction function in case (i) of Example 1.2)

$$R((N-1)x) = \begin{cases} 0 & \text{if} \quad z^* < 0, \\ L & \text{if} \quad z^* > L, \\ z^* & \text{otherwise,} \end{cases}$$

with $z^* = \big(A - c - B(N-1)x\big)/\big(2(B+e)\big)$.

Observe that the number of firms $N$ enters as a parameter, so we can study the stability conditions as $N$ is increased. The positive equilibrium is given by

$$\bar{x} = \frac{A-c}{B(N+1)+2e}$$

and the map $T$ is a contraction provided that $|T'(x)| < 1$, that is

$$0 < a\frac{BN+B+2e}{2(B+e)} < 2.$$

This implies that the positive equilibrium is always asymptotically stable for sufficiently small values of the adjustment speed $a$. Moreover, given $0 < a \leq 1$, asymptotic stability is obtained for

$$N < \frac{(4-a)\,B + 2\,(2-a)\,e}{aB}.$$

In the case of best reply dynamics, $a = 1$, the stability condition reads $N < (3B + 2e)/B$. In the case of linear costs, $e = 0$, we obtain the result by Theocharis stating that asymptotic stability is obtained for $N < 3$.

In the *semi-symmetric* case $(N-1)$ firms are assumed to be identical, whereas one firm differs with regard to its production costs and/or initial production quantity. Let firms $2, \dots, N$ be identical, then their production choices will coincide in each period, that is $x_k = x_2$ for all $k \geq 2$. Let us denote the production quantity of firm 1 by $x_1$, then

$$Q_1 = (N-1)x_2 \text{ and } Q_2 = x_1 + (N-2)x_2. \tag{1.44}$$

By using the reaction functions $R_1$ and $R_2 = \cdots = R_N$, we obtain a two-dimensional system with state variables $x_1$ and $x_2$. In (1.23) we set $c_2 = \cdots = c_N$, $e_2 = \cdots = e_N$, $a_2 = \cdots = a_N$, and $L_2 = \cdots = L_N$. Then the 2-dimensional model that governs the behavior of firm 1 and the common behavior of the identical firms $2, \dots, N$ becomes

$$T_N\!: \begin{cases} x_1(t+1) = (1-a_1)\,x_1(t) + a_1 R_1\left((N-1)\,x_2(t)\right), \\ x_2(t+1) = (1-a_2)\,x_2(t) + a_2 R_2\left(x_1(t) + (N-2)\,x_2(t)\right), \end{cases}$$

where (again refer to the reaction function in case (i) of Example 1.2)

$$R_1((N-1)x_2) = \begin{cases} 0 & \text{if} \quad z_1^* < 0, \\ L_1 & \text{if} \quad z_1^* > L_1, \\ z_1^* & \text{otherwise,} \end{cases}$$

with $z_1^* = \big(A - c_1 - B(N-1)x_2\big)/\big(2(B+e_1)\big)$ and

$$R_2(x_1 + (N-2)x_2) = \begin{cases} 0 & \text{if} \quad z_2^* < 0, \\ L_2 & \text{if} \quad z_2^* > L_2, \\ z_2^* & \text{otherwise,} \end{cases}$$

with $z_2^* = \big(A - c_2 - B\,(x_1 + (N-2)x_2)\big)/\big(2(B+e_2)\big)$.

The interior equilibrium is independent of $a_k$, $k = 1, 2$, but depends on the number of firms $N$. It is given by $E = (\bar{x}_1(N), \bar{x}_2(N))$ with

$$\bar{x}_1(N) = \frac{A(B+2e_2) - 2c_1e_2 + B\,(c_2(N-1) - c_1N)}{2B(N-2)(B+e_1) + 4(B+e_1)(B+e_2) - B^2(N-1)},$$

$$\bar{x}_2(N) = \frac{2(B+e_1)(A-c_2) - B\,(A-c_1)}{2B(N-2)(B+e_1) + 4(B+e_1)(B+e_2) - B^2(N-1)}.$$

The Jacobian matrix computed at the interior equilibrium is

$$\begin{pmatrix} 1-a_1 & -\,a_1\frac{B(N-1)}{2(B+e_1)} \\ -a_2\frac{B}{2(B+e_2)} & 1-a_2-a_2\frac{B(N-2)}{2(B+e_2)} \end{pmatrix},$$

from which the stability conditions can be obtained by applying conditions (1.40). Interesting stability results are obtained for the boundary equilibria, in the case when $B^2 > 4(B+e_1)(B+e_2)$ (illustrated in Fig. 1.3 for one possible situation). The Jacobian evaluated in the neighborhood of $E_1$ is either

$$\begin{pmatrix} 1-a_1 & -\,a_1\frac{B(N-1)}{2(B+e_1)} \\ 0 & 1-a_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1-a_1 & 0 \\ 0 & 1-a_2 \end{pmatrix}$$

or both, if the equilibrium is on the boundary between the two regions, since $R_2 \equiv 0$ here. The Jacobian evaluated in the neighborhood of $E_2$ is either

$$\begin{pmatrix} 1-a_1 & 0 \\ -a_2\frac{B}{2(B+e_2)} & 1-a_2-a_2\frac{B(N-2)}{2(B+e_2)} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1-a_1 & 0 \\ 0 & 1-a_2-a_2\frac{B(N-1)}{2(B+e_2)} \end{pmatrix}$$

or both, because $R_2 \equiv 0$ here.

As before, let $\boldsymbol{P} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, then the row norms of the Jacobians around $E_1$ generated by the matrix $\boldsymbol{P}$ are bounded by the row norm of the matrix

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-a_1 & a_1 \frac{B(N-1)}{2(B+e_1)} \\ 0 & 1-a_2 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-a_1 & a_1 \frac{Bx(N-1)}{2(B+e_1)} \\ 0 & 1-a_2 \end{pmatrix} \tag{1.45}$$

which is below one if

$$1 - a_1 + a_1 \frac{Bx(N-1)}{2(B+e_1)} < 1,$$

that is, when

$$x < \frac{2(B+e_1)}{B(N-1)}.$$

Hence the equilibrium $E_1$ is locally asymptotically stable for all values of $N$. Similarly, $E_2$ is locally asymptotically stable if there is a positive $x$ such that

$$\frac{a_2 B}{2x(B+e_2)} + \left| 1 - a_2 - a_2 \frac{B(N-2)}{2(B+e_2)} \right| < 1$$

which occurs if

$$-1 < 1 - a_2 \left( 1 + \frac{B(N-2)}{2(B+e_2)} \right) < 1.$$

Therefore, $E_2$ is stable provided that

$$0 < a_2 \frac{B(N-2) + 2(B+e_2)}{2(B+e_2)} < 2.$$

From this stability condition we can now derive several interesting results. First, as already shown before, in the case of duopoly ($N = 2$) the boundary equilibrium $E_2$ is also always stable, like $E_1$. Moreover, the boundary equilibrium $E_2$ is stable provided that $a_2$ is sufficiently small, which means that firms $2, \ldots, N$ have a high inertia in adjusting their quantities toward the best responses. Finally, increasing the number of firms has a destabilizing role. In fact the stability condition can be written as

$$N < 2 + \frac{2\,(2-a_2)\,(B+e_2)}{Ba_2},$$

so that for given cost parameters and adjustment speeds asymptotic stability is lost when the number of firms reaches a certain size.

To conclude this section, we study the global dynamics of the semi-symmetric model. Consider again the parameter values $A = 450$, $B = 30$ and $c_1 = c_2 = \ldots = c_N = 275$, $e_1 = e_2 = \cdots = e_N = -17$. For the adjustment speeds of the two firms we select $a_1 = 0.6$ and $a_2 = \cdots = a_N = 0.45$. For these parameter values the stability condition derived in the previous paragraph tells us that the boundary equilibrium $E_2$ is asymptotically stable if $N < 4$. In Fig. 1.12a we depict the basins

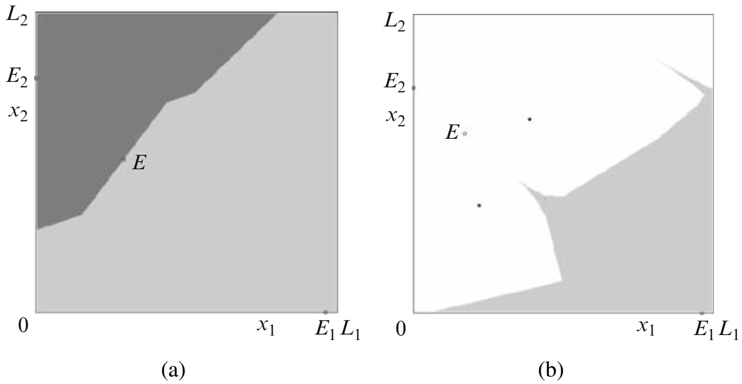


**Fig. 1.12** The Cournot oligopoly with linear demand/quadratic cost. Firms use partial adjustment towards the best response. Basins of attraction of the various equilibria for different values of the number of firms $N$. (**a**) The 3-firm case. Here both $E_1$ and $E_2$ are stable. *Dark grey* basin of $E_2$; *light grey* basin of $E_1$. (**b**) The 5-firm case. Now $E_1$ is stable, $E_2$ is unstable. *Light grey* basin of $E_1$; *white* basin of the two cycle

of the two boundary equilibria $E_1$ and $E_2$ for $N = 3$ firms. To guarantee non-negative prices, we have selected $L_1 = 7$ and $L_2 = L_3 = 4$. Both boundary equilibria are asymptotically stable, each with its own basin of attraction represented by the different shadings of grey. In Fig. 1.12b we show the situation for $N = 5$ firms where $L_1 = 7$ and $L_2 = \cdots = L_5 = 2$. Now only the boundary equilibrium $E_1$ is asymptotically stable, and its basin is represented by the light grey region. Points located in the white region converge to the 2-cycle represented by the two dots.

### *1.3.3 Cournot Duopoly Revisited: A Gradient Type Adjustment Process*

The local stability of an equilibrium and the global dynamics depend on the adjustment mechanism the firms use to update their production choices. We now reconsider the duopoly case analyzed in Sect. 1.3.1, but instead of assuming partial adjustment towards the best response, we now consider a discrete time adjustment process based on marginal profits, similar to the gradient adjustment process discussed in Sect. 1.2 (1.32). However we assume now that the *relative* variation in production quantities is proportional to the marginal profits, that is firm $i$ adjusts its output according to

$$\frac{x_i\,(t+1) - x_i\,(t)}{x_i\,(t)} = a_i \left( \frac{\partial \varphi_i}{\partial x_i} \right)$$

with $a_i > 0$. With these assumptions, the dynamics are now governed by the discrete time system

$$T_g\colon \begin{cases} x_1(t+1) = x_1(t) + a_1 x_1(t)\left[A - c_1 - 2(B+e_1)x_1(t) - Bx_2(t)\right], \\ \\ x_2(t+1) = x_2(t) + a_2 x_2(t)\left[A - c_2 - 2(B+e_2)x_2(t) - Bx_1(t)\right]. \end{cases} \tag{1.46}$$

It is easy to see that the interior steady state of the adjustment process based on marginal profits coincides with the unique interior Nash equilibrium $E = (\bar{x}_1, \bar{x}_2)$ given in (1.12). To study the local asymptotic stability of $E$, we consider the Jacobian matrix of (1.46). Since the Nash equilibrium is located at the intersection of the two reaction functions given in (1.34) and (1.35), we have $B\bar{x}_i = A - c_j - 2\left(B + e_j\right)\bar{x}_j \quad (i, j = 1, 2, \ i \neq j)$. Therefore, the Jacobian matrix evaluated at the interior equilibrium $E$ can be written as

$$\begin{pmatrix} 1 - 2a_1(B+e_1)\bar{x}_1 & -a_1 B\bar{x}_1 \\ -a_2 B\bar{x}_2 & 1 - 2a_2(B+e_2)\bar{x}_2 \end{pmatrix}. \tag{1.47}$$

We can check the stability conditions by use of the relations (1.40) with

$$q = (1 - 2a_1(B+e_1)\bar{x}_1)(1 - 2a_2(B+e_2)\bar{x}_2) - a_1 a_2 B^2 \bar{x}_1 \bar{x}_2,$$

and

$$p = -2 + 2a_1(B+e_1)\bar{x}_1 + 2a_2(B+e_2)\bar{x}_2.$$

By assuming that $B + e_k > 0$ for $k = 1, 2$, clearly $q < 1$. Notice that

$$p + q + 1 = 4a_1 a_2 (B+e_1)(B+e_2)\bar{x}_1\bar{x}_2 - a_1 a_2 B^2 \bar{x}_1 \bar{x}_2,$$

which is positive if $B^2 < 4(B+e_1)(B+e_2)$. Similarly,

$$-p+q+1 = 4 - 4a_1(B+e_1)\bar{x}_1 - 4a_2(B+e_2)\bar{x}_2 + 4a_1a_2\bar{x}_1\bar{x}_2(B+e_1)(B+e_2),$$

so this is positive, if

$$(4(B+e_1)(B+e_2) - B^2)\bar{x}_1\bar{x}_2 a_1 a_2 - 4(B+e_1)\bar{x}_1 a_1 - 4(B+e_2)\bar{x}_2 a_2 + 4 < 0\,. \tag{1.48}$$

If $B^2 < 4(B+e_1)(B+e_2)$ and the equilibrium $E$ is positive, then this additional condition can be used to determine a region of stability in the $(a_1, a_2)$-plane. In contrast to the adjustment process where firms partially adjust their quantities towards the best reply, here the speeds of adjustment are crucial for local asymptotic stability of the Nash equilibrium. As remarked earlier, the stabilizing role of sufficiently small values of the adjustment speeds has been observed before by many authors (see for example Fisher (1961), McManus and Quandt (1961), and Flam (1993)). In Fig. 1.13 we depict the stability region (shaded) in the $(a_1, a_2)$ plane obtained for the parameter values $A = 450$, $B = 30$, $c_1 = c_2 = 275$, $e_1 = e_2 = -11$. For values

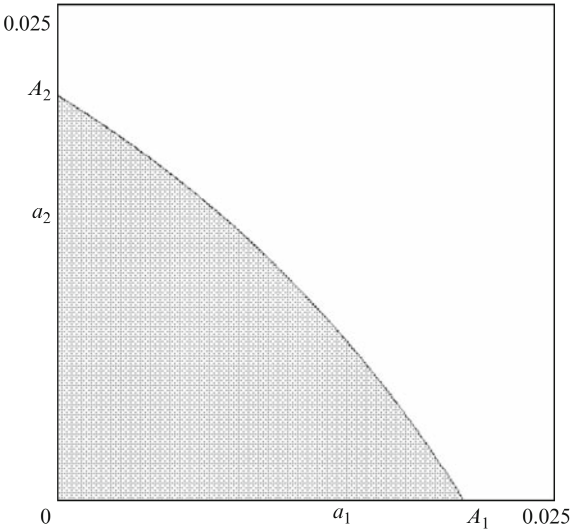


**Fig. 1.13** The Cournot duopoly with a gradient type adjustment process and linear demand/quadratic cost. The *hashed area* indicates the stability region of the interior Nash equilibrium $E$ in the $(a_1, a_2)$ plane of adjustment speeds

of $(a_1, a_2)$ inside the stability region, the Nash equilibrium $E$ is an asymptotically stable node. The boundary of this region represents a bifurcation curve at which $E$ loses asymptotic stability through a flip (or period doubling) bifurcation (see for example Guckenheimer and Holmes (1983), or Lorenz (1995)). This bifurcation curve intersects the axes in the points

$$A_1 = \left(\frac{1}{(B+e_1)\bar{x}_1}, 0\right) \quad \text{and} \quad A_2 = \left(0, \frac{1}{(B+e_2)\bar{x}_2}\right),$$

from which further information on the effects of the model's parameters on the local asymptotic stability of $E$ could be derived by further analysis.

So far we have only considered questions related to local asymptotic stability of the interior equilibrium. But what can we say about the global dynamics? That is, given that the interior Nash equilibrium is locally asymptotically stable, what can be said about its basin of attraction, defined as the set of feasible initial conditions which generate bounded and positive trajectories converging to $E$? In Fig. 1.14, obtained with parameters $A = 450$, $B = 30$, $c_1 = c_2 = 275$, $e_1 = e_2 = -11$ and speeds of adjustment $a_1 = 0.01, a_2 = 0.012$, the Nash equilibrium $E = (2.57, 2.57)$ is locally asymptotically stable and its basin of attraction (or feasible set) is represented by the white area. The region in grey represents the basin of infinity, denoted $B(\infty)$, that is the set of initial conditions that generates unbounded (and negative), therefore "infeasible", trajectories. The interior Nash equilibrium is not globally asymptotically stable since not all initial conditions in the strategy space

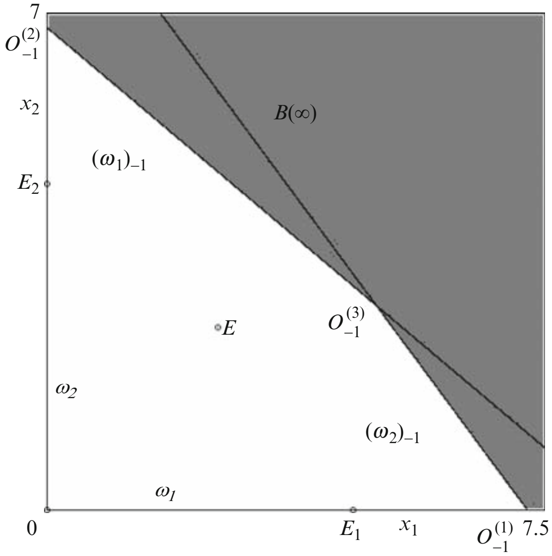


**Fig. 1.14** The Cournot duopoly with a gradient type adjustment process and linear demand/quadratic cost. The *white* region is the basin of attraction of the Nash equilibrium $E$, the *dark grey* region is $B(\infty)$. The basin of $E$ is bounded by the two segments $\omega_1$, $\omega_2$ and their rank-1 preimages $(\omega_1)_{-1}$, $(\omega_2)_{-1}$

are economically feasible. For all quantity choices in the basin of $E$, we obtain $x_1 + x_2 < A/B$. Therefore, non-negativity of prices is guaranteed. Note that for the set of parameters we have selected here, the interior equilibrium would be globally stable with respect to partial adjustment towards the best response.

For the set of parameters used to obtain Fig. 1.14, the set of initial conditions which lead to convergence to the Nash equilibrium $E$ is the interior of the quadrilateral $OO_{-1}^{(1)}O_{-1}^{(3)}O_{-1}^{(2)}$, where $O = (0,0)$ denotes the origin and the other three vertexes are the rank-1 preimages of $O$, meaning that for these points $T_g(O_{-1}^{(i)}) = O$ holds for $i = 1,2,3$ (Note that the mapping $T_g$ was defined in (1.46)). These points are given by

$$O_{-1}^{(1)} = (\frac{1 + a_1(A - c_1)}{2a_1(B + e_1)}, 0), \quad O_{-1}^{(2)} = (0, \frac{1 + a_2(A - c_2)}{2a_2(B + e_2)}) \tag{1.49}$$

and

$$O_{-1}^{(3)} = \Big(\frac{2a_2(B + e_2)\,(1 + a_1(A - c_1)) - a_1 B\,(1 + a_2(A - c_2))}{3B^2 a_1 a_2 + 4a_1 a_2(e_1 + e_2) + 4a_1 a_2 e_1 e_2}, \\ \frac{2a_1(B + e_1)\,(1 + a_2(A - c_2)) - a_2 B\,(1 + a_1(A - c_1))}{3B^2 a_1 a_2 + 4a_1 a_2(e_1 + e_2) + 4a_1 a_2 e_1 e_2}\Big), \tag{1.50}$$

which can be obtained by solving the fourth degree algebraic system (1.46) for $x_i(t)$, upon setting $x_i(t+1) = 0$, $(i = 1, 2)$. A simple strategy for obtaining the preimages of $O$ is to start from the dynamics of $T_g$ restricted to the axes. Since $x_i(t) = 0$ implies $x_i(t+1) = 0$, starting from an initial condition on a coordinate axis, the dynamics are "trapped" on this axis for all $t$. In other words, a monopoly prevails over time and the one-dimensional "monopoly dynamics" is obtained from (1.46) with $x_i = 0$, namely

$$x_j(t+1) = (1 + a_j(A - c_j))x_j(t) - 2(B + e_j)a_j x_j^2(t). \tag{1.51}$$

We also note that this map is conjugate to the standard logistic map $x(t+1) = \mu x(t)(1 - x(t))$ through the linear transformation $x_j = \frac{1+a_j(A-c_j)}{2a_j(B+e_j)}x$, from which the relation $\mu = 1 + a_j(A - c_j)$ can be obtained. The following results for our map can be directly derived from the properties of the logistic map, which is well-studied in the literature; see for example, Devaney (1989). The rank-1 preimages $O_{-1}^{(j)}$ given in (1.49) can now be easily derived from (1.51). Along the $x_j$-axis $(j = 1, 2)$, the one-dimensional restriction (1.51) gives bounded dynamics for $a_j(A - c_j) \leq 3$ provided that the initial conditions are taken inside the segment $\omega_j = OO_{-1}^{(j)}$. Observe that divergent trajectories along the invariant $x_j$ axis are obtained if the initial condition is out of the segment $\omega_j$ $(j = 1, 2)$. Let us now turn to the quadrilateral region bounded by the two segments $\omega_1$ and $\omega_2$ and their rank-1 preimages, say $(\omega_1)_{-1}$ and $(\omega_2)_{-1}$ respectively (see Fig. 1.14). The preimages $(\omega_1)_{-1}$ and $(\omega_2)_{-1}$ can be analytically computed as follows. Let $X = (x, 0)$ be a point of $\omega_1$. Its preimages are the real solutions $(x_1, x_2)$ of the algebraic system

$$\begin{cases} x_1\left[1 + a_1(A - c_1) - 2a_1(B + e_1)x_1 - a_1 B x_2\right] = x, \\ \\ x_2\left[1 + a_2(A - c_2) - a_2 B x_1 - 2a_2(B + e_2)x_2\right] = 0\,. \end{cases} \tag{1.52}$$

From the second equation it is easy to see that the preimages of the points of $\omega_1$ are either located on the same invariant axis $x_2 = 0$ or on the line represented by the equation

$$a_2 B x_1 + 2a_2(B + e_2)x_2 = 1 + a_2(A - c_2). \tag{1.53}$$

Analogously, the preimages of a point of $\omega_2$ belong to the same invariant axis $x_1 = 0$ or to the curve represented by equation

$$2a_1(B + e_1)x_1 + a_1 B x_2 = 1 + a_1(A - c_1). \tag{1.54}$$

It is now straightforward to see that the line (1.53) intersects the $x_2$ axis in the point $O_{-1}^{(2)}$ and the line (1.54) intersects the $x_1$ axis in the point $O_{-1}^{(1)}$. Moreover, the two lines intersect at the point $O_{-1}^{(3)}$. A summary of these observations leads to the following description of the basin of the asymptotically stable Nash equilibrium $E$ as shown in Fig. 1.14. The rank-1 preimages of the origin are the vertexes of the quadrilateral $OO_{-1}^{(1)}O_{-1}^{(3)}O_{-1}^{(2)}$. The sides of this region are given by $\omega_1, \omega_2$ and their respective rank-1 preimages $(\omega_1)_{-1}$ and $(\omega_2)_{-1}$ respectively. All points inside this quadrilateral region lead to convergence, all points outside cannot generate feasible trajectories. Points located to the right of $(\omega_2)_{-1}$ are mapped into points with negative value of $x_1$ after one iteration, as can be easily deduced from the first component of (1.46). Points located above $(\omega_1)_{-1}$ are mapped into points with negative value of $x_2$ after one iteration, as can be deduced from the second component of (1.46). The expressions in (1.53) and (1.54) can be used to determine the impact of parameter changes on the basin. Finally, observe that for these values of the parameters the basin of the unique interior Nash equilibrium is a rather simple and connected set.

### *1.3.4 Simple Basins and Critical Curves*

In this subsection we introduce the concept of critical curves (see also Appendix C). This subsection uses many concepts about dynamical systems that may not be familiar to some readers (such as noninvertible maps, critical sets, preimages of various ranks and so on). These concepts are reviewed in Appendix C, which the reader may need to study before working through this subsection.

Recall that in the previous subsection we have demonstrated how to obtain the boundaries of the feasible region by taking the preimages $(\omega_i)_{-1}$ $(i = 1, 2)$ of the coordinate axes. Since the map $T_g$ in (1.46) is a noninvertible map, as can be readily deduced from the fact that the origin has four preimages, there might be further preimages of $(\omega_i)_{-1}$ $(i = 1, 2)$, which have to be also considered in order to obtain the whole boundary of the feasible region. In order to determine if $(\omega_i)_{-1}$ $(i = 1, 2)$ have further preimages, we can use the critical curves of the map which can be used to identify regions in the feasible set (or strategy space) with a different number of preimages.

To begin with, let us consider a given point $\left(x_1', x_2'\right)$ in the strategy space. Then its preimages can be calculated by setting $x_1(t+1) = x_1', x_2(t+1) = x_2'$ in (1.46) and solving with respect to $x_1$ and $x_2$ the fourth degree algebraic system,

$$\begin{cases} x_1\left[1 + a_1\left(A - c_1 - 2(B + e_1)x_1 - Bx_2\right)\right] = x_1', \\ \\ x_2\left[1 + a_2\left(A - c_2 - 2(B + e_2)x_2 - Bx_1\right)\right] = x_2'. \end{cases} \tag{1.55}$$

Clearly, this algebraic system may have up to four real solutions, which are the rank-1 preimages of $\left(x_1', x_2'\right)$. We can now use this information to subdivide the strategy space into regions characterized by a different number of preimages. This is shown in Fig. 1.15a, which is obtained with the same parameters as Fig. 1.14. The regions $Z_k$ denote the sets of points which have $k$ real and distinct rank-1 preimages. For example, as shown above, the origin $O = (0, 0) \in Z_4$, because it has four

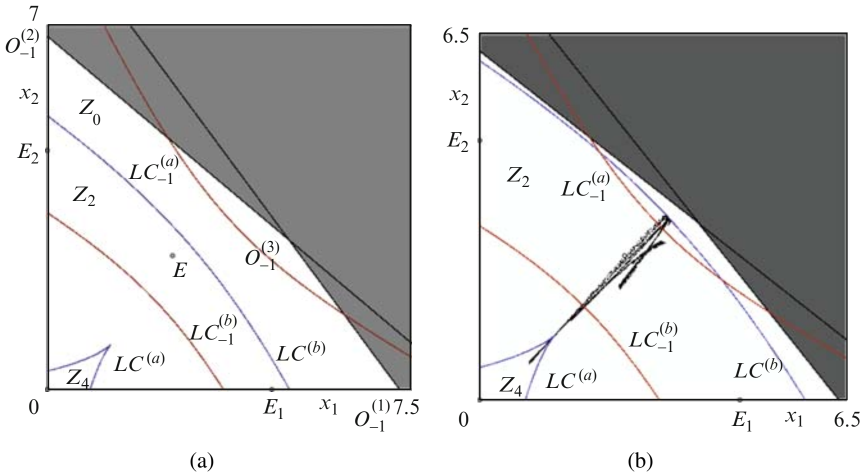


**Fig. 1.15** The Cournot duopoly with a gradient type adjustment process and linear demand/quadratic cost. Illustrating the regions of preimages of different ranks, the sets of points where the Jacobian vanishes ($LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$) and the critical curves $LC^a$ and $LC^b$. (**a**) The parameters are the same as in Fig. 1.13. (**b**) The speeds of adjustment are slightly higher, $E$ becomes unstable and a strange attractor emerges, but the basic structure of the basin remains the same as in (**a**). Note however that the critical curve $LC^{(b)}$ in now quite close to the boundary of the *white* and *grey* regions

rank-1 preimages, given by $O$ itself (since $T_g(0,0) = (0,0)$) and $O_{-1}^{(i)}$, $i = 1, 2, 3$ (since $T_g(O_{-1}^{(i)}) = (0,0)$ as well). The regions $Z_k$ are separated by segments of critical curves denoted as $LC^{(a)}$ and $LC^{(b)}$ in Fig. 1.15a.

An intuitive understanding of the importance of critical curves can be obtained by referring to the folding or unfolding mechanism of a map. The map (1.46) is noninvertible, which means that distinct points in the action set can be mapped into the same point by $T_g$. This can be geometrically envisioned by imagining a process which folds the action space onto itself (so that points which are in different locations are folded onto each other). A result from algebraic geometry tells us that the folding process can be characterized by a change of sign of the determinant of the Jacobian of the map: if the sign is positive, then the map is orientation preserving, whereas it is orientation reversing otherwise.[4] The folding curves where the sign change occurs is the locus of points where the determinant of the Jacobian of the map vanishes. Its *image* gives the so-called critical curve, which separates zones or regions with different numbers of *preimages* (this indicates the importance of the unfolding action of the map). To sum up, the following numerical procedure

[4] Consider a one-dimensional, continuously differentiable map $g(y)$. If $g'(y) > 0$, then for $x < y$, it follows that $g(x) < g(y)$. If, on the other hand, $g'(y) < 0$, the orientation is reversed. Obviously, the change of signs occurs exactly at the point where the derivative vanishes.

(see also Appendix C) can be used to obtain the critical curves (for a given set of parameters):

1. The map (1.46) is continuously differentiable, so the (folding) set $LC_{-1}$ can be obtained numerically as the locus of points $(x_1, x_2)$ for which the Jacobian determinant of $T_g$ vanishes.
2. The critical curves $LC$, which separate the regions $Z_k$, are obtained by computing the images of the points belonging to $LC_{-1}$, that is $LC = T_g(LC_{-1})$.

In Fig. 1.15a the set of points at which the Jacobian vanishes gives the curves denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$. It is formed by the union of the two branches of a hyperbola. Also the critical curve $LC = T_g(LC_{-1})$ is formed by two branches, denoted by $LC^{(a)} = T_g(LC_{-1}^{(a)})$ and $LC^{(b)} = T_g(LC_{-1}^{(b)})$. The curve $LC^{(b)}$ separates the region $Z_0$, whose points have no preimages, from the region $Z_2$, whose points have two distinct rank-1 preimages. The curve $LC^{(a)}$ separates the region $Z_2$ from $Z_4$, whose points have four distinct preimages.

Our analysis based on the critical curves of the map now reveals why the set of initial conditions that lead to convergence to the Nash equilibrium, bounded by $\omega_1$, $\omega_2$ and its preimages $(\omega_1)_{-1}$ and $(\omega_2)_{-1}$, is a rather simple set. It is due to the fact that only preimages of rank-1 of $\omega_1$ and $\omega_2$ exist. Note that $(\omega_1)_{-1}$ and $(\omega_2)_{-1}$ are entirely included in $Z_0$, that is a region of the feasible set whose points have no preimages. Therefore, the preimages $(\omega_i)_{-1}$ $(i = 1, 2)$ of the invariant axes, have no preimages of higher rank. Consequently, the whole boundary that separates the basin $B(E)$ and the infeasible set $B(\infty)$ is

$$\mathcal{F} = \left(\cup_{n=0}^{\infty} T_g^{-n}(\omega_1)\right) \bigcup \left(\cup_{n=0}^{\infty} T_g^{-n}(\omega_2)\right), \tag{1.56}$$

that is, the union of all the preimages of the segments $\omega_1$ and $\omega_2$ (see Appendix C), which is a rather simple set.

To conclude this subsection, we would like to stress the fact that the properties of the basin boundaries are related to the global dynamics of our duopoly model. Such a simple structure of the basin may be also maintained when the Nash equilibrium loses stability due to local (period-doubling) bifurcations. In Fig. 1.15b, obtained with the same parameters as before except that $a_1 = 0.015$ and $a_2 = 0.0165$, we depict a situation where (after the usual period-doubling sequence) a chaotic attractor describes the long run evolution of the production decisions of the duopolists. Despite the fact that the dynamic behavior can be considered as complex, the basin boundaries are still given by the same quadrilateral.

The reader should notice, however, that basins are not always as simple as in the examples presented so far in this book. Indeed, a closer look at Fig. 1.15b reveals that the critical curve $LC^{(b)}$ is rather close to a basin boundary. This indicates that a small shift of this curve due to a parameter variation may cause a contact, after which a portion of the set of infeasible points $B(\infty)$ crosses the critical curve and, consequently, enters the region $Z_2$. In the next subsection we will show that such contact bifurcations may have a considerable impact on the topological structure of the feasible set.

### 1.3.5 Disconnected Basins

In all the examples encountered up to now, the basins of the corresponding attractor were rather simple and were connected sets. As we shall now demonstrate, basins can have a quite complicated structure. For example, they can be pierced by many holes or may consist of areas without any connection. In such situations predicting the long run outcome of the duopoly game where players use certain adjustment processes to determine their production quantities over time is quite difficult. This becomes particularly relevant when stochastic influences play a role.

In Fig. 1.16a we depict the situation after an increase in the adjustment speeds from $a_1 = 0.015$, $a_2 = 0.0165$ (the values in Fig. 1.15b) to $a_1 = 0.015$, $a_2 = 0.017$. After the contact of the curve $LC^{(b)}$ with the boundary of $B(\infty)$, a set indicated as $H_0$ which belongs to the infeasible set $B(\infty)$ enters $Z_2$ (see the region indicated by the arrow in Figs. 1.16a, b).

This means that points belonging to $H_0$ have two distinct preimages, say $H_{-1}^{(1)}$ and $H_{-1}^{(2)}$, which are located on opposite sides of the curve $LC_{-1}^{(b)}$ (the preimages of points exactly on the curve $LC^{(b)}$ inside $B(\infty)$ are located on $LC_{-1}^{(b)}$). Obviously, since $H_0$ belongs to the set $B(\infty)$, initial conditions belonging to $H_{-1}^{(1)}$ and $H_{-1}^{(2)}$ also lead to infeasible trajectories, since they are mapped into the infeasible set after one iteration. The rank-1 preimages of $H_0$ constitute a so-called *hole* of $B(\infty)$ which is located entirely inside the feasible set (this hole is also called a "lake" in Mira et al. (1996)). Since this hole, also referred to as the *main hole*, again lies inside the region $Z_2$, it also has two preimages. These smaller holes, denoted as $H_{-2}^{(1)}$ and

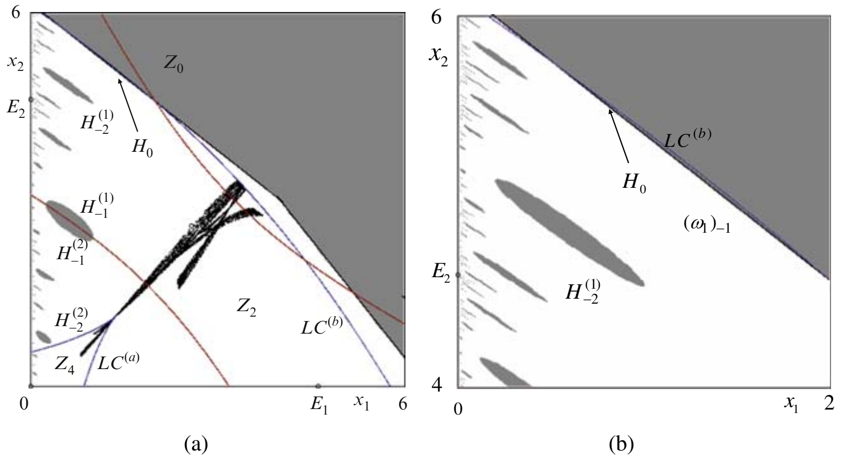


**Fig. 1.16** The Cournot duopoly with a gradient type adjustment process and linear demand/quadratic cost. Slightly higher speeds of adjustment than in the case of Fig. 1.15. The critical curve $LC^{(b)}$ has crossed the basin boundary and a disconnected basin of attraction now results. (**a**) The entire region. (**b**) A close up of the set $H_0$ and its preimages

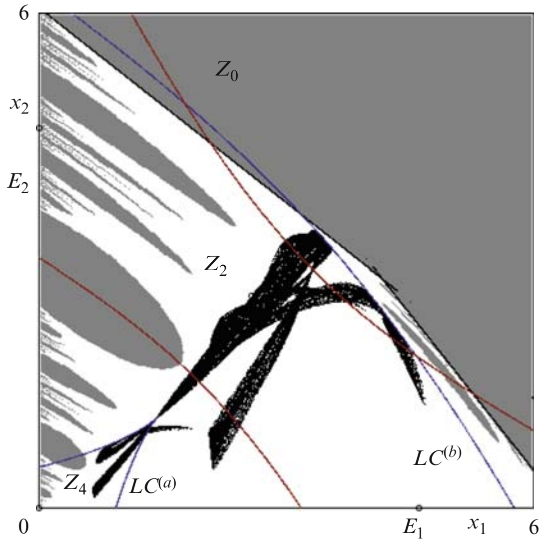


**Fig. 1.17** The Cournot duopoly with a gradient type adjustment process and linear demand/quadratic cost. The same situation as in Fig. 1.16, but with slightly higher speeds of adjustment. Note how the holes have become larger and connected along the vertical axis

$H_{-2}^{(2)}$, contain initial conditions which are mapped into the main hole and then into the infeasible set. The sets $H_{-2}^{(1)}$ and $H_{-2}^{(2)}$ are bounded by preimages of rank 3 of $\omega_1$. Since these smaller holes are again both inside $Z_2$, each of them has again two further preimages inside $Z_2$, and so on. Summarizing, we can conclude that the global bifurcation which we have just described transforms a *simply connected* basin into a *multiply connected* basin. The latter set has a countably infinite number of holes, called an *arborescent sequence of holes*, which belong to the infeasible set $B(\infty)$. As the speeds of adjustment are further increased, the holes become more pronounced and they become connected along the vertical axis as shown in Fig. 1.17.

Our numerical results show that the structure of the basins may become considerably more complex as the adjustment speeds are increased. The transition between qualitatively different structures of the boundary occur through so called *contact bifurcations* (see for example Mira et al. (1996)) and these bifurcations can be described in terms of contacts between the basin boundaries and arcs of the *critical curves*. To conclude this chapter, we would like to stress that in general there is no relation between the bifurcations which change the qualitative properties of the basins (global bifurcations) and the bifurcations which change the qualitative properties of the attractor (sequences of local bifurcations). The former is related to the global dynamics, whereas the latter focuses on the local (stability) properties. In later chapters we will encounter situations where the attractor is a rather simple set (that is, an equilibrium), but the structure of its basin is quite complex. As demonstrated above, in other situations exactly the opposite might be the case.