SMALL DENOMINATORS. I
MAPPINGS OF THE CIRCUMFERENCE ONTO ITSELF *

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In the first part of the paper it is shown that analytic mappings of the circumference, differing little from a rotation, whose rotation number is irrational and satisfies certain arithmetical requirements, may be carried into a rotation by an analytic substitution of variables. In the second part we consider the space of mappings of the circumference onto itself and the place occupied in this space by mappings of various types. We indicate applications to the investigation of trajectories on the torus and to the Dirichlet problem for the equation of the string.

Introduction

Continuous mappings of the circumference onto itself were studied by Poincaré (see [1], Chapter XV, pp. 165–191) in connection with the qualitative investigation of trajectories on the torus. The problem of Dirichlet for the equation of the string can be reduced to such mappings, but the topological investigation turns out here to be insufficient (see [5]). In the first portion of the present paper we attempt an analytic refinement of the Denjoy theorem completing the theory of Poincaré [2].

Suppose that $F(z)$ is periodic, $F(z+2\pi)=F(z)$, real on the real axis and analytic in its neighborhood, with $F'(z) \neq -1$ for $\operatorname{Im} z = 0$. Then to the mapping of a strip of the complex plane defined by $z \longrightarrow Az \equiv z + F(z)$ there corresponds an orientation-preserving homeomorphism B of the neighborhood of the points $w(z) = e^{iz}$:

$$w = w(z) \longrightarrow w(Az) \equiv Bw.$$

In this sense we say that A is an analytic mapping of the circumference onto itself.

Suppose that the rotation number * of A is equal to $2\pi\mu$. From Denjoy's theorem it follows that for irrational μ there exists a continuous inversible real function $\phi(z)$ of the real variable z, periodic in the sense that

$$\phi(z+2\pi) = \phi(z) + 2\pi$$

and such that

$$\phi(Az) = \phi(z) + 2\pi\mu. \tag{1}$$

*We assume that the reader is acquainted with the results of the papers [1] (pp. 165–191, 322–335) and [2], which appear in the textbooks [3] (pp. 65–76) and [4] (pp. 442–456).

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We shall say that ϕ is a new parameter and that when expressed in the parameter ϕ the transformation A becomes a rotation by the angle $2\pi\mu$. Such a function ϕ must be unique up to an additive constant.

In §1 it is shown that for certain irrational μ, in spite of the analyticity of $F(z)$, the function ϕ in (1) may turn out not to be absolutely continuous. The idea of this example consists of the following. Since under rotations of the circumference length is preserved, the reduction of a transformation to a rotation by an appropriate choice of parameter amounts to the determination of the invariant measure of the transformation. In the case of a rational rotation number the invariant measure is concentrated, as a rule, at separate points, the points of the cycles of the transformation. However, if the rotation number is irrational, but can be approximated extremely well by rationals, then the invariant measure retains its singular character, though it is distributed everywhere densely on the circumference.

The following conjecture appears to be plausible:

There exists a set $M \subseteq [0, 1]$ *of measure* 1 *such that for each* $\mu \in M$ *the solution of the equation* (1) *for any analytic transformation* A *with rotation number* $2\pi\mu$ *is analytic.*

At present this is proved only for analytic transformations sufficiently close to a rotation by the angle $2\pi\mu$ (§4, Theorem 2).* The proof consists in the construction of the solution of equation (1) by means of the solution of equations of the form

$$g(z + 2\pi\mu) - g(z) = f(z). \tag{2}$$

In the solution of this equation by the use of Fourier series, there appear small denominators, making the convergence difficult. The calculation of the successive corrections, adapting the solution of the equation (2) to the equation (1), is carried out by a method of the type of Newton's method, and the rapid convergence of this method guarantees the possibility of realizing not only all the approximations of the theory of perturbations, but also the passage to the limit.

*Note added in proof. As this paper was going to press the author learned of the work of A. Finzi [38], [39]. From the results of [38] it follows that if the rotation number of a sufficiently smooth mapping of the circumference onto itself satisfies certain arithmetical requirements, then the transformation may be converted into a rotation by a *continuously differentiable* change of variables. Thus the method of A. Finzi does not require that the transformation be close to a rotation. This partly confirms the conjecture stated above. A. Finzi notes, however, that he does not see how to extend his method to the case when a higher smoothness of the substitution of variables is required. The present paper contains a partial answer to some of the questions posed by Finzi. For a partial answer to some of the questions posed here the reader is referred to the Finzi papers.

Newton's method was applied for a similar purpose by A. N. Kolmogorov [6]. Theorem 2 of the present paper is in a way a discrete analogue of his theorem on the preservation of conditionally periodic motions under small changes of the Hamilton's function. In distinction to [6] we have no analytic integral invariants at our disposal, but rather we seek them. Moreover, we prove (in Theorem 2) the analyticity of the dependence on a small parameter ϵ, from which there follows the convergence of all the series in powers of ϵ that are usual in the theory of perturbations.

A direct proof of the convergence of these series has not been achieved, and A. N. Kolmogorov has even conjectured* (before studying the paper [7] of K. L. Siegel) that they might diverge.

Another conjecture of Kolmogorov, stated by him in the report [8], turned out to be true: questions in which small denominators play a role are connected with the monogenic functions of Borel [9]. For our case this is established in §§7,8 and used in §11.

Certain important problems with small denominators were solved by K. L. Siegel (see [7], [33], [34], [35]). There is a direct connection between mappings of the circumference and the problem of the center for the Schroeder equation: *is it possible to make an analytic substitution of variables* $\phi(z) = z + b_2z^2 + \cdots$ *which will convert a mapping of the neighborhood of the origin of the complex plane, given by the analytic function* $f(z) = e^{2\pi i\mu}z + a_2z^2 + \cdots$, *into a rotation by the angle* $2\pi\mu$?

The result of Siegel in [7] is analogous to our Theorem 2 and may be obtained by the same method. The problem of the center is a singular case of the problem of the mapping of a circumference whose radius, in the singular case, is equal to zero. In comparison with the general case the position here is simpler, since the solution (the Schroeder series) may be formally written down directly. The application of Newton's method also gives the Schroeder series; in distinction to Theorem 2, each coefficient of the solution will be exactly defined after a finite number of approximations.

In the second part of the paper we cite the classical mappings of the circumference onto itself and discuss the question of the typicality of various cases. In §9 we introduce the function $\mu(T)$ (rotation number) on the space of mappings of the circumference. Further we study, for rational (§10) and irrational (§11) μ, the level sets $\mu(T) = \mu$ from the point of view of their structure (Theorems 6 and 7) and density (Theorems 5 and 8). Of greatest importance from the topological

*In a report to the Moscow Mathematical Society on January 13, 1959.

point of view are the rough mappings (the word "rough" being taken in the sense of Andronov and Pontrjagin [10]) with normal cycles and rational rotation numbers; these mappings form an open everywhere dense set.* From the point of view of measure in finite-dimensional subspaces the ergodic case also is typical. In §12 we consider the two-dimensional subspace of mappings $x \to x + a + \epsilon \cos x$.

In §§13 and 14 the preceding results are applied to the qualitative investigation of trajectories on the torus and to the Dirichlet problem for the equation of the string.

I wish to express my thanks to A. N. Kolmogorov for his valuable advice and assistance.

Part I

On analytic mappings of the circumference onto itself

The basic content of the first part of this paper is contained in §§4–6 (Theorem 2). For an understanding of the proof of Theorem 2 (§§5,6) it is necessary to study subsections 2.1 and 2.3 of §2 and subsection 3.3 of §3. For the lemmas on implicit functions and on finite increments contained in §3 one may turn at need to the references. Each of §§1, 2, 7 may be read independently of all the rest. In §8 we prove a generalization of Theorem 2 (Theorem 3), used in the second portion of the paper.

§1. The case when the new parameter is not an absolutely continuous function of the old parameter

1.1. In this section we construct an analytic mapping A of the circumference C, subsets G_n $(n = 1, 2, \cdots)$ of the circumference and integers N_n $(n = 1, 2, \cdots)$ such that:

1. $\text{mes } G_n \to 0$ as $n \to \infty$.
2. $A^{N_n}(C \setminus G_n) \subset G_n$.
3. The rotation number μ of the transformation A is irrational.

This transformation A cannot be converted into a rotation by an absolutely continuous change of variables. Indeed, let ϕ be a continuous parameter in which the transformation becomes a rotation by the angle $2\pi\mu$ (ϕ exists from Denjoy's theorem). Suppose that $G \subset C$. The measure of the set $\phi(G)$ of values $\phi(x)$, $x \in G$, coincides with the measure of $\phi(A^N G)$, since these sets superpose under a rotation. Therefore it follows from condition 2 that:

$$2\pi - \text{mes}\,\varphi(G_n) \leqslant \text{mes}\,\varphi(G_n)$$

*Note added in proof. This result was also obtained by V. A. Pliss in the paper [43], published while this paper was being printed.

and

$$\operatorname{mes} \varphi(G_n) \geqslant \pi.$$

In view of condition 1, ϕ is not an absolutely continuous function on C.

1.2. For the construction we use the following lemmas.

Lemma α. *Let the transformation A of the circumference be semistable forward* and analytic in the neighborhood of the real axis, and suppose that the points $z_0, z_k = A(z_{k-1})$ $(0 < k < n)$ form a cycle, i.e., $A(z_{n-1}) = z_0$. Then for any $\epsilon > 0$ there is in the indicated neighborhood of the real axis a transformation A' differing from A by less than ϵ and having exactly one cycle, in fact $z_0, z_1, \cdots, z_{n-1}$.*

Proof. We construct a correction $\Delta(z)$ analytic in the strip in question, vanishing at the points $z_0, z_1, \cdots, z_{n-1}$ and positive on the remainder of the real points.

Put

$$A'(z) = A(z) + \varepsilon'\Delta(z);$$

for sufficiently small $\epsilon' > 0, |\epsilon'\Delta(z)| < \epsilon$ in the indicated strip and $A'(z)$ is a transformation of the circumference. Evidently the transformation $(A')^n$ moves forward all the points z not less than the transformation A^n; furthermore the points $z_0, \cdots, z_{n-1}$ move by $2\pi m$, and the remaining points by not less than $2\pi m$. Lemma α is proved.

Definition. Suppose that A is a transformation of the circumference C and that G is a set on C. We shall say that the transformation A has property 2 relative to G and N if $A^N(C \setminus G) \subset G$.

Lemma β. *Given a transformation A with the single cycle $z_0, \cdots, z_{n-1}$ and any $\epsilon > 0$, then A possesses property 2 relative to the set G_ϵ of points of the ϵ-neighborhood of the cycle and any N exceeding some $N_0(\epsilon)$.*

Proof. Suppose that $z_i < x < z_j$, where $x_i x_j$ is one of the arcs into which the cycle divides the circumference. The points $A^{kn}(x)$ $(n = 1, 2, \cdots)$ lie on the arc $z_i z_j$ and form a monotone sequence (for more details see §10). Therefore it follows that in the case when the transformation A is semistable forward (the case of backward semistability is completely analogous),

$$A^{kn}(x) \underset{k \to +\infty}{\longrightarrow} z_j.$$

Indeed, suppose that λ is the limit of the monotone sequence $A^{kn}(x)$. Then λ is invariant with respect to A^n and belongs to a cycle satisfying the inequalities

$$z_i < \lambda \leqslant z_j.$$

*This means that for some integers m, n and any real z, $A^n(z) \geq z + 2\pi m$, with equality attained.

Thus

$$\lim_{k\to\infty} A^{kn+l}(x) = A^l(z_j).$$

The same is true for the other intervals into which the cycle divides the circumference.

Consider the points $x_i = z_i + \epsilon$. By what has been proved, beginning with some $N_0(\epsilon)$, all the points $A^N x_i$ lie in an ϵ-neighborhood of the cycle. Evidently that N_0 is the one desired.

Lemma γ. *Suppose that the transformation has property 2 relative to G and N, and suppose that $\epsilon > 0$. Then there exists a $\delta > 0$ such that each transformation B differing from A by less than δ has property 2 relative to N and the ϵ-neighborhood of G.*

Proof. The lemma follows in an obvious way from the continuous dependence of A^N on A.

Lemma δ. *Suppose that A is a semistable forward transformation, $B(z) = A(z) + h$, $h > 0$. Then the rotation number μ of the transformation B is strictly larger than the rotation number m/n of the transformation A.*

Proof. Evidently $\mu \geq m/n$. In addition $B^n(z) > A^n(z)$ and therefore B does not have a cycle of order n. Hence $\mu > m/n$.

Lemma ϵ (degenerate case of Liouville's theorem). *If the inequality $|\alpha - m/n| < c/|n|$ for any $c > 0$ has an infinite set of irreducible solutions m/n, then the number α is irrational.*

Proof. If $\alpha = p/q$, then for $n > q$

$$\left|\frac{p}{q} - \frac{m}{n}\right| > \frac{\frac{1}{|q|}}{|n|},$$

since the quotient m/n is irreducible, so that $|pn - qm| \neq 0$ for $q < n$.

1.3. The transformation A is formed as a limit of a sequence of transformations A_n with rational rotation numbers. Beginning with the transformation $z \to A_1(z)$, we shall suppose that it has the following properties:

1_1. A_1 is analytic in the strip $|\operatorname{Im} z| < R$, and in this strip $|A_1(z)| < C/2$.

2_1. The rotation number of A_1 is rational: $\mu_1 = p_1/q_1$.

3_{1a}. A_1 is semistable forward.

3_{1b}. A_1 has exactly one cycle.

The existence of such an A_1 is evident: from each A_1'' with property 1_1 one may obtain, with an appropriate choice of $h > 0$, $A_1' = A_1'' + h$ with properties 1_1, 2_1 and 3_1, and then one may correct A_1' to A_1 using Lemma α. The subsequent transformations A_n are obtained from the preceding ones by using a process based on the following Induction Lemma.

Induction Lemma. *Suppose that* $\delta_n > 0$ *and suppose given transformations* A_k $(k = 1, 2, \cdots, n)$ *and* $R > 0$, $C > 0$ *such that*

1_n. *For* $|\mathrm{Im}\, z| < R$ *the* A_k *are analytic and satisfy the inequalities*

$$|A_k(z) - A_{k-1}(z)| < \frac{C}{2^k} \quad (A_0(z) \equiv 0).$$

2_n. *The rotation numbers of the* A_k *are rational and for* $k > 1$

$$\left| \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} \right| < \frac{1}{(k-1)^2 (\max\limits_{l<k} q_l)^2}$$

3_n. A_k *is semistable forward and has a unique cycle.*

Then one may construct a transformation A_{n+1} *such that the sequence* A_k $(k = 1, 2, \cdots, n+1)$ *will have properties* 1_{n+1}, 2_{n+1}, 3_{n+1} *and*

4_{n+1}. $\quad |A_{n+1}(z) - A_n(z)| < \delta_n$ *for* $\mathrm{Im}\, z = 0$.

Proof. Consider the transformation A_λ: $z \longrightarrow A_n(z) + \lambda$, $\lambda > 0$. Evidently there exists a $\lambda_0 > 0$ such that for $\lambda < \lambda_0$

$$|A_\lambda(x) - A_n(z)| < \frac{C}{2^{n+2}} \quad (|\mathrm{Im}\, z| < R),$$

$$|A_\lambda(z) - A_n(z)| < \frac{\delta_n}{2} \quad (\mathrm{Im}\, z = 0)$$

and the rotation number of A_λ is strictly larger than p_n/q_n (Lemma δ) and less than

$$\frac{p_n}{q_n} + \frac{1}{n^2 (\max\limits_{l \leqslant n} q_j)^2}$$

(continuity of the rotation number, see §9). Suppose that the rotation number of A_{λ_0} is μ. We select a rational number p_{n+1}/q_{n+1},

$$\frac{p_n}{q_n} < \frac{p_{n+1}}{q_{n+1}} < \mu.$$

Among all the λ for which the rotation number of A_λ is p_{n+1}/q_{n+1} we select the largest. Suppose that it is λ_1. The transformation A_{λ_1} has the properties 1_{n+1}, 2_{n+1}, 4_{n+1}, and, as is easily seen, is semistable forward. We apply Lemma α to it. Then we obtain a transformation A_{n+1} satisfying all the requirements of the Induction Lemma.

1.4. The transformation A_1 satisfies requirements 1_1, 2_1, 3_1 of the Induction Lemma for the same C, R. We shall describe the choice of δ_n in carrying out the induction from A_n to A_{n+1}. We denote by G_n^* the ϵ-neighborhood of the single cycle A_n, where $\epsilon > 0$ is chosen so that the measure of G_n^* is less than 2^{-n-2}. By Lemma β there is an N_n such that A_n has property 2 relative to G_n^* and N_n. By Lemma γ there exists a $\delta_n^* > 0$ for which the transformation A has property 2 relative to N_n and to a G_n-neighborhood of G_n^* of measure 2^{-n-1} if on the real axis

$$|A(z) - A_n(z)| < \delta_n^*.$$

Choose

$$\delta_{n+1} = \min\left(\frac{\delta_n}{2}, \frac{\delta_n^*}{2}\right)$$

(we formally take $\delta_0 = 0$). Applying the Induction Lemma, we obtain A_{n+1}.

If the transformations A_n, $n = 1, 2, \cdots$, are constructed in the way described, then, in view of property 1_n this sequence converges uniformly in the strip $|\operatorname{Im} z| < R$, so that the limit A is an analytic transformation. Evidently

$$|A(z) - A_n(z)| \leqslant \sum_{k=n}^{\infty} |A_{k+1}(z) - A_k(z)| \leqslant \sum_{k=n}^{\infty} \delta_n \left(\frac{1}{2}\right)^{n+1} \leqslant \delta_n \quad (\operatorname{Im} z = 0)$$

for any n and therefore A has property 2 relative to G_n and N_n, $n = 1, 2, \cdots$. From property 2_n and the continuity of the rotation number, we conclude on the basis of Lemma ϵ that the rotation number of A is irrational. Indeed, for any n

$$\left|\mu - \frac{p_n}{q_n}\right| \leqslant \sum_{k=n}^{\infty} \frac{1}{k^2 (\max\limits_{l \leqslant k} q_l)^2} \leqslant \sum_{k=n}^{\infty} \frac{1}{k^2 q_n^2} < \frac{2}{q_n^2}.$$

Thus all three properties of subsection 1.1 are satisfied, so that A is the desired transformation.

1.5. **Remark.** Considering the example just constructed, it is not difficult to see that a transformation A with the indicated properties may be found in any family of analytic transformations

$$z \to A_\Delta z \equiv z + \Delta + F(z)$$

and therefore in any neighborhood of any transformation with an irrational rotation number, given only that the family has the following property: among the transformations A_Δ^n there are no rotations. Probably the family $z \to z + \Delta + \frac{1}{2}\cos z$ has this property; in this case an example may be given by a simple analytic formula.

§2. On the functional* equation $g(z + 2\pi\mu) - g(z) = f(z)$

* Hilbert [12] gave this equation as an example of an analytic problem with a nonanalytic solution. It is encountered in investigations on the metric theory of dynamical systems (see [13], [14]), and is the simplest example of a problem with small denominators.

Added in proof. This paper was already in press when the author became acquainted with the paper [40] of A. Wintner in which this equation was apparently first studied from a modern point of view.

2.1. Suppose that $f(z)$ is a function of period 2π, μ a real number. It is required to define from the equation

$$g(z+2\pi\mu)-g(z)=f(z) \tag{1}$$

a function $g(z)$ having period 2π.

In case the equation (1) is solvable, evidently

$$\int_0^{2\pi} f(z)\,dz=0.$$

Furthermore, if $g(z)$ is a solution, then $g(z)+C$ is also a solution. Therefore we shall consider only right sides which are in the mean equal to zero and seek only solutions in the mean equal to zero. In each function $\phi(z)$ on $[0, 2\pi]$ we single out the constant part

$$\bar{\varphi}=\frac{1}{2\pi}\int_0^{2\pi}\varphi(z)\,dz$$

and the variable part

$$\tilde{\varphi}(z)=\varphi(z)-\bar{\varphi}.$$

The equation $\bar{f}=0$ is thus a necessary condition for the solvability of equation (1). By a solution of (1) we shall from now on always understand the variable part $g(z)$.

If $\mu=m/n$, i.e., is rational, then for the existence of a solution it is necessary that

$$\sum_{k=1}^{n} f\left(z+2\pi\frac{k}{n}\right)=0,$$

since this sum may be expressed in terms of the solution in the form

$$\sum_{k=1}^{n} g\left(z+2\pi\frac{m}{n}+2\pi\frac{k}{n}\right)-\sum_{k=1}^{n} g\left(z+2\pi\frac{k}{n}\right),$$

and in these two sums the terms are identical. If such a condition is satisfied, then a solution exists but it is defined only up to an arbitrary function of period $2\pi/n$, since such a function satisfies the homogeneous equation

$$g\left(z+2\pi\frac{m}{n}\right)-g(z)=0.$$

Now if μ is irrational, then the equation has a unique solution; in fact,

1) *For irrational* μ *equation* (1) *cannot have two distinct continuous solutions.*

Proof. The difference of two continuous solutions of equation (1) satisfies the equations

$$g(z+2\pi)-g(z)=0,$$
$$g(z+2\pi\mu)-g(z)=0;$$

i.e., this continuous function has two incommensurable periods. Such a function is a constant (see [15], pp. 55–56); it takes on one and the same value at all points $2\pi k + 2\pi\mu l$, which form an everywhere dense set. Since

$$\int_0^{2\pi} g(z)\,dz = 0,$$

then the constant in question is zero.

2) *For an irrational* μ *equation* (1) *cannot have two measurable solutions not coinciding almost everywhere.*

Proof. Again we consider the difference of two solutions of (1) and denote it by $g(z)$. It can be considered as a function on the circumference, since it has period 2π. By condition 1

$$g(z + 2\pi\mu) - g(z) = 0;$$

i.e., $g(z)$ does not change under a rotation through the angle $2\pi\mu$. Therefore the set E_a of points of the circumference where $g(z) > a$ is invariant under a rotation through the angle $2\pi\mu$. If the function $g(z)$ is constant almost everywhere, then this constant, as in case 1), is zero. If $g(z)$ is not constant, then for some a the set E_a has a measure satisfying $0 < \text{meas}\, E_a < 2\pi$. But it is well known that a set invariant with respect to rotations by an angle noncommensurable with 2π has measure zero or a complete measure (see, for example, [3]; for the proof it is sufficient to use the theorem on points of density). Thus $g(z) = 0$ almost everywhere.

If the function $f(z)$ is expanded into the Fourier series

$$f(z) = \sum_{n \neq 0} f_n e^{inz},$$

then for the Fourier coefficients of $g(z)$ we have

$$g_n e^{2\pi i\mu n} - g_n = f_n,$$

i.e.,

$$g_n = \frac{f_n}{e^{2\pi i\mu n} - 1}, \quad g(z) = \sum_{n \neq 0} g_n e^{inz}. \tag{2}$$

For rational μ some of the denominators vanish. For irrational μ there are arbitrarily small denominators. We note that

$$|e^{2\pi i\mu n} - 1| > |\mu n - m| \tag{3}$$

for any integer n and some integer m. Therefore the smallness of the denominators in (2) depends on the approximation of μ by rational numbers.

Lemma 1 (see [16]). *Suppose that* $\epsilon > 0$. *For almost every (in the sense of Lebesgue measure)* μ *with* $0 \le \mu \le 1$ *there exists a* $K > 0$ *such that*

$$|\mu n - m| \geqslant \frac{K}{n^{1+\varepsilon}} \tag{4}$$

for any integers m *and* $n > 0$.

Proof. We select any $K > 0$ and estimate the measure of the set E_K of points μ, $0 < \mu < 1$, not satisfying the inequality (4), which we rewrite in the form

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^{2+\varepsilon}}.$$

This set contains all the points m/n with circumferences of radius $K/n^{2+\epsilon}$. For a fixed n the number of these points will be equal to $n+1$, and the common length of the circumferences (on $[0, 1]$) will be equal to $K/n^{1+\epsilon}$. Therefore

$$\operatorname{mes} E_K \leqslant \sum_{n=1}^{\infty} \frac{K}{n^{1+\varepsilon}} = c(\varepsilon) K.$$

The set of points μ, for which the number K required in the lemma does not exist, is contained in E_K for any $K > 0$, so that this measure is less than $c(\epsilon)K$ for any K; i.e., it is equal to zero.

2.2. We shall show that for almost all μ small denominators worsen the convergence of the series (2) only a little.

Lemma 2 (see [17]). *The series*

$$S = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \frac{1}{|n\mu - m_n|} \tag{5}$$

converges for any $\epsilon > 0$ *and any integers* m_n, *if* μ *is such that*

$$|\mu n - m| \geqslant \frac{K}{n^{1+\varepsilon-\delta}} \quad (K > 0, \quad 0 < \delta < \varepsilon) \tag{6}$$

for all integers m *and* $n > 0$.

Proof. Without loss of generality we may suppose that $|\mu n - m_n| < 1$. We consider series S_i of the same type as S, but in which the summation is extended only over those indices $n = n_k^{(i)}$ for which

$$\frac{1}{2^{i+1}} \leqslant |\mu n_k^{(i)} - m_{n_k^{(i)}}| < \frac{1}{2^i} \quad (i = 0, 1, 2, \ldots; \; n_{k+1}^{(i)} > n_k^{(i)}). \tag{7}$$

The series S_i taken together contain all the terms of S, so that it is sufficient to prove that

$$\sum_{i=0}^{\infty} S_i < \infty.$$

To estimate S_i we note that from (6) the successive indices $n_k^{(i)}, n_{k+1}^{(i)}$ of terms of the series S_i are significantly far apart: since from (7) there follows the inequality

$$|\mu (n_k^{(i)} - n_{k+1}^{(i)}) - m| < \frac{1}{2^{i-1}},$$

from (6) we deduce

$$\frac{1}{2^{i-1}} > \frac{K}{N_i^{1+\varepsilon-\delta}},$$

where

$$N_i = \min_{0<k<\infty} (n_{k+1}^{(i)} - n_k^{(i)}).$$

Therefore we obtain

$$N_i > (2^{i-1}K)^{\frac{1}{1+\varepsilon-\delta}}. \tag{8}$$

Evidently $n_1^{(i)} > N_i$, and more generally $n_k^{(i)} > kN_i$, so that in view of (5), (7), (8) we have

$$S_i < \sum_{k=1}^{\infty} \frac{2^{i+1}}{(kN_i)^{1+\varepsilon}} = \frac{2^{i+1}}{N_i^{1+\varepsilon}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} = \frac{2^{i+1}}{2^{(i-1)\frac{1+\varepsilon}{1+\varepsilon-\delta}}} L(\varepsilon, K) \quad (L(\varepsilon, K) > 0),$$

$$S_i < 2^{1+\frac{1+\varepsilon}{1+\varepsilon-\delta}} L 2^{i\left(1-\frac{1+\varepsilon}{1+\varepsilon-\delta}\right)} = L'(\varepsilon, \delta, K)\theta^i.$$

Here

$$\theta = 2^{1-\frac{1+\varepsilon}{1+\varepsilon-\delta}} < 1,$$

so that

$$\sum_{i=0}^{\infty} S_i < \infty,$$

as was required to be proved.

As is well known, if $f(x)$ is a function $p+\epsilon$ times differentiable,* then its Fourier coefficients have an order of decrease

$$f_n = O\left(\frac{1}{n}\right)^{p+\varepsilon},$$

and if

$$f_n = O\left(\frac{1}{n}\right)^{p+1+\varepsilon}$$

then $f(x)$ is differentiable $p+\epsilon$ times. From this and from inequality (3) and Lemmas 1 and 2, applied to the series (2), we obtain the following result:

If the function $f(z)$ is $p+1+\epsilon+\delta$ times differentiable, then for almost all μ equation (1) *has a $p+\epsilon$ times differentiable solution.*

On the other hand, it is not hard to construct examples for which the number

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*I.e., a function whose pth derivative satisfies a Hölder condition of degree ϵ: $|f^{(p)}(x+h) - f^{(p)}(x)| < Ch^{\varepsilon}$.

μ can be approximated by rationals so well that in spite of the rapid decrease of the numerators f_n the series (2) converges slowly or not at all. So even if $f(z)$ is analytic there may appear cases where $g(z)$ is not analytic but is infinitely differentiable, or even only differentiable finitely many times, or only continuous, or even discontinuous, or the solution is not measurable (see [14], [17]).*

2.3. Consider the equation (1) in the class of analytic functions. To investigate this case we recall two lemmas concerning the Fourier coefficients of analytic functions.

Lemma 3. *If the function $f(z)$ of period 2π in the strip $|\mathrm{Im}\ z| \le R$ is analytic and in this strip $|f(z)| \le C$, then its Fourier coefficients satisfy the inequalities*

$$|f_n| \leqslant Ce^{-|n|R}.$$

Proof. By definition,

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(z)\, e^{-inz}\, dz.$$

From the periodicity of $f(z)e^{-inz}$,

$$\int_0^{i\tau} f(z)\, e^{-inz}\, dz = \int_{2\pi}^{2\pi+i\tau} f(z)\, e^{-inz}\, dz,$$

so that

$$f_n = \frac{1}{2\pi}\int_{0+i\tau}^{2\pi+i\tau} f(z)\, e^{-inz}\, dz$$

for any $\tau \in [-R, R]$. Integrating in the case $n > 0$ along the line $\tau = -R$ and for $n < 0$ along $\tau = R$, we obtain

$$|f_n| \leqslant \frac{1}{2\pi}\int_0^{2\pi} Ce^{-|n|R}\, dz,$$

as was required to be proved.

Lemma 4. *Suppose that the Fourier coefficients of $f(z)$ satisfy the inequalities $|f_n| \le Ce^{-|n|R}$. Then $f(z)$ is analytic and satisfies for $|\mathrm{Im}\ z| \le R - \delta$, $0 < \delta < R$, the inequality*

$$|f(z)| \leqslant \frac{2C}{1-e^{-\delta}},$$

and its derivative satisfies the inequality

$$|f'(z)| \leqslant \frac{2C}{(1-e^{-\delta})^2}.$$

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*A. N. Kolmogorov has conjectured that this last case is realized whenever the series $\sum_{n\neq 0} |f_n^2|/|e^{2\pi i\mu n}-1|^2$ diverges.

Proof. For $|\mathrm{Im}\, z| \leq R - \delta$, $0 < \delta < R$ it is evident that

$$|e^{inz}| \leqslant e^{|n|(R-\delta)}.$$

Therefore

$$|f_n e^{inz}| \leqslant Ce^{-|n|\delta}$$

and

$$\sum_{n=-\infty}^{\infty} |f_n e^{inz}| \leqslant 2 \sum_{n=0}^{\infty} Ce^{-n\delta} \leqslant \frac{2C}{1-e^{-\delta}}.$$

In the same way

$$\sum_{n=-\infty}^{\infty} |f_n ine^{inz}| \leqslant 2C \sum_{n=0}^{\infty} ne^{-n\delta} \leqslant \frac{2C}{(1-e^{-\delta})^2}.$$

In the strip $|\mathrm{Im}\, z| \leq R - \delta$ the series converge absolutely uniformly. The lemma is proved.

Now it is not difficult to investigate the analytic solutions of equation (1).

Theorem 1. *Suppose that $f(z) = \tilde{f}(z)$ is an analytic function of period 2π and that, for $|\mathrm{Im}\, z| \leq R$, $|f(z)| \leq C$. Let μ be irrational, $K > 0$ and*

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^3} \tag{9}$$

for any integers m and $n > 0$. Then the equation

$$g(z + 2\pi\mu) - g(z) = f(z)$$

has an analytic solution $g(z) = \tilde{g}(z)$, and for $|\mathrm{Im}\, z| \leq R - 2\delta$ and any $\delta < 1$, $0 < \delta < R/2$,

$$|g(z)| \leqslant \frac{4C}{K\delta^3}, \tag{10}$$

$$|g'(z)| \leqslant \frac{8C}{K\delta^4}. \tag{11}$$

Proof. Applying Lemma 3 for the estimate of the Fourier coefficients f_n of the function $f(z)$, and using inequalities (3) and (9), we obtain from (2)

$$|g_n| \leqslant \frac{C}{K} n^2 e^{-|n|R}. \tag{12}$$

We note the simple inequality

$$|n|^p \leqslant \left(\frac{p}{e}\right)^p \frac{e^{|n|\delta}}{\delta^p}, \tag{13}$$

valid for any $\delta > 0$. In fact $p \ln x < p \ln (p/e) + x$, since the function $p \ln x - x$ has its maximum at $x = p$. Putting $x = \delta |n|$, we obtain (13). Applying (13) to (12) (for $p = 2$), we have

$$|g_n| \leqslant \frac{Ce^{-|n|R} e^{|n|\delta}}{K\delta^2} = \frac{Ce^{-|n|(R-\delta)}}{K\delta^2},$$

so that from Lemma 4 we obtain in the strip $|\mathrm{Im}\, z| \leq R - 2\delta$:

$$|g(z)| \leqslant \frac{2C}{K\delta^2(1-e^{-\delta})}, \quad |g'(z)| \leqslant \frac{2C}{K\delta^2(1-e^{-\delta})^2}.$$

Since $|1-e^{-\delta}| > \delta/2$ for $\delta < 1$, we therefore obtain the inequalities (10) and (11). The theorem is proved.

Remark 1. Evidently the solution is real if $f(z)$ is real on the real axis.

Remark 2. If the function $f(z, \lambda)$ depends analytically on a parameter λ, then *the solution (under the conditions of Theorem 1) also depends analytically on that parameter.*

2.4. We consider equation (1) for complex μ. In this case the solution of the homogeneous equation

$$g(z+2\pi\mu) - g(z) = 0$$

is any doubly periodic function with periods 2π and $2\pi\mu$, so that the solution of the problem is certainly not unique. If we require that $g(z)$ be analytic in a strip *of width greater than* $|\mathrm{Im}\ 2\pi\mu|$, *then the solution of (1) is defined uniquely up to* a constant. Indeed, a strip of that width contains a parallelogram of periods, and a solution of the homogeneous equation analytic in it is bounded in the entire plane; i.e., it is a constant. The condition $\bar{g} = 0$ singles out the unique solution which is given by the series (2). This series converges for any nonreal μ, but we are interested in estimates, and thus we must exclude neighborhoods of rational μ. We shall denote by M_K^r the set of points μ of the rectangle in the complex plane $0 \le \mathrm{Re}\,\mu \le 1$, $|\mathrm{Im}\ \mu| \le r$ such that for all integer m, n the inequality

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3}$$

is satisfied. It is evident that along with μ the points $\bar{\mu}$, $1-\mu$ and $1-\bar{\mu}$ are also contained in M_K^r.

Instead of inequalities (3) we have

$$|e^{2\pi iz} - 1| \geqslant \min\left(\frac{1}{2}, \pi|z-m|\right) \tag{14}$$

for any complex z with some integer n. We shall prove (14). If $|e^{2\pi iz} - 1| \ge \frac{1}{2}$, then (14) is proved. If $|e^{2\pi iz} - 1| < \frac{1}{2}$, then we join the points 1 and $e^{2\pi iz}$ by a segment and consider the integral

$$\frac{1}{2\pi i}\int_1^{e^{2\pi iz}} \frac{dw}{w} = \frac{1}{2\pi i}(\ln e^{2\pi iz} - \ln 1) = z - m,$$

where $\ln w$ is one of the branches of the logarithm and $\ln 1 = 2\pi im$, m an integer. Since the segment of integration lies entirely in the circle

$$|w-1| < \frac{1}{2},$$

and in this circle $|w| > \frac{1}{2}$, we have

$$\left| \int_{1}^{e^{2\pi i z}} \frac{dw}{w} \right| \leqslant 2\,|e^{2\pi i z} - 1|.$$

Therefore

$$|z - m| \leqslant \frac{1}{\pi}\,|e^{2\pi i z} - 1|,$$

as was required to be proved.

If $\mu \in M_K^r$, then by applying (14) to $z = \mu n$ we find

$$|e^{2\pi i \mu n} - 1| \geqslant \min\left(\frac{1}{2}, \frac{\pi K}{n^2}\right).$$

Thus, *if* $\mu \in M_K^r$, *where* $K < 1/2\pi$, *then*

$$|e^{2\pi i \mu n} - 1| \geqslant \frac{\pi K}{n^2}.$$

Theorem 1′. *Suppose that* $f(z) = \tilde{f}(z)$ *is an analytic function of period* 2π *and that* $|f(z)| \leq C$ *for* $|\mathrm{Im}\, z| \leq R$, *and suppose that* $\mu \in M_K^r$, $K < 1/2\pi$. *Then the equation*

$$g(z + 2\pi\mu) - g(z) = f(z) \tag{1}$$

has an analytic solution $g(z) = \tilde{g}(z)$, *and for* $|\mathrm{Im}\,(z - 2\pi\mu)| < R - 2\delta$ *and any* $\delta < 1$, $0 < \delta < R/2$,

$$|g(z)| \leqslant \frac{4C}{\pi K \delta^3}, \quad |g'(z)| \leqslant \frac{8C}{\pi K \delta^4}. \tag{16}$$

Proof. From formula (2) and Lemma 3, we have

$$|g_n e^{inz}| \leqslant \frac{Ce^{-|n|R}}{e^{2\pi i \mu n} - 1}\, e^{in(z - 2\pi\mu + 2\pi\mu)}. \tag{17}$$

But for $|\mathrm{Im}\,(z - 2\pi\mu)| < R - 2\delta$

$$|e^{in(z - 2\pi\mu)}| < e^{|n|(R - 2\delta)},$$

so that it follows from (17) that

$$|g_n e^{inz}| \leqslant \frac{Ce^{-2\delta|n|}}{1 - e^{-2\pi i \mu n}}.$$

Since $1 - \mu \in M_K^r$, we have from (15),

$$|1 - e^{-2\pi i \mu n}| \geqslant \frac{\pi K}{n^2},$$

which means that

$$|g_n e^{inz}| \leqslant \frac{Ce^{-2\delta|n|} n^2}{\pi K}.$$

Hence from (13) it follows that the series $g(z)$ and $g'(z)$ converge, and accordingly the inequalities (16) are valid (see the proofs of Theorem 1 and Lemma 4).

Remark 1. Remark 2 to Theorem 1 applies also to Theorem 1′.

Remark 2. Let us fix the function f and the number z and consider the dependence of the solution just found on μ:

$$g(\mu) = \sum_{n \neq 0} \frac{f_n}{e^{2\pi i \mu n} - 1} e^{inz}. \tag{2}$$

The function $g(\mu)$ is analytic in the upper and lower half-planes, but the axis $\operatorname{Im} \mu = 0$ is a cut. On it the series (2) converges almost everywhere, but to an everywhere discontinuous limit. That does not prevent us in §7 from differentiating the solution with respect to μ even for $\operatorname{Im} \mu = 0$ if we make use of the ideas of Borel [9]. For the time being we shall take the formula

$$\frac{\partial g}{\partial \mu} = - \sum_{n \neq 0} \frac{2\pi i n e^{2\pi i \mu n} f_n}{(e^{2\pi i \mu n} - 1)^2} e^{inz}$$

to have a meaning only in the upper and lower half-planes separately.

§3. Lemmas necessary for the proof of Theorem 2

3.1. **Lemma 5.** *If at each point of the segment* $z_1 z_2$ *the function* $f(z)$ *is analytic and* $|df/dz| \leq L$, *then* $|f(z_2) - f(z_1)| \leq L\,|z_2 - z_1|$.

Proof. Indeed,

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} \frac{df(z)}{dz} dz,$$

from which it follows that

$$|f(z_2) - f(z_1)| \leqslant \int_{z_1}^{z_2} \left| \frac{df(z)}{dz} \right| |dz| \leqslant L |z_2 - z_1|.$$

Remark. The example $f(z) = e^{iz}$, $z_1 = 0$, $z_2 = 2\pi$ shows that in the complex domain the theorem on the finite increment in the form

$$f(z_2) - f(z_1) = \frac{df(\xi)}{dz} (z_2 - z_1)$$

or

$$|f(z_2) - f(z_1)| = \left| \frac{df(\xi)}{dz} \right| |z_2 - z_1|$$

is invalid.

3.2. **Lemma 6** (on implicit functions). *Suppose that the functions* $F(\epsilon)$, $\Phi(\epsilon, \Delta)$ *are analytic and that for* $|\epsilon| \leq \epsilon_0$, $|\Delta| \leq \Delta_0$

$$|F(\varepsilon)| \leqslant M_1, \quad |\Phi(\varepsilon, \Delta)| \leqslant M_2 |\Delta|,$$

where $M_1/(1 - M_2) < \Delta_0/3$ *and* $M_2 < 1/6$. *Then*

1. *The equation* $\Delta + F(\epsilon) + \Phi(\epsilon, \Delta) = 0$ *has analytic solution* $\Delta^*(\epsilon)$, *satisfying for* $|\epsilon| < \epsilon_0$ *the inequality* $|\Delta^*(\epsilon)| \leq M_1/(1 - M_2)$.

2. *The equation* $\Delta + F(\epsilon) + \Phi(\epsilon, \Delta) = \Delta_1$ *has a solution* $\Delta = \Delta(\Delta_1, \epsilon)$,

analytically depending on Δ_1 *and* ϵ, $|\Delta_1| < \Delta_0/6$, $|\epsilon| < \epsilon_0$, *where*

$$|\Delta(\Delta_1, \varepsilon) - \Delta^*(\varepsilon)| \leqslant 2|\Delta_1$$

Proof. The disk $|\Delta| < M_1/(1 - M_2)$ lies, when $M_1/(1 - M_2) < \Delta_0$, $|\epsilon| \leq \epsilon_0$, in the region where $|F(\epsilon)| \leq M_1$, $|\Phi(\epsilon, \Delta)| < M_2|\Delta|$, and therefore under the transformation $\Delta \longrightarrow -F(\epsilon) - \Phi(\epsilon, \Delta)$ is carried inside itself:

$$|F(\varepsilon) + \Phi(\varepsilon, \Delta)| \leqslant M_1 + \frac{M_1}{1 - M_2} M_2 = \frac{M_1}{1 - M_2}.$$

The fixed point of the transformation is the desired solution $\Delta^*(\epsilon)$. Analyticity follows from the usual theorem on implicit functions, since

$$\frac{\partial}{\partial \Delta}(\Delta + F(\varepsilon) + \Phi(\varepsilon, \Delta)) \neq 0,$$

which follows from the estimate of $\partial\Phi/\partial\Delta$ using Cauchy's integral formula: for $|\Delta| \leq 2\Delta_0/3$, $|\epsilon| < \epsilon_0$

$$\left|\frac{\partial \Phi}{\partial \Delta}\right| \leqslant \frac{M_2 \Delta_0}{\frac{\Delta_0}{3}} < \frac{1}{2}.$$

2. Under the transformation $w \longrightarrow w + \Phi(w, \epsilon)$ the point $\Delta^*(\epsilon)$ goes into $-F(\epsilon)$, and the point w of the disk $|w - \Delta^*(\epsilon)| \leq 2|\Delta_1|$ into the point

$$w + \Phi(\Delta^*(\varepsilon), \varepsilon) + [\Phi(w, \varepsilon) - \Phi(\Delta^*(\varepsilon), \varepsilon)].$$

Since under the conditions of the lemma

$$|\Phi(w, \varepsilon) - \Phi(\Delta^*(\varepsilon), \varepsilon)| \leqslant |\Delta_1|$$

for the points of this disk (Lemma 5), the image of the disk $|w - \Delta^*(\epsilon)| \leq 2|\Delta_1|$ contains the entire disk $|w + F(\epsilon)| \leq \Delta_1$ and has the point $\Delta(\Delta_1, \epsilon)$, going into $\Delta_1 - F(\epsilon)$. This point satisfies the inequality

$$|\Delta - \Delta^*| \leqslant 2|\Delta_1|$$

and the equation

$$\Delta = \Delta_1 - F(\varepsilon) - \Phi(\varepsilon, \Delta).$$

Uniqueness and analyticity follows from the inequality $|\partial\Phi/\partial\Delta| < \frac{1}{2}$.

Remark. It is easy to see that if under the conditions of Lemma 6 the functions $F(\epsilon)$ and $\Phi(\epsilon, \Delta)$ are real for real ϵ, Δ, then $\Delta^*(\epsilon)$ and $\Delta(\Delta_1, \epsilon)$ are real for real Δ_1, ϵ.

3.3. **Newton's method** (see [18], [19]). Suppose that we are seeking a solution of the equation $f(x) = 0$ (Figure 1). We determine x roughly as x_0 and find the point of intersection x_1 of the tangent at x_0 to the curve $y = f(x)$ with the x axis:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

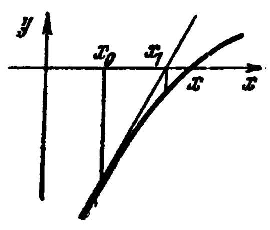

Figure 1

Further, we define successively

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

and estimate the rapidity of convergence of the process.* Suppose that x is the desired solution and $|x_0 - x| = \epsilon$. Then the deviation of the curve from its tangent at the point x_0 has order ϵ^2 at the point x, which means that $|x_1 - x|$ is a quantity of order ϵ^2. Thus after the nth step the error will be of order 2^n, which represents extraordinarily fast convergence.

We shall apply a similar method to the solution of a linear functional equation approximated by the equation considered in §2. The rapid convergence will paralyze the denominators appearing at each step.

§4. Theorem 2 and the Fundamental Lemma

4.1. Heuristic considerations. The transformation

$$z \to z + 2\pi\mu$$

is a rotation of the circumference. The transformation

$$z \to z + 2\pi\mu + \varepsilon F(z)$$

is a rotation perturbed by the term $\epsilon F(z)$, which is small along with ϵ. Its rotation number, even if $\bar{F} = 0$, may be different from $2\pi\mu$. However, we may seek $\Delta = \Delta(\epsilon)$ such that the transformation

$$z \to z + 2\pi\mu + \Delta + \varepsilon F(z)$$

will have a rotation number equal to $2\pi\mu$. We shall show that *for numbers μ that are normally approximable by rational numbers, and sufficiently small ϵ,*

1) *$\Delta(\epsilon)$ depends analytically on ϵ;*

2) *the transformation $z \to z + 2\pi\mu + \Delta + \epsilon F(z)$ may be converted into a rotation through the angle $2\pi\mu$ by an analytic substitution of variables $\phi(z) = z + g(z)$.*

Here $g(z)$ is a correction small with ϵ, and property 2) means that

$$\varphi(z + 2\pi\mu + \Delta(\varepsilon) + \varepsilon F(z), \varepsilon) = \varphi(z, \varepsilon) + 2\pi\mu.$$

or, what is the same thing (the dependence of g on ϵ is implied),

$$g(z + 2\pi\mu + \Delta + \varepsilon F(z)) - g(z) = -\Delta - \varepsilon F(z). \quad (1)$$

This equation differs from that considered in §2 only by small quantities of second order, and therefore it is natural in the first approximation to choose $\Delta = \Delta(\epsilon)$ so that the right side of equation (1) will be equal to zero in the mean:

$$\Delta_1 = -\varepsilon\bar{F}$$

*Here we cite no exact assumptions and estimates. They are given in the paper [18] in a very general form, which, however, does not include the arguments of the following sections.

and to seek $g_1(z)$ as the solution of the equation

$$g_1(z + 2\pi\mu) - g_1(z) = -\varepsilon \bar{F}(z).$$

The g_1 thus defined has order ϵ and in the variable $\phi_1 = z - g_1$ our transformation

$$z \to z + 2\pi\mu + \Delta_1(\varepsilon) + \varepsilon F(z)$$

has the form

$$\begin{aligned}\varphi_1(z + 2\pi\mu + \Delta_1(\varepsilon) + \varepsilon F(z)) &= z + 2\pi\mu + \Delta_1 + \varepsilon F \\ + g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) &= z + g_1(z) + 2\pi\mu \\ + [g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) &- g_1(z + 2\pi\mu)] \\ + [g_1(z + 2\pi\mu) - g_1(z) + \varepsilon \tilde{F}(z)] &+ (\Delta_1 + \varepsilon \bar{F}).\end{aligned}$$

The last two terms vanish because of the choice of Δ_1 and $g_1(z)$ and we obtain

$$\varphi_1(z) \to \varphi_1(z) + 2\pi\mu + F_2(z, \varepsilon).$$

Now the "perturbation" has the form

$$F_2(z, \varepsilon) = g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) - g_1(z + 2\pi\mu) = \frac{dg_1(\xi)}{dz}(\Delta_1 + \varepsilon F).$$

Here dg_1/dz, as also g_1, is a quantity of order ϵ, and, since the same relates to the second factor, the perturbation in the parameter ϕ_1 has order ϵ^2. With the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + F_2$$

one may proceed in the same way and define a "correction to the frequency" Δ_2 and a new parameter ϕ_2 such that the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2$$

in the parameter ϕ_2 goes into the transformation

$$\varphi_2 \to \varphi_2 + 2\pi\mu + F_3,$$

where $F_3 \sim \epsilon^4$. However, here in the parameter z the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2$$

will not have the form

$$z \to z + 2\pi\mu + \Delta + \varepsilon F.$$

Therefore we need to begin with the transformation

$$z \to z + 2\pi\mu + \Delta_1(\varepsilon) + \Delta_1'(\Delta_2) + \varepsilon F;$$

then with a proper choice of $\Delta_1'(\Delta_2)$ we may in the parameter ϕ_1 obtain the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2'(\varphi_1),$$

and in the parameter ϕ_2 the transformation

$$\varphi_2 \to \varphi_2 + 2\pi\mu + F_3',$$

and so forth. The rapid convergence of the method ($F_n \sim \epsilon^{2^{n-1}}$) makes it possible to carry out the limit transition and in the limit to obtain a new parameter

$\phi(z, \epsilon)$ and a final correction $\Delta(\epsilon)$ with the properties 1) and 2).

The usual method of solution of our problem in the theory of perturbations would consist in seeking $\Delta(\epsilon)$ and $\phi(z, \epsilon)$ in the form of series in powers of ϵ, while the coefficients of the series would be successively determined by equation (1) in the first approximation, in the second, and so forth. The proof of convergence of such series by direct estimates has not been achieved, though it results from the following fundamental theorem of this paper.

4.2. **Theorem 2.** *Suppose given a family of analytic transformations of the circumference, depending analytically on two parameters* ϵ, Δ;

$$z \to A(z, \varepsilon, \Delta) \equiv z + 2\pi\mu + \Delta + F(z, \varepsilon) \tag{2}$$

and numbers $R > 0$, $\epsilon_1 > 0$, $K > 0$, $L > 0$ *such that*

1) $F(z + 2\pi, \epsilon) = F(z, \epsilon)$;

2) *for* $\operatorname{Im} z = \operatorname{Im} \epsilon = 0$ *we always have* $\operatorname{Im} F(z, \epsilon) = 0$;

3) *for* $|\operatorname{Im} z| \leq R$, $|\epsilon| \leq \epsilon_0$

$$|F(z, \varepsilon)| \leqslant L|\varepsilon|; \tag{3}$$

4) *the irrational number* μ *for any integers* m *and* n *satisfies the inequality*

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3}. \tag{4}$$

Then there exist numbers ϵ' *and* R', $0 < \epsilon' \leq \epsilon_0$, $0 < R' \leq R$, *and functions* $\Delta(\epsilon)$, $\phi(z, \epsilon)$, *real for real* ϵ *and* z *and analytic for* $|\epsilon| < \epsilon'$, $|\operatorname{Im} z| < R'$, *such that*

$$\varphi(A(z, \varepsilon, \Delta(\varepsilon)), \varepsilon) = \varphi(z, \varepsilon) + 2\pi\mu. \tag{5}$$

This theorem is proved in §6 on the basis of the following lemma.

Fundamental Lemma. *Suppose given a family of analytic transformations of the circumference, depending analytically on the parameters* ϵ, Δ:

$$z \to A_0(z, \varepsilon, \Delta) \equiv z + 2\pi\mu + \Delta + F(z, \varepsilon) + \Phi(z, \varepsilon, \Delta) \tag{6}$$

and numbers $R_0 > 0$, $\epsilon_0 > 0$, $K > 0$, $\delta > 0$, $C > 0$, $0 < \Delta_0 < 1$ *such that*

1) $F(z + 2\pi, \epsilon) = F(z, \epsilon)$ $\Phi(z + 2\pi, \epsilon, \Delta) = \Phi(z, \epsilon, \Delta)$;

2) *for* $\operatorname{Im} z = \operatorname{Im} \epsilon = \operatorname{Im} \Delta = 0$ *always* $\operatorname{Im} F = \operatorname{Im} \Phi = 0$;

3) *for* $|\operatorname{Im} z| \leq R_0$, $|\epsilon| \leq \epsilon_0$, $|\Delta| \leq \Delta_0$

$$|F(z, \varepsilon)| \leqslant C < \delta^8, \tag{7}$$

$$|\Phi(z, \varepsilon, \Delta)| < \delta|\Delta|; \tag{8}$$

4) *the irrational number* μ *for any integers* m *and* n *satisfies the inequality* (4);

5) *the number* δ *satisfies the inequalities*

$$\delta < \frac{K}{64}, \quad \delta < \frac{R_0}{8}, \tag{9}$$

$$\delta < \frac{1}{36}, \tag{10}$$

and moreover

$$C < \frac{\Delta_0}{6}. \tag{11}$$

Then there exist analytic functions $z(\phi, \epsilon)$, $\Delta(\Delta_1, \epsilon)$, $F_1(\phi, \epsilon)$, $\Phi_1(\phi, \epsilon, \Delta_1)$ *such that*

1. *Identically*

$$z[A_1(\varphi, \varepsilon, \Delta_1), \varepsilon] = A_0[z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)], \tag{12}$$

where

$$A_1(\varphi, \varepsilon, \Delta_1) \equiv \varphi + 2\pi\mu + \Delta_1 + F_1(\varphi, \varepsilon) + \Phi_1(\varphi, \varepsilon, \Delta_1). \tag{13}$$

2. $F_1(\varphi + 2\pi, \varepsilon) = F_1(\varphi, \varepsilon)$, $\Phi_1(\varphi + 2\pi, \varepsilon, \Delta_1) = \Phi_1(\varphi, \varepsilon, \Delta_1)$; $z(\varphi+2\pi, \varepsilon) = z(\varphi, \varepsilon) + 2\pi$.

3. *For* $\operatorname{Im}\varphi = \operatorname{Im}\Delta_1 = \operatorname{Im}\varepsilon = 0$ *always* $\operatorname{Im} z = \operatorname{Im}\Delta = \operatorname{Im} F_1 = \operatorname{Im}\Phi_1 = 0$.

4. *For* $|\Delta_1| \leqslant C$, $|\operatorname{Im}\varphi| \leqslant R_0 - 7\delta$, $|\varepsilon| \leqslant \varepsilon_0$

$$|F_1(\varphi, \varepsilon)| \leqslant \frac{C^2}{\delta^6}, \tag{14}$$

$$|\Phi_1(\varphi, \varepsilon, \Delta_1)| \leqslant \delta^2 |\Delta_1|, \tag{15}$$

$$|z(\varphi, \varepsilon) - \varphi| \leqslant \frac{C}{\delta^4}, \quad \left|\frac{\partial z}{\partial \varphi}\right| < 2, \tag{16}$$

$$|\Delta(\Delta_1, \varepsilon)| \leqslant \Delta_0, \quad \left|\frac{\partial \Delta}{\partial \Delta_1}\right| < 2. \tag{17}$$

The Fundamental Lemma shows that small (of order C) perturbations of the rotation $z \to z + 2\pi\mu$ may be compensated by the change in the parameter $z \to \phi$ for $\Delta = \Delta(\Delta_1, \epsilon)$, so that in the new parameter the difference from a rotation will be of order C^2. The proof of the lemma is given in the next section.

4.3. In §11 we shall use the following assertion.

Corollary to Theorem 3. *Suppose that the irrational number* μ *satisfies inequality* (4) *of Theorem* 2, *and suppose that* $R > 0$. *Then there exists a* $C(R, K) > 0$ *such that if the transformation*

$$Az : z \to z + 2\pi\mu + F(z)$$

has a rotation number $2\pi\mu$ *and* $|F(z)| \leq C$ *for* $|\operatorname{Im} z| \leq R$, *then* Az *may be converted into a rotation by the angle* $2\pi\mu$ *by an analytic change of variables.*

Proof. Consider the function

$$F_1(z) = \frac{F(z)}{\max\limits_{|\operatorname{Im} z| \leqslant R} |F(z)|}$$

and the family of transformations

$$A_\varepsilon z : z \to z + 2\pi\mu + \varepsilon F_1(z),$$

satisfying the conditions of Theorem 2 for $L = 1$, since $|F_1(z)| \leq 1$ for $|\operatorname{Im} z| \leq R$. According to Theorem 2, there exists an $\epsilon'(R, K) > 0$ such that for $\epsilon < \epsilon'$ the transformation

$$z \to z + 2\pi\mu + \Delta(\varepsilon) + \varepsilon F_1(z)$$

can be converted into a rotation through the angle $2\pi\mu$. Choose $C(R, K) < \epsilon'$. Then, if $|F(z)| \geq C$ for $|\operatorname{Im} z| \leq R$, there exists a Δ such that

$$z \to z + 2\pi\mu + \Delta + F(z)$$

can be turned by an analytic transformation of coordinates into a rotation through the angle $2\pi\mu$, since

$$F(z) = \max_{|\operatorname{Im} z| \leqslant R} |F(z)| F_1(z),$$

and

$$\max |F(z)| \leqslant C < \varepsilon'.$$

But the rotation number of Az is equal to $2\pi\mu$, from which it follows that $\Delta = 0$ (see item 2 in the proof of Theorem 4 in §10, where it is shown that for an arbitrarily small Δ the rotation number of the transformation $z \to z + 2\pi\mu + \Delta + F(z)$ is larger than $2\pi\mu$). The corollary is proved.

The assertion of the corollary may be obtained directly as well, using constructions analogous to those of Theorem 2. Because of the absence of the parameters ϵ and Δ, these constructions will be less clumsy.

4.4. **Remark on the multidimensional case.** All the constructions of §§2–8 may be considered to be multidimensional if we replace a point of the circumference by a point of a torus of k variables. Condition 4) of Theorem 2 is replaced by the following condition of "incommensurability" for the vector $\vec{\mu}$:

$$|n_0 + (\vec{\mu}, \vec{n})| \geqslant \frac{K}{|\vec{n}|^\omega} \tag{18}$$

for any integer vector $\vec{n} = (n_0, \cdots, n_k)$. Here $(\vec{\mu}, \vec{n})$ is the scalar product

$$\sum_{i=1}^{k} \mu_i n_i, \quad |\vec{n}| = \sum_{i=0}^{k} |n_i|.$$

For sufficiently large ω condition (18) is satisfied for almost all vectors $\vec{\mu}$.

Without dwelling in detail on the formulations and proofs of all the inequalities, lemmas and theorems for the multidimensional case, we present only one result.

Multidimensional Theorem 2. *Suppose that $\vec{\mu} = (\mu_1, \cdots, \mu_k)$ is a vector with incommensurable components such that for any integer vector $\vec{n}$*

$$|n_0 + (\vec{\mu}, \vec{n})| > \frac{K}{|\vec{n}|^{k+1}} .$$

Then there exists an $\epsilon(R, C, k) > 0$ *such that for the vector field* $\vec{F}(\vec{z})$ *on the torus, analytic and sufficiently small,* $|\vec{F}(\vec{z})| < \epsilon$ *for* $|\text{Im}\, \vec{z}| < R$*, there exists a vector* $\vec{a}$ *for which the transformation*

$$\vec{z} \to \vec{z} + \vec{a} + \vec{F}(\vec{z})$$

of the torus into itself is converted into

$$\vec{\varphi} \to \vec{\varphi} + 2\pi\vec{\mu}$$

by an analytic substitution of variables.

§5 Proof of the Fundamental Lemma

5.1. Construction of $z(\phi, \epsilon)$, $\Delta(\Delta_1, \epsilon)$, $F_1(\phi, \epsilon)$ **and** $\Phi_1(\phi, \epsilon, \Delta_1)$. **The function** $z(\phi, \epsilon)$ **is constructed as the inverse to**

$$\varphi(z, \varepsilon) = z + g(z, \varepsilon), \tag{1}$$

and the function $\Delta(\Delta_1, \epsilon)$ **as the inverse to** $\Delta_1(\Delta, \epsilon)$. **In subsection 4.1 we saw these functions had to be chosen so that the expression**

$$g(A_0(z, \varepsilon, \Delta), \varepsilon) - g(z, \varepsilon) + F(z, \varepsilon) + \Delta + \Phi(z, \varepsilon, \Delta)$$

would be small. Without defining $\Delta(\Delta_1, \epsilon)$ **for the time being (i.e., considering** Δ **as an independent variable) we define** $g^*(z, \epsilon, \Delta)$ **as the solution of the equation**

$$g^*(z + 2\pi\mu, \varepsilon, \Delta) - g^*(z, \varepsilon, \Delta) = -\tilde{F}(z, \varepsilon) - \tilde{\Phi}(z, \varepsilon, \Delta). \tag{2}$$

Expressing the transformation A_0 **(see §4, formula (6)) in terms of the parameter**

$$\varphi^*(z, \varepsilon, \Delta) = z + g^*(z, \varepsilon, \Delta),$$

we obtain

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = z + 2\pi\mu + \Delta + F(z, \varepsilon) + \Phi(z, \varepsilon, \Delta) \\ + g^*(z + 2\pi\mu, \varepsilon, \Delta) + g^*(A_0(z, \varepsilon, \Delta)) - g^*(z + 2\pi\mu, \varepsilon, \Delta),$$

or, transforming the right side by means of (2),

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = z + g^*(z, \varepsilon, \Delta) + 2\pi\mu + \Delta + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta) \\ + g^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] - g^*(z + 2\pi\mu, \varepsilon, \Delta).$$

Thus from (1) we obtain

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = \varphi^*(z, \varepsilon, \Delta) + 2\pi\mu + \Delta + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta) + \\ + g^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] - g^*(z + 2\pi\mu, \varepsilon, \Delta).$$

We define $\Delta_0^*(\epsilon)$ **as the solution of the equation**

$$\Delta_0^*(\varepsilon) + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta_0(\varepsilon)) = 0 \tag{4}$$

and put

$$g^*(z, \varepsilon, \Delta_0^*(\varepsilon)) = g(z, \varepsilon). \tag{5}$$

Now the new parameter $\phi(z, \epsilon)$ is defined by equations (5) and (1). We represent (3) in the form

$$\varphi[A_0(z, \varepsilon, \Delta), \varepsilon] = \varphi(z, \varepsilon) + 2\pi\mu + \Delta_1(\varepsilon, \Delta) + \hat{F}_1(z, \varepsilon) + \hat{\Phi}_1(z, \varepsilon, \Delta), \tag{6}$$

where

$$\hat{F}_1(z, \varepsilon) = g(z_{\mathrm{I}}, \varepsilon) - g(z_{\mathrm{II}}, \varepsilon), \tag{7}$$

$$\hat{\Phi}_1(z, \varepsilon, \Delta) = g(z_{\mathrm{III}}, \varepsilon) - g(z_{\mathrm{I}}, \varepsilon), \tag{8}$$

$$\Delta_1(\varepsilon, \Delta) = \Delta + \overline{F}(\varepsilon) + \overline{\Phi}(\varepsilon, \Delta), \tag{9}$$

$$z_{\mathrm{I}} = z + 2\pi\mu + \widetilde{F}(z, \varepsilon) + \hat{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon)), \tag{10}$$

$$z_{\mathrm{II}} = z + 2\pi\mu, \tag{11}$$

$$z_{\mathrm{III}} = z + 2\pi\mu + \widetilde{F}(z, \varepsilon) + \Delta_1(\varepsilon, \Delta) + \widetilde{\Phi}(z, \varepsilon, \Delta). \tag{12}$$

We define $z(\phi, \epsilon)$ from (1), $\Delta(\Delta_1, \epsilon)$ from (9), and write

$$F_1(\varphi, \varepsilon) = \hat{F}_1(z(\varphi, \varepsilon), \varepsilon), \tag{13}$$

$$\Phi_1(\varphi, \varepsilon, \Delta_1) = \hat{\Phi}_1(z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)), \tag{14}$$

$$A_1(\varphi, \varepsilon, \Delta_1) = \varphi[A_0(z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)), \varepsilon]. \tag{15}$$

5.2. We shall prove that the functions just constructed are those sought. Assertions 1, 2, and 3 of the Fundamental Lemma are satisfied in an obvious way. The proof of assertion 4 is based on the following estimates.

1°. Estimate of $\Delta_0^*(\epsilon)$. On the basis of the inequalities (10), (11) of §4, Lemma 6 of §3 is applicable to equation (4). Here $M_1 = C$, $M_2 = \delta$, and since

$$\frac{C}{1-\delta} < \frac{\Delta_0}{3}, \quad \delta < \frac{1}{2}$$

(see formulas (10), (11) of §4),

$$|\Delta_0^*(\varepsilon)| < \frac{C}{1-\delta}.$$

Taking into account that $\delta < \frac{1}{2}$, we find for $|\epsilon| < \epsilon_0$ that

$$|\Delta_0^*(\varepsilon)| < 2C. \tag{16}$$

2°. Estimate of $g(z, \epsilon)$. Inequality (16) makes it possible to estimate the right side of equation (2). For $|\mathrm{Im}\, z| < R$, $|\epsilon| \le \epsilon_0$, $\Delta = \Delta_0^*(\epsilon)$, from (16) and inequalities (7), (8), (10) of §4 it follows that

$$|\widetilde{F}(z, \varepsilon) + \widetilde{\Phi}(z, \varepsilon, \Delta)| \leqslant 2C + 2\delta \cdot 2C < 4C. \tag{17}$$

Applying Theorem 1 of §2 to equation (2), we obtain on the basis of (5), (17) and condition 4) of the Fundamental Lemma that for $|\mathrm{Im}\, z| \le R_0 - 2\delta$, $|\epsilon| \le \epsilon_0$ and any $\delta < 1$, $0 < \delta < R_0/2$,

$$|g(z, \varepsilon)| < \frac{8\cdot 4C}{K\delta^3}, \quad \left|\frac{\partial g}{\partial z}\right| < \frac{16\cdot 4C}{K\delta^4},$$

from which, in view of inequality (9) of §4,

$$|g(z, \varepsilon)| < \frac{C}{\delta^4}, \quad \left|\frac{\partial g(z, \varepsilon)}{\partial z}\right| < \frac{C}{\delta^5}. \tag{18}$$

Since $C < \delta^8$ by inequality (7) of §4, it follows that

$$|g(z, \varepsilon)| < \delta.$$

Therefore under the mapping $z \to \phi(z, \epsilon) = z + g(z, \epsilon)$ the strip

$$|\mathrm{Im}\, z| \leqslant R_0 - 2\delta$$

goes into a region containing the strip

$$|\mathrm{Im}\, \varphi| \leqslant R_0 - 3\delta.$$

In the latter the inverse function is analytic, since $|\partial\phi/\partial z| > \frac{1}{2}$ for $|\mathrm{Im}\, z| < R_0 - 2\delta$. In the same way one proves inequality (16) of §4.

3°. Estimate of $F_1(\phi, \epsilon)$. Suppose that $|\mathrm{Im}\, z| < R_0 - 3\delta$, $|\epsilon| \le \epsilon_0$. Since, from inequality (16) and conditions 3) and 5) of the Fundamental Lemma,

$$F_1(\varphi, \varepsilon) = \hat{F}_1(z(\varphi, \varepsilon), \varepsilon),$$

the imaginary parts z_{I} and z_{II} (see (10) and (11)) do not exceed $R_0 - 2\delta$. Applying Lemma 5 of §3, we find on the basis of (17) and (18) that for $|\mathrm{Im}\, z| < R_0 - 3\delta$, $|\epsilon| \le \epsilon_0$

$$|\hat{F}(z, \varepsilon)| \leqslant \frac{4C^2}{\delta^5}. \tag{19}$$

We note that the appearance of C^2 in this inequality is the most essential feature of the proof of Theorem 2.

For $|\mathrm{Im}\, \phi| \le R_0 - 4\delta$ and $|\epsilon| \le \epsilon_0$ we have from 2°

$$|\mathrm{Im}\, z(\varphi, \varepsilon)| < R_0 - 3\delta,$$

and therefore estimate (14) of §4 follows from (19) in view of the definition of $F_1(\phi, \epsilon)$ and inequality (10) of §4.

4°. Estimate of $|\Delta(\Delta_1, \epsilon) - \Delta_0^*(\epsilon)|$. The equation

$$\Delta = \Delta_1 - \overline{F}(\varepsilon) - \overline{\Phi}(\varepsilon, \Delta),$$

defining $\Delta(\Delta_1, \epsilon)$, belongs to the type considered in Lemma 6 of §3. We have seen (see (16)) that $|\Delta_0^*(\epsilon)| < 2C$, from which, on the basis of formula (11) of §4, it results that

$$|\Delta_0^*(\varepsilon)| < \frac{\Delta_0}{3}. \tag{20}$$

Thus Lemma 6 is applicable, and for $|\Delta_1| \le C < \Delta_0/6$, $|\epsilon| \le \epsilon_0$

$$|\Delta(\Delta_1, \varepsilon) - \Delta_0^*(\varepsilon)| < 2|\Delta_1|. \tag{21}$$

Comparing (20) and (21), we find that for $|\epsilon| \le \epsilon_0$, $|\Delta_1| \le C$

$$|\Delta(\Delta_1, \varepsilon)| < \frac{2}{3}\Delta_0.$$

For $|\epsilon| < \epsilon_0$, $|\Delta| < (2/3)\Delta_0$, from Cauchy's formula we have

$$\left|\frac{\partial\Phi}{\partial\Delta}\right| < \frac{\delta\Delta_0}{\frac{\Delta_0}{3}} < \frac{1}{2}$$

(see inequalities (8), (10) of §4). Estimate (17) of §4 is proved since it is evident that

$$\left|\frac{\partial\Delta}{\partial\Delta_1}\right| = \left|\frac{1}{1 + \frac{\partial\Phi}{\partial\Delta}}\right| < 2.$$

5°. Estimate of $|\Phi_1(\phi, \epsilon, \Delta_1)|$. Let us set up the difference $z_{III} - z_I$. From formulas (12) and (10) it is equal to

$$\Delta_1 + \tilde{\Phi}(z, \varepsilon, \Delta(\Delta_1, \varepsilon)) - \tilde{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon)).$$

From Lemma 5 of §3, for $|\mathrm{Im}\, z| \le R_0$, $|\epsilon| \le \epsilon_0$, $|\Delta_1| < \Delta_0/6$

$$|\tilde{\Phi}(z, \varepsilon, \Delta(\Delta_1, \varepsilon)) - \tilde{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon))| < |\Delta - \Delta_0^*|,$$

since $|\partial\tilde{\Phi}/\partial\Delta| < 1$. Comparing the inequality just obtained with inequality (21), we have

$$|z_{III} - z_I| < 3|\Delta_1|. \tag{22}$$

Applying Lemma 5 of §3 to the right side of (8), on the basis of (22), (18) and inequalities (7), (10) of §4 we find that

$$|\hat{\Phi}_1(z, \varepsilon, \Delta)| < \frac{C}{\delta^5} 3|\Delta_1| < \delta^2|\Delta_1| \tag{23}$$

under the condition that $|\epsilon| \le \epsilon_0$, $|\Delta_1| < \Delta_0/6$,

$$|\mathrm{Im}(z + \Delta_1 + \tilde{F} + \tilde{\Phi})| \leqslant R_0 - 2\delta.$$

This last inequality is satisfied if

$$|\mathrm{Im}\, z| < R_0 - 6\delta, \quad |\Delta_1| < C, \quad |\varepsilon| < \varepsilon_0.$$

Indeed, then

$$|\tilde{F} + \tilde{\Phi}| < \delta + 2\delta\Delta_0 < 3\delta$$

(see formulas (7), (8), (17) of §4 and inequality (20)) in both the terms z_{III} and z_I. For $|\mathrm{Im}\,\phi| \le R_0 - 7\delta$, $|\Delta_1| < C$ we have, from 2°,

$$|\operatorname{Im} z| < R_0 - 6\delta.$$

Therefore estimate (15) of §4 follows from (23).

The Fundamental Lemma is proved.

§6. Proof of Theorem 2

6.1. Construction of $z(\phi, \epsilon)$ and $\Delta(\epsilon)$. We put $\Phi = 0$ in the Fundamental Lemma, and as $F(z, \epsilon)$ we take the function $F(z, \epsilon)$ of Theorem 2. We choose $\delta_1 > 0$ so that

1) $\sum_{n=1}^{\infty} \delta_n < \frac{R_0}{8}$, where $\delta_n = \delta_{n-1}^{1\frac{1}{2}}$ $(n = 2, 3, \ldots)$;

2) $\delta_1 < \frac{K}{64}$, $\delta_1 < \frac{1}{36}$.

Let $6\delta_1^{12} < \Delta_0 < 1$, $R = R_0$, K be the same as in the condition of the theorem. Let $L\epsilon' < C_1 = \delta_1^{12}$, $0 < \epsilon' < \epsilon_0$, C_1 and δ_1 be respectively ϵ_0, C and δ of the Fundamental Lemma. Then all the hypotheses of that lemma are satisfied, and for $|\operatorname{Im} \phi_1| \le R - 7\delta_1$, $|\epsilon| \le \epsilon'$, $|\Delta_1| \le C_1$, we find that

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_1 + F_1(\varphi_1, \varepsilon) + \Phi_1(\varphi_1, \varepsilon, \Delta_1),$$

where

$$|F_1(\varphi_1, \varepsilon)| \leqslant \delta_1^{18} = \delta_2^{12}, \tag{1}$$

$$|\Phi_1(\varphi_1, \varepsilon, \Delta_1)| \leqslant \delta_1^2 |\Delta_1| < \delta_2 |\Delta_1|, \tag{2}$$

$$|z(\varphi_1, \varepsilon) - \varphi_1| \leqslant \delta_1, \quad \left|\frac{\partial z}{\partial \varphi_1}\right| < 2, \tag{3}$$

$$|\Delta(\Delta_1, \varepsilon)| \leqslant \Delta_0, \tag{4}$$

$$\left|\frac{\partial \Delta}{\partial \Delta_1}\right| < 2. \tag{5}$$

More generally, if the functions*

$$\Delta_{k-1}(\Delta_k, \varepsilon), \quad F_k(\varphi_k, \varepsilon), \quad \Phi_k(\varphi_k, \varepsilon, \Delta_k), \quad \varphi_{k-1}(\varphi_k, \varepsilon), \quad A_k(\varphi_k, \varepsilon, \Delta_k),$$

are defined for $k = 1, 2, \cdots, n$ and satisfy the conclusion of the Fundamental Lemma with z replaced by ϕ_{k-1}^*, ϕ by ϕ_k, R_0 by R_{k-1}, $R_0 - 7\delta$ by $R_k = R_{k-1} - 7\delta_k$, Δ_0 by δ_{k-1}^*, A_0 by A_{k-1}, A_1 by A_k, δ by δ_k, C by $C_k = \delta_k^{12}$ for each $k = 1, 2, \cdots, n$, then we may introduce functions ϕ_{n+1} and Δ_{n+1} such that the conclusion of the Fundamental Lemma will be valid for them for $k = 1, 2, \cdots, n + 1$. Indeed, inequalities (9) and (10) are satisfied for δ_n from the definition of δ_1, (11) follows from the inequality $C_{k+1} = C_k^{3/2} < (1/6)\, C_k$, and all the other

* ϕ_0 denotes z, C_0 denotes Δ_0; $\Delta_{1-1}(\Delta_1, \varepsilon) = \Delta(\Delta_1, \varepsilon)$.

conditions of the lemma enter into the conclusion (of course, for the functions with the preceding index). Therefore we may consider all the functions indicated above as having been constructed. The functions $\phi_{n-1}(\phi_n, \epsilon)$, $\Delta_{n-1}(\Delta_n, \epsilon)$ ($n = N, N-1, \cdots, 1$) define the functions

$$z^{(N)}(\varphi_N, \varepsilon) = z(\varphi_1(\ldots(\varphi_N, \varepsilon)\ldots), \varepsilon), \tag{6}$$

$$\Delta_0^{(N)}(\Delta_N, \varepsilon) = \Delta(\Delta_1(\ldots(\Delta_N, \varepsilon)\ldots), \varepsilon). \tag{7}$$

Put $\Delta_N = 0$, and suppose that $\Delta_0^{(N)}(0, \epsilon) = \Delta^{(N)}(\epsilon)$. Then

$$\Delta(\varepsilon) = \lim_{N\to\infty} \Delta^{(N)}(\varepsilon),$$

$$z(\varphi, \varepsilon) = \lim_{N\to\infty} z^{(N)}(\varphi, \varepsilon).$$

For the basis of the convergence of $\Delta^{(N)}(\epsilon)$ and $z^{(N)}(\phi, \epsilon)$ we note first of all that from the definition of δ_n, for $\omega > 0$

$$\lim_{N\to\infty} 2^N \delta_N^{\omega} = 0.$$

6.2. Convergence of $\Delta^{(N)}(\epsilon)$. The functions $\Delta_0^{(N)}(\Delta_N, \epsilon)$, as follows from formula (7) and from inequality (17) of §4, are defined for $|\epsilon| \le \epsilon_0$, $|\Delta_N| \le \delta_N^{12}$. Since

$$\frac{\partial \Delta_0^{(N)}}{\partial \Delta_N} = \frac{\partial \Delta}{\partial \Delta_1} \cdots \frac{\partial \Delta_{N-1}}{\partial \Delta_N},$$

in the indicated region, on the basis of (5), the inequality

$$\left|\frac{\partial \Delta_0^{(N)}}{\partial \Delta_N}\right| < 2^N$$

is satisfied, and since

$$|\Delta_N\ [\Delta_{N+1}(\ldots(\Delta_M, \varepsilon)\ldots \varepsilon), \varepsilon]| \leqslant \delta_N^{12}$$

if $|\Delta_M| \le \delta_M^{12}$ ($M \ge N$), therefore from Lemma 5 of §3,

$$|\Delta_0^N [\Delta_N(\Delta_{N+1}\ldots(\Delta_M, \varepsilon)\ldots, \varepsilon), \varepsilon] - \Delta_0^{(N)}(0, \varepsilon)| < 2^N \delta_N^{12}.$$

Thus in view of (7) we deduce that

$$|\Delta^{(N)}(\varepsilon) - \Delta^{(M)}(\varepsilon)| < 2^N \delta_N^{12},$$

from which it immediately follows that $\Delta^{(N)}(\epsilon)$ converges for $|\epsilon| \le \epsilon_0$, and also that $\Delta(\epsilon)$ is analytic.

6.3. Convergence of $z^{(N)}(\phi, \epsilon)$. From the Fundamental Lemma, the functions $\phi_{n-1}(\phi_n, \epsilon)$ are defined for $|\mathrm{Im}\ \phi_n| \le R$, $|\epsilon| \le \epsilon_0$, and, in view of (3), differ from their arguments ϕ_n by less than δ_n, so that

$$|\mathrm{Im}\, \varphi_{n-1}(\varphi_n, \varepsilon)| < R_{n-1}.$$

Thus formula (6) defines $z^{(N)}(\phi, \epsilon)$ in the strip

$$|\operatorname{Im} \varphi| \leqslant R_n = R_0 - 7 \sum_{k=1}^{n} \delta_k.$$

From condition 1) on the choice of δ_1, all these strips contain the strip $|\operatorname{Im} \phi| \leq R/8$, so that all the functions $z^{(N)}(\phi, \epsilon)$ are defined in the latter.

Since

$$|\varphi_N(\varphi_{N+1} \cdots (\varphi_M, \varepsilon), \ldots, \varepsilon) - \varphi_M| < \sum_{k=N}^{M} \delta_k,$$

and this sum, from the definition of δ_n, is not larger than $2\delta_N$, we find from (6) that

$$|z^{(N)}(\varphi, \varepsilon) - z^{(M)}(\varphi, \varepsilon)| < \left|\frac{\partial z^{(N)}}{\partial \varphi}\right| 2\delta_N.$$

On the basis of (3),

$$\left|\frac{\partial z^{(N)}}{\partial \varphi}\right| < 2^N,$$

so that

$$|z^{(N)}(\varphi, \varepsilon) - z^{(M)}(\varphi, \varepsilon)| < 2^{N+1}\delta_N,$$

which proves the uniform convergence of $z^{(N)}(\phi, \epsilon)$ for $|\operatorname{Im} \phi| \leq R/8$, $|\epsilon| \leq \epsilon_0$.

6.4. We shall define $\phi(z, \epsilon)$ as the inverse to $z(\phi, \epsilon)$. From inequalities (1) and (2) and from the fact that $\delta_n \to 0$ as $n \to \infty$ it results that

$$\varphi(z, \varepsilon) \to \varphi(z, \varepsilon) + 2\pi\mu$$

when $z \to A(z, \epsilon, \Delta(\epsilon))$. Theorem 2 is proved.

§7. On monogenic functions

7.1. **The concept of monogeneity.** In the investigation of the dependence of the solutions of equation (1) of §2 on the parameter μ we encounter functions analytic in the upper and in the lower half-plane, and everywhere discontinuous on the real axis. All the functions, Δ_n, g_n, ϕ_n, F_n, Φ_n constructed in §6, considered as functions of μ, have these properties (see §8). These functions belong to the type called by Borel [9] monogenic.

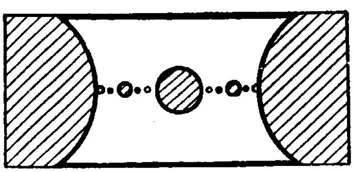

Figure 2

The monogenic functions of Borel are defined on the set $E = \bigcup_{k=1}^{\infty} E_k$, where $E_k \subseteq E_{k+1}$ are perfect compact subsets of the complex plane. In our case E_k is the set M_K^R of points μ of the rectangle $|\operatorname{Im} \mu| \leq R$, $0 \leq \operatorname{Re} \mu \leq 1$ of the complex plane, for which

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3} \quad \left(K = \frac{1}{k}\right),$$

i.e., the set formed by rejecting from the rectangle $|\operatorname{Im} \mu| \leq R$, $0 \leq \operatorname{Re} \mu \leq 1$ the circles $C_{m/n,K}$, shaded in Figure 2, of radii $K/|n|^3$ with centers at rational points m/n.

Definition. A function $f(\mu)$ is said to be *uniformly differentiable* on a perfect compact set F of the complex plane, and the function $g(\mu)$ its derivative, if for any $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$\left|\frac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} - g(\mu_3)\right| < \varepsilon,$$

whenever $|\mu_1 - \mu_3| < \delta$, $|\mu_2 - \mu_3| < \delta$, $\mu_1, \mu_2, \mu_3 \in F$.

A function is *monogenic* on $E = \bigcup_{k=1}^{\infty} E_k$ if it is uniformly differentiable on each E_k.

In particular, a uniformly differentiable function on E is monogenic on $E = \bigcup_{k=1}^{1} E_k$, and conversely a function monogenic on $E = \bigcup_{k=1}^{1} E_k$ is uniformly differentiable on E. Such functions will be called monogenic on E, in distinction to those that are monogenic on $E = \bigcup_{k=1}^{\infty} E_k$.

The following properties of monogenic functions are evident.

1) From monogenicity on $E = \bigcup_{k=1}^{\infty} E_k$ follows continuity of the derivative on E_k.

2) If Γ is a rectifiable curve joining two points α and β in E_k, then

$$\int_{\Gamma} f'(\mu)\, d\mu = f(\beta) - f(\alpha).$$

3) If a function is analytic in a neighborhood of each point of a set, it is monogenic on the set.

4) If E_k contains a region, then a function in it which is monogenic on $E = \bigcup_{k=1}^{\infty} E_k$ is analytic.

An example of a nonanalytic monogenic function was constructed in §2, as is proved in subsection 7.4 (see Lemma 10; the fact that $g(\mu)$ is not analytic for $\operatorname{Im} \mu = 0$ is left to the reader to prove).

Properties of monogenicity of a function may essentially depend on its region of definition $E = \bigcup_{k=1}^{\infty} E_k$ and on the decomposition of E into the E_k. If the rapidity of decrease of the components of the complements to the E_k is sufficiently great, then, as Borel proved, monogenic functions on $E = \bigcup_{k=1}^{\infty} E_k$ have many properties of analytic functions (Cauchy integral, infinite differentiability, uniqueness of the monogenic prolongation). The question as to which of these properties are preserved in our case will be left aside, since in the sequel

(§§8 and 11) we use only the definition of uniform differentiability.

The class of functions monogenic on $E = \bigcup_{k=1}^{\infty} E_k$ depends not only on E but also on E_k. However, if E is obtained by using another system of sets, $E = \bigcup_{k=1}^{\infty} F_k$, $F_k \subseteq F_{k+1}$, such that

$$E_{\alpha k} \subseteq F_k \subseteq E_{\beta k} \quad (\alpha < 1 < \beta),$$

then the class of functions monogenic on $E = \bigcup_{k=1}^{\infty} E_k$ and on $E = \bigcup_{k=1}^{\infty} F_k$ coincide. The sets M_K^R (Figure 2) are not convenient for the investigation of monogenic functions because of the complex character of the intersections of the disks $C_{m/n,K}$. Making use of the above remark, we replace these sets by another system of sets N_K^R such that:

1. $M_{2K}^R \subseteq N_K^R \subseteq M_{\frac{K}{2}}^R$.

2. The set N_K^R is obtained from the rectangle $|\operatorname{Im} \mu| \leq R$, $\operatorname{Re} \mu \in [0, 1]$ by deleting nonintersecting open disks.

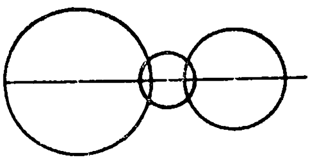

Figure 3

The construction of the N_K^R $(K < 1/9)$ is given in subsection 7.2; it is complex and may be omitted by the reader.

7.2. **Construction of the N_K^R.** The transformation of the M_K^R into the N_K^R consists of two operations. First the disks being deleted are diminished to disks $C'_{m/n,K}$ so that in the system $C'_{m/n,K}$ $(m = 0, 1, \cdots; n = 1, 2, \cdots)$ there are no "bridges" (see Figure 3), i.e., triples of disks of which the smallest intersects both the larger, while these latter do not intersect one another. Then the disks C' are increased to disks $C''_{m/n,K}$ so that two such disks either do not intersect or else one lies inside the other. Here it is necessary to order them so that

$$C_{\frac{m}{n}, K} \supseteq C'_{\frac{m}{n}, K} \supseteq C_{\frac{m}{n}, \frac{K}{2}},$$

$$C'_{\frac{m}{n}, K} \subseteq C''_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, 2K}$$

Then

$$C_{\frac{m}{n}, \frac{K}{2}} \subseteq C''_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, 2K}$$

and after deletion from the rectangle of the disks $C''_{m/n,K}$ there remains a set N_K^R, having both of the needed properties.

Lemma 7. *Suppose that the disks* $C_{m/n,K}$ *and* $C_{p/q,K}$ $(n \geq q)$ *intersect* *and* $K < 1/9$*. Then* $n > 2q^{4/3}$*; i.e., the smaller disk is much smaller than the larger.*

Proof. Indeed, the sum of the radii of the circles is larger than the distance between their centers, so that

$$\frac{K}{n^3} + \frac{K}{q^3} > \left| \frac{p}{q} - \frac{m}{n} \right|.$$

Since $pn - qm \neq 0$, then $\left| \frac{p}{q} - \frac{m}{n} \right| \geqslant \frac{1}{qn}$, and

$$K(n^3 + q^3) \geqslant q^2 n^2;$$

in view of the inequality $n \geq q$ we obtain

$$K(n^3 + q^3) \geqslant q^4,$$

or

$$n^3 > \frac{q^4}{K} - q^3.$$

Taking into account the fact that $K < 1/9$, we have

$$n^3 > 9q^4 - q^3 \geqslant 8q^4,$$

as was required to be proved.

Operation 1: Construction of the $C'_{m/n,K}$. This construction consists of an infinite number of successively realized stages such that after the nth stage disks $C'_{m/n,K}$ $(0 \leq m \leq n)$ have been constructed with the following properties:

A_n. No disk $C_{m_1/n_1,K}$ $(n_1 > n)$ can join a disk $C'_{m/n,K}$ to a disk $C'_{m_2/n_2,K}$ $(n_2 \leq n)$ if these disks $C'_{m/n,K}$ and $C'_{m_2/n_2,K}$ do not intersect each other.

B_n. $C_{\frac{m}{n},\frac{K}{2}} \subseteq C'_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},K}$.

We begin with the first stage. Suppose that $C'_{m/1,K} = C_{m/1,K}$. Then the property B_1 is satisfied. Property A_1 is also satisfied, since the diameter of the disk $C_{m_1/n_1,K}$ $(n_1 > 1)$ is less than

$$\frac{2K}{n_1^3} < \frac{2}{9 \cdot 8} \quad \left(K < \frac{1}{9}\right),$$

and the distance between the disks $C_{0/1,K}$ and $C_{1/1,K}$ is larger than

$$1 - 2K > \frac{2}{3}.$$

The first stage is done.

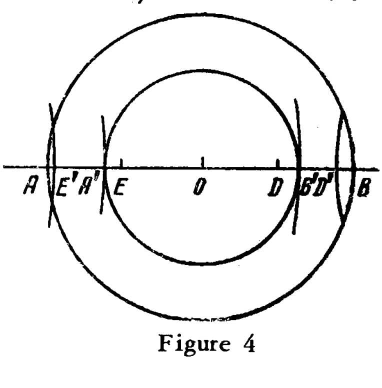

Figure 4

Now we suppose that we have successively performed $n-1$ stages. We consider any disk $C = C_{m/n,K}$ (Figure 4). Suppose that O is its center, AB the diameter lying on the real axis, E and D the means of AO and OB. The disk C can only intersect with those disks $C'_{m_2/n_2,K}$ ($n_2 < n$) for which $C_{m_2/n_2,K}$ intersects with C (because of property B_k, $k \le n-1$). Further, all such disks $C'_{m_2/n_2,K}$ intersect also one with another (as a consequence of property A_k, $k \le n-1$).

Now we arrange the disks in the order of decrease of n_2 (i.e., of the growth of the disks):

$$C_i = C_{\frac{m_{2,i}}{n_{2,i}}, K} \qquad (n = n_{2,0} > n_{2,1} > \ldots > n_{2,l} \geqslant 1).$$

From Lemma 7, $n_{2,i} \ge 2n_{2,i+1}$ $(0 \le i \le l-1)$, so that $n > 2^l$ and $l < \log_2 n$. Thus, the circumferences of the disks $C_{m_2/n_2,K}$ yield in their intersection with the diameter AB not more than $2\log_2 n$ points. Therefore among the portions into which these points divide the segment BD and the segment AE, there is at least one length larger than $K/4n^3\log_2 n$. Now the diameter of the circumference $C_{m_1/n_1,K}$ $(n_1 > n)$, which intersects with C, by Lemma 7 does not exceed

$$\frac{K}{8n^4} < \frac{K}{4n^3 \log_2 n}.$$

We take the ends B' and A' closest to O of the largest pieces of BD and AE, which we denote by $B'D'$ and $A'E'$, as the ends of the diameter of $C'_{m/n,K}$. Such a choice does not contradict property B_n. It is clear that if the circumference $C_1 = C_{m_1/n_1,K}$ $(n_1 > n)$ intersects $C'_{m/n,K}$, then it lies inside C, and among the disks $C_{m_2/n_2,K}$ $(n_2 \le n)$ can only intersect the C_i. But since the diameter of C_1 is less than the lengths of $B'D'$ and $A'E'$, therefore C_1 can only intersect those C_i which are intersected by $C'_{m/n,K}$. Therefore property A_n is also satisfied, and thus we have given the construction of the nth stage.

At the conclusion of all the stages one obtains a system of disks $C'_{m/n,K}$ with the following properties:

A. *No disk* $C_{\frac{m_1}{n_1},K}$ *can join* $C'_{\frac{m_2}{n_2},K}$ *and* $C'_{\frac{m_3}{n_3},K}$ *if* $n_1 > n_2$, $n_1 > n_3$ *and* $C'_{\frac{m_2}{n_2},K} \cap C'_{\frac{m_3}{n_3},K} = 0$.

B. $C_{\frac{m}{n},\frac{K}{2}} \subseteq C'_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},K}$.

Property B follows from B_n, and property A from A_{n_2} if $n_2 \ge n_3$, and from A_{n_3} if $n_3 \ge n_2$.

Operation 2: Construction of the $C''_{m/n,K}$. Now we shall enlarge the disks of the system $C'_{m/n,K}$.

By a *tail* $C = C'_{m/n,K}$ we shall mean the collection of all the $C'_{m_i/n_i,K}$ $(n_i > n)$ which may be joined to C by a monotone finite chain of pairwise intersecting disks $C'_{m_{j_k}/n_{j_k},K}$ $(0 \le k \le l_i)$:

$$\frac{m_{j_0}}{n_{j_0}} = \frac{m}{n}, \quad n_{j_k} < n_{j_{k+1}}, \quad C'_{\frac{m_{j_k}}{n_{j_k}},K} \cap C'_{\frac{m_{j_{k+1}}}{n_{j_{k+1}}},K} \neq 0, \quad \frac{m_{j_{l_i}}}{n_{j_{l_i}}} = \frac{m_i}{n_i}.$$

Obviously, if the disk C_1 enters into the tail of the disk C_2, then the tail of C_1 always enters into the tail of C_2. Moreover, if the tails of C_1 and C_2 intersect,* then one of the tails lies entirely in the other. We shall prove this fact. We suppose on the contrary that the disks C_1 and C_2 may be joined to a common disk of their tails, C_3, by monotone chains. Two such chains at the same time join C_1 and C_2. Of the chains joining C_1 and C_2 we select one consisting of the smallest number of disks. In this chain only successive disks intersect one another (see Figure 5; in the system of circles drawn there the shaded tail is the largest). If this chain is monotone, then our assertion is proved. If the chain is not monotone, then there is a disk in it which joins two preceding it, which contradicts property A of operation 1. Thus, if two tails intersect, then one of them contains the other.

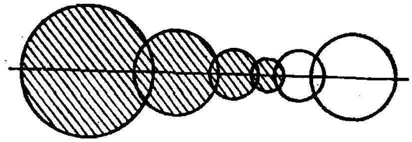

Figure 5

Suppose that α and β are the upper and lower bounds of the points of the real axis covered by the tail of the disk $C = C'_{m/n,K}$. The disk with diameter $\alpha\beta$ will also be a disk $C''_{m/n,K}$. From what has been stated above it follows that the circumferences of two such disks do not intersect.** Evidently $C''_{m/n,K} \supseteq C'_{m/n,K}$. We shall show that

$$C''_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},2K}$$

Indeed, on the basis of Lemma 7, it is easy to estimate the measure of the tail of C. Suppose that the disk C_1 belongs to the tail of C and the monotone chain joining C_1 to C consists of N disks. Since each of those following them is not less than 8 times smaller than the preceding one, the sum of their diameters does not exceed the diameter of C for any N. Therefore it is evident

* It is easy to see that if two tails intersect as point sets, then they have a common disk.

** But they can touch.

that α and β are distant from $C'_{m/n,K}$ by not more than 1/7 of the diameter of $C_{m/n,K}$, and from the center m/n by not more than 9/7 of the radius of $C_{m/n,K}$. Hence it follows that

$$C''_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},2K}$$

The construction of the N_K^R is complete.

7.3. **Differentiation of sequences.** The passage to the complex plane of μ was undertaken largely for the sake of the following lemma, which is not valid if by the set N_K^R is meant its part lying on the real axis.

Lemma 8. *Suppose that the sequence $f_n(\mu)$ of functions, monogenic on the set N_K^R, converges there uniformly to $f(\mu)$, and that the derivatives converge uniformly to $g(\mu)$. Then $f(\mu)$ is monogenic on N_K^R and $f'(\mu) = g(\mu)$.*

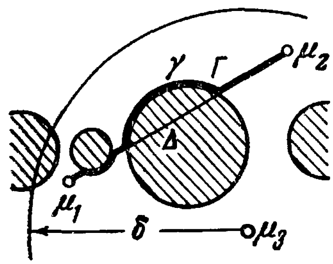

Figure 6

Proof. Suppose that $\epsilon > 0$. We may choose $\delta > 0$ so that

$$\left| \frac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} - g(\mu_3) \right| < \varepsilon$$

when $|\mu_1 - \mu_3| < \delta$, $|\mu_2 - \mu_3| < \delta$, $\mu_1, \mu_2, \mu_3 \in N_K^R$.

If $\delta > 0$ is sufficiently small, then all of these points lie in one component of N_K^R.

We shall show that in such a case the points μ_1, μ_2 may be joined in N_K^R by a rectifiable curve Γ such that the following conditions are satisfied:

1) for any point $\mu \in \Gamma$ $|\mu - \mu_3| < 2\delta$;

2) the length of Γ is less than $2|\mu_1 - \mu_2|$.

Indeed, let us join the points μ_1 and μ_2 by a segment $\mu_1\mu_2$ (see Figure 6). This segment may intersect certain disks C_i, by the deletion of which from the rectangle $|\mathrm{Im}\,\mu| \leq R$, $\mathrm{Re}\,\mu \in [0, 1]$ the set N_K^R was formed. These disks are disjoint and do not separate μ_1 from μ_2 in N_K^R, since the points μ_1 and μ_2 lie in one component. The disks C_i excise on $\mu_1\mu_2$ nonintersecting intervals Δ_i. We replace each such interval Δ_i by the smaller of the two arcs into which $\mu_1\mu_2$ divides the circumference C_i, and denote this arc by γ_i. The length of Δ_i is increased by such a substitution by not more than $\pi/2$ times, and therefore the length of Γ will be less than $2|\mu_1 - \mu_2|$. The distance $|\mu_1 - \mu_2|$, by hypothesis, does not exceed 2δ, so that all the points of γ_i are less than δ units distant from the midpoint of Δ_i. This last point, as well as all the points of the segment $\mu_1\mu_2$, lies in the disk $|\mu_1 - \mu_2| < \delta$, so that for any point $\mu \in \gamma_i$

$$|\mu - \mu_3| < 2\delta.$$

Thus the curve Γ is the one desired.

2. We have already noted that if $\phi(\mu)$ is monotone in N_K^R and Γ is a rectifiable curve with endpoints μ_1 and μ_2, then

$$\int_\Gamma \varphi'(\mu)\,d\mu = \varphi(\mu_2) - \varphi(\mu_1).$$

(For the proof it is only necessary to equate the integral to the integral sum.)

Applying this equation to the curve Γ constructed above and to the function $f_n(\mu)$, which is monogenic by hypothesis, we obtain

$$\int_\Gamma f_n'(\mu)\,d\mu = f_n(\mu_2) - f_n(\mu_1).$$

In view of the uniform convergence of the f_n to f and f_n' to g, we may pass to the limit on left and right:

$$\int_\Gamma g(\mu)\,d\mu = f(\mu_2) - f(\mu_1).$$

3. Now we shall estimate

$$\left|\frac{f(\mu_2) - f(\mu_1)}{\mu_2 - \mu_1} - g(\mu_3)\right|.$$

To this end we consider the integral

$$\int_\Gamma (g(\mu) - g(\mu_3))\,d\mu = f(\mu_2) - f(\mu_1) - (\mu_2 - \mu_1)\,g(\mu_3).$$

We have

$$\left|\int_\Gamma (g(\mu) - g(\mu_3))\,d\mu\right| \leqslant \int_\Gamma |g(\mu) - g(\mu_3)|\,|d\mu| \leqslant \max_{\mu\in\Gamma} |g(\mu) - g(\mu_3)|\cdot 2\,|\mu_2 - \mu_1|,$$

since the length of Γ is less than $2\,|\mu_2 - \mu_1|$.

Thus,

$$\left|\frac{f(\mu_2) - f(\mu_1)}{\mu_2 - \mu_1} - g(\mu_3)\right| \leqslant 2 \max_{\mu\in\Gamma} |g(\mu) - g(\mu_3)|.$$

The right side of the last inequality, from property 1) of the curve Γ, is twice the increment of $g(\mu)$ on a segment of length less than 2δ, and, in view of the uniform continuity of the function $g(\mu)$ on the compactum N_K^R, tends to zero along with δ. Lemma 8 is thus proved.

7.4. **Functions of several variables and operations on them.** In what follows we shall need functions analytic in one variable and monogenic in the others.

Suppose that the variable z is angular (varies in the strip $\mathrm{Im}\, z \in (ab)$)* and has period 2π,** the variables ϵ and Δ vary in the neighborhood of zero, and

* The boundaries may depend on μ.

** I.e., as z increases by 2π, functions of z have increments of 0 or 2π.

$\mu \in N_K^R$.

Definition. The function $f(z, \epsilon, \Delta, \mu)$ is *analytic in* z, ϵ, *and* Δ, *and monogenic in* $\mu \in N_K^R$ if the series

$$f(z, \varepsilon, \Delta, \mu) = \sum f_{kmn}(\mu) e^{ikz} \varepsilon^m \Delta^n,$$

in which the coefficients are monogenic functions of $\mu \in N_K^R$, converges along with its derivative with respect to μ uniformly for $\mu \in N_K^R$ and z, ϵ, Δ, varying in the indicated regions.

Evidently such a function is continuous, while

a) for fixed μ it is analytic in z, ϵ, Δ and

b) for fixed z, ϵ, Δ it is monogenic in $\mu \in N_K^R$.

Property b) follows from Lemma 8.

Lemma 9. *Suppose that the functions* $h_i(z, \epsilon, \Delta, \mu)$ *are monogenic with respect to* $\mu \in E$ *and analytic in* z, ϵ, Δ. *Then the following functions have the same property in the corresponding regions:*

1) *the functions*

$$h_1(z, \varepsilon, \Delta, \mu) + h_2(z, \varepsilon, \Delta, \mu), \quad h_1(z, \varepsilon, \Delta, \mu) h_2(z, \varepsilon, \Delta, \mu),$$
$$h_1(h_2(z, \varepsilon, \Delta, \mu), \varepsilon, \Delta, \mu), \quad h_1(z, \varepsilon, h_2(z, \varepsilon. \Delta, \mu), \mu);$$

2) *the solution* $\phi(z, \epsilon, \Delta, \mu)$ *of the equation* $h(\phi, \epsilon, \Delta, \mu) = z$;

3) *the solution* $\gamma(z, \epsilon, \Delta, \mu)$ *of the equation* $h(z, \epsilon, \gamma, \mu) = \Delta$;

4) *the partial derivatives of* h *with respect to* z, ϵ, Δ;

5) *the integral with respect to a parameter* $\int_0^{2\pi} h(z, \epsilon, \Delta, \mu)\, dz$;

while in all these cases the usual rules of differentiation apply; for example, in case 2)

$$\frac{\partial \varphi}{\partial \mu} = -\frac{\frac{\partial h}{\partial \mu}}{\frac{\partial h}{\partial \varphi}}.$$

The proof repeats well-known arguments from standard analysis and will be omitted.

Lemma 10. *Suppose that the function* $f(z, \epsilon, \Delta, \mu) = \tilde{f}$ *is analytic with respect to* z *in the region* $|\mathrm{Im}\, z| \le R$; ϵ, $|\epsilon| \le \epsilon_0$; $|\Delta| \le \Delta_0$ *and is monogenic with respect to* $\mu \in N_K^R$, *and suppose that in the indicated region*

$$|f| \leqslant C, \quad \left|\frac{\partial f}{\partial \mu}\right| \leqslant L.$$

Then the solution of the equation

$$g(z + 2\pi\mu, \varepsilon, \Delta, \mu) - g(z, \varepsilon, \Delta, \mu) = f(z, \varepsilon, \Delta, \mu)$$

is monogenic with respect to $\mu \in N_K^R$ *and analytic with respect to* z *in the* *region* $|\mathrm{Im}(z-2\pi\mu)| \le R-2\delta$, ϵ, $|\epsilon| \le \epsilon_0$, Δ, $|\Delta| \le \Delta_0$, *while in this region*

$$|g| \leqslant \frac{4C}{K\delta^3}, \quad \left|\frac{\partial g}{\partial z}\right| \leqslant \frac{8C}{K\delta^4}, \quad \left|\frac{\partial^2 g}{\partial z^2}\right| \leqslant \frac{10C}{K\delta^5},$$

$$\left|\frac{\partial g}{\partial \mu}\right| \leqslant \frac{C+L}{K^2}\frac{10^3}{\delta^6}, \quad \left|\frac{\partial^2 g}{\partial z \partial \mu}\right| \leqslant \frac{C+L}{K^2}\frac{10^3}{\delta^7}.$$

Proof. The solution is given for fixed μ by the series (2) of §2:

$$\sum_{n \neq 0} \frac{f_n(\mu, \varepsilon, \Delta)}{e^{2\pi i n \mu} - 1} e^{inz},$$

of which it is required to establish the uniform convergence for $|\mathrm{Im}(z-2\pi\mu)| \le R-2\delta$, since

$$f_n(\mu, \varepsilon, \Delta) = \sum f_{nkl}(\mu)\, \varepsilon^k \Delta^l.$$

But the uniform convergence of this series has been established in §2 along with the desired estimates of g and $\partial g/\partial z$ in the proof of Theorem 1′, since

$$N_K^{\frac{1}{2\pi}} \subseteq M_{\frac{K}{2}}^{\frac{1}{2\pi}}.$$

Estimates of the other derivatives are obtained by differentiation of the series using the usual formulas and taking account of inequality (13) of §2.

§8. On the dependence of the constructions of Theorem 2 on μ

8.1. We have seen, in subsection 7.4, that the solution of the linear equation (1) of §2 depends on μ monogenically. In the present section we shall prove the monogenicity with respect to μ of the functions Δ_n, F_n, Φ_n, g_n, $\Delta^{(n)}$ constructed in §6.

It turns out that the region of monotonicity contracts as n increases (by $|\mathrm{Im}\, 2\pi\mu|$ at each step) and the author has not been able to establish whether the solution of equation (1) of §4 depends monogenically on μ.

The monogenicity of $\Delta^{(n)}$ with respect to μ for real μ is used in §11. There we shall also make use of the smallness (uniformly with respect to n) of $\partial\Delta^{(n)}/\partial\mu$ for small ϵ.

In order to shorten complicated expressions in this section the argument ϵ will be dropped in all functions. This is similar to the way in which we earlier ignored the dependence on μ and took only z, ϕ, ϵ, Δ as arguments.

The construction of $\Delta^{(n)}(\mu)$ was carried out in the following way.

Step by step we constructed new parameters $\phi_n = \phi_n(\phi_{n-1}, \mu)$ and quantities $\Delta_{n-1} = \Delta_{n-1}(\Delta_n, \mu)$ such that the transformation

$$\varphi_{n-1} \to \varphi_{n-1} + 2\pi\mu + \Delta_{n-1}(\Delta_n, \mu) + F_{n-1}(\varphi_{n-1}, \mu) + \Phi_{n-1}(\varphi_{n-1}, \Delta_{n-1}(\Delta_n, \mu)\, \mu)$$

is converted into the transformation

$$\varphi_n \to \varphi_n + 2\pi\mu + \Delta_n + F_n(\varphi_n, \mu) + \Phi_n(\varphi_n, \Delta_n, \mu)$$

with significantly smaller F and Φ, where $\phi_0 = z$, $F_0 = F$, $\Phi_0 = 0$, $\Delta_0 = \Delta$.

Further, we constructed $\Delta^{(n)}(\mu)$ such that the transformation

$$z \to z + 2\pi\mu + \Delta^{(n)}(\mu) + F(z)$$

converts, in the variable ϕ_n, into the transformation

$$\varphi_n \to \varphi_n + 2\pi\mu + F_n(\varphi_n, \mu) + \Phi_n(\varphi_n, 0, \mu),$$

to which end we put

$$\begin{aligned} &\Delta_k^{(n)}(\mu) = \Delta_k(\Delta_{k+1}^{(n)}(\mu), \mu) \quad (k = 0, 1, \ldots, n-1), \\ &\Delta_n^{(n)}(\mu) = 0. \end{aligned} \tag{1}$$

Thus we obtained

$$\Delta_0^{(n)}(\mu) = \Delta^{(n)}(\mu).$$

Theorem 3. *Under the conditions of Theorem 2, for sufficiently small* $\epsilon > 0$, $0 < K < 1/9$

$$\Delta(\mu) = \lim_{n\to\infty} \Delta^{(n)}(\mu),$$

where the functions $\Delta^{(n)}(\mu)$ *are monogenic with respect to* $\mu \in N_K^{r_n}$ $(r_n > 0)$ *and under these conditions* $|\partial\Delta^{(n)}/\partial\mu| < 6L|\epsilon|$.

The proof of this theorem rests on the following lemma, which repeats the Fundamental Lemma (see §§4 and 5).

Lemma 11. *Suppose we are given a family of analytic mappings of the circumference, depending analytically on* Δ *and monogenically on* $\mu \in N_K^r$,

$$z \to A_0(z, \Delta, \mu) = z + 2\pi\mu + F(z, \mu) + \Delta + \Phi(z, \Delta, \mu)$$

and numbers $R_0 > 0$, $1/9 > K > 0$, $\delta > 0$, $C > 0$, $0 < \Delta_0 < 1$, $0 < r < 1/2\pi$, $2\pi r \leq R_0 - 5\delta$ *such that*

1) $F(z + 2\pi, \mu) = F(z, \mu)$, $\Phi(z + 2\pi, \Delta, \mu) = \Phi(z, \Delta, \mu)$;

2) *for* $\operatorname{Im} z = \operatorname{Im} \mu = \operatorname{Im} \Delta = 0$ *always* $\operatorname{Im} F = \operatorname{Im} \Phi = 0$;

3) *for* $|\operatorname{Im} z| \leqslant R_0$, $\mu \in N_K^r$, $|\Delta| \leqslant \Delta_0$

$$|F(z, \mu)| \leqslant C, \tag{2}$$

$$\left|\frac{\partial F(z, \mu)}{\partial\mu}\right| \leqslant C, \tag{3}$$

$$|\Phi(z, \mu, \Delta)| \leqslant \delta^2 |\Delta|, \tag{4}$$

$$\left|\frac{\partial\Phi(z, \mu, \Delta)}{\partial\mu}\right| \leqslant \delta^2 |\Delta|; \tag{5}$$

4) *the number* δ *satisfies the inequality*

$$\delta < \frac{K^2}{5\cdot 10^4}; \tag{6}$$

5) $C = \delta^{27}$, $\Delta_0 = \delta^{26}$.

Then there exist functions $z(\phi, \mu)$, $\Delta(\Delta_1, \mu)$ *analytic in* ϕ, Δ_1 *and mono-*

genic in $\mu \in N_K^r$ *such that*

1. *Identically*

$$z(A_1(\varphi, \mu, \Delta_1), \mu) = A_0(z(\varphi, \mu), \Delta(\Delta_1, \mu), \mu),$$

where

$$A_1(\varphi, \mu, \Delta_1) \equiv \varphi + 2\pi\mu + \Delta_1 + F_1(\varphi, \mu) + \Phi_1(\varphi, \mu, \Delta_1).$$

2. $F_1(\varphi + 2\pi, \mu) = F_1(\varphi, \mu), \quad \Phi_1(\varphi + 2\pi, \mu, \Delta_1) = \Phi_1(\varphi, \mu, \Delta_1),$

$$z(\varphi + 2\pi, \mu) = z(\varphi, \mu) + 2\pi.$$

3. *For* $\operatorname{Im}\varphi = \operatorname{Im}\Delta_1 = \operatorname{Im}\mu = 0$ *always* $\operatorname{Im} z = \operatorname{Im}\Delta = \operatorname{Im} F_1 = \operatorname{Im}\Phi_1 = 0$

4. *For* $|\Delta_1| \leqslant \delta^{28}$, $|\operatorname{Im}\varphi| \leqslant R_0 - 7\delta - |\operatorname{Im} 2\pi\mu|$, $\mu \in N_K^r$ *the functions constructed above are analytic in* ϕ, Δ_1, *monogenic in* $\mu \in N_K^r$, *and the following relations hold*:

$$|F_1| \leqslant \frac{C^2}{\delta^6}, \tag{7}$$

$$|\Phi_1| \leqslant \frac{C}{\delta^6}|\Delta_1|, \tag{8}$$

$$\left|\frac{\partial F_1}{\partial \mu}\right| \leqslant \frac{C^2}{\delta^{13}}, \tag{9}$$

$$\left|\frac{\partial \Phi_1}{\partial \mu}\right| \leqslant \frac{C}{\delta^{13}}|\Delta_1|, \tag{10}$$

$$\left|\frac{\partial z}{\partial \mu}\right| \leqslant \frac{C}{\delta^7}, \tag{11}$$

$$\left|\frac{\partial \Delta}{\partial \mu}\right| \leqslant 4C, \tag{12}$$

$$|\Delta(\Delta_1, \mu)| \leqslant \Delta_0, \tag{13}$$

$$|z(\varphi, \mu) - \varphi| \leqslant \frac{C}{\delta^4}, \tag{14}$$

$$\left|\frac{\partial \Delta}{\partial \Delta_1}\right| \leqslant 2, \tag{15}$$

$$\left|\frac{\partial z}{\partial \varphi}\right| \leqslant 2. \tag{16}$$

8.2. The proof of Lemma 11 is more complicated than the proof of the Fundamental Lemma. The construction repeats the considerations of subsections 5.1 with the difference that μ changes from a fixed real number to an independent complex variable. In the construction of $\Delta(\Delta_1)$, $z(\phi)$, g, F_1 and Δ_1, following subsections 5.1, one uses integration with respect to z, the solution of equation (1) of §2, the construction of an inverse function and the substitution of a function into a function. From the lemmas of subsection 7.4 all of these operations do not lead out of the class of functions monogenic in $\mu \in N_K^r$ and analytic with respect to z, Δ, ϕ, Δ_1 in the corresponding regions.

Therefore special attention need be directed only to inequalities (9), (10), (11), and (12), which are not in the Fundamental Lemma. Their proof is based on the following estimates.

1°. **Estimate of $\partial g^*/\partial\mu$.** On the basis of subsections 5.1, 7.4, and in view of the conditions of the lemma, for $|\mathrm{Im}\, z| \le R_0$, $\mu \in N_K^r$, $|\Delta| \le \Delta_0$

$$\left|\frac{\partial\widetilde{F}}{\partial\mu}\right| \leqslant 2C, \quad \left|\frac{\partial\widetilde{\Phi}}{\partial\mu}\right| \leqslant 2\delta^2|\Delta_0| \leqslant 2C.$$

Thus the right side of equation (2) of §5 has a derivative with respect to μ not exceeding $4C$. Applying Lemma 10, we find that

$$|g^*| \leqslant \frac{16C}{K\delta^3}, \tag{17}$$

$$\left|\frac{\partial g^*}{\partial z}\right| \leqslant \frac{32C}{K\delta^4}, \tag{18}$$

$$\left|\frac{\partial^2 g^*}{\partial z^2}\right| \leqslant \frac{40C}{K\delta^5}, \tag{19}$$

$$\left|\frac{\partial g^*}{\partial\mu}\right| \leqslant \frac{5\cdot 10^3 C}{K^2\delta^6}, \tag{20}$$

$$\left|\frac{\partial^2 g^*}{\partial z\partial\mu}\right| \leqslant \frac{5\cdot 10^3 C}{K^2\delta^7} \tag{21}$$

for $|\mathrm{Im}(z-2\pi\mu)| \leqslant R-2\delta$, $\mu \in N_K^r$, $|\Delta| \leqslant \Delta_0$.

2°. **Estimate of $\partial\Delta_0^*/\partial\mu$.** From equation (4) of §5 and subsection 7.4 it follows that

$$\frac{\partial\Delta_0^*(\mu)}{\partial\mu} = -\frac{\dfrac{\partial\overline{F}}{\partial\mu} + \dfrac{\partial\overline{\Phi}}{\partial\mu}}{1+\dfrac{\partial\overline{\Phi}}{\partial\Delta}}.$$

Estimating Δ_0^* as in 1° of subsection 5.2, we find that

$$|\Delta_0^*| < 2C < \frac{\Delta_0}{2}.$$

For $|\Delta| \le \Delta_0/2$, using Cauchy's integral formula, we find from (4) that

$$\left|\frac{\partial\Phi}{\partial\Delta}\right| \leqslant \frac{\delta^2\Delta_0}{\frac{\Delta_0}{2}} = 2\delta < \frac{1}{2}, \quad \left|\frac{\partial\overline{\Phi}}{\partial\Delta}\right| < \frac{1}{2}, \quad \left|\frac{\partial\widetilde{\Phi}}{\partial\Delta}\right| < 1.$$

Accordingly $|1 + \partial\overline{\Phi}/\partial\Delta| > 1/2$ for $|\Delta| \le \Delta_0/2$. Therefore, on the basis of (3), (5), and Lemma 9,

$$\left|\frac{\partial\Delta_0^*}{\partial\mu}\right| < 2(C+\delta^2\Delta_0).$$

In view of (6), $\delta^2\Delta_0 < C$, so that

$$\left|\frac{\partial\Delta_0^*}{\partial\mu}\right| < 4C \tag{22}$$

for $\mu \in N_N^r$.

3°. **Estimate of $\partial g/\partial\mu$.** From subsections 7.4 and 5.1,

$$\frac{\partial g}{\partial\mu} = \frac{\partial g^*}{\partial\mu} + \frac{\partial g^*}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}, \tag{23}$$

$$\frac{\partial^2 g}{\partial z \partial \mu} = \frac{\partial^2 g^*}{\partial z \partial \mu} + \frac{\partial^2 g^*}{\partial z \partial \Delta} \frac{\partial \Delta_0^*}{\partial \mu}. \tag{24}$$

First we shall estimate $\partial g^*/\partial\Delta$ and $\partial^2 g^*/\partial z \partial\Delta$. We note that the equation

$$g^*(z + 2\pi\mu, \Delta, \mu) - g^*(z, \Delta, \mu) = -\widetilde{F}(z, \mu) - \widetilde{\Phi}(z, \Delta, \mu)$$

on differentiation with respect to Δ gives the equation

$$\frac{\partial g^*}{\partial \Delta}(z + 2\pi\mu, \Delta, \mu) - \frac{\partial g^*}{\partial \Delta}(z, \Delta, \mu) = -\frac{\partial \widetilde{\Phi}}{\partial \Delta}$$

of the same form with respect to $\partial g^*/\partial\Delta$, and we may use Lemma 10. To this end we estimate $\partial\widetilde{\Phi}/\partial\Delta$ using Cauchy's integral formula: for $|\mathrm{Im}\, z| \le R, |\Delta| \le \Delta_0/2$

$$\left|\frac{\partial \widetilde{\Phi}}{\partial \Delta}\right| \leqslant \frac{2\delta^2 \Delta_0}{\frac{\Delta_0}{2}} < 4\delta^2.$$

From Lemma 10, for $|\mathrm{Im}(z - 2\pi\mu)| \le R_0 - 2\delta$, $|\Delta| \le \Delta_0/2$, $\mu \in N_K^r$

$$\left|\frac{\partial g^*}{\partial \Delta}\right| < \frac{4}{K\delta^3} 4\delta^2,$$
$$\left|\frac{\partial^2 g^*}{\partial \Delta \partial z}\right| < \frac{8}{K\delta^4} 4\delta^2.$$

Substituting these estimates, and also estimates (20), (21) and the estimate of Δ_0^* from point 2°, into formulas (23) and (24), we find that

$$\left|\frac{\partial g}{\partial \mu}\right| < \frac{5C}{K^2}\frac{10^3}{\delta^6} + \frac{16}{K\delta} 4C < \frac{C10^4}{K^2\delta^6},$$
$$\left|\frac{\partial^2 g}{\partial z \partial \mu}\right| < \frac{5 \cdot 10^3 C}{K^3\delta^7} + \frac{32}{K\delta^2} 4C < \frac{C10^4}{K^2\delta^7}$$

for $|\mathrm{Im}\,(z - 2\pi\mu)| \le R - 2\delta$, $\mu \in N_K^r$.

4°. **Estimate of $\partial\Delta(\Delta_1, \mu)/\partial\mu$.** Analogously to subsection 2°, we have

$$\frac{\partial \Delta}{\partial \mu} = -\frac{\frac{\partial \overline{F}}{\partial \mu} + \frac{\partial \overline{\Phi}}{\partial \mu}}{1 + \frac{\partial \overline{\Phi}}{\partial \Delta}},$$

and if $|\Delta| \le \Delta_0/2$, then, as in point 2°, we obtain

$$\left|\frac{\partial \Delta}{\partial \mu}\right| < 4C.$$

In order that the inequality $|\Delta| < \Delta_0/2$ should be satisfied it is sufficient that $|\Delta_1| \le \delta^{27}$. For then, as was shown in §5, $|\Delta_0^*| \le 2C$, $|\Delta - \Delta_0^*| \le 2|\Delta_1|$, and since $C = \delta^{27}$, then for $|\Delta_1| \le \delta^{27}$ we have

$$|\Delta(\Delta_1, \mu)| \leqslant 4\delta^{27} < \frac{\delta^{26}}{2} = \frac{\Delta_0}{2}.$$

Thus, for $|\Delta_1| \le \delta^{27}$, $\mu \in N_K^r$,

$$\left|\frac{\partial \Delta(\Delta_1, \mu)}{\partial \mu}\right| < 4C. \tag{25}$$

At the same time we have shown that for $|\Delta_1| \leq \delta^{27}$ the estimates of point 1° are valid.

5°. Estimate of $\partial \hat{F}_1/\partial\mu$. From subsections 5.1 and 7.4 we have

$$\frac{\partial \hat{F}_1(z,\mu)}{\partial\mu} = \left[\frac{\partial g(z_{\mathrm{I}},\mu)}{\partial\mu} - \frac{\partial g(z_{\mathrm{II}},\mu)}{\partial\mu}\right] + \left[\frac{\partial g(z_{\mathrm{I}},\mu)}{\partial z} - \frac{\partial g(z_{\mathrm{II}},\mu)}{\partial z}\right]2\pi$$
$$+ \frac{\partial g(z_{\mathrm{I}},\mu)}{\partial z}\left[\frac{\partial\widetilde{F}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\Delta}\frac{\partial\overset{*}{\Delta}_0}{\partial\mu}\right], \tag{26}$$

where

$$z_{\mathrm{I}} = z + 2\pi\mu + \widetilde{F}(z,\mu) + \widetilde{\Phi}(z,\mu,\overset{*}{\Delta}_0(\mu)), \tag{27}$$
$$z_{\mathrm{II}} = z + 2\pi\mu. \tag{28}$$

The first two brackets on the right side of (26) may be estimated by using the lemma on finite increments, Lemma 5 of §3. We have

$$\left|\frac{\partial g(z_{\mathrm{I}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{II}})}{\partial\mu}\right| \leqslant |z_{\mathrm{I}} - z_{\mathrm{II}}|\left|\frac{\partial^2 g}{\partial\mu\,\partial z}\right|;$$

putting their estimates in place of $z_{\mathrm{I}} - z_{\mathrm{II}}$ and $\partial^2 g/\partial\mu\partial z$, we obtain

$$\left|\frac{\partial g(z_{\mathrm{I}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{II}})}{\partial\mu}\right| \leqslant \frac{4\cdot 10^4 C^2}{K^2\delta^7}$$

and analogously

$$\left|\frac{\partial g(z_{\mathrm{I}})}{\partial z} - \frac{\partial g(z_{\mathrm{II}})}{\partial z}\right| \leqslant \left|\frac{\partial^2 g}{\partial z^2}\right| |z_{\mathrm{I}} - z_{\mathrm{II}}| \leqslant \frac{40\,C}{K\delta^5}4C = \frac{160\,C^2}{K\delta^5}.$$

The last term on the right side of (26) may be estimated using inequalities (3), (5), (22), (18) and does not exceed

$$\frac{32\,C}{K\delta^4}(4\,C + 2\,C) < \frac{200\,C^2}{K\delta^4}.$$

Thus

$$\left|\frac{\partial\hat{F}_1}{\partial\mu}\right| < C^2\left[\frac{4\cdot 10^4}{K^2\delta^7} + 2\pi\frac{160}{K\delta^5} + \frac{200}{K\delta^4}\right] < \frac{5\cdot 10^4}{K^2\delta^7}C^2.$$

All of these estimates are valid if the arguments z_{I} and z_{II} do not leave the region $|\operatorname{Im}(z - 2\pi\mu)| \leq R_0 - 2\delta$, where the estimates of g and its derivatives operate. To this end it suffices, for example, that $|\operatorname{Im} z| \leq R_0 - 3\delta$. Indeed, then

$$|\widetilde{F}(z,\mu) + \widetilde{\Phi}(z,\mu,\overset{*}{\Delta}_0(\mu))| \leqslant 2C < \delta,$$

i.e.,

$$|\operatorname{Im}(z_{\mathrm{I}} - 2\pi\mu)| < R_0 - 2\delta.$$

Thus, for $|\operatorname{Im} z| \leq R_0 - 3\delta$, $\mu \in N_K^r$, $|\Delta_1| < \delta^{27}$

$$\left|\frac{\partial \hat{F}_1}{\partial \mu}\right| < \frac{5 \cdot 10^4}{K^2 \delta^7} C^2. \tag{29}$$

6°. **Estimate of** $(\partial/\partial\mu)(\Delta - \Delta_0^*)$. **We have**

$$\frac{\partial}{\partial \mu}(\Delta(\Delta_1, \mu) - \Delta_0^*(\mu)) = \frac{\partial \Delta(\Delta_1, \mu)}{\partial \mu} - \frac{\partial \Delta(0, \mu)}{\partial \mu};$$

by the lemma on finite increments,

$$\left|\frac{\partial}{\partial \mu}(\Delta - \Delta_0^*)\right| \leqslant \left|\frac{\partial^2 \Delta(\Delta_1, \mu)}{\partial \Delta_1 \partial \mu}\right| |\Delta - \Delta_0^*|.$$

We estimate $|\partial^2\Delta(\Delta_1, \mu)/\partial\Delta_1\partial\mu$, using the Cauchy integral, as the derivative of $\partial\Delta/\partial\mu$. For $|\Delta_1| \leq \delta^{27}$, as follows from (25), $|\partial\Delta/\partial\mu| < 4C$. Therefore in the disk $|\Delta_1| \leq \delta^{27}/2$ always

$$\left|\frac{\partial^2 \Delta}{\partial \Delta_1 \partial \mu}\right| < \frac{4C}{\frac{\delta^{27}}{2}} = 8.$$

In particular, $|\partial^2\Delta/\partial\Delta_1\partial\mu| < 8$ when $|\Delta_1| \leq \delta^{28}$. Since

$$|\Delta - \Delta_0^*| \leqslant 2|\Delta_1|,$$

then for $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$

$$\frac{\partial}{\partial \mu}(\Delta(\Delta_1, \mu) - \Delta_0^*(\mu))| < 16|\Delta_1|. \tag{30}$$

7°. **Estimate of** $(\partial/\partial\mu)[\tilde{\Phi}(\Delta(\Delta_1, \mu)) - \tilde{\Phi}(\Delta_0^*(\mu))]$. **This derivative is equal to**

$$\frac{\partial\tilde{\Phi}(\Delta)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\mu} + \frac{\partial\tilde{\Phi}}{\partial\Delta}\frac{\partial\Delta(\Delta_1)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}.$$

The first difference may be estimated using the lemma on finite increments: for $|\Delta| \leq \Delta_0/2$, $\mu \in N_K^r$, $|\mathrm{Im}\, z| \leq R$

$$\left|\frac{\partial\tilde{\Phi}(\Delta)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta^*)}{\partial\mu}\right| \leqslant \left|\frac{\partial^2\tilde{\Phi}}{\partial\mu\partial\Delta}\right| |\Delta - \Delta_0^*| \leqslant 8\delta^2 |\Delta_1|$$

(here $|\partial^2\tilde{\Phi}/\partial\mu\partial\Delta|$ is estimated using the Cauchy integral: $|\partial^2\tilde{\Phi}/\partial\mu\partial\Delta| < 2\delta^2 |\Delta_0|/\frac{1}{2}|\Delta_0| = 4\delta^2$).

The second difference may be written in the form

$$\frac{\partial\tilde{\Phi}(\Delta)}{\partial\Delta}\left(\frac{\partial\Delta(\Delta_1)}{\partial\mu} - \frac{\partial\Delta_0^*}{\partial\mu}\right) + \frac{\partial\Delta_0^*}{\partial\mu}\left(\frac{\partial\tilde{\Phi}(\Delta)}{\partial\Delta} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\Delta}\right), \tag{31}$$

where the first term is estimated with the use of inequality (30) and does not exceed $16|\Delta_1|$, since $|\partial\tilde{\Phi}/\partial\Delta| < 1$ (see point 2°) and the second term, from the lemma on finite increments, does not exceed

$$\left|\frac{\partial\Delta^*}{\partial\mu}\right| \left|\frac{\partial^2\tilde{\Phi}}{\partial\Delta^2}\right| |\Delta(\Delta_1) - \Delta_0^*| \leqslant 4C\frac{16}{\delta^{24}} 2|\Delta_1|.$$

Here the only new estimate is that of $\partial^2\tilde{\Phi}/\partial\Delta^2$. To find it we employ the expression for the second derivative obtained from the Cauchy integral:

$$\left|\frac{\partial^2\tilde{\Phi}}{\partial\Delta^2}\right| \leqslant 2\frac{2\delta^2\Delta_0}{\left(\frac{\Delta_0}{2}\right)^2} = \frac{16}{\delta^{24}}$$

for $|\Delta| \leq \Delta_0/2$, for which, as we have seen in point 4°, it is sufficient that the inequality $|\Delta_1| \leq \delta^{27}$ should be satisfied. Comparing all three estimates, we find that

$$\left|\frac{\partial}{\partial\mu}[\tilde{\Phi}(\Delta) - \tilde{\Phi}(\Delta_0^*)]\right| < 8\delta^2|\Delta_1| + 16|\Delta_1| + 128\,\delta^3|\Delta_1|.$$

Finally we have

$$\left|\frac{\partial}{\partial\mu}[\tilde{\Phi}(\Delta(\Delta_1, \mu)) - \tilde{\Phi}(\Delta_0^*(\mu))]\right| < 100|\Delta_1| \tag{32}$$

for $|\Delta_1| \leq \delta^{28}$, $|\mathrm{Im}\, z| \leq R_0$, $\mu \in N_K^r$.

8°. **Estimate of** $(\partial/\partial\mu)\,\hat{\Phi}_1(z, \mu, \Delta(\Delta_1, \mu))$. It is convenient for us first to consider the function of z, μ and Δ_1, and not of z, μ, and Δ. We have

$$\frac{\partial\hat{\Phi}_1}{\partial\mu} = \left[\frac{\partial g(z_{\mathrm{III}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{I}})}{\partial\mu}\right] + \left[\frac{\partial g(z_{\mathrm{III}})}{\partial z} - \frac{\partial g(z_{\mathrm{I}})}{\partial z}\right]\frac{\partial z_{\mathrm{I}}}{\partial\mu} + \frac{\partial g(z_{\mathrm{III}})}{\partial z}\left[\frac{\partial z_{\mathrm{III}}}{\partial\mu} - \frac{\partial z_{\mathrm{I}}}{\partial\mu}\right], \tag{33}$$

where

$$z_{\mathrm{I}} = z + 2\pi\mu + \overline{F}(z, \mu) + \tilde{\Phi}(z, \mu, \Delta_0^*(\mu)), \tag{27}$$

$$z_{\mathrm{III}} = z + 2\pi\mu + \Delta_1 + \overline{F}(z, \mu) + \tilde{\Phi}(z, \mu, \Delta(\Delta_1, \mu)) + \Delta_1. \tag{34}$$

The first two brackets on the right side of (33) may be estimated as in point 5°:

$$\left|\frac{\partial g(z_{\mathrm{III}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{I}})}{\partial\mu}\right| \leqslant \left|\frac{\partial^2 g}{\partial\mu\,\partial z}\right| |z_{\mathrm{III}} - z_{\mathrm{I}}| \leqslant \frac{C\,10^4}{K^2\delta^7}3|\Delta_1|,$$

since

$$z_{\mathrm{III}} - z_{\mathrm{I}} = \Delta_1 + \tilde{\Phi}(z, \mu, \Delta) - \tilde{\Phi}(z, \mu, \Delta^*(\mu))$$

and, using the estimate (22) of §5,

$$|z_{\mathrm{III}} - z_{\mathrm{I}}| \leqslant 3|\Delta_1|.$$

Analogously,

$$\left|\left(\frac{\partial g(z_{\mathrm{III}})}{\partial z} - \frac{\partial g(z_{\mathrm{I}})}{\partial z}\right)\frac{\partial z_{\mathrm{I}}}{\partial\mu}\right| \leqslant \left|\frac{\partial^2 g}{\partial z^2}\right| |z_{\mathrm{III}} - z_{\mathrm{I}}| \left|\frac{\partial z_{\mathrm{I}}}{\partial\mu}\right|$$

$$\frac{40\,C}{K\delta^5}3|\Delta_1|\left|2\pi + \frac{\partial\overline{F}}{\partial\mu} + \frac{\partial\tilde{\Phi}}{\partial\mu} + \frac{\partial\tilde{\Phi}}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}\right| \leqslant \frac{40\,C}{K\delta^5}3|\Delta_1|(2\pi + 6\,C) \leqslant \frac{1600\,C}{K\delta^5}|\Delta_1|,$$

where the factor $|\partial z_{\mathrm{I}}/\partial\mu|$ is estimated using conditions 3) of Lemma 11 and estimate (22), taking account of the fact that $C < 1$. It remains for us to estimate

$(\partial/\partial\mu)\,(z_{III} - z_I)$. We have

$$z_{III} - z_I = \Delta_1 + \tilde{\Phi}(z, \mu, \Delta(\Delta_1, \mu)) - \tilde{\Phi}(z, \mu, \Delta_0^*(\mu)).$$

From estimate (32) we find that

$$\frac{\partial}{\partial\mu}(z_{III} - z_I) \leqslant 100\,|\Delta_1|,$$

where $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$.

Thus,

$$\left|\frac{\partial g(z_{III})}{\partial z}\left(\frac{\partial z_{III}}{\partial\mu} - \frac{\partial z_I}{\partial\mu}\right)\right| \leqslant 100\,|\Delta_1|\,\frac{32\,C}{K\delta^4} \leqslant \frac{10^4 C}{K\delta^4}\,|\Delta_1|.$$

Comparing the estimates of all three terms of the right side of equation (33), we find that

$$\left|\frac{\partial}{\partial\mu}\hat{\Phi}_1(z, \mu, \Delta(\Delta_1, \mu))\right| \leqslant \frac{C\,10^4}{K^2\delta^7}\,3\,|\Delta_1| + \frac{1600\,C}{K\delta^5}\,|\Delta_1| + \frac{C\,10^4}{K\delta^4}\,|\Delta_1| \leqslant \frac{C\,10^5}{K^2\delta^7}\,|\Delta_1|.$$

All of these estimates have been derived under the hypothesis that $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$ and that z_I, z_{III} do not leave the strip $|\mathrm{Im}\,(z - 2\pi\mu)| \leq R_0 - 2\delta$, where Lemma 10 operates. For this it is sufficient, for example, that $|\mathrm{Im}\, z| \leq R_0 - 4\delta$, since then

$$|\Delta_1 + \tilde{F}(z, \varepsilon) + \tilde{\Phi}(z, \varepsilon, \Delta)| \leqslant \delta + 2C + 2C < 2\delta,$$
$$|\mathrm{Im}\,(z_{III} - 2\pi\mu)| \leqslant R_0 - 4\delta + 2\delta = R_0 - 2\delta.$$

6°. **Estimate of $\partial z/\partial\mu$.** The function $g(z, \mu)$ is defined for

$$|\mathrm{Im}\,(z - 2\pi\mu)| \leqslant R_0 - 2\delta.$$

Therefore the following function is also defined in that strip:

$$\varphi(z, \mu) = z + g(z, \mu).$$

Since in that strip $|g(z, \mu) < \delta|$ (see (6), (17)), then the image of this strip as $z \to \phi$ contains the strip

$$|\mathrm{Im}\,(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta,$$

which as $\phi \to z$ goes into a region containing the strip

$$|\mathrm{Im}\,(z - 2\pi\mu)| \leqslant R_0 - 4\delta.$$

From subsections 5.1 and 7.4 it follows that

$$\frac{\partial z}{\partial\mu} = -\frac{\dfrac{\partial g}{\partial\mu}}{1 + \dfrac{\partial g}{\partial z}}.$$

From inequality (18) and conditions 4), 5) of Lemma 11, $|\partial g/\partial z| < \frac{1}{2}$. Thus, applying estimate (23), we obtain

$$\left|\frac{\partial z}{\partial\mu}\right| \leqslant \frac{10^4 C}{K^2\delta^6}$$

for $|\text{Im}(z-2\pi\mu)| \leq R_0 - 2\delta$, $\mu \in N_K^r$ and, in particular, for

$$|\text{Im}(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta.$$

10°. **Estimate of $(\partial/\partial\mu) F_1(\phi, \mu)$, $(\partial/\partial\mu)\Phi_1(\phi, \mu, \Delta_1)$.** From subsection 5.1,

$$F_1(\varphi, \mu) = \hat{F}_1(z(\varphi, \mu), \mu),$$
$$\Phi_1(\varphi, \mu, \Delta_1) = \hat{\Phi}_1(z(\varphi, \mu), \mu, \Delta(\Delta_1, \mu)).$$

The function $z(\phi, \mu)$ is defined for $|\text{Im}(\phi - 2\pi\mu)| \leq R_0 - 3\delta$, $\mu \in N_K^r$, and if

$$|\text{Im}(z - 2\pi\mu)| \leqslant R_0 - 4\delta,$$

then for this z there exists a ϕ such that $z = z(\phi, \mu)$ and

$$|\text{Im}(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta.$$

The functions $\hat{F}_1(z)$, $\hat{\Phi}_1(z)$ are defined for $|\text{Im}\, z| \leq R_0 - 4\delta$ and therefore the functions $F_1(\phi, \mu)$, $\Phi_1(\phi, \mu, \Delta_1)$ are defined for

$$|\text{Im}\,\varphi| \leqslant R_0 - |\text{Im}\, 2\pi\mu| - 5\delta$$

under the hypothesis that $|\text{Im}\, 2\pi\mu| \leq R_0 - 5\delta$, i.e., that $2\pi\mu \leq R_0 - 5\delta$. In this region

$$\frac{\partial F_1}{\partial\mu} = \frac{\partial \hat{F}_1}{\partial\mu} + \frac{\partial \hat{F}_1}{\partial z}\frac{\partial z}{\partial\mu}, \qquad \frac{\partial \Phi_1}{\partial\mu} = \frac{\partial \hat{\Phi}_1}{\partial\mu} + \frac{\partial \hat{\Phi}_1}{\partial z}\frac{\partial z}{\partial\mu},$$

where in the calculation of $\partial\hat{\Phi}_1/\partial\mu$ the independent variables are taken to be z, μ and Δ_1, as in point 8°.

For the estimation of $\partial\hat{F}_1/\partial z$ and $\partial\hat{\Phi}/\partial z$ we use the Cauchy integral. Staying at a distance δ from the boundary of the strip, where the estimates of $\hat{F}_1$ and $\hat{\Phi}_1$ are known, we see from the estimates of 3° and 5° of §5 that

$$\left|\frac{\partial \hat{F}_1}{\partial z}\right| \leqslant \frac{4C^2}{\delta^6}, \quad \left|\frac{\partial \hat{\Phi}_1}{\partial z}\right| \leqslant \frac{3C\,|\Delta_1|}{\delta^6}$$

for $|\text{Im}\, z| \leq R_0 - 5\delta$. Applying estimates 5°, 8° and 9°, we find from (35) that

$$\left|\frac{\partial F_1}{\partial\mu}\right| \leqslant \frac{5\cdot 10^4 C^2}{K^2\delta^7} + \frac{10^4 C}{K^2\delta^6}\frac{4C^2}{\delta^6},$$
$$\left|\frac{\partial \Phi_1}{\partial\mu}\right| \leqslant \frac{C\cdot 10^5\,|\Delta_1|}{K^2\delta^7} + \frac{3C\,|\Delta_1|}{\delta^6}\frac{10^4 C}{K^2\delta^6}.$$

Thus, for

$$|\Delta_1| \leqslant \delta^{28}, \quad \mu \in N_K^r, \quad 2\pi r \leqslant R_0 - 5\delta, \quad |\text{Im}\,\varphi| \leqslant R_0 - |\text{Im}\, 2\pi\mu| - 6\delta,$$

we have

$$|F_1(\varphi, \mu)| \leqslant \frac{C^2}{\delta^6}, \qquad |\Phi_1(\varphi, \mu, \Delta_1)| < \frac{|\Delta_1|}{\delta^6}$$
$$\left|\frac{\partial F_1}{\partial\mu}\right| \leqslant \frac{C^2}{\delta^{13}}, \qquad \left|\frac{\partial \Phi_1}{\partial\mu}\right| \leqslant \frac{C\,|\Delta_1|}{\delta^{13}},$$

since

$$\frac{5\cdot 10^4}{K^2}\delta < 1. \tag{6}$$

In exactly the same way all the remaining estimates 1°–9°, in view of conditions 4) and 5) of Lemma 11, may be brought into the form (7)–(16).

Lemma 11 is proved.

8.3. **Proof of Theorem 3.** Theorem 3 is derived from Lemma 11 in the same way as Theorem 2 was derived from the Fundamental Lemma in §6.

We choose $\delta_1 > 0$ such that

1) $\sum_{n=1}^{\infty} \delta_n < \frac{R}{8}$, where $\delta_n = \delta_{n-1}^{1\frac{1}{2}}$ $(n = 2, 3, \dots)$,

2) $\delta_1 < \frac{K^2}{5\cdot 10^4}$.

Let $R = R_0$, K be the same as in condition of Theorem 2, $\mu \in N_K^{\frac{K}{16\pi(n+1)}}$, $\Delta_0 = \delta_1^{26}$, $L\varepsilon_0 < C_1$, where

$$C_1 = \delta_1^{27}, \tag{35}$$

and C_1, δ_1 are respectively the C and δ of Lemma 11. Then from inequalities (7)–(16) we obtain

$$|F_1| < \frac{\delta_1^{54}}{\delta_1^{13}} < \delta_1^{40,5} = (\delta_1^{1\frac{1}{2}})^{27} = \delta_2^{27},$$

$$\left|\frac{\partial F_1}{\partial \mu}\right| < \delta_2^{27},$$

$$|\Phi_1| \leqslant \frac{\delta_1^{27}}{\delta_1^{13}} |\Delta_1| < \delta_1^3 |\Delta_1| = \delta_2^2 |\Delta_1|,$$

$$\left|\frac{\partial \Phi_1}{\partial \mu}\right| < \delta_2^2 |\Delta_1|$$

for

$$|\Delta_1| < \delta_2^{26} = \delta_1^{39} < \delta_1^{28}, \quad |\operatorname{Im} \varphi_1| \leqslant R_0 - 7\delta_1 - |\operatorname{Im} 2\pi\mu| = R_1, \quad \mu \in N_K^{\frac{R}{16\pi(n+1)}}$$

Thus, we again find ourselves in the conditions of Lemma 11, but with a decrease of $7\delta_1 + R/8(n+1)$ in the radius of R_1. Since

$$\sum_{n=1}^{\infty} \delta_n < \frac{R}{8},$$

then we may carry out n successive approximations, and the last will operate for

$$|\operatorname{Im} \varphi_n| \leqslant \frac{R}{8(n+1)}, \quad \mu \in N_K^{\frac{R}{16\pi(n+1)}}, \quad |\Delta_n| < \delta_{n+1}^{26}.$$

Omitting the usual (see §6) proof of the convergence of the approximations for real μ, we estimate $|\partial\Delta^{(n)}/\partial\mu|$.

From subsection 8.1 it follows that

$$\frac{\partial\Delta_k^{(n)}}{\partial\mu}=\frac{\partial\Delta_k}{\partial\mu}+\frac{\partial\Delta_k}{\partial\Delta_{k+1}}\frac{\partial\Delta_{k+1}^{(n)}}{\partial\mu}.$$

Putting $C_k=\delta_k^{27}$, on the basis of Lemma 11 we find that

$$\left|\frac{\partial\Delta_k^{(n)}}{\partial\mu}\right|\leqslant 4C_{k+1}+2\left|\frac{\partial\Delta_{k+1}^{(n)}}{\partial\mu}\right|.$$

If

$$\left|\frac{\partial\Delta_{k+1}^{(n)}}{\partial\mu}\right|<C_{k+1},$$

then

$$\left|\frac{\partial\Delta_k^{(n)}}{\partial\mu}\right|<6C_{k+1}<C_k.$$

Since

$$\left|\frac{\partial\Delta_n^{(n)}}{\partial\mu}\right|=0,$$

then

$$\left|\frac{\partial\Delta_0^{(n)}}{\partial\mu}\right|<6C_1.$$

Theorem 3 is proved.

Remark. In exactly the same way we may prove the monogenicity of the functions g_n, F_n, Φ_n, ϕ_n and obtain analogous estimates.

Part II

On the space of mappings of the circumference onto itself

The problem of studying the dependence of the rotation number on the coefficients of the equation was posed by Poincaré [1]. The consideration of the rotation number as a function on the space of mappings makes it possible to elucidate questions concerning typical and exceptional cases.

Angular coordinates of a point on a circumference will be denoted by small greek letters; ϕ and $\phi+2\pi$ are one and the same point of the circumference. We shall use capital letters to denote transformations:

$$T:\phi\longrightarrow T\phi.$$

We shall consider only continuous one-to-one direct (orientation-preserving) transformations. As an example one may cite the rotation through the angle $\theta:\ \phi\longrightarrow\phi+\theta$. To each transformation we assign a "displacement," namely a function on the circumference showing how much each point is displaced. We shall denote a displacement by the same letter as the transformation, but in lower case:

$$T : \varphi \to T_\varphi = \varphi + t(\varphi).$$

Here $t(\phi)$ is the displacement. If T is a rotation through the angle θ, then $t(\phi) \equiv \theta$. Generally speaking, a shift, as also ϕ, is defined only up to a multiple of 2π. However, if we define $t(\phi)$ at one point, we may uniquely continue it by continuity.

If T is a smooth transformation, then $t(\phi)$ is a smooth periodic function:

$$t(\varphi + 2\pi) = t(\varphi).$$

We denote by

$$T^n\varphi = \varphi + t^{(n)}(\varphi)$$

the nth power of the transformation of T. In connection with this notation we suppose that branches of $t^{(n)}(\phi)$ are chosen to correspond to the branches of $t(\phi)$:

$$t^{(n)}(\varphi) = t^{(n-1)}(\varphi) + t(T^{n-1}(\varphi)) \quad (n = 2, 3, \ldots).$$

Under this condition $t^{(n)}(\phi)$ is called a displacement with n steps.

§9. The function $\mu(T)$ and its level sets

We consider the spaces

$$C \supset C^1 \supset C^2 \supset \ldots \supset C^n \supset \ldots \supset C^\infty \supset A$$

of one-to-one direct mappings of the circumference onto itself, continuous, continuously and infinitely differentiable, and analytic in the neighborhood of the real axis, with the topologies usual in these spaces. Each successive topology is stronger than the preceding one and each of the spaces is everywhere dense in the preceding one.*

Poincaré [1] defined for each transformation $T \in C$ the rotation number $2\pi\mu$. Thus on the space C there is given a function $\mu(T)$. The following theorem was stated by Poincaré without proof.

Theorem 4. *The function* $\mu(T)$ *is continuous at each point of* C.

Proof. We shall show that $\mu(T)$ is continuous at the point T_0.

Suppose given a point $\epsilon > 0$. We choose a number $n > 2/\epsilon$ such that

$$\frac{m}{n} < \mu(T_0) < \frac{m+1}{n}.$$

*If T lies in one of the spaces $C^1, C^2, \cdots, A$ without distinction as to which one, then we shall call T a *smooth* transformation.

Then under the transformation

$$T_0^n : \varphi \to \varphi + t_0^{(n)}(\varphi)$$

each point is shifted by more than $2\pi m$. Indeed, if some point were shifted less, and another point more, then there would be a point shifted by exactly $2\pi m$, i.e., a point which is fixed for T_0^n. Then, evidently, in spite of the choice of n, we would have

$$\mu = \frac{m}{n}.$$

If all the points were shifted by less than $2\pi m$, then we would have $\mu \leq m/n$, which again contradicts the choice of n.

Analogously one proves that each point is shifted in the course of n steps through less than $2\pi(m+1)$. Thus

$$2\pi m < t_0^{(n)}(\varphi) < 2\pi(m+1).$$

In view of the continuity of $t_0^{(n)}(\phi)$,

$$2\pi m + \eta < t_0^{(n)}(\varphi) < 2\pi(m+1) - \eta$$

for some $\eta > 0$, and in view of the continuous dependence of $T^{(n)}$ on T there exists a $\delta > 0$ such that

$$|t^{(n)}(\varphi) - t_0^{(n)}(\varphi)| < \eta,$$

as soon as the transformation T differs from T_0 by less than δ:

$$|t(\varphi) - t_0(\varphi)| < \delta.$$

For such T

$$2\pi m < t^{(n)}(\varphi) < 2\pi(m+1),$$

so that

$$\frac{m}{n} < \mu(T) < \frac{m+1}{n}.$$

Thus, $|\mu(T) - \mu(T_0)| < \epsilon$ for $|t(\phi) - t_0(\phi)| < \delta$. The theorem is proved.

Remark. Even in very nice cases the function $\mu(T)$ is only continuous. For example, consider the family of transformations

$$T_h : \varphi \to \varphi + h + 0{,}1 \sin^2 \varphi,$$

where h is a parameter. By what has been proved, $\mu(T_h)$ is a continuous function of h. With increasing h the function $\mu(T_h)$ grows, but with a lag at each rational value of μ. To this value there corresponds a whole segment $[h_1 h_2]$ of values of h. On the other hand, for $h > h_2$ the function $\mu(T_h)$ increases with extreme rapidity. E. G. Belaga showed that, for example, in the neighborhood of the origin $\mu(T_h)$ grows at least as fast as $C\sqrt{h}/-\log h$.

A level set of $\mu(T)$ is a set of transformations with the same rotation number $2\pi\mu$. To such transformations there belong the rotation through the angle $2\pi\mu$, transformations convertible into rotations through the angle $2\pi\mu$ by an appropriate change of variables, and possibly other transformations.

The structure of the level set $\mu(T) = \mu$ essentially depends on whether μ is rational or irrational.

§10. The case of rational μ

10.1. If $\mu(T) = m/n$, then, as Poincaré showed, T^n has fixed points: $t^{(n)}(a) = 2\pi m$. The set of these points is invariant relative to T and closed, as the level set of the continuous function $t^{(n)}(a)$. The points $a, Ta, \cdots, T^{n-1}a$ are called a *cycle*. In the investigation of cycles it is convenient to consider the graph of the transformation T^n and the graph of the function $t^{(n)}(\phi)$ (see Figure 7; on this drawing we have shown the graph of $T(\phi) = \phi + \frac{1}{2}\cos\phi$ and we have indicated the image of 0 for several iterations of T). A cycle is called *isolated* if in some neighborhood of its points there are no points of other cycles. An isolated cycle is *stable* if one of its points, and thus all of its points, has arbitrarily small neighborhoods which are taken into their own interiors by the transformation T^n. It is easy to see that as $n \to +\infty$ the points of such a neighborhood tend to points of the cycle, which explains the usage. A stable cycle of the transformation T^{-1} is called an *unstable* cycle of T. An isolated cycle is *semistable forward* (*backward*) if all the points of some neighborhood of a point of the cycle (the point itself excluded) are moved forward (backward) by the transformation T^n, i.e., if in this neighborhood

$$t^{(n)}(\varphi) - 2\pi m > 0 \quad (< 0).$$

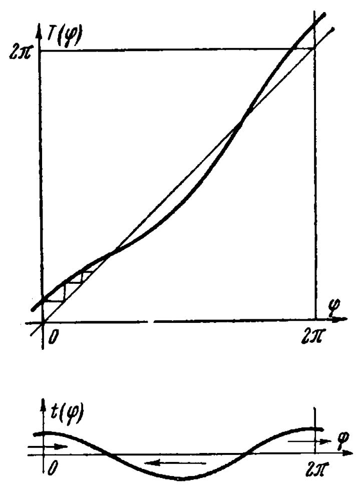

Figure 7

A transformation $T \in C^1$ is *normal* if at the points of its cycles

$$\frac{dt^{(n)}(\varphi)}{d\varphi} \neq 0.$$

Evidently, a normal transformation has a finite number of cycles, while all of these cycles are stable or unstable. Indeed, those roots of $t^{(n)}(\phi) - 2\pi m$, where $dt^{(n)}/d\phi < 0$, are points of stable cycles, and those where $dt^{(n)}/d\phi > 0$ are points of unstable cycles. Therefore it follows that the points of stable and unstable cycles of a normal transformation alternate.

10.2. **Theorem 5.** *Normal transformations form a set open in C^1 and everywhere dense in A.*

Proof. 1. The points of a cycle are the points where $t^{(n)}(\phi) = 2\pi m$. At these points $dt^{(n)}(\phi)/d\phi \neq 0$. Therefore for a small, along with the first derivative, variation of $t^{(n)}(\phi)$ the function $t^{(n)}(\phi) - 2\pi m$ does not acquire any new roots and the old ones do not disappear, but rather are displaced continuously, while the derivatives at the roots preserve sign. This means that the transformation T with such a variation of the function $t^{(n)}(\phi)$ becomes normal. In view of the continuous dependence of $t^{(n)}(\phi)$ on T, the first assertion of the theorem is proved.

2. We shall show that *arbitrarily close to any transformation there is an analytic transformation with a cycle.* Evidently it is sufficient to prove this for an analytic transformation and analytic proximity. Suppose that T is an analytic transformation with an irrational rotation number, and suppose that $\epsilon > 0$. Among the points $\phi_n = T^n\phi_0$ is one displaced from ϕ_0 by less than ϵ, for example, backward:

$$2\pi m - \varepsilon < t^{(n)}(\varphi_0) < 2\pi m$$

(Denjoy's theorem). We consider a family of analytic transformations T_λ ($\lambda \geq 0$, $T_0 = T$):

$$T_\lambda : \varphi \to \varphi + t(\varphi) + \lambda.$$

It is not hard to see that for $\lambda = \epsilon$ T_λ^n displaces ϕ_0 ahead:

$$t_\lambda^{(n)}(\varphi_0) \geqslant 2\pi m.$$

Hence, in view of the continuity of $t_\lambda^{(n)}(\phi_0)$ in λ, it follows that for some $\lambda_0 \leq \epsilon$ T_{λ_0} has a cycle $\phi_0, T_{\lambda_0}\phi_0, \cdots$:

$$t_{\lambda_0}^{(n)}(\varphi_0) = 2\pi m.$$

3. **An analytic transformation with a cycle can be converted into a normal transformation by an arbitrarily small variation.** Indeed, suppose that T is an analytic transformation, and among its cycles there is no stable cycle (and therefore also no unstable cycle). We choose a cycle $\phi_0, \phi_1, \cdots, \phi_{n-1}$ and introduce an analytic function $\Delta(\phi)$, vanishing at these points and having there negative derivatives. The transformation

$$T_\theta : t_\theta(\varphi) = t(\varphi) + \theta\Delta(\varphi)$$

for small θ is arbitrarily proximate to T and has at least one stable cycle ϕ_0, $\phi_1, \cdots, \phi_{n-1}$. Therefore it is sufficient to consider the case when the desired transformation T has a stable cycle. We shall construct over T an analytic function $\delta(\phi)$ which

1) is equal to zero and has a negative (positive) derivatives at the points of the stable cycles of T;

2) is positive (negative) at the points of the cycles of T which are semistable forward (backward).

The existence of such a function is obvious, since the number of all cycles of T is finite, because the analytic function $t^{(n)}(\phi) - 2\pi m$ has an isolated root and therefore is not identically zero.

Consider the transformation $T_\theta : \phi \longrightarrow \phi + t(\phi) + \theta\delta(\phi)$. For small θ this transformation is normal. The formal proof of the fact that the stable cycles of T for small θ are only somewhat shifted, while the roots of $t^{(n)}(\phi) - 2\pi m$ become multiple, and the semistable cycles vanish, is left to the reader. For sufficiently small θ the transformation T_θ is the desired one.

Theorem 5 is proved.

10.3. The construction of a normal transformation may be easily perceived from the graph of the function $t^{(n)}(\phi) - 2\pi m$. Its roots are the points of the cycles of the transformation and divide the circumference into arcs. Each such arc $\alpha\beta$ is bounded at one end by a point α of a stable cycle and at the other end by a point β of an unstable cycle. For $n \longrightarrow +\infty$ the points of the arc wind around onto the stable cycle, and for $n \longrightarrow -\infty$ onto the unstable cycle, i.e.,

$$\lim_{k\to\infty} T^{kn}(\gamma) = \alpha \,(\mathrm{mod}\, 2\pi), \qquad \lim_{k\to-\infty} T^{kn}(\gamma) = \beta \,(\mathrm{mod}\, 2\pi),$$

where $\gamma \in (\alpha, \beta)$. Assertions of this type are well known in the qualitative theory of differential equations, and we omit the proof.

Thus a topologically normal transformation is characterized by three integers m, n, k, where m/n is the rotation number and k the number of stable (and therefore of unstable) cycles. Two transformations with the same m, n, k are similarly arranged in the sense that one of them can be converted into the other by a continuous change of variables on the circumference (i.e., $T_2 = \Phi T_1 \Phi^{-1}$, where $\Phi \in C$). In addition the derivative $dt^{(n)}(\phi)/d\phi$ at the points of the cycle, which characterizes the rapidity of winding around onto the cycle, is an invariant under a smooth change of variables. Probably there are no other invariants, but I have not been able to prove this.

Theorem 6. *The set $E_{m/n}$ at the level $\mu = m/n$ in any of the spaces $C^1, \cdots$ $\cdots, A$ is connected and consists of*

1) *a kernel $\bigcup_{k=1}^{\infty} E^k_{m/n}$ of normal transformations dense in $E_{m/n}$ and open in $C^p(A)$. The kernel consists of connected components $E^k_{m/n}$ of transformations with k stable and k unstable cycles. Two transformations of one component $E^k_{m/n}$ may be converted one into the other by a continuous change of variables;*

2) *the boundaries of* $E_{m/n}$ *and* $E^k_{m/n}$. *The boundary of* $E_{m/n}$ *consists of transformations* T *for which* $t^{(n)}(\phi) - 2\pi m$ *does not change sign. Its parts* $F_+(t^{(n)}(\phi) - 2\pi m \geq 0)$ *and* $F_-(t^{(n)}(\phi) - 2\pi m \leq 0)$ *contain transformations semi-stable forward and backward, and are connected and intersect in a connected set* F_0. *Transformations from* F_0 *change under a smooth substitution of variables into rotations.* F_0 *lies in the boundary of each component of* $E^k_{m/n}$.

Proof. 1. The sets $E_{m/n}$, F_+, F_- are connected. For the proof we join, without leaving the set in question, any transformation $T \in E_{m/n}(F_+, F_-)$ with the rotation T_2 through the angle $2\pi m/n$ by an arc T_θ $(0 \leq \theta \leq 2,\ T_0 = T)$. Suppose that $\phi_0, \cdots, \phi_{n-1}$ is a cycle of T. Making the smooth substitution of variables

$$\varphi \to \Psi\varphi = \varphi + \psi(\varphi)$$

we transfer the points $\phi_0, \cdots, \phi_{n-1}$ into $2\pi ml/n$ $(0 \leq l \leq n-1)$. Put

$$\Psi_\theta\varphi = \varphi + \theta\psi(\varphi)$$

and consider

$$T_\theta\varphi = \Psi_\theta T \Psi_\theta^{-1}\varphi = \varphi + t_\theta(\varphi) \quad (0 \leqslant \theta \leqslant 1).$$

This transformation is the transformation T described in the variable Ψ_θ, and belongs to $E_{m/n}(F_+, F_-)$.

Now we consider the segment joining T_1 to T_2:

$$T_\theta\varphi = \varphi + (\theta - 1)2\pi\frac{m}{n} + (2 - \theta)t_1(\varphi) \quad (1 \leqslant \theta \leqslant 2).$$

The points $2\pi ml/n$ $(0 \leq l \leq n-1)$ form a cycle T_θ for all $1 \leq \theta \leq 2$ and therefore the curve T_θ lies entirely in $E_{m/n}$ (respectively F_+, F_-). The connectedness is proved.

2. The set $E^k_{m/n}$ **of normal transformations with given** m, n, k **is connected in any of the spaces** $C^1, \cdots, A$. For the proof we join in the space in question the transformations T_0, T_2 by the arc T_θ $(0 \leq \theta \leq 2)$. We carry out a smooth substitution of variables

$$\Psi\varphi = \varphi + \psi(\varphi),$$

taking the points of the cycles T_0 into the corresponding points of the cycles T_2 (which is not hard to do since the number of these points is the same and they follow in the same order). The transformation $T_1 = \Psi T_0 \Psi^{-1}$ operates on the points of the cycles of T_2 in the same way as the transformation T_2; it is easy to see that it does not have other cycles. Putting

$$\Psi_\theta(\varphi) = \varphi + \theta\psi(\varphi)$$

and

$$T_\theta = \Psi_\theta T_0 \Psi_\theta^{-1} \quad (0 \leqslant \theta \leqslant 1),$$

we join T_0 to T_1 by a curve lying in $E^k_{m/n}$.

Consider the transformation

$$T_1(\varphi) = \varphi + t_1(\varphi), \quad T_2(\varphi) = \varphi + t_2(\varphi).$$

The functions $t_1(\phi)$ and $t_2(\phi)$ coincide at the points of the cycles, and therefore all the transformations

$$T_\theta(\varphi) = \varphi + (2-\theta)\, t_1(\varphi) + (\theta - 1)\, t_2(\varphi) \quad (1 \leqslant \theta \leqslant 2)$$

have the same cycles. Accordingly, the curve T_θ $(0 \leq \theta \leq 2)$, joining T_0 to T_2, lies entirely in $E^k_{m/n}$.

3. The proof of the fact that the set $E^k_{m/n}$ is open and that the set $\bigcup_k E^k_{m/n}$ of normal transformations with the rotation number m/n is everywhere dense in $E_{m/n}$ is analogous to the proof of Theorem 5 (subsections 1 and 3).

4. If $T_1, T_2 \in E^k_{m/n}$, then we may carry out a continuous change of variables $\Psi = \phi + \psi(\phi)$ such that T_1 goes into T_2: $T_2 = \Psi T_1 \Psi^{-1}$. Indeed, we denote the points of the stable cycles of T_1 by a^l_i $(1 \leq l \leq k,\ 1 \leq i \leq n,\ T_1 a_i = a_{i+1},\ a_{n+1} = a_1)$ and the points of the unstable cycles of T_1 by b^l_i (by l we denote the number of the cycle in the order in which it follows on the circumference). Here there are no points of the cycles on the arc $a^1_1 b^1_1$ (thus the same is true of each arc $a^l_i b^l_i$ and $b^l_i a^{l+1}_i$).*

Suppose further that c^l_i and d^l_i are the points of the stable and unstable cycles of T_2, enumerated in an analogous way. The substitution of variables Ψ carries the points a^l_i, b^l_i into c^l_i, d^l_i, and it remains for us to complete the definition of Ψ to the arcs $a^l_i b^l_i$, $b^l_i a^{l+1}_i$. We choose the points x and y inside the arcs $a^1_1 b^1_1$ and $c^1_1 d^1_1$. The points $T^n_1 x$ and $T^n_2 y$ lie in the same arcs closer to a^1_1 and c^1_1 respectively. We map the arc $(x, T^n_1 x)$ onto the arc $(y, T^n_2 y)$ homeomorphically and directly using Ψ: $x \to y$, $T^n_1 x \to T^n_2 y$. Evidently under the transformations T^p_1 the images of the arc $[x, T^n_1 x]$ (or of the arc $[y, T^n_2 y]$ under the transformations T^p_2) entirely cover the whole arc $a^1_i b^1_i$ $(1 \leq i \leq n)$ (the whole arc $c^1_i d^1_i$). Thus we define $\Psi(\phi)$ on the arc $T^p_1 x$, $T^{p+n}_1 x$ as $T^p_2 \Psi T^{-p}_1$. An analogous construction is possible on $a^l_i b^l_i$ and $b^l_i a^{l+1}_i$. The proof of the fact that the substitution of variables just found is the desired one is not complicated and we omit it.

5. **The structure of the boundary.** If $t^{(n)}(\phi) - 2\pi m$ changes sign, then T is an interior point of $E_{m/n}$ since under a small variation of T, $t^{(n)}(\phi) - 2\pi m$ will change sign as before, and T preserves the cycle. Therefore the boundary $E_{m/n}$ enters into the sum of F_+ ($T \in F_+$ if $t^{(n)}(\phi) - 2\pi m \geq 0$) and F_-. In order to convert the transformation $T \in F_0 = F_+ \cap F_-$ into a rotation, we need to carry the points of one cycle into $2\pi ml/n$ by a smooth substitution of variables and then to

*By $l+1$ for $l = k$ we understand 1.

redefine the parameters on all the arcs $[2\pi ml/n, 2\pi(ml+1)/n]$, except one ($l=0$), according to the formula

$$\Psi(\varphi) = 2\pi\frac{ml}{n} + T^{-l}(\varphi).$$

By a small variation of a rotation through the angle $2\pi m/n$ one may convert it into a transformation in any $E^k_{m/n}$, roughly as was done in the proof of Theorem 5 of subsection 3. From the preceding considerations it follows that the same is true also for all transformations of F_0, which proves the last assertion of Theorem 6.

10.4. From Theorem 6 (point 4 of the proof) it follows that normal transformations are *rough* in the sense of Andronov-Pontrjagin [10]. Since, by Theorem 5, the set of all normal transformations is everywhere dense, no nonnormal transformation can be rough.

From the topological point of view normal transformations fill out a predominant part of the space of transformations, namely an everywhere dense open set. In the following section it will be proved that from the point of view of measure the typical case is also the ergodic case.

§11. The case of irrational μ

11.1. Consider now the set E_μ of irrational level μ. In the spaces $C^2, \cdots, A$, by Denjoy's theorem, each transformation $T \in E_\mu$ may be converted into a rotation through the angle $2\pi\mu$ by a *continuous* change of variables. We are also concerned with transformations which can be converted into a rotation by a *smooth* change of variables. The set of such transformations will be denoted by $E^{C^p}_\mu$ (respectively by E^A_μ; the common notation is E'_μ).

Theorem 7. 1°. *The set E^A_μ is everywhere dense in E_μ in the topology of C. All sets E'_μ are connected.*

2°. *If μ is such that $|\mu - m/n| > K/|n|^3$ for any integers m and n not equal to zero, then the set E^A_μ is open in E_μ in the topology of A.*

Proof. 1°. Suppose that T_0 denotes a rotation through the angle $2\pi\mu$, and suppose that $T_1 \in E'_\mu$. Then there exists a smooth substitution of variables

$$\Psi(\varphi) = \varphi + \psi(\varphi)$$

such that $T_1 = \Psi T_0 \Psi^{-1}$. The substitution

$$\Psi_\theta(\varphi) = \varphi + \theta\psi(\varphi) \quad (0 \leqslant \theta \leqslant 1)$$

converts T_0 into $T_\theta = \psi_\theta T_0 \psi_\theta^{-1}$; thus the curve T_θ joining T_0 to T_1 lies entirely in E'_μ. The connectedness of E'_μ is proved.

We shall construct in E^A_μ a transformation T^* in a given neighborhood of $T \in E_\mu$. By Denjoy's theorem there exists a continuous substitution of variables

$\Psi(\phi)$ such that $T = \Psi T_0 \Psi^{-1}$. We shall construct an analytic substitution $\Psi^*(\phi)$ of variables ϕ such that Ψ and Ψ^*, Ψ^{-1} and Ψ^{*-1} differ only slightly in the metric of C. Then $T^* = \Psi^* T_0 \Psi^{*-1}$ approximates T in the metric of C and lies in E_μ^A. Assertion 1° is completely proved.

2°. The fact that the set E_μ^A is open in $E_\mu \cap A$ follows from Theorem 2. Evidently it is sufficient to show that some neighborhood of the rotation T_0 in $E_\mu \cap A$ lies in E_μ^A. The transformation $T \in E_\mu \cap A$ may be written in the form

$$\varphi \to \varphi + 2\pi\mu + F(\varphi),$$

while the neighborhood $U_{R,C}$ of the transformation T_0 is given by the inequality $|F(\phi)| < C$ for $|\mathrm{Im}\,\phi| < R$. But by the Corollary to Theorem 3 (see subsection 4.3), for a given R there exists a C such that all the transformations $T \in U_{R,C} \cap E_\mu$ analytically reduce to rotations. Theorem 7 is proved.

11.2. In turning to the question of typicality from the point of view of measure (see [8]) we encounter the absence of a reasonable measure in functional spaces and therefore we are forced to restrict ourselves to finite-dimensional subspaces.

Consider the two-dimensional space of analytic transformations

$$A_{a,b}: z \to z + a + F(z, b),$$

where for $|\mathrm{Im}\,z| < R$, $|b| < b_0$ $F(z, b)$ is an analytic function satisfying the inequality $|F(z, b)| < L|b|$.

Theorem 8.

$$\lim_{\theta \to 0} \frac{\mathrm{mes}\, E_\theta}{2\pi\theta} = 1, \tag{1}$$

where E_θ is the set of points of the set (ab), $a \in [0, 2\pi]$, $b \in [0, \theta]$, such that the transformation A_{ab} converts into a rotation by an analytic substitution of the coordinate z.

Proof. 1. Consider the set M_K, namely the compact set of points $0 < \mu < 1$ satisfying the inequality

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^3}$$

for all $m, n > 0$. By Theorem 2, for any $\mu \in M_k$ there exist $C = C(K, R) > 0$ and a function $\Delta(b, \mu)$, analytic in b, such that the transformation $A_{2\pi\mu+\Delta(b,\mu),\, b}$ for $\mu \in M_K$, $|b| < C$ may be converted into a rotation by an analytic change of parameter: $(2\pi\mu + \Delta(b, \mu), b) \in E_\theta$. We denote by $M_K(b)$ the set of points $\mu + \Delta(b, \mu)/2\pi$, $\mu \in M_K$ for a fixed b. Then the transformation $D_b: \mu \to \mu + \Delta(b,\mu)/2\pi$ carries M_K into the set $M_K(b)$.

Put $\epsilon > 0$ and choose $K > 0$ so that $M_{2K} > 1 - \epsilon/3$ (from Lemma 1 of §2 this is possible). We shall show that for sufficiently small b the inequality

$$\mathrm{mes}\, M_{\frac{K}{2}}(b) > 1 - \epsilon$$

is valid, from which Theorem 8 will follow immediately, since it is evident that

$$2\pi\theta \geqslant \operatorname{mes} E_0 \geqslant 2\pi \int_0^{\theta} \operatorname{mes} M_{\frac{K}{2}}(b)\, db.$$

2. In §7 we constructed a perfect set $N_K^0 = N_K$, $M_{2K} \subseteq N_K \subseteq M_{K/2}$. Evidently it is sufficient to show that for sufficiently small b

$$\operatorname{mes} N_K(b) > 1 - \varepsilon. \tag{2}$$

(Since $K > 0$ is fixed, we may now drop the index K: $N_K = N$.)

From Theorem 3, the mapping $D_b : N \longrightarrow N(b)$ is the limit of a uniformly converging sequence of monogenic mappings

$$D_b^n : \mu \to \mu + \frac{1}{2\pi} \Delta^n (b, \mu).$$

We shall show that for any $\epsilon > 0$ there exists a $b(\epsilon)$ such that for $b < b(\epsilon)$ and any n

$$\operatorname{mes} D_b^n (N) > 1 - \varepsilon. \tag{3}$$

From Theorem 3, there exists a $b(\epsilon)$ such that for n, $b < b(\epsilon)$, $\mu \in N$ the following inequality will hold:

$$\left| \frac{\partial \Delta^n}{\partial \mu} \right| < \frac{\varepsilon}{3};$$

i.e., under the mapping D_b^n, N maps almost without dilation.

We shall show that $b(\epsilon)$ has the desired property (the index n will be dropped everywhere, since the argument is always carried out for n fixed). Suppose $b < b(\epsilon)$. From the definition of monogenicity, for $\epsilon/3$ there exists a $\delta > 0$ such that

$$\left| \frac{\Delta(\mu_1) - \Delta(\mu_2)}{\mu_1 - \mu_2} - \frac{\partial \Delta(\mu_3)}{\partial \mu} \right| < \frac{\varepsilon}{3}$$

if $|\mu_1 - \mu_3| < \delta$, $|\mu_2 - \mu_3| < \delta$, $\mu_1, \mu_2, \mu_3 \in N$. Then under the same conditions

$$\left| \frac{\Delta(\mu_1) - \Delta(\mu_2)}{\mu_1 - \mu_2} \right| < \frac{2\varepsilon}{3}, \tag{4}$$

in view of the choice of $b(\epsilon)$.

3. We decompose N into nonintersecting (of course, measurable) parts N^i, $\bigcup_{i=1}^{L} N^i = N$, the diameter of each of which is less than δ, and suppose that $N^i(b)$ are their images under the transformation D_b^n. Since under this transformation the distance between two points of N^i cannot decrease, as follows from (4), by more than $1 - 2\epsilon/3$ times, therefore

$$\operatorname{mes} N^i(b) > \left(1 - \frac{2\varepsilon}{3}\right) \operatorname{mes} N^i,$$

from which it follows that

$$\sum_{i=1}^{L} \operatorname{mes} N^i(b) > \left(1 - \frac{2\varepsilon}{3}\right) \sum_{i=1}^{L} \operatorname{mes} N^i.$$

Thus

$$\operatorname{mes} N(b) > \left(1 - \frac{2\varepsilon}{3}\right) \operatorname{mes} N,$$

and since

$$\operatorname{mes} N > 1 - \frac{\varepsilon}{3},$$

we obtain

$$\operatorname{mes} N(b) > \left(1 - \frac{2\varepsilon}{3}\right)\left(1 - \frac{\varepsilon}{3}\right) > 1 - \varepsilon,$$

and inequality (3) is proved. Inequality (2) follows from this, since the following lemma is valid.

Lemma. *Suppose that $E \subseteq [0, 1]$ is a perfect set and that f_n is a sequence of continuous mappings of this set onto $F_n \subseteq [0, 1]$, uniformly converging to the mapping f: $E \to F$, and suppose $0 \leq \Delta < 1$. If* mes $F_n > 1 - \Delta$ *for all n, then* mes $F \geq 1 - \Delta$.

Proof. Suppose that $\epsilon > 0$. We consider the set D_ϵ of contiguous intervals of F larger than ϵ. There will be a finite number of them, and for a sufficiently large n these intervals will be arbitrarily little different from the corresponding contiguous intervals of F_n. The sum of the length of the latter for any n is less than Δ, since mes $F_n > 1 - \Delta$. Therefore the total length of D_ϵ does not exceed Δ. In view of the arbitrariness of the choice of $\epsilon > 0$, the measure of the entire complement to F is also not larger than Δ, as was required to be proved.

Putting $E = N$, $f_n = D_b^n$, $F_n = D_b^n(N)$, $\Delta = \epsilon$, we obtain inequality (2) from (3). Theorem 8 is proved.

§12. Example

We consider the two-dimensional space of mappings of the circumference onto itself of the form

$$\varphi \to \varphi + a + \varepsilon \cos \varphi \equiv T_{a,\varepsilon}(\varphi). \tag{1}$$

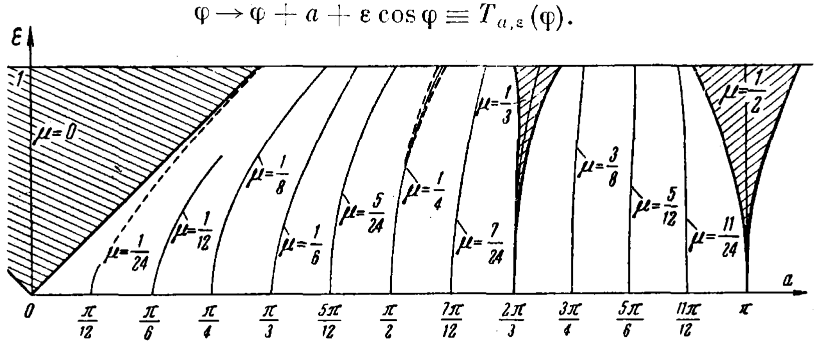

Figure 8

For $\epsilon = 0$ we obtain $T_{a,0}$, namely a rotation through the angle a. For $|\epsilon| < 1$ formula (1) defines a direct one-to-one continuous mapping of the circumference onto itself.

The level sets of the function

$$\mu(a, \varepsilon) = \mu(T_{a,\varepsilon})$$

continuous for $|\epsilon| \leq 1$ may be studied from two points of view. First, we may seek those points (a, ϵ) of the plane for which μ is rational; the boundaries of such regions are given by the conditions of semistability of the cycle. For example, the point (a, ϵ) enters into the level set $\mu = 0$ if the equation

$$\varphi = \varphi + a + \varepsilon \cos \varphi$$

has a real solution, i.e., the boundary of the region $\mu = 0$ is the straight line $a = \pm\epsilon$. In the same way we find the regions $\mu = m/n$. They approach the line $\epsilon = 0$ with ever narrowing tongues (Figure 8); two boundaries of the region $\mu = m/n$ have contact of $(n-1)$st order. For example, the regions $\mu = 1/2$ and $\mu = 1/3$ have bounding curves

$$a = \pi \pm \frac{\varepsilon^2}{4} + O(\varepsilon^4), \tag{2}$$

$$a = \frac{2\pi}{3} + \frac{\sqrt{3}}{12}\varepsilon^2 \pm \frac{\sqrt{7}}{24}\varepsilon^3 + O(\varepsilon^4). \tag{3}$$

Therefore one obtains approximate formulas, valid also for not very small ϵ: for $\epsilon = 1$ formula (2) gives $\pi \pm 0.25$ instead of $\pi \pm 0.23237\cdots$.

The second approach to the determination of the level sets $\mu(a, \epsilon)$ consists in using Newton's method for the approximate determation of the curves of irrational level μ. After two steps of Newton's method we obtain the following approximate equation for the level lines:

$$a = 2\pi\mu + \frac{\varepsilon^2}{4}\operatorname{ctg}\pi\mu - \frac{\varepsilon^4}{32}\operatorname{ctg}^3\pi\mu + \frac{\varepsilon^4}{32}\operatorname{ctg}2\pi\mu\,(1 + \operatorname{ctg}^2\pi\mu), \tag{4}$$

which works well when the cotangents are not large. Figure 9 gives an idea of the character of the convergence of the approximations and on the relation of this result to the preceding one. On this drawing we have shown the graph of the function $\mu(a) = \mu(a, 1)$. We have denoted the zeroth approximation of Newton's method by 0, the first by I, and the second by II. The horizontal segments for $\mu = 0, 1/2, 1/3$ are determined independently in accordance with formulas (2) and (3). For the number a given by

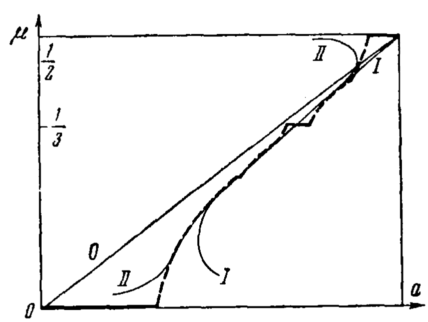

Figure 9

formula (4) the substitution of variables

$$\psi(\varphi) = \varphi - \frac{\varepsilon}{2}\frac{\sin(\varphi - \pi\mu)}{\sin \pi\mu} + \frac{\varepsilon^2}{4}\frac{\sin(2\varphi - \pi\mu)}{\sin \pi\mu \sin 2\pi\mu}$$

converts the transformation (1) into the transformation

$$\psi \to \psi + 2\pi\mu + F_2(\psi, \varepsilon, \mu),$$

where $F_2 \sim \epsilon^4$.

Remark. In the theory of oscillations the phenomenon of "locking in" is well known. This phenomenon corresponds to zones with rational rotation numbers.

Transformations of type (1) and diagrams of the type of Figure 8 describe a certain regime of the work of a generator of relaxation oscillations, synchronized by a sinusoidal impulse (see [25]). Another problem of a similar sort connected also with the mappings of a circumference onto itself is considered in the book [37] (pp. 221–231 of 2nd ed.).

§13. On trajectories on the torus*

13.1. Suppose that we are given on the torus $x, y \in [0, 2\pi]$ a differential equation

$$\frac{dy}{dx} = F(x, y) \quad (F(x + 2\pi k, y + 2\pi l) = F(x, y) > 0)$$

and that the usual conditions of existence and uniqueness theorems are satisfied. Through each point y_0 of the meridian $x = 0$ there passes a trajectory

$$y(x, y_0), \quad y(0, y_0) = y_0.$$

Following Poincare, we make correspond to the point y_0 the point $y(2\pi, y_0)$. Then we obtain a mapping of the circumference $x = 0$ onto itself, direct, one-to-one, continuous, and smooth or analytic for sufficiently smooth or analytic right side. If now the function $F(x, y)$ differs by little from a constant, then this mapping will be close to a rotation. All the properties of the transformation $y_1(y_0)$ reflect the corresponding properties of the solutions of equation (1), and we need only formulate the results of the preceding sections in the new terms.

If the mapping $y_1(y_0)$ is converted by the change of variables from y to $\phi(y)$ into a rotation through the angle $2\pi\mu$, then this substitution may be extended in a natural way to the whole torus if at the point $(x, y(x, y_0))$ we set

$$\varphi(x, y) = \varphi(y_0) + \mu x.$$

Evidently, if $\phi(y)$ is a smooth, or analytic, substitution, then the substitution $\phi(x, y)$ on the whole torus will also be smooth or analytic. In the x, ϕ coordinates the trajectories are written in the form

$$\varphi = \varphi_0 + \mu x$$

*See [1] – [4], [14], [19] and [20].

and one therefore says that a substitution of this kind *straightens out,* or *rectifies, the trajectories.* An analytic rectification of trajectories was obtained by A. N. Kolmogorov [14] in the case of the presence of an analytic integral invariant. On the basis of Theorem 2 we may now assert that *if the function* $F(x, y)$ *is analytically close to a constant and if the rotation number* μ *satisfies the usual arithmetic conditions, then the trajectories may be analytically rectified.* Thus it follows that the dynamical system

$$\frac{dy}{dt} = F(x, y), \quad \frac{dx}{dt} = 1$$

has an analytic integral invariant with invariant measure equal to the area in the x, ϕ coordinates.

On the other hand, in the same way as in the example of §1 one may construct an analytic function $F(x, y)$ such that the invariant measure of the system is not absolutely continuous relative to the area $dxdy$, although the rotation number μ is irrational and the system ergodic.*

13.2. Suppose that on the torus we are given a system of differential equations

$$\frac{dx}{dt} = A(x, y), \quad \frac{dy}{dt} = B(x, y) \quad (A(x, y) > 0, \quad B(x, y) > 0) \qquad (1)$$

with analytic right side. Consider the equation

$$\frac{dy}{dx} = \frac{B(x, y)}{A(x, y)},$$

which has the same integral curves as the system. If these may be rectified in accordance with subsection 13.1, then in the new coordinates the system has the form

$$\frac{dx}{dt} = A'(x, \varphi), \quad \frac{d\varphi}{dt} = \mu A'(x, \varphi),$$

where $A'(x, \phi) = A(x, y(x, \phi))$. This system has the analytic integral invariant $1/A'(x, \phi)$, and in the paper [14] it was shown, with the usual hypotheses on μ, how to convert it to the system

$$\frac{du}{dt} = 1, \quad \frac{dv}{dt} = \mu$$

by an analytic substitution of variables.

The contrary possibility, both in the case of an equation and in the case of a system, is the presence of limit cycles [20]. The decomposition of the space

*Added in proof. The contrary assertion in the review [41], which appeared while this paper was being printed, is mistaken.

of right sides of the system (1) into level sets for the rotation number, the characterization of rough systems and the consideration of the question as to the typicality, are analogous to the considerations of §§9–11. It results that

1. The case of normal cycles (it is still rough) is topologically predominant.* The corresponding set of right sides is open and everywhere dense; however, in systems with an integral invariant this case cannot happen at all.

2. The ergodic case (the case of irrational μ) is typical as well if one uses measures in finite-dimensional spaces as the point of departure for judging typicality. For systems with an analytic integral invariant this case is predominant.

In the multidimensional case, in the absence of an integral invariant, the rotation number is not defined. Nevertheless, by making use of the remark of subsection 4.4, we may obtain the following assertion.

Theorem 9. *Suppose that $\vec{\mu} = (\mu_1, \cdots, \mu_n)$ is a vector with noncommensurable components such that for any integer $\vec{k}$*

$$|(\vec{\mu}, \vec{k})| > \frac{C}{|\vec{k}|^n}.$$

Then there exists an $\epsilon(R, C, n) > 0$ such that for any analytic vector field $\vec{F}(\vec{x})$ on the torus, i.e., a field with $\vec{F}(\vec{x} + 2\pi\vec{k}) = \vec{F}(\vec{x})$), which is sufficiently small, $|\vec{F}(\vec{x})| < \epsilon$ for $|\mathrm{Im}\, \vec{x}| < R$, there exists a vector $\vec{a}$ for which the system of differential equations

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}) + \vec{a}$$

converts into

$$\frac{d\vec{u}}{dt} = 2\pi\vec{\mu}$$

by an analytic change of variables.

§14. Dirichlet's problem for the equation of the string

14.1. Suppose that D is a region on the plane, convex in the coordinate directions; i.e., its boundary Γ intersects each line $x = c$, $y = c$ at not more than two points.

The *Dirichlet problem* for the equation $\partial^2 u/\partial x \partial y = 0$ on D consists in *finding on D a function $u(x, y) = \phi(x) + \psi(y)$ which on Γ is transformed into a given function $f(a)$ $(a \in \Gamma)$: $u|_\Gamma = f$.*

Here one may impose various requirements of smoothness, analyticity and so

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* In the paper [19], to judge from the review [21], it is asserted that a necessary and sufficient condition for roughness is the presence of *one* stable cycle. This is not true.

forth on f, ϕ, ψ, Γ.

In the case when D is the rectangle $0 \le x + y \le l$, $0 \le y - x \le t$, it is convenient to refer to the coordinates $\xi = x + y$, $\tau = y - x$. Then our equation becomes the equation of the string, and the problem may be interpreted as the problem of finding the motion of the string with respect to two instantaneous photographs and the motion of the ends. From physical considerations (standing waves) it is clear that with commensurable l and t the problem is not always solvable, and if it is solvable, not always uniquely. This problem has been the object of a series of papers, e.g., [22], [23], [5], [24], [17], [28]. There are difficulties of an analogous order in the solution of certain other problems, e.g., [25]–[27].

14.2. Uniqueness theorems (see [5]). We shall associate with the boundary certain of its mappings onto itself (see Figure 10). Suppose that P is a transformation carrying the point $a \in \Gamma$ into the point $Pa \in \Gamma$ with the same coordinate x, and that Q is a transformation carrying the point $a \in \Gamma$ into the point $Qa \in \Gamma$ with the same coordinate y. These transformations are continuous, one-to-one, and change the orientation of the contour Γ. We write $QP = T$. Evidently

$$P^2 = Q^2 = E, \quad PQ = T^{-1}.$$

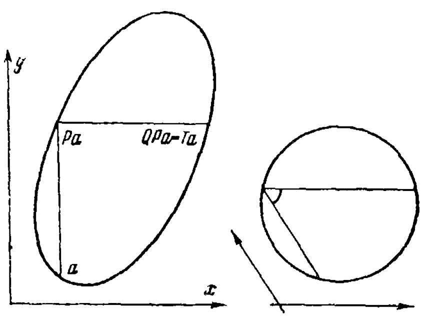

Figure 10

T is a direct homeomorphic mapping.

Theorem 10 (see [5]). *If the contour Γ is such that for some point $a_0 \in \Gamma$ the set $T^n a_0$ $(n = 0, 1, 2, \cdots)$ is everywhere dense on Γ, then the Dirichlet problem for Γ cannot have more than one continuous solution.*

Proof. The solution $u(x, y) = \phi(x) + \psi(y)$ defines functions $\phi(x)$, $\psi(y)$ up to a constant. We shall show that under the conditions of the theorems, knowing $\phi(x)$ at one point $a \in \Gamma$ makes it possible to determine $\phi(T^n a)$, $\psi(T^n a)$ at all the points $T^n a$ $(n = 0, 1, \cdots)$ (we write $\phi(a)$ and $\psi(a)$ for $\phi(x)$, $\psi(y)$, where x, y are the coordinates of the point $a \in \Gamma$).

Knowing $\phi(a)$, it is easy to find

$$\psi(Pa) = f(Pa) - \varphi(a),$$

since the abscissae of a and Pa are the same. Then we may determine

$$\varphi(Ta) = f(Ta) - \psi(Pa),$$

using the fact that the ordinates of the points Pa and Ta coincide. Further, in the same way we obtain ϕ, ψ at all the points $T^n Pa$, $T^n a$. They form a set everywhere dense on Γ, so that continuous functions which coincide at these points of Γ coincide everywhere. The theorem is proved.

In the case when D is the rectangle $0 \le x + y \le l$, $0 \le y - x \le t$, the transformation T is, in particular, a rotation. Indeed, if we introduce on the contour Γ a parameter

$$\vartheta = \frac{2\alpha\pi}{\sqrt{2}(l+t)},$$

where α is the length measured along the contour from the point 0 to a (Figure 11), then our transformation

$$T : T\vartheta = \vartheta + \frac{2\pi t}{t+l}$$

is a rotation through the angle $2\pi t/(t + l)$. If D is an ellipse, then it is not difficult to introduce on Γ a parameter such that in it the transformation may be written as a rotation. Indeed, we map the ellipse affinely onto a disk. The straight lines in the coordinate directions go into two families of parallel lines, while two lines of different families form an angle of $\pi\mu$, in general not a right angle. Evidently, when the ellipse is subjected to the transformation T, the circumference rotates by an angle $2\pi\mu$ (Figure 10).

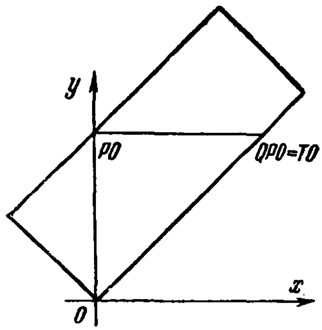

Figure 11

If Γ is a curve of bounded curvature, then T is a twice differentiable transformation, from which, by Denjoy's theorem, we have the result that for an irrational rotation number μ the mapping T of the set $T^n a$ is everywhere dense on Γ. Hence we have the following theorem.

Theorem 11 (see [5], [24]). *If Γ has bounded curvature and μ is irrational, then the Dirichlet problem can have only one continuous solution.*

Remark. Using the theorem on points of density, it is easy to prove that under the conditions of our theorem there can be only one measurable solution. On the other hand, the method of proof of Theorem 10 makes it possible, for irrational μ, to construct as many solutions as desired, but, generally speaking, nonmeasurable ones.

14.3. **Detailed investigation of the rectangle.**

Theorem 12 (see [23], [17]). *Suppose that on the boundary Γ of the rectangle $0 \le x + y \le l$, $0 \le y - x \le t$, there is given a function $f(\vartheta)$ which is $(p + \epsilon)$ times differentiable along the boundary. Then for all $\mu = t/(t + l) \in M_k$ satisfying the inequality $|\mu - m/n| > K/|n|^3$ for any m and n and some $K > 0$, the Dirichlet problem with the indicated boundary functions has a $p - 1$ times differentiable* *solution, and the problem relative to $f(\vartheta)$ is correctly posed. In the case of analyticity of f the solution for the same μ is analogous.*

For certain irrational μ, even in spite of the analyticity of the function $f(\vartheta)$, the solution may turn out to be

1) *only infinitely differentiable,*
2) *differentiable k, but not $k+1$, times,*
3) *only continuous,*
4) *discontinuous,*
5) *nonmeasurable.*

Proof. If

$$f(\vartheta) = \sum_{n\neq 0} a_n e^{in\vartheta}, \quad \varphi(\vartheta) = \sum_{n\neq 0} b_n e^{in\vartheta}, \quad \psi(\vartheta) = \sum_{n\neq 0} c_n e^{in\vartheta},$$

then, since $\phi(\vartheta)$ depends only on x, and $\psi(\vartheta)$ only on y, we have

$$\varphi(\vartheta) = \varphi(-2\pi\mu - \vartheta), \quad b_n = b_{-n} e^{in2\pi\mu},$$
$$\psi(\vartheta) = \psi(-\vartheta), \qquad c_n = c_{-n}.$$

Since $f(\vartheta)$ is real and therefore $a_n = \bar{a}_{-n}$, from the equation $f(\vartheta) = \phi(\vartheta) + \psi(\vartheta)$ we find that

$$b_n + c_n = a_n, \quad b_n e^{-in2\pi\mu} + c_n = \bar{a}_n,$$

or

$$b_n = \frac{\bar{a}_n - a_n}{e^{-2\pi i\mu n} - 1}, \quad c_n = a_n - b_n. \tag{1}$$

Now, when the formal solution is found, the rest of the proof may be carried out by an exact repetition* of the argument of §2.

Remark. It is clear from formula (1) that for all μ it is possible, by truncating the series, to construct an "approximate solution," the degree of approximation of which is greater in proportion as the commensurability of l and t is less. For rational μ the approximation is not higher than the limit imposed by μ, but for strongly noncommensurable l and t we have Theorem 11. This meaning of correctness with respect to a region was introduced by N. N. Vahanija in the paper [28].

We may assert that the dependence of the solution on μ is *monogenic* (see §7).

14.4. **General case.** If the boundary of D is such that the transformation T may be represented as a rotation in a parameter which is a smooth function of the

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*Added in proof. In an article [42] by P. P. Mosolov, published while the present article was at the press, the statement analogous to that of Theorem 12 was proved for an arbitrary linear differential equation with constant coefficients in which all the derivatives are of even order.

point on the boundary, then evidently for each such contour all the arguments of subsection 14.3 are applicable, and in the case of a "sufficiently irrational μ" the Dirichlet problem has a smooth solution.

As an example there is the ellipse, for which the parameter was constructed in subsection 14.2. Now in the general case of irrational μ, in spite of the arbitrary degree of smoothness of Γ, one cannot guarantee the smoothness of the parameter in which the transformation T becomes a rotation, although by Denjoy's theorem such a parameter exists. F. John [5] showed that with a *continuous* change of variables x, y of the form $x \to u(x)$, $y \to v(y)$ ("preserving the equation $\partial^2 w/\partial x \partial y = 0$") it is possible to map a region for which T has an irrational μ onto a rectangle or onto an ellipse with the same μ. However this substitution, generally speaking, is only continuous, and it may convert smooth boundary conditions on the curve into nonsmooth boundary conditions on the ellipse.

We note that if Γ is an analytic curve, then P and Q, and thus T and T^n, are analytic mappings. But if Γ is also analytically close to an ellipse, then in an appropriate parameter the transformation will be analytically close to a rotation. Therefore it follows from Theorem 2 that *among the curves for which $\mu \in M_k$, all curves sufficiently close to the ellipse are analogous to the ellipse in respect to the solvability of the Dirichlet problem.*

In exactly the same way one may formulate other problems on mappings of the circumference in these terms. In particular, if the transformation T has a cycle, then the Dirichlet problem with zero boundary conditions has a nonzero solution (at least piecewise constant; for more details see [24]).

The Dirichlet problem for the string equation is a problem on eigenvalues for the two-dimensional Sobolev equation

$$\frac{\partial^2 \Delta u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(see [24], [27], [29], [30]). The values of λ which belong to the spectrum are those for which the mapping T_λ, constructed for the curves Γ_λ, has a cycle (here by Γ_λ we mean the curve Γ subjected to a dilation depending on λ).

From the results of §10 it follows that if the cycle is stable, then all the curves close to Γ_λ yield an analogous cycle, and accordingly the point λ belongs to the spectrum, together with a neighborhood. An example of a curve Γ generating a transformation with a stable cycle was constructed by R. A. Aleksandrjan [24]. On the basis of §10 we may show that such curves may lie in any neighborhood of any curve Γ.

The Dirichlet problem for the wave equation with given values on the ellipsoid was recently investigated by R. Dencev [32].

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