VLADIMIR I. ARNOLD

Collected Works

VOLUME I

Representations of Functions,
Celestial Mechanics, and KAM Theory
1957 – 1965

Springer

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Vladimir I. Arnold, 1961
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Preface

Vladimir Igorevich Arnold is one of the most influential mathematicians of our time. V.I. Arnold launched several mathematical domains (such as modern geometric mechanics, symplectic topology, and topological fluid dynamics) and contributed, in a fundamental way, to the foundations and methods in many subjects, from ordinary differential equations and celestial mechanics to singularity theory and real algebraic geometry. Even a quick look at a partial list of notions named after Arnold already gives an overview of the variety of such theories and domains:

KAM (Kolmogorov–Arnold–Moser) theory,
The Arnold conjectures in symplectic topology,
The Hilbert–Arnold problem for the number of zeros of abelian integrals,
Arnold's inequality, comparison, and complexification method in real algebraic geometry,
Arnold–Kolmogorov solution of Hilbert's 13th problem,
Arnold's spectral sequence in singularity theory,
Arnold diffusion,
The Euler–Poincaré–Arnold equations for geodesics on Lie groups,
Arnold's stability criterion in hydrodynamics,
ABC (Arnold–Beltrami–Childress) flows in fluid dynamics,
The Arnold–Korkina dynamo,
Arnold's cat map,
The Arnold–Liouville theorem in integrable systems,
Arnold's continued fractions,
Arnold's interpretation of the Maslov index,
Arnold's relation in cohomology of braid groups,
Arnold tongues in bifurcation theory,
The Jordan–Arnold normal forms for families of matrices,
The Arnold invariants of plane curves.

Arnold wrote some 700 papers, and many books, including 10 university textbooks. He is known for his lucid writing style, which combines mathematical rigour with physical and geometric intuition. Arnold's books on *Ordinar*y *differential equation*s and *Mathematical method*s *of classical mechanics* became mathematical bestsellers and integral parts of the mathematical education of students throughout the world.

Some Comments on V.I. Arnold's Biography and Distinctions

V.I. Arnold was born on June 12, 1937 in Odessa, USSR. In 1954–1959 he was a student at the Department of Mechanics and Mathematics, Moscow State University. His M.Sc. Diploma work was entitled "On mappings of a circle to itself." The degree of a "candidate of physical-mathematical sciences" was conferred to him in 1961 by the Keldysh Applied Mathematics Institute, Moscow, and his thesis advisor was A.N. Kolmogorov. The thesis described the representation of continuous functions of three variables as superpositions of continuous functions of two variables, thus completing the solution of Hilbert's 13th problem. Arnold obtained this result back in 1957, being a third year undergraduate student. By then A.N. Kolmogorov showed that continuous functions of more variables can be represented as superpositions of continuous functions of three variables. The degree of a "doctor of physical-mathematical sciences" was awarded to him in 1963 by the same Institute for Arnold's thesis on the stability of Hamiltonian systems, which became a part of what is now known as KAM theory.

After graduating from Moscow State University in 1959, Arnold worked there until 1986 and then at the Steklov Mathematical Institute and the University of Paris IX.

Arnold became a member of the USSR Academy of Sciences in 1986. He is an Honorary member of the London Mathematical Society (1976), a member of the French Academy of Science (1983), the National Academy of Sciences, USA (1984), the American Academy of Arts and Sciences, USA (1987), the Royal Society of London (1988), Academia Lincei Roma (1988), the American Philosophical Society (1989), the Russian Academy of Natural Sciences (1991). Arnold served as a vice-president of the International Union of Mathematicians in 1999–2003.

Arnold has been a recipient of many awards among which are the Lenin Prize (1965, with Andrey Kolmogorov), the Crafoord Prize (1982, with Louis Nirenberg), the Lobachevsky Prize of Russian Academy of Sciences (1992), the Harvey prize (1994), the Dannie Heineman Prize for Mathematical Physics (2001), the Wolf Prize in Mathematics (2001), the State Prize of the Russian Federation (2007), and the Shaw Prize in mathematical sciences (2008).

One of the most unusual distinctions is that there is a small planet Vladarnolda, discovered in 1981 and registered under #10031, named after Vladimir Arnold. As of 2006 Arnold was reported to have the highest citation index among Russian scientists.

In one of his interviews V.I. Arnold said: "The evolution of mathematics resembles the fast revolution of a wheel, so that drops of water fly off in all directions. Current fashion resembles the streams that leave the main trajectory in tangential directions. These streams of works of imitation are the most noticeable since they constitute the main part of the total volume, but they die out soon after departing the wheel. To stay on the wheel, one must apply effort in the direction perpendicular to the main flow."

With this volume Springer starts an ongoing project of putting together Arnold's work since his very first papers (not including Arnold's books.) Arnold continues to do research and write mathematics at an enviable pace. From an originally planned 8 volume edition of his Collected Works, we already have to increase this estimate to 10 volumes, and there may be more. The papers are organized chronologically. One might regard this as an attempt to trace to some extent the evolution of the interests of V.I. Arnold and cross-fertilization of his ideas. They are presented using the original English translations, whenever such were available. Although Arnold’s works are very diverse in terms of subjects, we group each volume around particular topics, mainly occupying Arnold’s attention during the corresponding period.

Volume I covers the years 1957 to 1965 and is devoted mostly to the representations of functions, celestial mechanics, and to what is today known as the KAM theory.

Acknowledgements. The Editors thank the Göttingen State and University Library and the Caltech library for providing the article originals for this edition. They also thank the Springer office in Heidelberg for its multilateral help and making this huge project of the Collected Works a reality.

March 2009

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Contents

ON THE REPRESENTATION OF FUNCTIONS OF TWO VARIABLES IN THE FORM $\chi[\phi(x) + \psi(y)]^\star$

V.I. Arnol'd
translated by Gerald Gould

1. Kolmogorov proved [1] that the set of functions of two variables representable as a certain combination of continuous functions of one variable and addition is everywhere dense in the space $C(E^2)$ of continuous functions defined on the square E^2. It follows immediately from our result proved below that this is not true for the simplest combinations: the set of functions of the form $\chi[\phi(x) + \psi(y)]$ even turns out to be nowhere dense in $C(E^2)$.

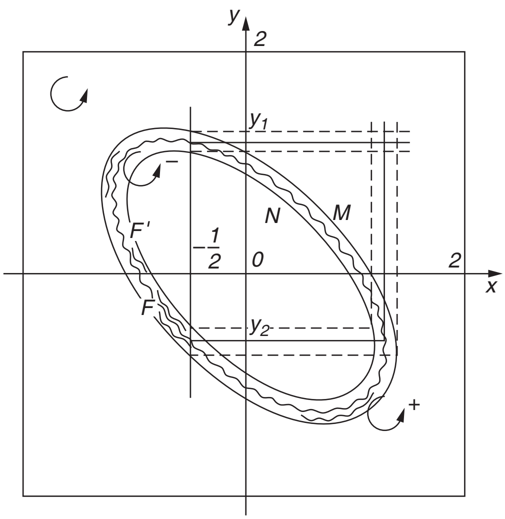

Fig. 1.

We shall indicate *a closed subset* N *of the square* $|x| \leqslant 2$, $|y| \leqslant 2$ (*Fig. 1*) *such that for any continuous function* $f(x, y)$ *vanishing on* (*and only on*) N *there exists* $\delta(f) > 0$ *such that* $|f(x, y) - \chi[\phi(x) + \psi(y)]| \geqslant \delta$ *at some point of this square for any continuous functions* χ, ϕ *and* ψ; *every function having*

* Uspekhi Math. Nauk **12**, No. 2, 119–121 (1957)

N as its level set is 'with a neighbourhood' non-representable in the form $\chi[\phi(x)+\psi(y)]$. An example of such a set N is the ellipse $(x+y)^2+\frac{(x-y)^2}{4}=1$.

We shall prove this. Since $f(x,y)$ is of constant sign outside the ellipse we can assume that $f(x,y)>0$ there. Then clearly there exists $\delta>0$ such that $f(x,y)>2\delta$ at all points in the region $G \stackrel{\text{def}}{=} (x+y)^2+\frac{(x-y)^2}{4}>\frac{5}{4}$, that is, outside the ellipse $M \stackrel{\text{def}}{=} (x+y)^2+\frac{(x-y)^2}{4}=\frac{5}{4}$. Suppose that there exist continuous functions $\phi(x)$, $\psi(y)$, $\chi(z)$ such that $|f(x,y)-\chi[\phi(x)-\psi(y)]|<\delta$ † for all (x,y), $2\leqslant x,\ y\leqslant 2$. Then the inequality $\chi[\phi(x)+\psi(y)]<\delta$ holds on N and the inequality $\chi[\phi(x)+\psi(y)]>\delta$ holds on M.

The largest open connected sets $G^-\supset N$ and $G^+\supset G$,⋆ where $\chi[\phi(x)+\psi(y)]<\delta$ and $\chi[\phi(x)+\psi(y)]>\delta$, respectively, are separated by the closed set F where $\chi[\phi(x)+\psi(y)]=\delta$ (that is, each continuum intersecting G^- and G^+ also intersects F), because the continuous function $\chi[\phi(x)+\psi(y)]$ on a continuum takes all values between any two given values. By a well-known theorem (Theorem E in [2]) the boundary of G^+ has a component $F'\subseteq F$ already separating G^- and G^+, and hence M and N. We claim that the continuous function $\phi(x)+\psi(y)$ is constant on F'. Indeed, suppose that, on the contrary, $z_1=\phi(x)+\psi(y)|_a<\phi(x)+\psi(y)|_b=z_2$, where $a,b\in F'$. Then in a sufficiently small neighbourhood of a there is a point $a'\in G^+$ where $\phi(x)+\psi(y)<z_1+\frac{z_2-z_1}{3}$, and in a sufficiently neighbourhood of b there is a point $b'\in G^+$ where $\phi(x)+\psi(y)>z_2-\frac{z_2-z_1}{3}$. Therefore on a polygonal line joining a' and b' in G^+ there is a point c where $\phi(x)+\psi(y)=\frac{z_1+z_2}{2}$; also there is a point c on the continuum F' where $\phi(x)+\psi(y)=\frac{z_1+z_2}{2}$. Consequently, $\chi[\phi(x)+\psi(y)]|_{c'}=\chi[\phi(x)+\psi(y)]|_c$, which contradicts the conditions $c'\in G^+$, $c\in F'$.

We denote by z the unique value of $\phi(x)+\psi(y)$ at points of F'. Then on the intervals $x=-\frac{1}{2}$, $y\in[1.1,1.22]$ and $x=-\frac{1}{2}$, $y\in[-0.62,-0.5]$ intersecting M and N there are points $(-\frac{1}{2},y_1)$ and $(-\frac{1}{2},y_2)$ at which $\phi(x)+\psi(y)=z$. There is such a point (x_1,y_2) on the interval on which the line $y=y_2$ intersects the strip between M and N for $x>0$.

It follows from the equalities⋆⋆

$$\begin{aligned}\phi(-\tfrac{1}{2})+\psi(y_1)&=z\,,\\ \phi(-\tfrac{1}{2})+\phi(y_2)&=z\,,\\ \phi(x_1)+\psi(y_2)&=z\end{aligned}$$

that $\phi(x_1)+\psi(y_1)=z$ and $\chi[\phi(x_1)+\psi(y_2)]=\delta$. However, it is easy to see that the point (x_1,y_1) lies in G, therefore $\chi[\phi(x_1)+\psi(y_2)]>\delta$. This contradiction proves the 'stable' non-representability of $f(x,y)$ in the form $\chi[\phi(x)+\psi(y)]$;

† *Translator's note*: This should be $|f(x,y)-\chi[\phi(x)+\psi(y)]|<\delta$.

⋆ *Translator's note*: This should be $G^+\supset M$.

⋆⋆ *Translator's note*: The second of these inequalities contains a misprint. It should read $\phi(-\frac{1}{2})+\psi(y_2)=z$.

in particular, for the function $f(x,y) = (x+y)^2 + \frac{1}{4}(x-y)^2 - 1$ we can choose $\delta > \frac{1}{4}$.

2. I.A. Weinstein proved that the class of continuous functions of the form $\chi[\phi(x)+\psi(y)]$ that are strictly monotone in each variable is a closed subset of $C(E^2)$. Here the strict monotonicity is essential: we claim that *the function xy is not representable in the form $\chi[\phi(x)+\psi(y)]$ even though it is the uniform limit of the sequence of functions* $\exp\big(\ln(x+\frac{1}{n})+\ln(y+\frac{1}{n})\big)$, *which do have the form* $\chi[\phi(x)+\psi(y)]$ (where $\phi_n(x) = \psi_n(x) = \ln(x+\frac{1}{n})$ and $\chi(z) = \exp(z)$).

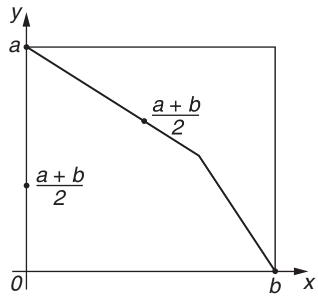

Fig. 2.

In fact, if $\chi[\phi(x)+\psi(y)] = xy$ everywhere in the square $x, y \in [0,1]$, then the function $\phi(x)+\psi(y)$ would take the same value at the points $(0,0)$, $(0,1)$, and $(1,0)$. Indeed, any two of these three points can be joined by a polygonal line having no common points with the set $xy = 0$ apart from the end points, and also by a polygonal line lying entirely in this set. If $\phi(x) + \psi(y)$ took different values a and b at these end points (see Fig. 2), then the intermediate value $\frac{a+b}{2}$ would be taken both on the set $xy = 0$ and outside this set, which would mean that $\chi(\frac{a+b}{2}) = 0$ and $\chi(\frac{a+b}{2}) > 0$ simultaneously. This contradiction proves that $\phi(0)+\psi(0) = \phi(0)+\psi(1) = \phi(1)+\psi(0)$; hence $\phi(0)+\psi(0) = \phi(1)+\psi(1)$ and therefore

$$0 = \chi[\phi(0)+\psi(0)] = \chi[\phi(1)+\psi(1)] = 1\,.$$

In other words, there do not exist any functions $\phi(x)$, $\psi(y)$, $\chi(z)$ such that $\chi[\phi(x)+\psi(y)] = xy$.

We also point out that the first example of a continuous function not representable in the form $\chi[\phi(x)+\psi(y)]$ (obtained simultaneously by A.A. Kirillov and the author), namely, the function $f(x,y) = \min(x,y)$ (where $x, y \in [0,1]$) can also be approximated to arbitrary precision by functions of the form $\chi[\phi(x)+\psi(y)]$.

Received 26 December 1956

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[2] Krondov, A.S.: On functions of two variables. Usp. Mat. Nauk **5**, No, 1 (1950).

ON FUNCTIONS OF THREE VARIABLES*

V.I. ARNOL'D

In the present paper there is indicated a method of proof of a theorem which yields a complete solution of the 13th problem of Hilbert (in the sense of a denial of the hypothesis expressed by Hilbert).

Theorem 1. *Every real, continuous function $f(x_1,x_2,x_3)$ of three variables which is defined on the unit cube E^3 can be represented in the form*

$$f(x_1,x_2,x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1,x_2), x_3], \tag{1}$$

where the functions h_{ij} and φ_{ij} of two variables are real and continuous.

A.N. Kolmogorov [1] obtained recently the representation

$$f(x_1,x_2,x_3) = \sum_{i=1}^{3} h_i[\varphi_i(x_1,x_2), x_3], \tag{2}$$

where the functions h_i and φ_i are continuous, the function h_i is real, and the function φ_i takes on values which belong to some tree Ξ. In the construction of A.N. Kolmogorov (for the case of functions of three variables), the tree Ξ can be taken not as a universal tree, but such that all of its points have a branching index not greater than 3. For this, the functions u^r_{km} of the fundamental lemma [1] (for $n = 2$) must be chosen so that in addition to the indicated five properties they must have the following properties.

(6) *The boundary of each level set of each function u^r_{km} divides the plane into not more than 3 parts.*

(7) *For every r, $G^r_{11} \supseteq E^2$.*

On the basis of this remark, Theorem 1 is a consequence of the existence of the representation (2) and of the next theorem.

Theorem 2. *Let F be any family of real equicontinuous functions $f(\xi)$ defined on a tree Ξ all of whose points have a branching index $\leqslant 3$. One can realize the tree as a subset X of the three-dimensional cube E^3 in such a way that any function of the family F can be represented in the form*

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1,x_2,x_3)$ is the image of $\xi \in \Xi$ in the tree X; the $f_k(x_k)$ are continuous real functions of one variable, while the f_k depend continuously

* Editor's note: translation into English published in Amer. Math. Soc. Transl. (2) 28 (1963), 51–54. Translation of V.I. Arnol'd: On functions of three variables Dokl. Akad. Nauk SSSR 114:4 (1957), 679–681

on f *(in the sense of uniform convergence).*

We will introduce certain auxiliary concepts. Let K be a finite complex of segments contained in E^3 and consisting of segments which are not parallel to any coordinate plane.

Definition 1. A system of points

$$a_0 \neq a_1 \neq \dots \neq a_{n-1} \neq a_n$$

belonging to K will be called a *zigzag* (lightning) if the segments $\overline{a_{i-1}a_i}$ are perpendicular to the axes X_{α_i}, respectively, and

$$\alpha_1 \neq \alpha_2 \neq \dots \neq \alpha_{n-1} \neq \alpha_n .$$

The finite system of the pairwise distinct points $a_{i_1 i_2 \dots i_n}$ tagged by the corteges of indices $i_1 i_2 \dots i_n$, will be called a *branching scheme* if (1) there exists only one point a_0 tagged with one index; (2) the presence of $a_{i_1 i_2 \dots i_{n-1} i_n}$ in the system implies the presence of $a_{i_1 \dots i_{n-1}}$ in the system.

Definition 2. A branching system of points $a_{i_1 \dots i_n}$ contained in K will be called a *generating scheme* if for a given cortege $i_1 \dots i_n$ the set of points of the form $a_{i_1 \dots i_n i_{n+1}}$ lies on the plane passing through $a_{i_1 \dots i_n}$ and perpendicular to some coordinate axis $x_{\alpha_{i_1 \dots i_n}}$, and contains all points of intersection of this plane with K, that are distinct from $a_{i_1 \dots i_n}$.

The tree Ξ can be represented in the form

$$\Xi = \overline{\bigcup_{n=1}^{\infty} D_n}, \quad D_n \subset D_{n+1},$$

where D_n is a finite tree, D_1 is a simple arc, and D_{n+1} is obtained from D_n by attaching segments S_n at certain points p_n that are not branch points or endpoints of d_n [2].

We will denote by ω_n the upper boundary of the oscillations of the functions $f \in F$ on the components of the difference $\Xi \setminus D_n$. It is easy to see that

$$\omega_n \to 0 \quad \text{when} \quad n \to \infty.$$

Therefore, one can select a sequence

$$n_1 < n_2 < \dots < n_r < \dots,$$

so that

$$\omega_n \leqslant \frac{1}{r^2} \quad \text{when} \quad n \geqslant n_r .$$

The realization X of the tree Ξ in E^3 is constructed in the form

$$X = \overline{\bigcup_{n=1}^{\infty} D'_n},$$

where D'_n is a complex of segments which realize D_n in such a way that the images S'_n of the arcs S_n are segments that are not perpendicular to the coordinate axes.

The inductive construction of D'_n is performed so that $\overline{\bigcup_{n=1}^{\infty} D'_n}$ is a tree [2], and that the following conditions are satisfied.

(1) Every function $f \in F$ can be represented on D_n in the form

$$f(\xi) = \sum_{k=1}^{3} f_k^n(x_k), \tag{3}$$

where the $f_k^n(x_k)$ depend continuously on f.

(2) The tree D'_n has for every point a_0 a generating system issuing from a_1, and whose initial direction α_0 can be chosen arbitrarily.

(3) Let A_n be the set of points D'_n which is the image of the branch points of Ξ. There exists a denumerable set $B_n \subseteq D'_n$, $B_n \cap A_n = 0$ such that the zigzag $a_0 \ldots a_m$, which begins at $a_0 \in D'_n \setminus B_n$, has no points in common with A_n and no coincident points $a_i = a_j$, $i \neq j$.

(4) If $n_r < n \leqslant n_{r+1}$, then

$$| f_k^n(x_k) - f_k^{n_r}(x_k) | \leqslant \left(3 + \frac{n - n_r}{n_{r+1} - n_r} \right) \frac{1}{r^2}. \tag{4}$$

This proof of the possibility of the inductive construction of the trees D'_n, and of the functions f_k^n with properties (1) to (4), is too complicated to be given here. Roughly speaking, at each step the attached segment S'_{n+1} is chosen of very short length; its direction, and the way of mapping of S_{n+1} on S'_{n+1} are selected so as to guarantee the fulfillment of properties (2) and (3) by D'_{n+1}. The preservation of equality (3), in the transition from n to $n + 1$, on the newly attached segment S_{n+1}, requires the introduction of a correction $f_k^{n+1} - f_k^n$, for at least one of the functions f_k^n, on the projection S'_{n+1} on the axis x_k. For the preservation of equality (3) on the earlier constructed tree D'_n, it is necessary to compensate for this correction by means of new corrections for the functions f_k^n on a number of other segments. The exact method of the introduction of these corrections, we will not present here. We only note the following: these corrections must be such that inequality (4) will be preserved for $n' = n + 1$; if S'_{n+1} is chosen small enough, and if its direction is chosen appropriately, it must be possible to produce it for every function f_k^n on a finite system of non-intersecting segments of the axis x_k. In the proof of this possibility one makes use of the fact that the tree D_n' has properties (2) and (3).

The proof of the existence of the continuous function

$$f_k(x_k) = \lim_{n\to\infty} f_k^n(x_k)$$

and of the validity of the equation

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k)$$

on the entire X, is not complicated.

I express my sincere thanks to A.N. Kolmogorov for the aid and advice I have received from him in the preparation of this work.

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Translated by:
H.P. Thielman

THE MATHEMATICS WORKSHOP FOR SCHOOLS AT MOSCOW STATE UNIVERSITY*

V.I. Arnol'd
Moscow
translated by Gerald Gould

The mathematics workshop for schools at Moscow State University in the name of Lomonosov came into existence in 1935. The organizers of the workshop were: the now-deceased Corresponding Member of the Academy of Sciences of the USSR L.G. Shnirel'man, Professor L.A. Lyusternik (now Corresponding Member of the Academy of Sciences of the USSR), and Doctor I.M. Gel'fand (now Corresponding Member of the Academy of Sciences of the USSR).

The activities of the workshop proceed in two streams: twice a month (on Sundays) *lectures on mathematics* are given by professors and instructors at Moscow State University and other institutes (separately for the pupils of the 7–8 class and for pupils of the 9–10 class) and *sections of the mathematics workshop* meet weekly under the guidance of students and (more rarely) post-graduate students of the university.[1] The annual *Mathematical Olympiad* is, in a certain sense, the culmination of the activities of the circle; here the directors of the mathematics workshop traditionally play a large role in bringing this about.

General information on the activities of the mathematics workshop in the 1955/56 academic year is given in the preceding issue of "Matematicheskiĭ Prosveshchenie"; there one can find the list of lectures given in that year.[2] The series "Popular lectures on mathematics" published by Gostekhizdat will give an idea of the character of these lectures.[3] The main part of this series of books by Moscow authors consists in expositions of the lectures given in the mathematical circle for schools at Moscow State University. Here we wish to shed light on the activities of the sections of the circle (the early part of these

* Mat. Prosveshchenie **2**, 241–245 (1957)

[1] It was only at the very beginning of the activities of the workshop that professors of Moscow State University were also involved in the work of the sections.

[2] Dynkin, E.B., Girsanov, I.V.: The nineteenth School Mathematical Olympiad in Moscow. Mat. Prosveshch. **2**, No. 1, 187 (1957).

[3] *Editor's note*: See the paper by N.B. Beskin on pp. 275–290 of this issue.

activities is well reflected in the series of books "Library of the mathematical circle", also published by Gostekhizdat).

The 'lessons' of a section take place in the form of a discussion: the supervisor of the section introduces the topic of study to the participants; 5–10 minutes is set aside for each problem; then the solution is explained and the supervisor continues his talk on the topic being studied. Each individual problem is not difficult (most of the pupils manage it in 5–10 minutes). At the end of the lesson the pupils are given (usually more difficult and sometimes very difficult) homework problems, which are collected at the beginning of the next lesson.

Below we give a summary account of two lessons of the workshop (a section for 10 pupils) on the themes "Variation of a curve" and "Harmonic functions".

Variation of a curve

We are given a line segment AB of length 1. If this line segment is illuminated by parallel rays, then the length of the shadow thrown onto various lines will vary from 0 to 1. More precisely, the length of the projection of the segment onto lines lying in the same plane will, in general be different for different lines; however in all cases it will be between 0 and 1. The length of the projection of AB onto a line l is called the *variation of the segment AB in the direction l* (Fig. 1); we shall denote it by $V_l(AB)$ or simply by V_l if it is clear which segment we are referring to.

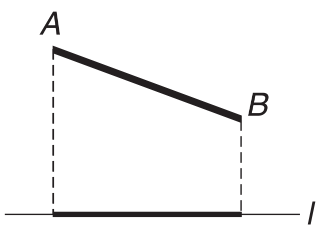

Fig. 1.

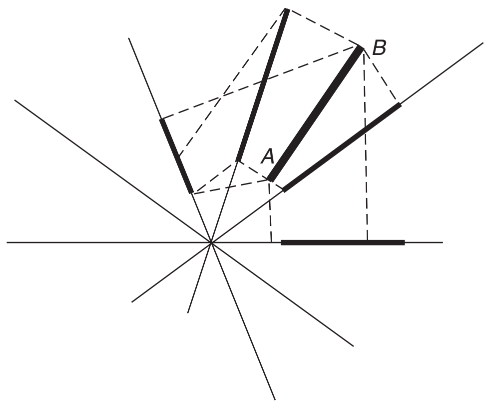

Fig. 2.

It is intuitively obvious that the mean value of the 'shadow' over all directions exists and that it is between 0 and 1. More precisely, this means that if we divide the 360° angle into n equal parts, and take the arithmetic mean

$$V_n = \frac{V_{l_1} + V_{l_2} + \cdots V_{l_n}}{n}$$

of the variations of the segment AB in the directions $l_1, l_2, \ldots, l_n$ (Fig. 2), then the limit

$$\lim_{n\to\infty} V_n = K$$

exists and K lies between 0 and 1.

This number K is called the *mean variation* or simply the *variation* of the unit segment AB.

This number is not very difficult to find;[4] it is equal to $\frac{2}{\pi} \approx 0.637$. However, we shall not find it now, but calculate it later via an indirect route (Problem 7) Nevertheless, we shall use the fact that this limit exists from the very outset.

Problem 1. What is the variation of a segment of length a?

Solution. Since, clearly, the variation of such a segment in any given direction is a times as large as that of a unit segment parallel to it, and the limit of this quantity, that is, the mean variation of the segment of length a, is equal to Ka.

We define the *variation of a polygonal line in some direction* to be the sum of the lengths of the projections of its component line-segments ('links') in this direction (Fig. 3).

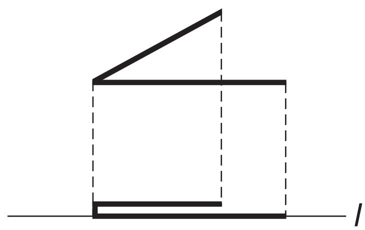

Fig. 3.

Problem 2. Determine the variation of a square of side 1 in the directions of its sides and its diagonals.

Solution. Clearly, the variation of the square in the direction of each side is equal to 2, and in the direction of a diagonal is equal to $2\sqrt{2}$.

The mean variation of a polygonal line over all directions, or simply the *variation of the polygonal line* over all directions is defined, as above, via the passage to the limit: $V = \lim_{n\to\infty} V_n$, where V_n is the arithmetic mean of the variations of the polygonal line along the n directions of the sides of a regular n-gon.

[4] See, for example, the book Yaglom, A.M., Yaglom, I.M.: Elementary problems in a non-elementary setting. Gostekhizdat, Moscow (1954), Problem 147b.

Problem 3. Determine the variation of a polygonal line of length a.

Solution. Clearly the variation of a polygonal line in each direction is the sum of the variations of the projections of its links in this direction, and since the mean value of a sum is equal to the sum of its mean values,[5] the variation of the polygonal line is the sum of the variations of its links. Since, by Problem 1 the variation of each link is equal to the product of the length of this link by K, the variation of the polygonal line is Ka.

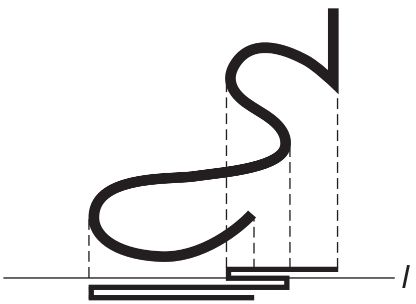

Fig. 4.

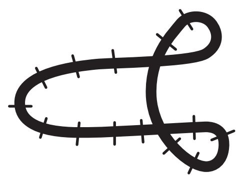

Fig. 5.

In order to transfer the definition of variation to *curves* we need to make precise the notion of a curve. This is difficult to do in the general case. However, we shall assume that the curve is either convex or can be divided into finitely many convex pieces. Then when one projects the curve in any given direction one can divide it into finitely many pieces each of which is intersected just once by each of the projecting lines.[6] Then *the variation of the curve in the chosen direction is, by definition, the sum of the lengths of the projections of its pieces in this direction* (Fig. 4). It can be shown that there exists a mean value of this quantity over all directions. We call this the mean variation or simply *variation* of the curved line.

It is clear that if the curve is a polygonal line, then we arrive at the previous definition.

[5] The precise meaning of this phrase is as follows: the arithmetic mean of the variations of a polygonal line over n directions is equal to the sum of the arithmetic means of the variations of its links over these directions. Therefore the limit as $n \to \infty$ of arithmetic means of the variations of the polygonal line over the different directions is equal to the sum of the limits of the arithmetic means of the variations of the individual links.

[6] Here we do not rule out the case when such a piece is a straight-line segment, so that when projecting in one of the directions the straight-line segment lies entirely in the projecting line.

Problem 4. Find the variation of a circle of diameter D.

Solution. First we choose some direction. The diameter having this direction divides the circle into two pieces, namely, into two arcs each of which is intersected by any line perpendicular to the chosen direction in at most one point. Hence the variation of the circle in the chosen direction is equal to $2D$. Clearly the variation in any other direction is the same, therefore the mean variation of the circle is equal to $2D$.

We now select several points on the curve and join them successively by straight lines (Fig. 5). Then we obtain a polygonal line. It can be shown that for sufficiently good curves (for example, for all convex curves) the limit of the lengths of these polygonal lines exists, provided that as these polygonal lines vary the length of the largest link of the lines tends to zero. This limit is called the *length of the curve*.

It can also be proved that for curves that can be divided into finitely many convex pieces the limit of the variations of these polygonal lines exists as the length of the largest link tends to zero.

Problem 5. Find the limit which the variation of a polygonal line inscribed in a "sufficiently good" curve of length a tends to when the polygonal line varies so that the length of its largest link tends to zero.

Solution. Since for each polygonal line of length a_n the variation is equal to Ka_n and $a_n \to a$ for "sufficiently good" curves, the limit of the variations of the polygonal lines is equal to Ka.

Problem 6. Prove that the variation of a ('sufficiently good') curve of length a is equal to Ka.

Solution. It suffices to observe that one can inscribe in such a curve a polygonal line with arbitrarily small links whose variation along each of the n given directions coincides with the variation of the curve. Therefore, once the limit in Problem 5 exists it is equal to the variation of the curve.

Problem 7. Find the numerical value of K, that is, the variation of a segment of length 1.

Solution. Since, on the one hand, a circle of diameter D has length D and hence variation $K\pi D$ (Problems 5 and 6) while, on the other hand (Problem 4), the variation of this circle is equal to $2D$, it follows that $K = \frac{2}{\pi}$.

By the *width of a curve with respect to a given direction* we mean the smallest distance between two lines of this direction that enclose the curve.

A curve has *constant width* if its width with respect to all directions is the same. Examples of a curve of constant width are the circle and the so-called *Rello triangle* consisting of three equal arcs of a circle (Fig. 6).[7] With the help

[7] There is a lot of information about curves of constant width in the book: Yaglom, I.M., Boltyanskii, V.G.: Convex figures. Gostekhizdat, Moscow (1951).

of variation one can obtain a very elegant proof of the following *Barbier's Theorem*:

Problem 8. Prove that all curves of constant width h have the same length πh.

Solution. This follows from the fact that the variation of each such curve in any direction is equal to $2h$; see the reults of Problems 6 and 7.

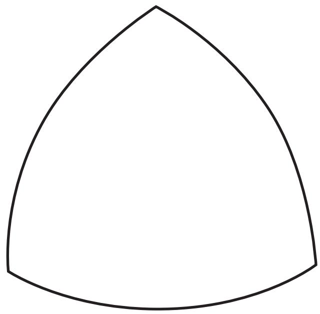

Fig. 6.

Here is another problem which at first glance appears to be rather complicated:

Problem 9. A curve L of length 22 is contained in a circle C of radius 1. Prove that there is a line intersecting L in at least 8 points.

Solution. The variation of L is equal to $\frac{2}{\pi} \cdot 22 > 14$ (Problems 6 and 7). On the other hand, the length of the projection of L in any direction does not exceed 2 (L is contained in C!). Hence for some directions certain parts of the projection of L will be covered by the projections of separate arcs of L more than 7 times (that is, at least 8 times). This completes the proof.

We now turn to an account of the lesson devoted to the topic "Harmonic functions".

The conclusion of this article will appear in the next issue

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(Окончание)

Гармонические функции

Две первые задачи не имели отношения к основной теме. Для полноты освещения занятия кружка мы приводим их; близкая к ним по методу решения третья задача являлась подготовительной к четвертой, с которой, по существу, и начиналась тема.

Задача 1. Найти наибольшее и наименьшее значения выражения

$$a \sin \varphi + b \cos \varphi \ (a \text{ и } b \text{ положительны}).$$

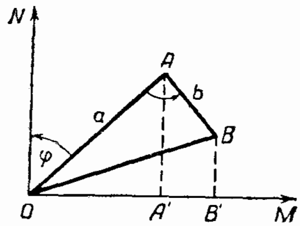

Рис. 1.

Решение. Проведем два взаимно-перпендикулярных луча OM и ON и построим прямоугольный треугольник OAB с катетами $OA = a$ и $AB = b$, расположив их так, как на рис. 1 (прямые углы MON и OAB ориентированы против часовой стрелки). Обозначим угол AON через φ, тогда, проектируя ломаную OAB на ось OM (проекции направленные!), получаем [1]:

$$(OB') = \text{пр.}\, OB = \text{пр.}\, OA + \text{пр.}\, AB = a \sin \varphi + b \cos \varphi.$$

Если вращать треугольник OAB вокруг вершины O, то угол φ изменяется; наибольшее и наименьшее значения проекции (OB') достигаются, когда отрезок OB коллинеарен OM, т. е. когда $\operatorname{tg} \varphi = \frac{a}{b}$; они равны $\sqrt{a^2 + b^2}$ и $-\sqrt{a^2 + b^2}$.

Задача 2. Доказать, что если

$$a_1 \cos \varphi_1 + a_2 \cos \varphi_2 + \ldots + a_n \cos \varphi_n = 0 \tag{1}$$

и

$$a_1 \cos (\varphi_1 + 1) + a_2 \cos (\varphi_2 + 1) + \ldots + a_m \cos (\varphi_m + 1) = 0 \tag{2}$$

(все коэффициенты a_i положительны), то и при любом α

$$a_1 \cos (\varphi + \alpha) + a_2 \cos (\varphi_2 + \alpha) + \ldots + a_m \cos (\varphi_m + \alpha) = 0. \tag{3}$$

[1]) (OB') — величина направленной проекции.

* Editor's note: V.I. Arnol'd: The school mathematical circle at Moscow State University: harmonic functions. Published in Mat. Prosveshchenie 3 (1958), 241–250

Решение. Выберем в плоскости луч OM и построим ломаную линию $OA_1A_2 \ldots A_m$ (на рис. 2 $m=3$), где $OA_1=a_1$, $A_1A_2=a_2$, ..., $A_{m-1}A_m=a_m$, причем векторы $\overline{OA_1}$, $\overline{A_1A_2}$, ..., $\overline{A_{m-1}A_m}$ образуют с лучом OM соответственно углы φ_1, φ_2, ..., φ_m. Легко видеть, что условие (1) означает, что $OA_m \perp OM$, а условие (2) — что $O\tilde{A}_m \perp OM$, где $O\tilde{A}_m$ получается из OA_m вращением

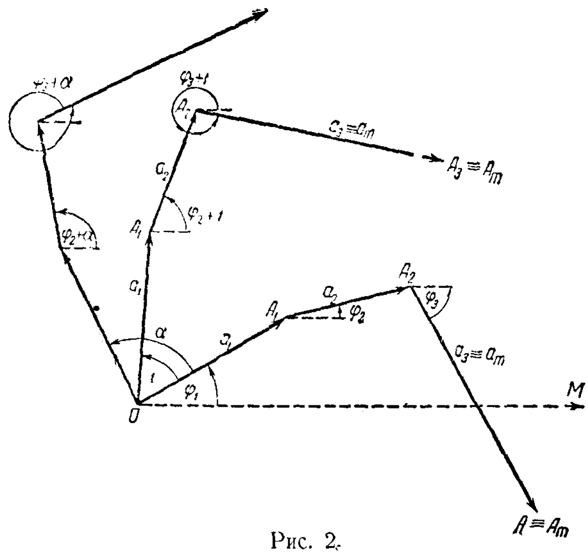

Рис. 2.

против часовой стрелки (при обычном направлении отсчета углов) на угол 1 (радиан). Оба условия вместе означают поэтому, что $\overline{OA_m}=0$, т. е. A_m совпадает с O. Но в таком случае проекция вектора $\overline{OA_m}$, повернутого на угол α [т. е. выражение $\sum_{i=1}^{m} a_i \cos(\varphi_i+\alpha)$], тоже равна нулю, что и доказывает (3).

Задача 3. Вычислить сумму m векторов с общим началом в центре правильного m-угольника и с концами в его вершинах (рис. 3, *а*).

Было предложено три решения.

Решение 1. Пусть сумма этих векторов — вектор $\overline{OA}$. Повернем многоугольник вокруг точки O на угол $\frac{2\pi}{m}$. Каждый вектор-слагаемое повернется на $\frac{2\pi}{m}$; тогда и сумма $\overline{OA}$ повернется на тот же угол, приняв положение $\overline{OA'}$. Вместе с тем каждый вектор перейдет при таком повороте в следующий, так что сумма не изменится, следовательно, $\overline{OA'}=\overline{OA}$. Но эти векторы образуют угол $\frac{2\pi}{m}$. Это может быть лишь при условии $\overline{OA}=0$.

Решение 2. Складывая векторы по «правилу треугольника» в порядке следования вершин, получим, очевидно, m-звенную ломаную, все звенья которой равны (они равны радиусу окружности, описанной около многоугольника) и все внешние углы равны (они равны $\frac{2\pi}{m}$, рис. 3, *б*). Отсюда следует что ломаная образует правильный m-угольник; так как он замкнут, то искомая сумма равна нулю.

Решение 3. Достаточно доказать это для правильного m-угольника, расположенного в комплексной плоскости так, что его вершины изобра-

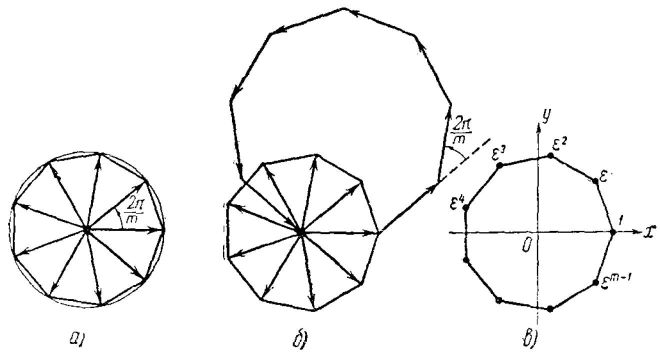

Рис. 3.

жают все корни m-й степени из 1: 1, ε, ε^2, ..., ε^{m-1} (рис. 3, *в*). Такой правильный m-угольник мы в дальнейшем будем называть основным m-угольником. Центр основного m-угольника изображает число 0, а одна из вершин — число 1.

Как известно, вершины основного m-угольника изображают все решения уравнения $z^m - 1 = 0$. По теореме Виета, сумма этих решений равна нулю, ибо коэффициент при z^{m-1} в этом уравнении равен нулю. Но комплексные числа складываются по правилу сложения изображающих их векторов. Следовательно, сумма векторов, о которых говорится в условии задачи, равна нулю.

Задача 4. Вычислить предел K

$$K = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)}{n}$$

— *среднее значение* функции $y = \sin x$ на отрезке $0 \leqslant x \leqslant \pi$.

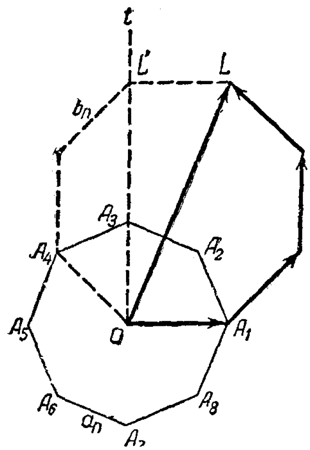

Рис. 4.

Решение. Рассмотрим снова правильный m-угольник, о котором говорилось в предыдущей задаче; на этот раз будем считать радиус описанной окружности равным 1, а число его сторон четным: $m = 2n$ (на рис. 4 $m = 8$). Сложим теперь только «правую половину» векторов: $\overline{OA_1} + \overline{AO_2} + \ldots + \overline{OA_n} = \overline{OL}$. Замыкающая OL рассматриваемой суммы будет совпадать с диаметром D_n окружности, описанной около нового m-угольника. Легко видеть, что если вектор $\overline{OA_1}$ направить горизонтально, то эта замыкающая при большом m близка к ее проекции OL' на вертикальную прямую Ot. А так как проекции единичных векторов $\overline{OA_1}$, $\overline{OA_2}$, ..., $\overline{OA_n}$ на эту вертикаль равны как раз

$$\sin 0 = 0, \quad \sin\frac{2\pi}{m} = \sin\frac{\pi}{n}, \quad \sin\frac{4\pi}{m} = \sin\frac{2\pi}{n}, \quad \ldots, \quad \sin\frac{(n-1)\pi}{n},$$

то среднее значение K равно пределу, к которому стремится частное $\frac{D_n}{n}$. Но из подобия m-угольников, изображенных на рис. 4, ясно, что $\frac{b_n}{a_n} = \frac{D_n}{2}$ (радиус $OA_1 = 1$), где $a_n = 2\sin\frac{\pi}{n}$, а $b_n = 1$. Следовательно,

$$\lim_{n\to\infty}\frac{\sum\limits_{k=0}^{n-1}\sin\left(\frac{k\pi}{n}\right)}{n} = \lim_{n\to\infty}\frac{|OL|}{n} = \lim_{n\to\infty}\frac{D_n}{n} = \lim_{n\to\infty}\frac{2}{n\cdot 2\sin\frac{\pi}{2n}} =$$

$$= \lim_{n\to\infty}\left(\frac{2}{\pi} : \frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right) = \frac{2}{\pi} : \lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} = \frac{2}{\pi} : 1 = \frac{2}{\pi}\ ^{1)}.$$

Замечание. Полученный результат имеет следующий геометрический смысл: предел, к которому стремится площадь ступенчатой фигуры, изображенной на рис. 5, между полуволной синусоиды и осью абсцисс, равен 2.

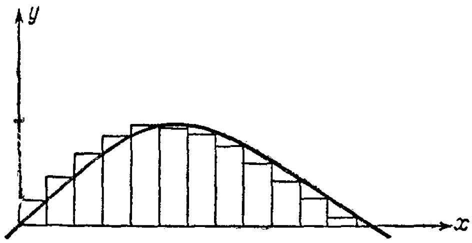

Рис. 5.

Задача 5. Доказать, что среднее значение произвольного многочлена с комплексными коэффициентами

$$P_k(z) = z^k + a_1 z^{k-1} + \ldots + a_k \tag{1}$$

в n вершинах правильного n-угольника на комплексной плоскости, при $n > k$, равно значению многочлена в центре этого многоугольника.

Решение производится в три этапа.

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[1]) Таким образом, этот предел оказался равным тому значению K, который мы раньше (см. «Математическое просвещение», вып. 2, стр. 242) назвали *средней вариацией* единичного отрезка. Это не случайно; решение всего цикла задач о вариациях кривых может рассматриваться как косвенное вычисление указанного в этой задаче предела.

1°. Пусть сначала $P_k(z) = z^k$ и правильный n-угольник является **основным** (см. решение 3 задачи 3). Тогда сумма k-х степеней комплексных чисел, изображаемых вершинами n-угольника, равна нулю при любом $k < n$.

В самом деле, при замене каждого числа z на $z\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right) = z\cdot\varepsilon$ многоугольник переходит в себя $\left(\text{он поворачивается на } \frac{2\pi}{n}\right)$, а каждое значение z^n умножается на $\varepsilon^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} \neq 1$. Значит, сумма значений z^k в вершинах n-угольника не меняется и в то же время умножается на ε^k. Следовательно, она может равняться только нулю.

Это же рассуждение непосредственно переносится и на случай $P_k(z) = az^k$.

2°. Так как среднее значение $P_l(z) = az^l$ $(1 \leqslant l \leqslant n)$ в вершинах основного n-угольника равно нулю, то и среднее значение суммы $z^k + a_1 z^{k-1} + \ldots + a_{k-1}z$ равно нулю. Следовательно, среднее значение многочлена $P_1(z)$ в его вершинах равно a_k, т. е. $P_k(0)$.

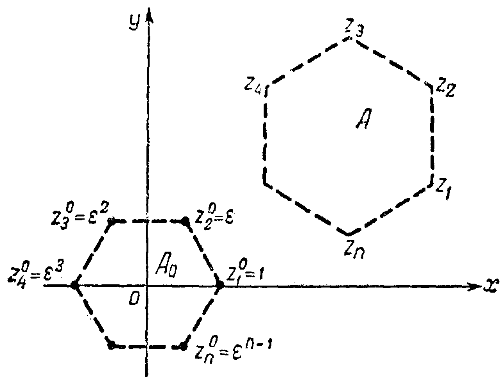

Рис. 6.

3°. Обозначим теперь комплексные числа — вершины основного n-угольника A_0 через $z_1^0 = 1$, $z_2^0 = \varepsilon$, $z_3^0 = \varepsilon^2$, $\ldots$, $z_n^0 = \varepsilon^{n-1}$ (рис. 6) и рассмотрим произвольно расположенный одноименный правильный многоугольник A с вершинами $z_1, z_2, \ldots, z_n$. Очевидно, правильный многоугольник A можно получить из A_0 поворотом, гомотетическим расширением (или сжатием) и параллельным переносом. Другими словами, найдутся два таких комплексных числа α и β, что

$$z_i = \alpha z_i^0 + \beta \qquad (i = 1, 2, \ldots, n).$$

Здесь модуль α равен отношению сторон многоугольников A и A_0, аргумент — углу поворота, а β — комплексное число, изображаемое центром многоугольника A.

Теперь заметим, что среднее значение многочлена (1) в вершинах $z_1, z_2, \ldots, z_n$ равно

$$\frac{\sum\limits_{i=1}^{n} P_k(z_i)}{n} = \frac{\sum\limits_{i=1}^{n} P_k(\alpha z_i^0 + \beta)}{n} = \frac{\sum\limits_{i=1}^{n} Q_k(z_i^0)}{n},$$

где

$$Q_k(z) = P_k(\alpha z + \beta) = (\alpha z + \beta)^k + a_1(\alpha z + \beta)^{k-1} + \ldots + a_{k-1}(\alpha z + \beta) + a_k$$

есть многочлен k-й степени относительно z [той же, что и $P_k(z)$], принимающий в точках $z_1^0, z_2^0, \ldots, z_n^0$ соответственно значения $P(z_1)$, $P(z_2)$, ..., $P(z_n)$. Поэтому среднее значение $P_k(z)$ в точках $z_1, z_2, \ldots, z_n$ равно среднему значению $Q_k(z)$ в точках $z_1^0, z_2^0, \ldots, z_n^0$, т. е. (см. этап 2°) равно

$$Q_k(0) = \beta^k + \alpha_1\beta^{k-1} + \ldots + a_{n-1}\beta + a_n.$$

Но $Q_k(0)$ совпадает с значением многочлена $P_k(z)$ в центре многоугольника A, что и завершает доказательство.

Пусть $f(z)$ — некоторая функция комплексного переменного z. Рассмотрим последовательность правильных n-угольников ($n = 3, 4, 5, \ldots$), вписанных в определенную окружность комплексной плоскости, и последовательность средних арифметических $f(z)$ в вершинах этих многоугольников. Если при $n \to \infty$ эти средние арифметические стремятся к определенному пределу, не зависящему от выбора вписанных в окружность многоугольников, то этот предел

$$\lim_{n\to\infty} \frac{f(z_1) + f(z_2) + \ldots + f(z_n)}{n}$$

называется *средним значением функции $f(z)$ по окружности.*

Из задачи 5 следует, что *среднее значение произвольного многочлена по любой окружности равно значению этого многочлена в центре окружности.*

Можно говорить не только о среднем значении функции в смысле среднего арифметического, но и о *среднем геометрическом* функции $f(z)$ по некоторой окружности. Под этим понимается действительное неотрицательное число

$$\lim_{n\to\infty} \sqrt[n]{|f(z_1)|\,|f(z_2)|\ldots|f(z_n)|}$$

(значение корня арифметическое!), где z_i — также вершины правильного n-угольника, вписанного в окружность.

Рассмотрим задачу, связанную с понятием среднего геометрического функции на окружности.

Задача 6. Доказать теорему: если многочлен $P_k(z)$ степени k не имеет корней внутри или на окружности, то его среднее геометрическое на этой окружности равно модулю его значения в центре окружности.

Решение проведем снова в три этапа.

1°. Пусть сначала окружность есть окружность $|z| = 1$, правильные n-угольники — основные, а многочлен $P_1(z) = z + a$. (Очевидно, $|a| > 1$, так как иначе корень $P_1(z)$ лежал бы внутри окружности.)

Рассмотрим произведение

$$(z_1 + a)\ a) + (z \ldots (z_n + a).$$

Это комплексное число есть значение многочлена

$$F_n(z) = [z - (-z_1)][z - (-z_2)]\dots[z - (-z_n)]$$

в точке $z = a$. «Старший коэффициент» многочлена $F_n(z)$ при z^n равен единице, а корни его равны $-z_1, -z_2, \dots, -z_n$; поэтому $F_n(z) \equiv z^n - (-1)^n$. Следовательно,

$$F_n(a) = a^n - (-1)^n \text{ и } |a^n| - 1 \leqslant |F_n(a)| \leqslant |a^n| + 1.$$

Но так как $|a| > 1$, то

$$\lim_{n\to\infty} \sqrt[n]{|a|^n + 1} = \lim_{n\to\infty} \sqrt[n]{|a|^n - 1} = |a|,$$

откуда $\lim\limits_{n\to\infty} \sqrt[n]{F_n(a)} = |a|$, что и доказывает теорему в этом частном случае.

2°. Так как, очевидно, *среднее геометрическое произведения равно произведению средних геометрических*, то доказываемая теорема справедлива и для любого многочлена $P_n(z)$, все корни которого по модулю больше 1, так как такой многочлен есть произведение сомножителей вида $(z + a_i)$, где $-a_i$ — корни $P_n(z)$.

3°. Наконец, пусть данная окружность S — произвольная, имеющая центр в точке, изображающей комплексное число β, а радиус α; ее уравнение $|z - \beta| = \alpha$. Рассмотрим преобразование комплексной плоскости

$$w = \alpha z + \beta.$$

Оно переводит единичную окружность $|z| = 1$ и круг $|z| \leqslant 1$ соответственно в окружность S и в ограничиваемый его круг.

Подставим в данный многочлен $P_n(z)$ вместо z выражение $\alpha z + \beta$. Получим:

$$P_n(\alpha z + \beta) = Q_n(z);$$

при этом значения многочлена $Q_n(z)$ в вершинах основного n-угольника равны значениям $P_n(z)$ в вершинах n-угольника, вписанного в S (ср. с решением задачи 5). Все корни $Q(z)$ лежат вне круга $|z| \leqslant 1$ [все корни $P(z)$ лежат вне круга, ограниченного S); среднее геометрическое $P_n(z)$ по окружности S равно среднему геометрическому $Q_n(z)$ по окружности $|z| = 1$. Но это последнее среднее вычислено в п. 2°; оно равно $|Q_n(0)| = |P_n(\beta)|$, ч. т. д.

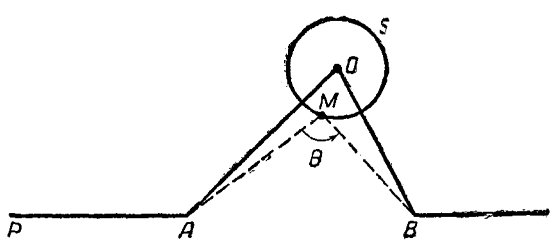

Рис. 7.

Задача 7. На плоскости имеются две фиксированные точки A и B (рис. 7). Рассмотрим функцию $\theta = f(M)$ точки M этой плоскости, равную углу θ (наименьшему, отсчитываемому против часовой стрелки), на который поворачивается луч MA до совмещения с MB. Доказать, что среднее значение функции $f(M)$

по любой окружности S, не пересекающей лучей AP и BQ, равно значению $f(M)$ в центре O окружности[1]).

Решение. Пусть $M_1\, M_2\, \ldots\, M_n$ — правильный n-угольник, вписанный в окружность S. Обозначим углы AM_iB через θ_i; угол AOB через θ_0. Нужно доказать, что

$$\lim_{n\to\infty} \frac{\theta_1+\theta_2+\ldots+\theta_n}{n} - \theta_0 = \lim_{n\to\infty} \frac{(\theta_1-\theta_0)+(\theta_2-\theta_0)+\ldots+(\theta_n-\theta_0)}{n} = 0.$$

Но, очевидно, для любой точки M на окружности S

$$\theta - \theta_0 = \angle OBM - \angle OAM;$$

таким образом, требуется доказать, что

$$\lim_{n\to\infty}\left(\frac{\angle OBM_1+\angle OBM_2+\ldots+\angle OBM_n}{n} - \frac{\angle OAM_1+\angle OAM_2+\ldots+\angle OAM_n}{n}\right) = 0, \qquad (1)$$

т. е. что среднее значение угла OBM на окружности S равно среднему значению угла OAM на этой окружности. Предположим теперь, что $n=2m$ четно и $(2m)$-угольник $M_1M_2\ldots M_{2m}$ имеет прямую OA осью симметрии, проходящей через середины сторон M_1M_{2m} и M_mM_{m+1}. В этом случае, очевидно,

$$\angle OAM_1 + \angle OAM_{2m} = 2\pi,$$

$$\angle OAM_2 + \angle OAM_{2m-1} = 2\pi, \ldots, \quad \angle OAM_n + \angle OAM_{m-1} = 2\pi$$

и, следовательно,

$$\frac{\sum_{i=1}^{2m} \angle OAM_i}{2m} = \pi$$

независимо от значения n.

Отсюда вытекает, что *если только среднее значение функции* $\angle OAM$ *существует* (а это мы будем предполагать, не задерживаясь на доказательстве), то оно равно π. Точно так же равно π и среднее значение по окружности функции $\angle OBM$, что и доказывает (1) и требуемую теорему.

Функции, среднее значение которых на каждой окружности равно значению в центре окружности, называются гармоническими.

Из задачи 5 вытекает, что действительная часть и коэффициент при мнимой части любого многочлена от комплексного переменного (точки комплексной плоскости) являются гармоническими функциями; задача 6 связана с гармоничностью логарифма модуля многочлена (в об-

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[1]) Приведенное ниже решение задачи 7 заимствовано из заметки В. А. Успенского «Геометрический вывод основных свойств гармонических функций», Успехи матем. наук 4, вып. 2 (30), стр. 201—205, в которой эта задача кладется в основу теории гармонических функций.

ласти, где многочлен не имеет корней), задача 7 дает геометрический пример гармонической функции.

Гармонические функции играют выдающуюся роль в математике, физике и технике. Для примера упомянем здесь о задаче нахождения распределения температур в произвольной плоской однородной пластинке. Ясно, что если распределение температур — установившееся, т. е. самопроизвольного перераспределения температур не происходит, то оно дается гармонической функцией, ибо если бы среднее значение температуры на малой окружности было, например, больше температуры в центре O, то точка O нагревалась бы.

Очевидно, что *заданная в некоторой области гармоническая функция может принимать наибольшее и наименьшее значения лишь на границе этой области,* ибо если бы наибольшее значение достигалось во внутренней точке O, то среднее значение по окружности с центром в O не могло бы совпадать со значением в O. Это предложение называется *принципом максимума* и играет большую роль в теории гармонических функций. Из него следует, что значения гармонической функции в области полностью определяются ее значениями на границе этой области: так, например, распределение температур на пластинке определяется температурами на крае пластинки. Действительно, если бы существовали две разные гармонические функции, то их разность (которая, очевидно, тоже будет гармонической функцией) была бы равна нулю на границе области и отлична от нуля где-то внутри нее; но это противоречит тому, что гармоническая функция принимает наибольшее и наименьшее значения на границе.

Функции, заданные в отдельных точках плоскости, например в центрах квадратов бумаги «в клетку», называются *функциями на сетке*. *Гармонической функцией на сетке* называется такая, у которой значение в каждой точке равно среднему арифметическому ее значений в соседних точках. Как и для гармонических функций на плоскости, здесь можно показать, что наибольшее и наименьшее значения гармоническая на сетке функция принимает на границе сетки и что значения гармонической функции на сетке однозначно определяются ее значениями в граничных узлах сетки.

При математическом приближенном решении задач, связанных с гармоническими функциями, их часто заменяют гармоническими на сетке функциями. Таким образом, например, можно вычислить температуру в точке однородной плоской пластинки, если известна температура на краю. Для этого пластинка делится на мелкие квадратики, где температура предполагается неизменной, и выписывается условие гармоничности на сетке, состоящей из центров квадратиков: среднее арифметическое температур соседей данного квадратика равно его собственной температуре; решение задачи удобно проводить методом последовательных приближений.

Легкая задача 8 касается одного важного свойства гармонических функций на сетке [см. также задачу **20** на стр. 269. — *Ред.*].

Задача 8. В каждой клетке бесконечного листа клетчатой бумаги написано натуральное число, равное среднему арифметическому чисел, стоящих в четырех соседних клетках. Доказать, что во всех клетках написано одно и то же число.

Решение. Четыре соседа-числа в такой таблице, как указано в условии, не могут быть все больше его и не могут быть все меньше его. Вместе с тем среди любого количества натуральных чисел всегда есть наименьшее n. Все четыре его соседа равны n, так как они не меньше n, и если хотя бы одно было больше, то среднее арифметическое тоже было бы больше n, тогда как по условию оно равно n.

Точно так же соседи этих соседей равны n и т. д. Так мы убеждаемся, что все числа в клетках равны n.

ON THE REPRESENTATION OF FUNCTIONS OF SEVERAL VARIABLES AS A SUPERPOSITION OF FUNCTIONS OF A SMALLER NUMBER OF VARIABLES*

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translated by Gerald Gould

In this paper we wish to give an account of several recent papers by Moscow mathematicians devoted to the question in the title of this paper. §1 contains the definition of superposition of functions and the statement of Hilbert's 13th problem relating to superpositions. §2 is devoted to superpositions of smooth functions. In §3 we present several very recent papers, in spite of the fact that the content of that section is now perhaps only of historical interest. The principal topic there is the description given by Kronrod of "the tree of components of a function of several variables", which is a concept whose popularization would seem to be very desirable (although the connection between this concept and the problems considered in our paper has proved to be less close than it originally appeared). The reader interested only in the strongest (and, moreover, the simplest in its method of proof) result relating to the representation of continuous functions of several variables as superpositions of functions of a smaller number of variables can, after looking at the introductory §1 go straight to §4, missing out §2–3. In addition, the smaller print in this paper means, as usual, that the corresponding material is auxiliary and omitting it will not affect the reader's understanding of what follows.

1. One of the problems of the famous problem book by Pólya and Szegö[1] begins as follows:

"*Do functions of three variables exist at all?*"

The meaning of this question is as follows. Suppose that we have two functions of two variables $u(x, y)$ and $v(y, z)$. We now consider a new function of two variables $w(u, v)$ and substitute our functions in place of u and v. Then the function $w[u(x, y), v(y, z)]$ now depends on the three variables x, y and z. Thus, by substituting in place of the arguments u and v of the function of two variables $w(u, v)$ the new functions of two variables we obtain

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* Mat. Prosveshchenie **3**, 41–61 (1958)

[1] Pólya, G., Szegö, G.: Problems and theorems of analysis, part I. Moscow, Section II, Problems 119 and 119a.

a function of three variables (one can even obtain a function of four variables $w[u(x,y),v(z,t)]$; we call this function a *single superposition* formed from the functions of two variables u, v and w. It is clear that all the properties of this function are determined by our three functions of two variables. Pólya and Szegö's question (which, however, was not formulated in their book in all its breadth) is as follows: can all functions of three variables be reduced to such a superposition (or a somewhat more complicated superposition) of functions of two variables, or do there in fact exist functions that are "essentially of three variables" which cannot be reduced to functions of two variables.

Note first of all that if one can also use *discontinuous* functions, then the answer to Pólya and Szegö's question is clearly negative.[2] Thus the only question of interest is whether or not all *continuous* functions of three variables are representable as superpositions of *continuous* functions of two variables.

In fact, a discontinuous function $u = \phi(x,y)$ enables one to map the (x,y) plane bijectively onto the line u [the fact that the set of pairs (x,y) of numbers and the set of numbers u have the same cardinality means precisely that these sets can be bijectively mapped onto each other]. We now choose *any* function of three variables $F(x,y,z)$ and *define* the function $\psi(u,z)$ by the equality

$$\psi[\phi(x,y),z] = F(x,y,z)\,;$$

this is possible because each pair of values (x,y) corresponds to a unique value $u = \phi(x,y)$ and we can take $\psi(u,z)$ to be equal to the corresponding value of $F(x,y,z)$.[3]

For the *simplest* continuous functions of three variables it is not hard to find representations of them as superpositions of continuous functions of two variables. For example, the function

$$F(x,y,z) = xy + yz$$

can be represented in the form

$$F = w[u(x,y),v(y,z)]\,,$$

where

$$w(u,v) = u + v, \quad u(x,y) = x + y, \quad v(y,z) = yz\,.$$

For the somewhat more complicated function

$$F(x,y,z) = xy + yz + zx$$

it is already impossible to represent it as a *simple* superposition of functions of two variables;[4] However, it is possible to represent it as a *double* superposition of functions of two variables, that is, in the form

[2] See the solution of problem 119 in Pólya and Szegö's book.

[3] It suffices to require that no two distinct pairs (x,y) correspond to the same value $u = \phi(x,y)$; here, for values $\bar{u}$ not belonging to the range of the function $\phi(x,y)$ the function $\psi(\bar{u},z)$ can be defined arbitrarily.

[4] See Pólya and Szegö's book.

$$w\{u[p(x,y),q(y,z)],\ v[r(y,z),s(z,x]\}\,;$$

it suffices merely to set

$$w(u,v)=u+v$$

and

$$u(p,q)=p+q,\quad p(x,y)=xy,\quad q(y,z)=yz,\quad v(r,s)=s,\quad s(z,x)=zx\,.$$

In general, all *entire rational functions* of several variables can by definition be obtained from their arguments by means of a multiple application of the operations of addition and multiplication, that is, they are the result of a multiple superposition of functions of not more than two variables

$$\phi(x,y)=x+y,\quad \psi(x,y)=xy,\quad f(x)=x+a,\quad g(x)=ax\,,$$

that is, the result of a multiple substitution of the arguments of these functions by more complex expressions formed by means of the same functions. By analogy with this, the *rational functions* are obtained as superpositions of six of the simplest functions of not more than two variables:

$$\phi(x,y)=x+y,\quad \psi(x,y)=xy,\quad \chi(x,y)=\frac{x}{y},$$
$$f(x)=x+a,\quad g(x)=ax,\quad h(x)=\frac{a}{x}\,.$$

If a segment of x is a function of known segments $a,b,c,\ldots$, then in order to be able to construct it using a ruler and compasses, it is necessary and sufficient that this function be homogeneous of the first dimension and that it be a superposition of these same simplest functions and the function $y=\sqrt{x}$. All the *elementary functions* can be represented as superpositions obtained via those same functions and in addition certain special functions of one variable, such as

$$\sqrt[n]{x},\quad e^x,\quad \ln(x),\quad \sin(x),\quad \text{and others.}$$

The simplest examples of algebraic functions going outside the limits of the class of elementary functions are provided by the roots of algebraic equations; the arguments of these functions are the values of the coefficients of the equations. But the roots of equations of the first, second, third and fourth degrees are, as is well known, elementary functions of the coefficients obtained as the result of superposition of those same functions of two variables, the sum, the difference, the product and the quotient, and (for equations of these 2nd–4th degrees) functions of the single variable $\sqrt[n]{x}$ (here $n=2$ in the case of a quadratic equation and can be equal to 2 or 3 in the case of equations of the 3rd and 4th degrees). For equations of the 5th and higher degrees such a representation is not possible in general; this was shown by Abel. However, the roots of equations of the 5th and 6th degrees can be expressed in terms of the coefficients by means of superpositions of certain more complex analytic functions of two variables; these representations can be used for the practical calculation of the roots of equations; in particular for nomographic solution of equations.

Attempts to obtain a representation of roots of 7th-degree equations as a superposition of suitable functions have not been crowned with success. Using algebraic transformations the general 7th-degree equation

$$x^7 + a_1x^6 + a_2x^5 + a_1x^6 + a_3x^4 + a_4x^3 + a_6x + a_7 = 0$$

can be reduced to the form

$$x^7 + ax^3 + bx^2 + cx + d = 0$$

where a, b and c are elementary (algebraic) functions of the coefficients $a_1, a_2, \ldots, a_7$ of the original equation, therefore they are expressed in terms of these coefficients as superpositions composed of simple functions of two variables. Thus, the question of the possibility of representing the roots of a 7th-degree equation by superpositions of functions of two variables reduces to the problem of finding such a representation for the special function of three variables a, b, c of the roots of the equation written above.

To date it is not known whether this function of three variables (which is easily seen to be analytic) can be represented as a superposition of analytic functions of two variables. Nevertheless, Hilbert managed to show that *certain* analytic functions of three variables are not such superpositions.

Hilbert's result is in connection with the following situation. If a function of *three* variables is a superposition of functions of two variables, then among the partial derivatives of the superposition and the functions of which it is composed there exist fully determined analytic relations. Therefore if *all* analytic functions of three variables are representable in such a form, then the space of coefficients of the series expansion of the functions of two variables involved in this superposition can be mapped analytically onto the space of coefficients of the expansion of functions of three variables; but this is not possible, since the latter space has a greater dimension (here we are restricted by the definite but large number of first coefficients of the expansion, that is, the first partial derivatives).

In his lecture at the 1900 International Mathematical Congress held in Paris the celebrated German mathematician David Hilbert posed 23 problems awaiting solution.[5] The thirteenth of these "Mathematical problems" of Hilbert's was as follows:

Can the roots of the equation

$$x^7 + ax^3 + bx^2 + cx + 1 = 0$$

be represented as superpositions of continuous functions of two variables?

[5] Hilbert, D.:Mathematische Problemen; Gesammelte Abhandlungen, vol.3, No.17 (1935). [*Editor's note*: Translation of this work of Hilbert's will appear in the next issues of Mat. Prosveshch.]

Hilbert conjectured that the answer to this question would turn out to be negative; in that case the more general question of whether all functions of three variables are superpositions of continuous functions of two variables would be solved at the same time.

2. The first results touching on Hilbert's 13th problem were obtained under the assumptions that the superpositions have some special form, for example, under conditions restricting the 'single' superpositions; they all supported Hilbert's conjecture.[6] The strongest result here is the result of A.G. Vitushkin who succeeded in constructing a polynomial such that neither the polynomial itself nor any function sufficiently close to it (in the sense of uniform convergence) can be decomposed into a simple superposition of continuous functions of two variables in any region or in any system of coordinates.

Further results are in connection with restrictions imposed on the functions involved in the superposition. As already recalled, Hilbert had noted earlier that it was impossible to obtain all the *analytic* functions of three variables as superpositions of *analytic* functions of two variables. Important results in this direction were obtained by Vitushkin,who by using his theory of multidimensional variations of functions and sets showed that not all l times continuously differentiable functions of three variables can be represented as superpositions of $\left[\frac{2}{3}l\right]$ times[7] differentiable functtons of two variables all of whose derivatives of order $\left[\frac{2}{3}l\right]$ satisfy Lipschitz condititons.[8]

In Kolmogorov's interpretation[9] Vitushkin's results are connected with the difference of the 'dimensions' of the corresponding function spaces. As Pontryagin and Shnirel'man had already proved in 1932, the dimension of a compact metric space (for example, a cube in Euclidean space) can be defined in the following way. We cover our space with 'small' sets of diameter ε. Clearly, the number $N(\varepsilon)$ of sets required to do this will increase as ε gets smaller; here it can be shown that $N(\varepsilon)$ increases as $\frac{1}{\varepsilon^n}$, where n is the

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[6] The simplest examples of this kind already appear in the book of Pólya and Szegö; a number of other examples (due to A.Ya. Dubovitskiĭ and R.A. Minlos) are given in the book:Vitushkin, A.G.: On multidimensional variations. Moscow (1955).

[7] Here the square brackets indicate the integer part.

[8] It also follows from this result that there exists in a three-dimensional cube an analytic function (of three variables) satisfying a Lipschitz condition with Lipschitz constant 1 such that no functions close to it (including the function itself) can be represented as an s-fold superposition of two variables satisfying a Lipschitz condition with some constant L_1 (s and L_1 are given in advance), and there exists an unbounded differentiable function satisfying a Lipschitz condition with Lipschitz constant 1 which is not a superposition of functions of two variables satisfying a Lipschitz condition. See Vitushkin's book referred to in footnote 6.

[9] Kolmogorov, A.N.: Estimates of the minimum number of elements of ε-nets in various function classes and their application to the question of the representation of functions of several variables as superpositions of functions of a smaller number of variables. Usp. Mat. Nauk **10**, No.1, 192–195 (19??).

dimension of the space; thus the dimension n can be *defined* as the limit

$$\liminf_{\varepsilon\to 0}\left[-\frac{\log N(\varepsilon)}{\log\varepsilon}\right].$$

For infinite-dimensional spaces this limit is equal to infinity. However, in a number of cases the number $N(\varepsilon)$ can increase as the function $1 : \exp(z^k)$, where k is some constant which one can provisionally call the "dimension of the infinite-dimensional space". Thus for infinite-dimensional spaces the role of dimension is played by the limit[10]

$$\liminf_{\varepsilon\to 0}\left[-\frac{\log\log N(\varepsilon)}{\log\varepsilon}\right].$$

For the space of functions $f(x_1, x_2, \ldots, x_n)$ of n arguments defined on an n-cube, where the functions are l times differentiable in all their arguments and are such that all the partial derivatives of order l satisfy a Hölder condition of order α,[11] the above-defined dimension can be considered to be equal to

$$\frac{n}{l+\alpha}.$$

Hence it follows, in particular, that the set of l times differentiable functions of three arguments is in a certain sense 'richer in its elements' than the set of $\left[\frac{2}{3}l\right]$ times differentiable functions of two arguments satisfying a Lipschitz condition (that is, a Hölder condition of order 1); hence it follows that it is impossible to express all the first functions as superpositions of the last ones.

---

[10] Instead of the number $N(\varepsilon)$ of sets of diameter ε completely covering the (compact) space one could choose the number $M(\varepsilon)$ of points of an ε-net, that is, the smallest number of points such that each point of the space is at a distance of at most ε from at least one of the chosen points, or the maximum number $K(\varepsilon)$ of points such that the distance between any two of them is greater than ε. It is curious to note that the same definition of the dimension of function spaces was arrived at (almost at the same time) by Shannon [Shannon, C.E.: The mathematical theory of communication, Urbana (1949); in the Russian translation of Shannon's work (in the collection "Theory of transmission of electric signals in the presence of noise". Inost. Lit., Moscow (1953)) the corresponding place was omitted for some reason] which started from arguments relating to "the theory of information": in the space of the transmitted information $K(\varepsilon)$ is the maximum number of 'ε-different signals' that cannot be confused by the receiver provided that the distortion of the information in the transmitter does not exceed ε.

[11] A function $f(x)$ satisfies a Hölder condition of order α if there exists a number C such that for each x_1, x_2 in the domain of the function

$$|f(x_1) - f(x_2)| < C|x_1 - x_2|^2.$$

A function of several variables is said to satisfy a Hölder condition if it satisfies this condition as a function of each of its variables.

3. However, in the domain of all continuous functions Hilbert's conjecture has proved to be false.

In the spring of 1956 Kolmogorov succeeded in showing that every continuous function of n variables ($n \geqslant 4$) defined on an n-cube is a superposition of continuous functions of the three variables.[12] The main tool in his construction is the one-dimensional *tree of components of level sets of a function introduced by Kronrod.*[13]

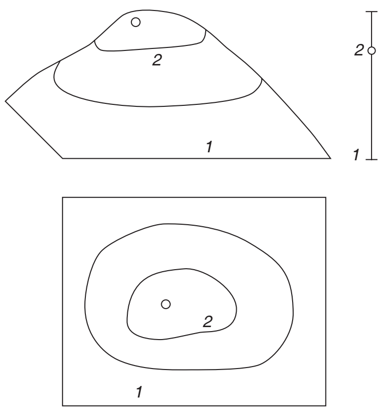

Fig. 1.

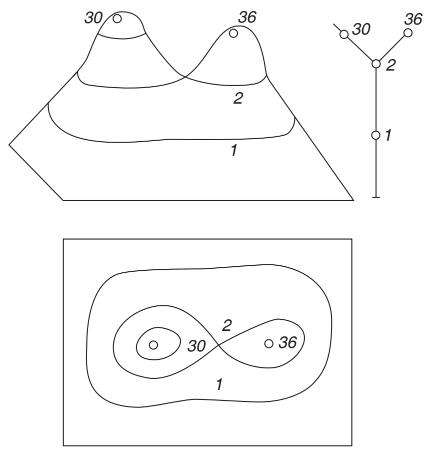

Fig. 2.

By the *level set* of a function we mean the collection of all points in the domain of the function at which the function takes some fixed value. For example, if the function of a point of part of the land surface represents the height at this point above sea level, then the level set will consist of all points of the locality having the same height above sea level; in topography these level sets are called *contour lines*. In Figs. 1 and 2 we have depicted simple functions of two variables and the 'maps' of the level sets of these functions (that is, a partition of the squares on which the functions are defined into

[12] Kolmogorov, A.N.: On the representation of continuous functions of several variables by superpositions of continuous functions of a smaller number of variables. Dokl. Akad. Nauk SSSR **108**, 179–182(1956); English transl. in Amer. Math Soc. transl. Ser. 2, vol. 17, 369–373 (1961).

[13] Kronrod, A.S.: On functions of two variables. Usp. Mat. Nauk **5**, No.1, 24–134 (1950).

their separate level sets). A level set can consist of a single piece (for example, all the level sets of the function depicted in Fig. 1 or the 1-level set of the function depicted in Fig 2; or it may consist of several connected pieces or components (for example, the 3-level set in Fig. 2 consists of the two pieces 3a and 3b). To study the structure of the set of components of a level set of a continuous function Kronrod proposed that one use the notion of a tree.

In topology, by a *tree* we mean a curve ('one-dimensional locally connected continuum') not containing any closed arcs ('homeomorphic images of a circle'). A tree can be represented in the following way. From the base of the 'trunk' of the tree there emerge 'branches' at the 'branch points' (the number of branch points can be denumerable and from each such point there can be denumerably many branches coming out of it); in turn, from each branch there can emerge new branches (we can call them 'twigs'), and so on (Fig. 3). In general a tree can be somewhat complex; however, as the celebrated Austrian (now American) mathematician Karl Menger showed, there exists in the plane a *universal tree* such that any other tree is a part of it (more precisely, such that any tree is homeomorphic to a part of the universal tree).[14]

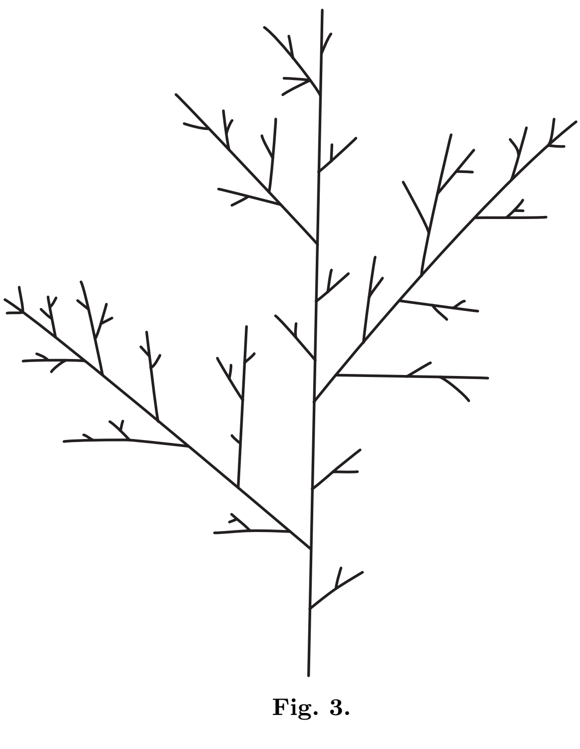

Fig. 3.

[14] Menger, K.: Kurventheorie, Ch. X. Berlin–Leipzig (1932).

Kronrod showed that *the set of components of all level sets of a continuous function of several variables is naturally representable as a tree.*

Thus, for example, the set of components of the level set of the functions depicted in Fig. 1 corresponds to a segment (the set of level 1 corresponds to the point 1 of this segment and the set of level 2 corresponds to the point 2); the set of components of the level sets of the somewhat more complicated function depicted in Fig. 2 corresponds to a Y-shaped tree (the set of level 1 corresponds to point 1 of the tree, the "figure 8" set of level 2 corresponds to the branch point 2: the components $3a$ and $3b$ of the set of level 3 correspond to the points $3a$ and $3b$ of the tree).

In more precise terms, one can introduce on the set of components a natural topology after which it becomes a topological space T which Kronrod called the *one-dimensional tree of the function.*

A study of the structure of this space can be carried out in the following way. First, T is the continuous image of an n-dimensional cube and therefore T is a locally connected continuum. Second, under the map d of the cube onto T the inverse image of each point of T is a component, that is, a closed connected set. We call such maps *monotone.*[15] Visually they can be represented as a contraction without gluing: for example, the projection of a square onto one of its sides is a monotone map, while the formation of a cylinder from a square by gluing is not a monotone map. One can prove that simple connectedness is preserved under a monotone map; therefore T, which is the monotone image of a cube, is *a simply connected set.* Finally, under a mapping of T onto a segment different components of the same level are taken to each point of the segment, that is, a zero-dimensional subset of T (not containing connected pieces) and, as is well known, under a map with zero-dimensional inverse images the dimension is not lowered. Therefore T is *one-dimensional.* Thus T is a one-dimensional and simply connected locally connected continuum. Hence T is a *tree.*

We can regard each function $f(x_1, x_2, \ldots, x_n)$ as a superposition of two maps: 1) a map $d(x_1, x_2, \ldots, x_n)$ of the domain of definition *onto the tree of components of the level sets* of f; under the map d the image of each point of the domain of definition is the component of the level set containing this point; 2) the map $f(d)$ of the set of components onto the segment that is the range of the function $f(x_1, x_2, \ldots, x_n)$. Under this map all the components of the level set $f(x_1, x_2, \ldots, x_n) = t$ are taken to the point t.

Thus, for example, the function of two variables $f(x, y) = xy$ defined on the square $-1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1$ can be represented as a superposition of two maps: the map of the square onto the X-shaped tree of the components of the level sets of this function (Fig. 4) [under which all the points of the 'cross' $xy = 0$ or one of the branches of the hyperbola $xy = const$ are taken to a single point of the tree], and the map of this tree onto the segment $-1 \leqslant t \leqslant 1$ [under which two points of the tree corresponding to branches of the same hyperbola (or one branch point corresponding to the cross $xy = 0$)are taken to the same point of the segment.

15 *Editor's note*: Since (non-strictly) monotone continuous functions of a single variable have this property [see the remark by Keldysh on p.261 of the current issue].

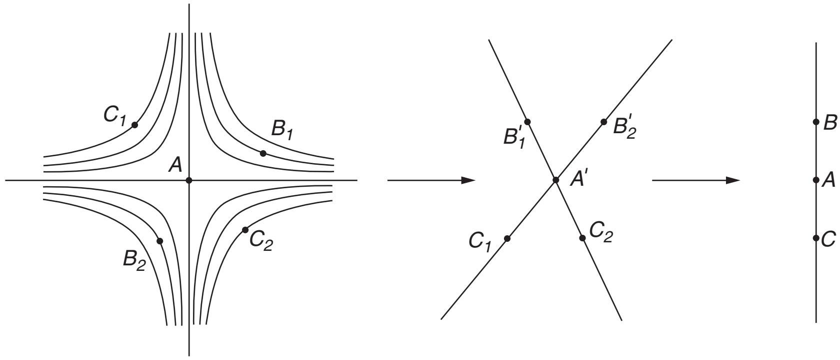

Fig. 4.

Thus, each function $f(x_1, x_2, \ldots, x_n)$ of n variables can be represented as a superposition of two new functions: the function $d(x_1, x_2, \ldots, x_n)$, which defines a map of the domain of definition of $f(x_1, x_2, \ldots, x_n)$ onto the tree of components of the level sets of this function, and $f(d)$, which is the map of the tree onto a segment (since each point d of the tree belonging to a given level set corresponds to a single value of $f(d)$ of the function f. Since a tree can be embedded in a plane, the points of this plane can be defined by the coordinates $u(d)$ and $v(d)$; this means that the second map $f(d)$ can be regarded as a function of two variables $f(u, v)$, while the first map $d(x_1, x_2, \ldots, x_n)$ can be regarded as two functions of n variables $u(x_1, x_2, \ldots, x_n)$ and $v(x_1, x_2, \ldots, x_n)$.

Kolmogorov managed to represent each function of n variables as a sum of $n+1$ functions each of which has *standard* (that is, not dependent on the function in question) components of the level sets:

$$f(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{n+1} f^r(x_1, x_2, \ldots, x_n);$$

thus, each function of two variables $f(x, y)$ can be represented as a function of three functions $f^1(x, y), f^2(x, y)$ and $f^3(x, y)$ where the 'maps' of the level sets of these three functions do not depend on f, but have some predetermined form, as illustrated in Fig. 5. Here for each function $f^r(x_1, x_2, \ldots, x_n)$ ($r = 1, 2, \ldots, n+1$) the map $d^r(x_1, x_2, \ldots, x_n)$ of the domain of definition onto the tree will not depend on the function f; on the other hand, the second map $f^r(d)$ of the tree onto the range of f^r does depend on f.

We now regard the function of n variables $f(x_1, x_2, \ldots, x_n)$ as a *one-parameter* (depending on the parameter x_n!) *family of functions of* $n-1$ *variables*:

$$f(x_1, x_2, \ldots, x_n) = f_{x_n}(x_1, x_2, \ldots, x_{n-1}).$$

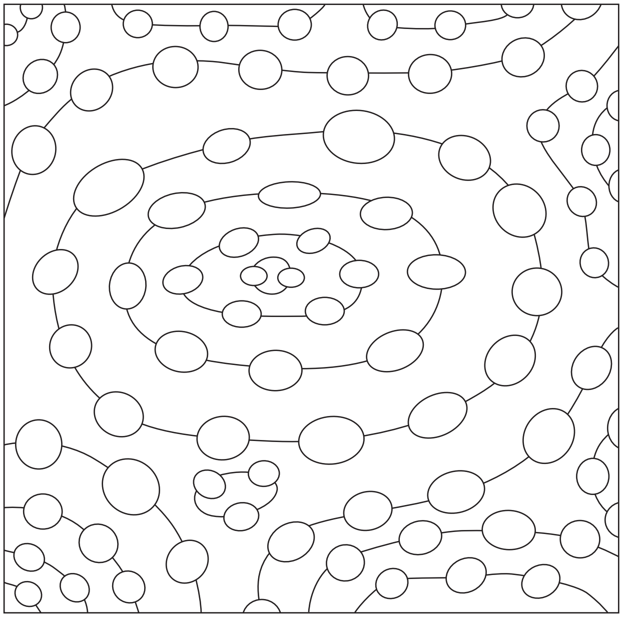

Fig. 5.

In this case we have

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= f_{x_n}(x_1, x_2, \ldots, x_{n-1}) \\ &= \sum_{r=1}^{n+1} f^r_{x_n}(x_1, x_2, \ldots, x_{n-1}) \\ &= \sum_{r=1}^{n+1} f^r_{x_n}(d^r(x_1, x_2, \ldots, x_{n-1})) \\ &= \sum_{r=1}^{n+1} f^r(d^r(x_1, x_2, \ldots, x_{n-1}), x_n)\,, \end{aligned} \tag{1}$$

where $d^r(x_1, x_2, \ldots, x_{n-1})$ is a map of the domain of definition of the function $f^r_{x_n}(x_1, x_2, \ldots, x_{n-1})$ which, as we have said, is *independent of the value of the parameter* x_n (the components of the level sets of the function f^r are standard!) and $f^r_{x_n}(d^r) = f^r(d^r, x^n)$ is the map of the point of the 'standard tree' d^r onto the range of f^r (which now depends on x_n). By introducing the system of coordinates (u^r, v^r) onto the plane of the tree d^r we obtain:

$$f(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{n} f^r(u^r(x_1, x_2, \ldots, x_{n-1}),\ v^r(x_1, x_2, \ldots, x_{n-1})\,, x_n)\,; \tag{2}$$

in other words, we have a *representation of an arbitrary function f of n variables as a sum of n functions each of which can be represented as a superposition of a function of three variables $f^r(u^r, v^r, x_n)$ and two functions $u^r(x_1, x_2, \ldots, x_{n-1})$ and $v^r(x_1, x_2, \ldots x_{n-1})$ of $n-1$ variables.* In the case when $n > 3$ we can apply the same process to the functions u^r and v^r of $n-1$ variables, so that we can eventually represent *a function of n variables $f(x_1, x_2, \ldots, x_n)$ as a superposition of functions of three variables.* Thus, the function $f(x_1, x_2, x_3, x_4)$ can now be represented in the form

$$f(x_1, x_2, x_3, x_4) = \sum_{r=1}^{4} f^r(u^r(x_1, x_2, x_3), v^r(x_1, x_2, x_3), x_4)\,; \tag{2a}$$

[we recall once more that the function of four variables $f = f^1 + f^2 + f^3 + f^4$ can be obtained as a superposition consisting of a single function of two variables $\phi(f^1, f^2) = f^1 + f^2$]. For $n = 3$ we obtain in this way only the representation

$$f(x, y, z) = \sum_{r=1}^{3} f^r(d^r(x, y)\ \ z)\,, \tag{3}$$

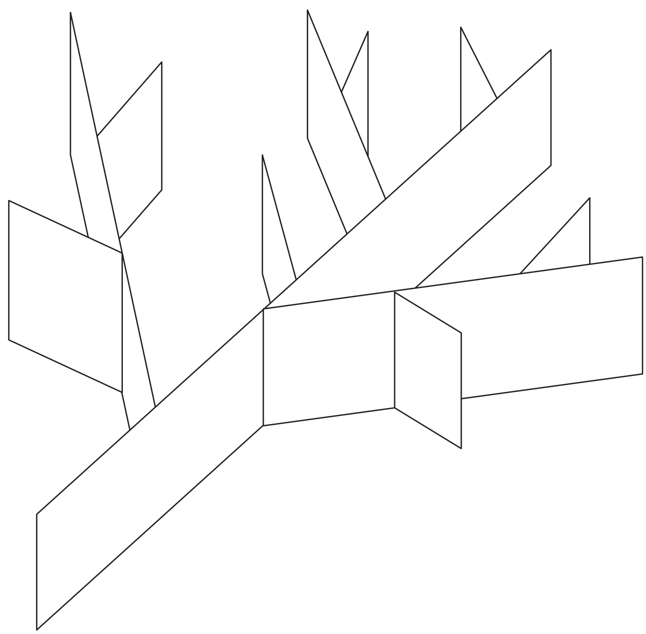

Fig. 6.

where $d^r(x, y)$ is a map of the square (x, y) onto the tree (which can be defined by two functions of two variables) and the $f^r(d^r, z)$ are defined on the set of pairs (d^r, z) where z ranges over the segment and d^r ranges over the tree, that is, functions of three variables that can, however, be defined on some special *two-dimensional* set, which is the direct product of the tree and the segment (see Fig. 6).

Recently it became clear[16] that the results of Kolmogorov can be improved: *any continuous function of n variables can be represented as a sum of $3n$ functions each of which can be represented as a superposition obtained by substituting in the function of two variables in place of one of the arguments the function of $n-1$ variables.*

The proof of this result is based on the fact that the trees d^r featuring above can be located in a three-dimensional cube (u, v, w) so that each function defined on any of them can be decomposed into a sum of three functions depending only on one of the coordinates

$$f^r(d^r) = \phi^r(u^r) + \psi^r(v^r) + \chi^r(w^r)\,. \tag{4}$$

Hence from (1) we obtain:

$$\begin{aligned}
f(x_1, x_2, \ldots, x_n) &= \sum_{r=1}^{n} f^r_{x_n}(d^r(x_1, x_2, \ldots, x_n)) \\
= \sum_{r=1}^{n} [\phi^r_{x_n}(u^r(x_1, x_2, \ldots, x_{n-1})) &+ \psi^r_{x_n}(v^r(x_1, x_2, \ldots, x_{n-1})) \\
&+ \chi^r_{x_n}(w^r(x_1, x_2, \ldots, x_{n-1})) \\
= \sum_{r=1}^{n} [\phi^r(u^r(x_1, x_2, \ldots, x_{n-1}), x_n) &+ \psi^r(v^r(x_1, x_2, \ldots, x_{n-1}), x_n) \\
&+ \chi^r(w^r(x_1, x_2, \ldots, x_{n-1}), x_n)\,.
\end{aligned}$$

In particular, as applied to functions of three variables we obtain instead of (3):

$$f(x, y, z) = \sum_{r=1}^{3} [\phi^r(u^r(x, y), z) + \psi^r(v^r(x, y), z) + \chi^r(w^r(x, y), z)]\,. \tag{5}$$

Thus, *each continuous function of three variables can be represented as a sum of 9 functions each of which is a single superposition of functions of two variables.* This then is the answer to the question posed by Hilbert.

In the proof of the decomposition (4) an essential role is played by the fact that in Kolmogorov's construction one can, as it turns out, avoid only trees having exceptional branch points of the third order (that is, points at which a single branch emerges from the main 'trunk'). Next it is easy to see that the simplest 'Y-shaped' tree can be arranged in the square (u, v) so that any function $f(u, v)$ defined on it

[16] Arnold, V.I.: On functions of three variables. Dokl. Akad. Nauk SSSR **114**, 679–681 (1957).

can be represented as the sum of two functions of a single variable: in fact, if in Fig. 7a we define arbitrarily the function $\phi(u)$ on the interval $\left(0, \frac{1}{2}\right)$, then we can define the function $\psi(v)$ on the interval $\left(0, \frac{1}{2}\right)$ since the sum $\phi(u)+\psi(v)$ on OA of the tree coincides with $f(u,v)$; next, we define the function $\psi(v)$ on the interval $\left(\frac{1}{2}, 1\right)$ so that the sum of $\phi(u)+\psi(v)$ on the interval AB of the tree coincides with $f(u,v)$; finally, we can define $\phi(u)$ on the interval $\left(\frac{1}{2}, 1\right)$ so that the sum $\phi(u)+\psi(v)$ on the interval AC of the tree coincides with $f(u,v)$; thus the function $f(u,v)$ defined on the tree can be represented as the sum $\phi(u)+\psi(v)$. If the tree has two branch points, that is, it has the form depicted in Fig. 7b, then the function $f(u,v)$ defined on it can also be represented as a sum $\phi(u)+\psi(v)$; we merely need to start from the definitions of the functions $\phi(u)$ and $\psi(v)$ on the interval $\left(\frac{3}{4}, 1\right)$, assuming that on the interval DC of the tree the sum $\phi(u)+\psi(v)$ coincides with $f(u,v)$, and then define the functions $\phi(u)$ and $\psi(v)$ in the same way as before, so that the sum $\phi(u)+\psi(v)$ on the entire tree coincides with the function $f(u,v)$. In the same way, any function defined on a tree with *finitely many* third-order branch points can be represented as a sum of two functions of one variable. For functions defined on a tree with *infinitely many* branch points, the above procedure fails; nevertheless, such a tree can be located in a *three-dimensional* cube such that a function defined on it can be represented as a sum of three functions depending on the separate coordinates.

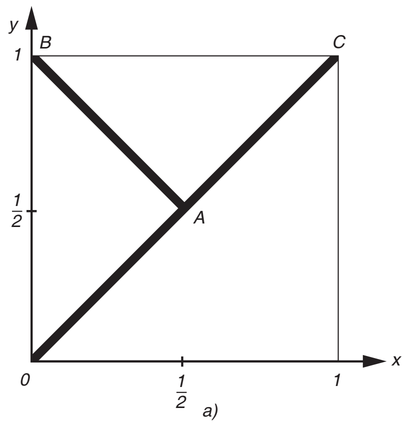

Fig. 7a.

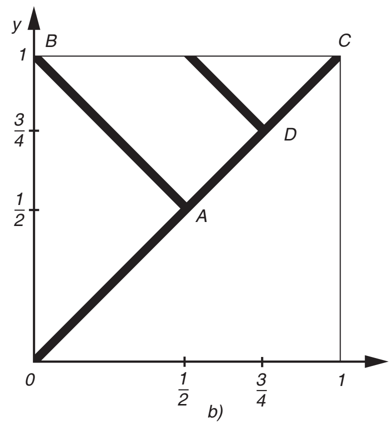

Fig. 7b.

It turns out that the complicated constructions that we have been talking about are superfluous for obtaining the final result. In the next section we give a much more direct route enabling one to obtain stronger theorems.

4. The above discussion enables one to answer *in the negative* the question posed by Pólya and Szegö whether there exist functions of three variables; more precisely this means that all continuous functions of three variables can be reduced to superpositions of continuous functions of two variables and all the properties of a function of three variables $f(x_1, x_2, x_3)$ are completely determined by certain functions of two variables, namely, the nine functions $u^r, v^r, w^r (r = 1, 2, 3)$ and the nine functions ϕ^r, ψ^r, χ^r $(r = 1, 2, 3)$ featuring in the representation (5). It is now natural to pose the question: *do there exist functions of two variables ?*

The precise meaning of the latter question is as follows. A superposition

$$F[f_1(f_2(\dots f_{n-1}(f_n))) \dots]$$

of any number of functions of one variable is, of course, a function of one variable and one cannot obtain functions of more than one variable in this way. However, if we add to our supply of functions of one variable just one 'standard' function of two variables, say, the sum

$$g(x, y) = x + y,$$

then superpositions composed of $g(x, y)$ and functions of one variable can now be functions of any number of variables; thus, for example, the $(n-1)$-fold superposition of the function g

$$g(g(g \dots g(g(x_1, x_2), x_3), \dots, x_{n-1}), x_n) = x_1 + x_2 + \dots + x_{n-1} + x_n$$

is a function of n variables. Here there arises the question: *can all continuous functions of two or more variables be represented as superpositions of this kind ?* This is the question we have in mind when we ask whether there exist ('artificial') functions of two variables. [More precisely, here we could ask: is our supply of functions of two variables essentially exhausted by one such function $g(x, y) = x + y$?]

If we restrict ourselves to the simplest representations of functions of two variables as a superposition of the function $g(x, y)$ and continuous functions of one variable, then the answer to the question of the possibility of obtaining all functions of two variables will be negative; thus, one can show by quite elementary means that the set of functions defined on a square that are representable in the form $f[\phi(x) + \psi(y)]$ (f, ϕ, ψ are continuous functions of one variable) not only fails to coincide with the set of all continuous functions, but is even nowhere dense and non-closed.[17] On the other hand, Kolmogorov had proved even before he had obtained the representation (2) that any continuous function of n variables can be *approximated to within any degree of accuracy* by a superposition of continuous functions of one variable and the sum $g = x + y$; thus, for example, any function $f(x, y)$ of two variables can be approximated arbitrarily closely by an expression

$$P_1(x) \cdot Q[R_1(x) + y] + P_2(x) \cdot Q[R_2(x) + y]\,,$$

[17] See Arnold, V.I.: On the representation of functions of two variables in the form $\chi[\phi(x) + \psi(y)]$. Usp. Mat. Nauk **12**, No.2, 119–121 (1957).

where $P_1(x), P_2(x); R_1(x), R_2(x); Q(u)$ are specially chosen polynomials of one variable.[18]

In more recent times, in his attempts to simplify the methods by which he had obtained the representations (2) and (5), Kolmogorov turned his attention to more elementary considerations that led him to the above result. Along these lines he succeeded in proving by extraordinary elementary and elegant means that *each continuous function of n variables defined on the unit cube of n-dimensional space E_n can be represented in the form*

$$f(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{2n+1} h_q \left[\sum_{p=1}^{n} \phi_q^p(x_p) \right], \tag{6}$$

where the $h_q(u)$ are continuous and the $\phi_p^q(x_p)$ are, in fact, standard, that is, they do not depend on the choice of the function f; in particular, each continuous function of two variables can be represented in the form

$$f(x, y) = \sum_{q=1}^{5} h_q[\phi_q(x) + \psi_q(y)]. \tag{6a}$$

For $n = 3$ it follows from (6) that

$$f(x, y, z) = \sum_{q=1}^{7} h_q[\phi_q(x) + \psi_q(y) + \chi_q(z)]] = \sum_{q=1}^{7} F_q[g_q(x, y)z],$$

where we have set

$$F_q(u, z) = h_q[u + \chi_q(z)], \quad g_q(x, y) = \phi_q(x) + \psi_q(y).$$

This last formula is even stronger than (5), since here the function of three variables $f(x, y, z)$ is representable in the form of seven (and not nine, as in (5)) terms that are single superpositions of functions of two variables; here these functions of two variables themselves have a special simple structure, and the inner function $g_q(x, y)$ (and the functions $\chi_q(z)$ occurring in the definition of $F_q(u, z)$) are, moreover, standard [so that all the properties of the functions $f(x, y, z)$ are completely determined by the seven functions of one variable $h_q(v)$].

The proof of (6) is so simple and beautiful that we shall reproduce it here almost in its entirety, referring those interested in the details to the author's more formalized account.[19] Since all the ideas of the proof occur quite clearly already in the case $n = 2$, we shall merely talk about the representation (6a)

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[18] See Kolmogorov's paper mentioned in the footnote 9 on page 5.

[19] See Kolmogorov, A.N.: On the representation of continuous functions of several variables as superpositions of continuous functions of one variable. Dokl. Akad. Nauk SSSR **114**, 953–956 (1957).

of an arbitrary continuous function $f(x, y)$ of two variables x and y. The possibility of such a representation is proved in several stages.

1°. The 'inner' functions $\phi_q(x)$ and $\psi_q(y)$ of the representation (6a) are completely independent of the function $f(x, y)$ to be decomposed.

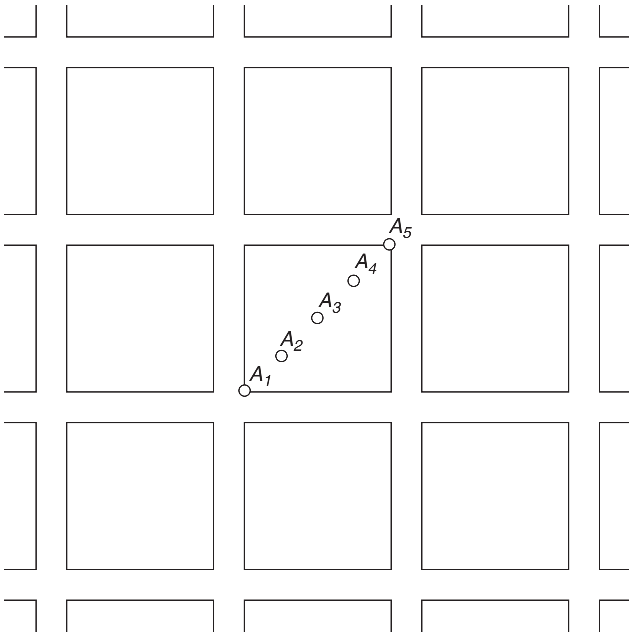


Fig. 8.

To define these functions we require certain preliminary constructions. We consider a 'town' consisting of a system of identical 'blocks' (non-intersecting closed squares) separated by narrow 'streets' all of the same width; see Fig. 8. We homothetically reduce our 'town' N times; for the centre of the homothety we can take, for example, the point A_1; we obtain a new 'town', which we call 'a town of rank 2'. The 'town of rank 3' is obtained in exactly the same way from the 'town of rank 2' by a homothetic reduction with homothety coefficient $\frac{1}{N}$: the 'town of rank 4' is obtained by a homothetic N-fold reduction by the 'town of rank 3', and so on. In general, the 'town of rank k' is obtained from the original 'town' (which we call 'the town of the first rank') by an N^k-fold reduction (with the centre of the homothety at A_1; incidentally the choice of the centre of the homothety is of no importance in what follows).

We call the system of 'towns' constructed above the 1st system. The 'town of the first rank of the qth system' ($q = 2, \ldots, 5$) is obtained from the 'town' depicted in Fig. 7† by moving the point A_1 to the point A_q by a parallel translation. It is not difficult to see that the 'streets' of the 'town' can be chosen sufficiently narrow so that a point of the plane will be covered by at least three blocks of our five 'towns of the first rank'. In the same way, the 'town of the kth rank' of the qth system ($k = 2, 3, \dots : q = 2, \dots, 5$) is obtained from the 'town of the kth rank of the first system' by a parallel translation taking the point A_1^k to the point A_q^k, where A_1^k and A_q^k are obtained from the points A_1 and A_q by a homothety taking the 'town of the first rank' of the first system (that is, our original 'town') to the 'town of the kth rank' of the same first system; here *each point of the plane will belong to 'blocks' of at least three of the five 'towns' of any fixed rank k.*

We define the function

$$\Phi_q(x, y) = \phi_q(x) + \psi_q(y) \quad (q = 1, 2, \dots, 5)$$

so that it divides any two 'blocks' of each 'town' of the system q, that is, so that the set of values taken by $\Phi_q(x, y)$ on a certain 'block' of the 'town of kth rank' (here k is an arbitrary fixed number) of the qth system does not intersect the set of values taken by $\Phi_q(x, y)$ on any other 'block' of the same 'town'. Here, of course, it suffices to consider the function $\Phi_q(x, y)$ on the unit square (and not on the entire plane).

In order that the function $\Phi_q(x, y) = \phi_q(x) + \psi_q(y)$ divide the 'blocks' of the 'town of the first rank' we can require, for instance, that on the projections of the 'blocks' of the 'town' onto the x axis $\phi_q(x)$ differs very slightly from the various integers and on the projections of the 'blocks' on the y axis $\psi_q(x)$ differs very slightly from the various multiples of $\sqrt{2}$ (because $m + n\sqrt{2} = m' + n'\sqrt{2}$ for integers m, n, m', n', only if $m' = m, n' = n$). Here, these conditions do not, of course, determine the functions $\phi_q(x)$ and $\psi_q(y)$ (on the 'streets' the function $\Phi_q = \phi_q + \psi_q$ can in general be defined completely arbitrarily for the moment); using this we can select limits on the values of $\phi_q(x)$ and $\psi_q(y)$ on the 'blocks' of the 'town of the second rank' so that the function $\Phi_q(x, y) = \phi_q(x) + \psi_q(y)$ divides not only the 'blocks' of the 'town of the first rank' but also the 'blocks' of the 'town of the second rank'.[20] In similar fashion, by bringing into consideration 'towns' of subsequent ranks and refining each time the values of the functions $\phi_q(x)$ and $\psi_q(y)$, in the limit we obtain continuous functions $\phi_q(x)$ and $\psi_q(y)$ (one can even require that they be monotone) satisfying the conditions in question.

2° By contrast, the functions $h_q(u)$ of the decomposition (6a) depend essentially on the original function $f(x, y)$.

To construct these functions we prove first of all that *any continuous function $f(x, y)$ of two variables x and y defined on the unit square can be represented in the form*

† *Translator's note*: This should be Fig. 8.

[20] The designated programme can be carried out if N is sufficiently large (so that the blocks of subsequent ranks do not join on to blocks of the previous ones). Kolmogorov chose $N = 18$.

$$f(x,y) = \sum_{q=1}^{5} h_q^{(1)} * \Phi_q(x,y)) + f_1(x,y), \tag{7}$$

where the $\Phi_q(x,y) = \phi_q(x) + \psi_q(y)$ *are the functions constructed above, and*

$$M_1 = \max|f_1(x,y)| \leqslant \frac{5}{6}\max|f(x,y)| = \frac{5}{6}M\,, \tag{7a}$$

$$\max|h_q^{(1)}(|\Phi_q(x,y)| \leqslant \frac{1}{3}M, \quad q = 1,\ldots,5\,. \tag{7b}$$

We choose the rank k sufficiently large so that the oscillation[21] of the function $f(x,y)$ on each 'block' of any of the 'towns of rank k' does not exceed $\frac{1}{6}M$; this, of course, is possible since as k increases the 'blocks' decrease without limit. Next, let $p_1^{(ij)}$ be a certain 'block' of a 'town of the first system' (and of the chosen rank k); then on this 'block' the (continuous) function $\Phi_1(x,y)$ takes values belonging to a certain segment $\Delta_1^{(ij)}$ of the real line (where, in view of the definition of the function Φ_1, this segment does not intersect segments of values taken by Φ_1 on any of the other 'blocks'). We now define the function $h_1^{(1)}$ on the segment $\Delta_1^{(ij)}$ to be a constant equal to one third of the value taken by $f(x,y)$ on any interior point $M_1^{(ij)}$ of the block $p_1^{(ij)}$ (it does not matter which). (We call this point the 'centre of the block'.) In similar fashion we define the function $h_1^{(1)}$ on each of the other segments defined by the values of $\Phi_1(x,y)$ on the 'block' of the 'town of rank k' of the first system; here all the values of $h_1^{(1)}$ will be at most $\frac{1}{3}M$ in modulus (since the value of $f(x,y)$ at the 'centre' of any 'block' will not exceed M in modulus). We now define in arbitrary fashion the function $h_1^{(1)}(u)$ at those values of the argument u at which it has not already been defined, with the proviso that it be continuous and that inequality (7b) should hold; we define all the other functions $h_q^{(1)}(u)(q = 2,\ldots,5)$ in similar fashion.

We now prove that the difference

$$f_1(x,y) = f(x,y) - \sum_{q=1}^{5} h_q^{(1)}(\Phi_q(x,y))$$

satisfies condition (7a), that is,

$$|f_1(x_0,y_0)| \leqslant \frac{5}{6}M\,,$$

where (x_0,y_0) is an arbitrary point of the unit square. This point belongs (as indeed do all the points of the plane) to at least three blocks of 'towns of rank k'; therefore there certainly exist three of the five functions $h_1^{(1)}(\Phi_q(x,y))$ taking at the point (x_0,y_0) a value equal to one third of the value of $f(x,y)$

[21] that is, the difference between the largest and smallest values

at the 'centre' of the corresponding 'block', that is, differing from $\frac{1}{3}f(x_0, y_0)$ by not more than $\frac{1}{18}M$ (since the oscillation of $f(x, y)$ on each block does not exceed $\frac{1}{6}M$); the sum of these three values $h_q^{(1)}(\Phi_q(x_0, y_0))$ differs from $f(x_0, y_0)$ in modulus by at most $\frac{1}{6}M$. But since each of the remaining two numbers $h_q^{(1)}(\Phi_q(x_0, y_0))$ does not exceed $\frac{1}{3}M$ in modulus (in view of (7)), we obtain

$$|f_1(x_0, y_0)| = \Big|f(x_0, y_0) - \sum_{q=1}^{5} h_q^{(1)}(\Phi(x_0, y_0))\Big| \leqslant \frac{1}{6}M + \frac{2}{3}M = \frac{5}{6}M\,,$$

which proves (7a).

We now apply the same representation (7) to the function $f_1(x, y)$ featuring in (7); we obtain

$$f_1(x, y) = \sum_{q=1}^{5} h_q^{(2)}(\Phi_q(x, y)) + f_2(x, y)$$

or

$$f(x, y) = \sum_{q=1}^{5} h_q^{(1)}(\Phi_q(x, y)) + \sum_{q=1}^{5} h_q^{(2)}(\Phi_q(x, y) + f_2(x, y))\,,$$

where

$$M_2 = \max|f_2(x, y)| \leqslant \frac{5}{6}M_1 \leqslant \left(\frac{5}{6}\right)^2 M$$

and

$$\max|h^{(2)}(\Phi_q(x, y)) \leqslant \frac{1}{3}M_1 \leqslant \frac{1}{3}\cdot\frac{5}{6}M \quad (q = 1, 2, \ldots, 5)\,.$$

Next we apply the decomposition (7) to the function $f_2(x, y)$ so obtained, and so on; after an n-fold application of this decomposition we obtain

$$f(x, y) = \sum_{q=1}^{5} h_q^{(1)}(\Phi_q(x, y)) + \sum_{q=1}^{5} h_q^{(2)}(\Phi_q(x, y)) + \cdots + \sum_{q=1}^{5} h_q^{(n-1)}(\Phi_q(x, y)) + f_2(x, y)\,,$$

where

$$M_2 = \max|f_n(x, y)| \leqslant \left(\frac{5}{6}\right)^n M$$

and

$$\max|h_q^{(s)}(\Phi_q(x, y)| \leqslant \frac{1}{3}\left(\frac{5}{6}\right)^{s-1} M \quad (q = 1, 2, \ldots, 5;\ s = 1, 2, \ldots, n-1))\,.$$

The last estimates show that as $n \to \infty$

$$f(x,y) = \sum_{q=1}^{5} h_q^{(1)}(\Phi_q(x,y)) + \sum_{q=1}^{5} h_q^{(2)}(\Phi_q(x,y)) + \cdots$$
$$+ \sum_{q=1}^{5} h_q^{(n)}(\Phi_q(x,y)) + \cdots ,$$

where the infinite series on the right hand side converges uniformly, as does each of the five series

$$h_q^{(1)}(\Phi_q(x,y)) + h_q^{(2)}(\Phi_q(x,y)) + \cdots + h_q^{(n)}(\Phi_q(x,y)) + \cdots \quad (q = 1, 2, \ldots, 5) .$$

This enables us to introduce the notation

$$h_q(u) = h_q^{(1)}(u) + h_q^{(2)}(u) + \cdots + h_q^{(n)}(u) + \cdots \quad (q = 1, 2, \ldots, 5) .$$

Thus, we finally obtain

$$f(x,y) = \sum_{q=1}^{5} h_q(\Phi_q(x,y)) = \sum_{q=1}^{5} h_q[\phi_q(x) + \psi_q(y)] ,$$

which is the required decomposition (6).

In conclusion we note that the representations (2), (5) and (6) are of purely theoretical interest, since they use essentially non-smooth functions such as the Weierstrass function;[22] therefore for practical purposes these representations are, it would seem, useless (in contrast with the representations (recalled earlier) of roots of equations of the 5th and 6th degrees as superpositions of functions of two variables). Thus the results that we have obtained do not remove the problem of finding convenient representations of, say, roots of 7th degree equations.

It is also unclear to what extent the decomposition (6) can be further improved; for example, the question of the uniqueness of the choice of the function h has not been solved. Also there are no methods enabling one to represent a given smooth function as a superposition of functions that are also relatively smooth; the strongest result in this direction remains the purely negative results of Vitushkin. Positive results of this kind would be of enormous interest.

We note one further result of Kolmogorov that goes in another direction. He proved that for each function of two variables defined on a square there exists a sum

[22] In view of the results of Bari (see Bari, N.K.: Mémoire sur la représentation finie des fonctions continues. Math. Ann. **103**, 145-248 and 590–653 (1930), one can represent each continuous function of one variable as a superposition of absolutely continuous functions. It therefore follows from (6) that each continuous function of n variables can be represented as a superposition of *monotone* functions of one variable and the sum function $g(x,y) = x + y$; however, these monotone functions are also essentially non-smooth.

of the form $\phi(x) + \psi(y)$ that *best approximates* this function. It can also be shown that for any (even everywhere discontinuous) bounded real function f defined on a compact set and any continuous function g defined on the same set there is a continuous function ϕ such that the deviation of $\phi(g)$ from f is a minimum. In particular, for each bounded function $f(x)$ there is a continuous function $\phi(x)$ best approximating it (in the sense of uniform convergence).

REPRESENTATION OF CONTINUOUS FUNCTIONS OF THREE VARIABLES BY THE SUPERPOSITION OF CONTINUOUS FUNCTIONS OF TWO VARIABLES*

V. I. ARNOL'D

Contents

Introduction

The present work is devoted to the proof of the following theorem, which was stated in an earlier note [1].

Theorem 1. *Every real continuous function* $f(x_1, x_2, x_3)$ *of three variables, defined on the unit cube* E^3, *can be represented in the form*

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3],$$

* Editor's note: translation into English published in Amer. Math. Soc. Transl. (2) 28 (1963), 61–147 Translation of V.I.Arnol'd: On the representation of continuous functions of three variables by superpositions of continuous functions of two variables, Mat. Sb. (n.S.) 48 (90):1 (1959), 3–74 Corrections in Mat. Sb. (n.S.) 56 (98):3 (1962), 392

where h_{ij} *and* φ_{ij} *are real continuous functions of two variables.*

For the proof of this theorem in note [1], use was made of two theorems whose complete proofs were not given in that paper. Here are these theorems.

Theorem 2. *Every continuous function* $f(x_1, x_2, x_3)$ *defined on* E^3 *can be represented in the form*

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} h_i[\varphi_i(x_1, x_2), x_3],$$

where h_i *and* φ_i *are continuous functions; the functions* h_i *are real and are defined on the product* $\Xi \times E^1$ *of the tree* (see [3], Chapter X) Ξ *by the interval* E^1, *while the functions* $\varphi_i(x_1, x_2)$ *are defined on a square and have for their values points of* Ξ. *Here* Ξ *is a tree, whose points have a branching index not greater than* 3.

Theorem 3. *Let* F *be any family of real, equi-continuous functions* $f(\xi)$ *defined on the tree* Ξ *all of whose points have a branching index* $\leqslant 3$. *Then one can realize the tree in the form of its homeomorphic image* X, *a subset of the three-dimensional unit cube* E^3, *in such a way that every function* f *of the family* F *can be represented in the form*

$$f(x) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1, x_2, x_3)$ *is the image in* X *of the element* $\xi \in \Xi$, $f(x) = f(\xi)$, *and the* $f_k(x_k)$ *are continuous real functions of one variable. Here* f_k *depends continuously on* f *in the sense of uniform convergence.*

Theorem 2 (with the exclusion of the last phrase) is contained in a work of A.N. Kolmogorov [2]. Its proof is also outlined there, but the proofs of the lemmas used there were not published. In Part I of the present work there are presented the proofs of these lemmas for the case when the branching index of the points of the obtained tree is not greater than 3. After that, the Theorem 2 given above is derived from these lemmas.

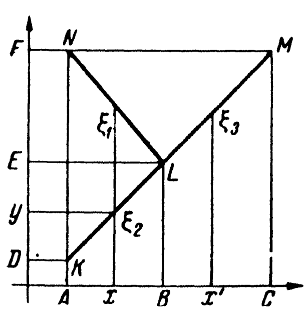


Figure 1. Representation in the form $\varphi(x) + \psi(y)$ of a function given on a Y-type tree.

For greater explicitness, let us consider the case $n = 2$ of the lemmas of the note [2]. The proofs (as well as the formulations) of these lemmas are somewhat different from those given by A.N. Kolmogorov. This is due to the introduction of the items 6) and 7) into the fundamental lemma, and to our desire to obtain

Theorem 2 in the formulation given above.

Theorem 3 is proved in the second part of this work. The ideas behind this theorem are quite simple.

Let a continuous function $f(\xi)$ $(\xi \in \Delta)$ be given on a Y-type tree (Figure 1). Then there exist continuous functions $f_1(x)$ and $f_2(y)$ such that $f_1(x) + f_2(y) = f(\xi)$ if x and y are the coordinates of the point $\xi \in \Delta$.

The proof can be accomplished, for example, as follows.

Suppose that the function $f_1(x)$ on AB is equal to $f(\xi_1)$ for a point $\xi_1 \in LN$ whose abscissa is x. In order that $f = f_1 + f_2$ on KL, one has to define $f_2(y)$ on DE as $f_2(y) = f(\xi_2) - f_1(x)$, where $\xi_2 \in KL$ is the point with coordinates x, y. Hereby, $f_2 = 0$ at the point E. Let $f_2(y) = 0$ on EF also. Finally, in order that $f = f_1 + f_2$ on LM, one has to set $f_1(x') = f(\xi_3)$, where $\xi_3 \in LM$ is the point of LM with abscissa x'. It is easily seen that the constructed functions $f_1(x)$ and $f_2(y)$ are the desired ones.

It is easy to devise an analogous construction for the function given on a more complicated tree (Figure 11). In general, we have the following type of theorem.

Every finite tree, whose branch points are of index not greater than 3, can be mapped homeomorphically onto a flat segment-like complex K such that every continuous function $f(\xi)$ is representable on K in the form $f(\xi) = f_1(x) + f_2(y)$, where x and y are the coordinates of the point $\xi \in K$.***

Theorem 3 asserts that an analogous result holds in the three-dimensional space for any tree whose points have a branching index not greater than three. The proof is very involved, but can be reduced in essence to the considerations given above, and to the transition to the infinite tree from finite trees.

Theorem 1 is a direct consequence of Theorems 2 and 3. Taking the risk of possibly confusing the reader, who could derive the proof himself, we nevertheless present a simple argument.

From Theorem 2 it follows that one can express the function $f(x_1, x_2, x_3)$ as the sum of three functions $h_i(\xi_i, x_3)$ $(i = 1, 2, 3)$ from the product of the tree $(\xi_i \in \Xi)$, none of whose points have a branching index greater than 3, by the segment $(x \in E^1)$: $(\xi_i, x_3) \in \Xi \times E^1$. Theorem 3 asserts that the function $h(\xi)$ on such a tree can be expressed as the sum of three continuous

* A tree with a finite number of points.

** The reader can easily construct the proof of this theorem after he reads §3-7. Whether it is possible to give an analogous representation for an infinite tree, is not known.

functions $h_j(x_j)$ $(j = 1,2,3)$ of the coordinates x_j of some realization $x_j(\xi)$ of the tree Ξ in the three-dimensional space. These functions $h_j(x_j)$ depend continuously on the decomposed function $h(\xi)$ (in the sense of uniform convergence) if the function h belongs to the same family F of equi-continuous functions on the tree Ξ for which the realization is constructed. The functions $h_i(\xi_i, x_3)$ that are obtained from Theorem 2 can be considered to be such a family of functions $h_i(\xi)$ on the tree Ξ, which depend continuously on the parameter $x_3 \in E^1$, and they are, therefore, equi-continuous. Applying Theorem 3, we find a realization of Ξ in the form $X \subset E^3$.

In the decomposition $f(x_1, x_2, x_3) = \sum_{i=1}^{3} h_i(\xi_i, x_3)$, $\xi_i = \varphi_i(x_1, x_2)$ is a point of the tree Ξ and depends continuously on x_1 and x_2 (Theorem 2). Hence, after the realization of Ξ in the form X, every coordinate $x \in X$ becomes a real, continuous function of x_1 and x_2. If $\xi_i = \varphi_i(x_1, x_2)$ and the jth coordinate of the point x that is realized by ξ_i is $\varphi_{ij}(x_1, x_2)$, then, in view of Theorem 3, the decomposition of $h_i(\xi_i, x_3)$, as a function of $h_{i x_0}(\xi_i)$, into the sum $\sum_{j=1}^{3} h_{ij x_0}(x_j(\xi_i))$ can be written in the form

$$h_i[\xi_i(x_1, x_2), x_3] = \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3].$$

Therefore,

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3],$$

which is the assertion of Theorem 1.

About two months after the completion of our work [1], A.N. Kolmogorov [2] strengthened the Theorem 1 by showing that every continuous function on the three-dimensional cube is representable in the form

$$f(x_1, x_2, x_3) = \sum_{i=1}^{7} h_i[\varphi_{i1}(x_1) + \varphi_{i2}(x_2) + \varphi_{i3}(x_3)],$$

where the functions h_i and φ are continuous; the functions φ_{ik} are, however, selected once for all independently of f. From this result of A.N. Kolmogorov it follows that the three-dimensional cube can be imbedded in a seven-dimensional space so that any continuous function on the cube will be expressible as the sum of continuous functions of (seven-dimensional) coordinates. According to the work [2], an analogous representation for a square can be realized in a five-dimensional space. From this it follows directly that in a five-dimensional space we can place our tree Ξ, once for all, so that any function continuous on it is expressible as the sum of continuous functions of the coordinates (while in our Theorem 3 the representation in the three-dimensional space depended on the family F). But by modifying the methods of the note [2], one can obtain a representation of the tree Ξ which is valid for all continuous functions f in the three-dimensional space also.

In the constructions of the first and second parts of the present work, use is being made of the tree of the components of the level sets, which was introduced by A.S. Kronrod. The essential information about this tree can be found in the Appendix. The Appendix and each of the two parts of this work are independent of each other.

I take this opportunity to thank my teachers A.G. Vituškin and A.N. Kolmogorov for their constant attention, counsel and help. In particular, I am indebted to A.N. Kolmogorov for the final formulation of the fundamental "inductive lemma" of the second part.

PART I

Proof of Theorem 2

Here we shall prove Theorem 2. The fundamental lemma of the work [2] and Lemma 2 are proved in such a formulation that the tree Ξ, under consideration in Theorem 3, consists of points whose branching index does not exceed 3.

The following notations will be used:

R^2 is the plane of the (x, y) points; E^2 is the closed unit square in this plane, i.e., the set of points (x, y) with $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

The metric in the plane is defined as the distance

$$\rho((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|).$$

$U_d(A)$ denotes a d-neighborhood of the set A, i.e. the set of all points in the plane whose distance from the set A is less than d $(d > 0)$.

$\bar{A}$ is the closure of A.

A polygon is a closed broken line that does not intersect itself. An open polygon Q is the part of the plane lying inside a polygon, while a closed polygon $\bar{Q}$ is the closure of the open polygon.

An open polygonal band is the part of the plane bounded by two nonintersecting polygons, one of which lies inside the other (is separated by the other from infinity). A closed polygonal band is the closure of an open one.

The set of the level c of a function $u(x,y)$ is the set of points (x,y) such that $u(x,y) = c$.

A list of the topological terms used in this work is given at the end of the Appendix.

§1. Fundamental lemma

Suppose that we are given a finite number of nonintersecting regions g_m in a plane, and that over each region there is a hill u_m. The set of hills form a "mountain country" G. Suppose that we are given not only one mountain country G (Figure 2) but an infinite sequence Γ of such mountain countries,

$$G_1, G_2, \ldots, G_k, \ldots,$$

where the "country of rank k" G_k consists of some finite number m_k of hills u_{km} of rank k $(m = 1, \ldots, m_k)$ over the regions g^r_{km}; no two regions of a given mountain country intersect each other (Figure 2). For large k, the country G_k has more hills, but their bases, the regions g^r_{km}, are smaller.

Finally, let us suppose (and this is not shown in Figure 2) that we are given three such sequences of countries Γ^r $(r = 1, 2, 3)$, namely, three systems Γ^r. Each of them consists of mountain countries G^r_k $(k = 1, 2, \ldots)$, and each mountain country G^r_k consists of hills u^r_{km} $(m = 1, \ldots, m_k)$.

In the fundamental lemma there are constructed three such systems of hills u^r_{km} satisfying a number of requirements. For example, every hill u^r_{km} is constructed in such a way that over every region $g^r_{k'm'}$ $(k' > k)$ it possesses a horizontal plane (requirement 5).

Fundamental lemma. *It is possible to define on the plane R^2 a system of real functions $u^r_{km}(x,y)$, with indices lying within the limits*

$$1 \leqslant r \leqslant 3, \quad 1 \leqslant k < \infty, \quad 1 \leqslant m \leqslant m_k,$$

and having the following properties:

1) $0 \leqslant u^r_{km} \leqslant 1$.

2) $u^r_{km} \neq 0$ *just on the region* g^r_{km} *whose diameter is less than* $d_k > 0$; $d_k \to 0$ *as* $k \to \infty$; $u^r_{km} = 1$ *on the set* $g^r_{k+1\, m*}$ *only.*

3) *Two sets* g^r_{km} *and* $g^r_{km'}$ *with the same indices* r *and* k, *but* $m \neq m'$, *do not intersect.*

4) *For any given* k, *and for every point of the square* E^2, *it is true that*

$$0 < c \leqslant \sum_{r=1}^{3} \sum_{m=1}^{m_k} u_{km}^r \leqslant C,$$

where c *and* C *are constants independent of* k.

5) *The function* u_{km}^r *is constant on each set* $g_{k'm'}^r$ *with the same index* r *when* $k' > k$ *but* m *and* m' *arbitrary.*

6) *The boundary of each level set of the function* u_{km}^r *is connected and divides the plane* R^2 *into three parts at most.*

7) *For every* r, $g_{11}^r \supset E^2$.

The functions u_{km}^r and the sets g_{km}^r with the same index $r = r_0$ will be called functions and sets of the one system r_0, while those with the same index k (and arbitrary r and m) will be said to be functions and sets of the same rank. The index m will be called *number*. Obviously, for any N the totality of functions (sets) of rank not higher than N in each system will be finite.

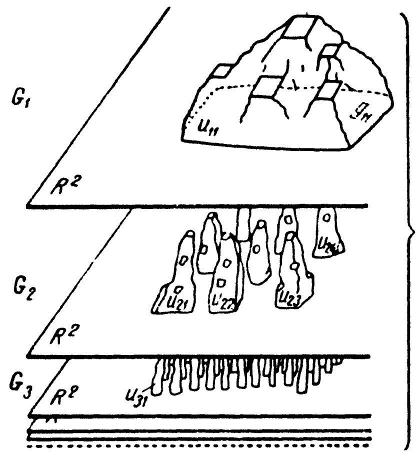


Figure 2. System of mountain countries. All the horizontal planes R^2 are actually in the same plane.

It is known that for every $\varepsilon > 0$, the bounded region $E \supset E^2$ of the plane R^2 can be enclosed (covered) by means of closed squares $P_{\varepsilon m}$, whose sides are parallel to the coordinate axes, in such a way that the set of squares can be divided into three systems $P_{\varepsilon m}^r$, $1 \leqslant r \leqslant 3$, whereby the distance between any two squares of one system will be greater than $\varepsilon / 2$ (Lebesgue covering, Figure 3). These squares are the cells of the regions g_{km}^r.

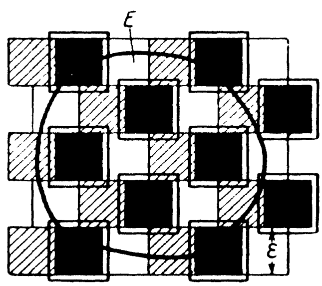


Figure 3. Lebesgue covering. The squares of one system are lined, those of another system are black, those of the third one are white. The functions $Q_{\varepsilon_{km}}^r$ are constructed for the black squares $P_{\varepsilon_{km}}^r$.

All the successive constructions for each r are done independently. During each of the constructions of the functions u_{km}^r, r is kept fixed.

The sets g_{km}^r $(m = 1, \ldots, m_k)$ are obtained from the squares $P^r_{\varepsilon_k m}$, where $\varepsilon_k > 0$. The selection of the number ε_k will be described later. The regions g^r_{km} will be obtained by means of a "dilatation" of the $P^r_{\varepsilon_k m}$ in such a way that $P^r_{\varepsilon_k m} \subset g^r_{km} \subseteq Q^r_{\varepsilon_k m}$, where $Q^r_{\varepsilon_k m}$ is the closure of the square which is an $(\varepsilon_k/6)$-neighborhood of $P^r_{\varepsilon_k m}$: $\overline{U}_{\varepsilon_{k/6}}(P^r_{\varepsilon_k m}) = Q^r_{\varepsilon_k m}$ (see Figure 3).

It is obvious that if $m_1 \neq m_2$, $\rho(Q^r_{\varepsilon_k m_1}, Q^r_{\varepsilon_k m_2}) > \varepsilon_{k/6}$. Therefore $\rho(\overline{g}^r_{km_1}, \overline{g}^r_{km_2}) \geqslant \varepsilon_{k/6}$.

This means that by this construction the requirement 3) of the fundamental lemma will be satisfied.

In order to fulfil the requirement 2), it is obviously necessary that $\varepsilon_k \to 0$ as $k \to \infty$. It will become obvious that this condition will be fulfilled by the construction given below.

This construction is divided into several stages. Everything that is constructed at the nth stage will carry the superscript n together with that of the system r.

In general, all notations are constructed so that $A^{rn}_{\varepsilon_k m}$ should be read as follows: the object A is constructed for the function u (or the set g) of the system r of rank k and number m, i.e. for u^r_{km} (g^r_{km}) at the nth stage. The letters have the following designations:

P is the square cell.

Q is an approximation to g from within.

$\hat{Q}$ is an approximation to g from without.

${}^xO$ is an approximation to the set of the level $u = x$ $(0 < x < 1)$ and to the boundary of the set of the level $u = x$ when $x = 0$ and $x = 1$.

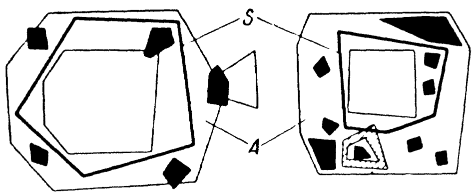
${}^x\Theta$ is an approximation to the boundary of the set of the level $u = x$ $(0 < x < 1)$.

For example, ${}^{x_i}\Theta^{rn}_{\varepsilon_k m}$ denotes the approximation to the boundary of the set of the level $u^r_{km} = x_i$ constructed at the nth stage.

We start the construction of the g^r_{km} at the kth stage, but at the nth stage $(n \geqslant k)$ we construct the $(n - k + 1)$st approximation to g^r_{km} from within and from without: $Q^{rn}_{\varepsilon_k m} \subseteq g^r_{km} \subset \hat{Q}^{rn}_{\varepsilon_k m}$. Hereby $Q^{rn+1}_{\varepsilon_k m} \supseteq Q^{rn}_{\varepsilon_k m}$ and g^r_{km} is determined as $\bigcup_{n=k}^{\infty} Q^{rn}_{\varepsilon_k m}$, i.e. as the sum of the dilated approximations from within.

The functions u^r_{km} are constructed with the aid of their level sets. The construction is begun at the kth stage where one constructs the first approximation ${}^0O^{rk}_{\varepsilon_k m} = \hat{Q}^{rk}_{\varepsilon_k m} \setminus Q^{rk}_{\varepsilon_k m}$ to the set of the zero level. At the next stage one constructs the first approximations to the sets of the levels 1/2 and 1, and to certain other levels, and the second approximations to the zero level. At each stage there appear first approximations to new levels, and one makes successive approximations to the earlier used levels. Each approximation is a closed polygonal band imbedded in the preceding approximation, while the level set itself is the intersection of all its approximations. The values of the function u on each of such level sets are selected so that u^r_{km} is continuous, positive on g^r_{km}, larger than 1/2 on $P^r_{\varepsilon_{km}}$ but does not exceed 1 anywhere. The requirements 1) and 4) of the fundamental lemma will thus be satisfied.

We shall make use of an elementary geometric lemma whose proof will be omitted. It is sufficient to examine Figure 4 to convince oneself of the truth of this lemma.



Case 1) Case 2)

Figure 4. The polygons Q_m are black.
The band B is lined.

Geometric lemma. *Let* A *be a closed polygonal band whose width (i.e. the smallest distance between the boundaries of the polygons) is greater than a positive number* d. *Let the* Q_m $(m = 1,\dots,M)$ *be closed nonintersecting polygons.*

1) *If the diameter of each of the polygons* Q_m *does not exceed* d, *then it is possible to construct a polygon* S *which is strictly inside the band* A, *separates the boundary of the band* A, *and does not intersect the polygons* Q_m $(m = 1,\dots,M)$.

2) *If another closed polygonal band* B *lies strictly inside the band* A, *and if the polygons* Q_m *do not intersect the boundaries of* A *and* B, *then the polygon* S, *which separates the boundaries of* A *and does not intersect* Q_m, *can be drawn strictly within the band* A *so that its intersection with* B *will be an interval (segment).*

We now begin the construction at the first stage.

In order to fulfil the requirement 7) of the fundamental lemma, we set

$\varepsilon_1 = 1$, $m_1 = 1$, $P^r_{11} = E^2$. We construct the squares (Figure 5)

$$Q^r_{11} = \hat{Q}^{r1}_{11} = \overline{U_{\frac{1}{6}}(P^r_{11})} \text{ and } Q^{r1}_{11} = U_{\frac{1}{12}}(P^r_{11}).$$

This is the first approximation to g^r_{11}, for we see that $Q^{r1}_{11} \subseteq g^r_{11} \subseteq \hat{Q}^{r1}_{11}$. $\hat{Q}^{r1}_{11} \setminus Q^{r1}_{11} = {}^0O^{r1}_{11}$ is called the first approximation to the boundary g^r_{11}. This is a closed polygonal band of width 1/12.

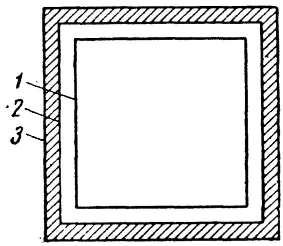


Figure 5. *1* is the boundary of P^r_{11}, *2* is the boundary of Q^{r1}_{11}, *3* is the boundary of $\hat{Q}^{r1}_{11}$. The shaded band ${}^0O^{r1}_{11}$ is the first approximation to the boundary of the region g^r_{11}.

If $\varepsilon_2 < (1/12)\cdot(3/4)$, then the squares $Q^r_{\varepsilon_2 m}$ $(m = 1, \ldots, m_2)$ can be taken for the Q_m in the geometric lemma,* while the first approximation to the boundary of g^r_{11} plays the role of A.

With this selection of ε_2, we start the second stage (Figure 6). For this ε_2 we construct the squares $P^r_{\varepsilon_2 m}$, $Q^r_{\varepsilon_2 m} = \hat{Q}^{r2}_{\varepsilon_2 m} = \overline{U_{\varepsilon_2/6}(P^r_{\varepsilon_2 m})}$; $Q^{r2}_{\varepsilon_2 m} = U_{\varepsilon_{2/12}}(P^r_{\varepsilon_2 m})$ $(m = 1, \ldots, m_2)$. The $Q^{r2}_{\varepsilon_2 m}$ are the first approximations to the regions g^r_{2m}, while the ${}^0O^{r2}_{\varepsilon_2 m} = \hat{Q}^{r2}_{\varepsilon_2 m} \setminus Q^{r2}_{\varepsilon_2 m}$ are the first approximations to their boundaries.

It will be convenient to perform the construction so that the boundaries of the regions g^r_{km} and $g^r_{k'm'}$ do not intersect. It can happen that this requirement is not fulfilled for the first approximation: the band ${}^0O^{r1}_{11}$ may intersect the squares $Q^r_{\varepsilon_2 m}$. However, on the basis of the geometric lemma one can draw a polygon within this band which separates Q^{r1}_{11} from infinity and winds among the squares $Q^r_{\varepsilon_2 m}$ without touching them. This polygon, naturally, can be enclosed in the closed polygonal band ${}^0O^{r2}_{11}$ which will be the second approximation to the boundary of g^r_{11} or to the boundary of the set of the level $u^r_{11} = 0$. (This explains the use of the left 0 superscript.) The band ${}^0O^{r2}_{11}$ determines the second approximation Q^{r2}_{11} to g^r_{11} and can be represented in the form $\hat{Q}^{r2}_{11} \setminus Q^{r2}_{11}$.

At the second stage we construct also the first approximations to certain other level sets of the function u^r_{11}. It is easy to see that, since $\varepsilon_2 < (3/4)\cdot(1/12)$, one can find a square $Q^r_{\varepsilon_2 m^*}$ which will lie entirely within P^r_{11}. It is the first approximation to the set of the level 1 for the function

* The construction of the squares is described after the formulation of the fundamental lemma (see Figure 3). For the region E, which occurs there, one should take Q^r_{11}.

u^r_{11}, while the band ${}^1O^{r2}_{11} = \hat{Q}^{r2}_{\varepsilon_2 m^*} \setminus Q^2_{\varepsilon_2 m^*}$ is the first approximation to the boundary of this set.

Next, in order to satisfy the requirement 4), we construct the set of the level 1/2. The boundaries of P^r_{11} and Q^{r1}_{11} are at a distance of 1/12 from each other, while $\varepsilon_2 < (3/4)(1/12)$. Therefore, applying the geometric lemma to the band between P^r_{11} and Q^{r1}_{11}, we construct a polygon, and then a closed band ${}^{1/2}O^{r2}_{11}$, which winds among the squares $\hat{Q}^{r2}_{\varepsilon_2 m}$ $(m = 1, \ldots, m_2)$ without touching them, lies within Q^{r1}_{11}, and separates P^r_{11} from infinity. The band ${}^{1/2}Q^{r2}_{11}$ becomes the first approximation to the set of the level 1/2 for the function u^r_{11}. The successive approximations ${}^{1/2}\,Q^{rn}_{11}$ $(n > 2)$ are constructed within this band.

Finally, one constructs at the second stage the first approximations to the sets of the levels of the function u^r_{11} that contain g^r_{2m}, and one determines the values u^r_{11} on these sets.

First of all we discard forever those squares $Q^r_{\varepsilon_2 m}$ which were found to lie outside Q^{r2}_{11} (and, hence, outside $\hat{Q}^{r2}_{11}$). The remaining squares $Q^r_{\varepsilon_2 m}$ $(m \in M^{r2}_{11})$ (excluding $Q^r_{\varepsilon_2 m^*}$) *lie in the ring-shaped regions* into which Q^{r2}_{11} is divided by the *finished* bands ${}^0O^{r2}_{11}$, ${}^{1/2}O^{r2}_{11}$ and ${}^1O^{r2}_{11}$. Each ring-shaped region is an open polygonal band which separates $Q^r_{\varepsilon_2 m^*}$, and everything that lies within it, from infinity.

Let us consider any one square $Q^r_{\varepsilon_2 m_0}$ $(m_0 \in M^{r2}_{11},\ m \neq m^*)$. We take the closure of the polygonal band in which the given square lies, for the band A of the geometric lemma; the remaining squares $Q^r_{\varepsilon_2 m}$ $(m \neq m_0)$ we take for the Q_m, and the band ${}^0Q^{r2}_{\varepsilon_2 m} = \hat{Q}^{r2}_{\varepsilon_1 m_0} \setminus Q^{r2}_{\varepsilon_2 m_0}$ for the band B. In accordance with this lemma, we now draw the polygon S, which intersects ${}^0O^{r2}_{\varepsilon_2 m_0}$ in an interval, separates $\overline{Q^r_{\varepsilon_2 m^*}}$ from infinity, lies inside the open polygonal band between the finished bands, and does not touch the squares $\overline{Q^r_{\varepsilon_2 m}}$ $(m \neq m_0)$. This polygon S can be enclosed in a closed polygonal band d, which has the same properties, in such a way that $d \cup Q^r_{\varepsilon_2 m_0}$ is also a closed polygonal band (Figure 6). It is ${}^{x_0}O^{r2}_{11}$, the first approximation to the level set, for the function u^r_{11}, that contains $g^r_{2m_0}$. The value x_0 of the function u^r_{11} on this level set is determined below.

Adding the band ${}^{x_0}O^{r2}_{11}$ to the finished ones, we choose from the M^{r2}_{11} a new $m \neq m_0$, $m \neq m^*$, and construct by the same method an ${}^{x_1}O^{42}_{11}$, and so on, until the set M^{r2}_{11} is exhausted and every square $Q^r_{\varepsilon_2 m}$ $(m \in M^{r2}_{11})$ is enclosed in the first approximation to some level set of the function u^r_{11}. These approximations are polygonal closed nonintersecting bands. The sets ${}^{x_i}\Theta^{r2}_{11} = {}^{x_i}O^{r2}_{11} \setminus O^{r2}_{\varepsilon_2 m_i}$ are called Θ-type closed bands. Each of them divides the plane into three parts: the $Q^{r2}_{\varepsilon_{2m_i}}$, the part that contains ${}^{1}O^{r2}_{11}$, and the part that contains ${}^{0}O^{r2}_{11}$ (Figure 6). They are the first approximations to the boundaries of the level sets of u^r_{11} containing $g^r_{2m_i}$ $(m_i \in M^{12}_{11}\ ;\ m_i \neq m^*)$.

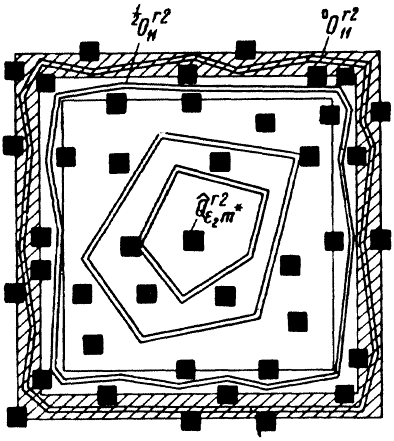


Figure 6. The thin lines are first stage constructions. The black squares are the $Q^r_{\varepsilon_{2m}}$. There should be many more of them but then one would not be able to see anything on the figure. The black square at the center is $Q^r_{\varepsilon_{2m}*}$. Only a few of the bands containing the squares are shown; the second stage is not completed.

Finally, let us determine the values x_i.

Between the boundary of the set g^r_{11} and ${}^{1/2}O^{r2}_{11}$, the function u^r_{11} will increase from 0 to 1/2, while between ${}^{1/2}O^{r2}_{11}$ and ${}^{1}O^{r2}_{11}$, from 1/2 to 1. The bands ${}^{x_i}O^{r2}_{11}$ are divided by ${}^{1/2}O^{r2}_{11}$ into two classes: p_1 outer bands lying outside ${}^{1/2}O^{r2}_{11}$, and p_2 inner ones. Let us reorder them by means of an index $j = j(i)$ in the order determined by their separation from infinity: the outer ones from 1 to p_1, the inner ones from $p_1 + 1$ to $p_1 + p_2$. Let us spread out the increase of u from 0 to 1/2 uniformly among the outer bands, by letting the jth band be an approximation to the set of the level $u^r_{11} = j/2(p_1 + 1)$. For the inner bands of uniform increase from 1/2 to 1, we let the jth band be the approximation to the set of the level $1/2 + (j - p_1)/2(p_2 + 1)$.

Thus we have obtained the following objects at the second stage:

1) The first approximations $Q^{r2}_{\varepsilon_{2m}}$ and ${}^{0}O^{r2}_{\varepsilon_{2m}}$ to the sets g^r_{2m} and their boundaries.

2) The second approximations Q^{r2}_{11} and ${}^{0}O^{r2}_{11}$ to the g^r_{11} and its boundary.

3) The first approximations to the set of the level 1 of the function u^r_{11} and to its boundary, to the set of the level 1/2, and to the sets of the levels of u^r_{11} on which the $g^r_{2m_i}$ $(m_i \neq m^*;\ m \in M^{r2}_{11})$ lie, and also to the boundaries of these sets, $\hat{Q}^{r2}_{\varepsilon_{2m}*}$, ${}^{1}O^{r2}_{11}$, ${}^{1/2}O^{r2}_{11}$, ${}^{x_i}O^{r2}_{11}$, ${}^{x_i}\Theta^{r2}_{11}$.

4) The values x_i of the function u^r_{11} on the $g^r_{2m_i}$, and on the level sets that contain them (not yet constructed).

The approximations to the open sets are open polygons containing the preceding approximations, while the approximations to the closed sets are closed polygons, polygonal bands, and Θ-type bands contained within the preceding approximations.

We note that the construction at the second stage of the functions and sets of rank 2, is exactly the same procedure (if one disregards the scale ε_2) as that used at the first stage for the construction of the functions and sets of rank 1.

In general, after the nth stage we will have:

1) the first stage of the construction of the functions and sets of the nth rank, the second stage of the construction of the functions and sets of the $(n-1)$st rank, and so on up to the $(n-1)$st stage of the construction of u^r_{2m} and g^r_{2m};*

2) the nth approximations Q^{rn}_{11} and ${}^{0}O^{rn}_{11}$ to the set g^r_{11} and its boundary, respectively;

3) the $(n-1)$st approximation to the level sets of the function u^r_{11}, which we began to construct at the second stage, and the $(n-2)$nd approximations to those level sets which we began to construct at the third stage, and so on up to the first approximations ${}^{x_i}O^{rn}_{11}$ and ${}^{x_i}\Theta^{rn}_{11}$ to the level sets of u^r_{11} that contain the $g^r_{nm_i}$, and to the boundaries of these level sets. Here $m_i \in M^{rn}_{11}$, i.e. m_i runs through those values from 1 to m_n for which the corresponding squares $Q^r_{\varepsilon_n m_i}$ do not lie within $Q^{rn}_{\varepsilon_k m'}$ $(1 < k < n;\ m' \leqslant m_k)$, but lie inside Q^{rn}_{11};

4) the values x_i of the function u^r_{11} on g^r_{nm} $(m \in M^{rn}_{11})$.

We have the following results.

1°. The approximations to the open sets are open polygons whose boundaries do not intersect each other (nor, in particular, the small squares $Q^r_{\varepsilon_n m} = \hat{Q}^{rn}_{\varepsilon_n m}$). These approximations contain the preceding ones.

2°. The approximations to the closed sets are closed polygons, closed polygonal, or polygonal Θ-type bands enclosed in the preceding approximations. The polygons that are the boundaries of these approximations do not intersect the other polygons constructed at the nth stage (nor, in particular, the boundaries of the small squares $Q^r_{\varepsilon_n m}$).

3°. Each one of the bands ${}^{x_i}O^{rn}_{11}$, and each of the ${}^{x_i}\Theta^{rn}_{11}$ $(m_i \in M^{rn}_{11})$ contained in it, separates $\hat{Q}^{rn}_{\varepsilon_2 m^*}$ from infinity, while ${}^{x_i}\Theta^{rn}_{11}$, besides that, separates from the rest of $Q^{rn}_{\varepsilon_n m_i} \subset {}^{x_i}O^{rn}_{11}$ the first approximation to the set

* We call attention to the fact that the notation always reflects the number of the stage at which an object is constructed and not the number of the approximation. For example, $Q^{rn}_{\varepsilon_n m}$ is the first approximation to $g^r_{\varepsilon_n m}$.

$g^r_{nm_i}$ which lies on the set of the level $u^r_{11} = x_i$.

4°. The values of u^r_{11} are uniformly distributed on g^r_{nm} ($m \in M^{rn}_{11}$).

The last phrase has the following meaning by definition.

Let the bands A and B be constructed at the nth stage of the approximation to the set of the levels a and b of the function u^r_{11}, where a and b are determined up to the nth stage. Suppose that at the $(n-1)$st stage there was no band (of the approximation to the set of the level u^r_{11}) between A and B, but at the nth stage such bands C_i $(i = 1, \ldots, p)$ were constructed (the numbering of the C_i is from A to B). If the value x_i of the function u^r_{11} on the level set for which C_i is the first approximation is equal to $i(b-a)/(p+1)$ then the values of u^r_{11} on g^r_{nm} are said to be distributed uniformly between A and B. The condition 4) requires that the values of u^r_{11} on g^r_{nm} be so constructed between any two bands A and B of the indicated type.

The $(n+1)$st stage begins with the selection of an ε_{n+1}. Since any two of the polygons that bound the nth stage approximations to all level sets of all the functions u^k_{rm} $(k \leqslant n)$ and to their boundaries do not intersect (provided they are not identical), there exists a positive number d such that the distance between any two distinct polygons is greater than d. We choose ε_{n+1} so that $\varepsilon_{n+1} < 3d/4$. This ε_{n+1} permits us to carry out the first stage of the construction of the sets $g^r_{n+1\,m}$ and of the functions $u^r_{n+1\,m}$, the second stage of the construction of g^r_{nm} and u^r_{nm}, and so on up to the nth stage of the construction of g^r_{2m} and u^r_{2m}.

Since we now assume that we have gone through the stages of rank less than $n+1$ for u^r_{11}, and since they are entirely analogous for the remaining g^r_{km} and u^r_{km} $(k \leqslant n)$, we consider only, as an example, the first stage of the construction of the sets $g^r_{n+1\,m}$ and of the functions $u^r_{n+1\,m}$.

For ε_{n+1} we construct a Lebesgue covering with the squares $P_{\varepsilon_{n+1}m}$ of the nth approximations $\hat{Q}^{rn}_{11}$ to g^r_{11} from without. We divide this covering into three systems $P^r_{\varepsilon_{n+1}m}$, and construct with them the first approximations to g^r_{nm} from within and from without,

$$Q^{r\,n+1}_{\varepsilon_{n+1}m} = U_{\frac{\varepsilon_{n+1}}{12}}(P^r_{\varepsilon_{n+1}m}), \quad Q^r_{\varepsilon_{n+1}m} = \hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m} = \overline{U_{\frac{\varepsilon_{n+1}}{6}}(P^r_{\varepsilon_{n+1}m})}$$

$$(m = 1, \ldots, m_{n+1})$$

and the first approximations to the boundaries of g^r_{nm}

$${}^0O^{r\,n+1}_{\varepsilon_{n+1}m} = \hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m} \setminus Q^{r\,n+1}_{\varepsilon_{n+1}m} \quad (m = 1, \ldots, m_{n+1}).$$

(The squares $Q^r_{\varepsilon_{n+1}m}$ will be called the small squares.)

Since $\varepsilon_{n+1} < 3d/4$, one can now proceed with the second stage of the construction of g^r_{nm} and u^r_{nm}, and so on to the nth stage of the construction g^r_{2m} and u^r_{2m}.

Suppose all this has been done. Then one has to carry out the $(n+1)$st stage of the construction of g^r_{11} and u^r_{11}.

Let us consider any closed band ${}^xO^{rn}_{11}$ which is an approximation to the set of the level x of the function u^r_{11}. If $x = 0$, or $x = 1/2$, then ${}^xO^{rn}_{11}$ will not intersect the sets $\hat{Q}^{rn}_{km}$ $(k \leqslant n)$. It can intersect the squares $Q^r_{\varepsilon_{n+1}m} = \hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m}$, but their diameters are less than d, which is less than or equal to the width of the band. Therefore, applying the geometric lemma, and expanding the polygon S up to the closed polygonal band which winds within the band ${}^xO^{rn}_{11}$ without touching the small squares, we obtain the bands ${}^0O^{r\,n+1}_{11}$ and ${}^{1/2}O^{r\,n+1}_{11}$ that satisfy all the requirements 1° to 4°.

If $x = 1$, then ${}^1O^{r\,n+1}_{11}$ will be ${}^1O^{r\,n+1}_{\varepsilon_{2m^*}}$, a band that already has been constructed, since we assume that the nth stage of the construction of the functions $u^r_{2m^*}$ has been completed.

If $x \neq 0$, $1/2$, or 1, then the band ${}^{x_i}O^{rn}_{11}$ contains the approximation $\hat{Q}^{rn}_{\varepsilon_{km_i}}$ to $g^r_{km_i}$ $(k \leqslant n)$, which was constructed at the nth stage, and this band contains, therefore, also the band ${}^0O^{r\,n+1}_{\varepsilon_{km_i}}$ that has been constructed at the $(n+1)$st stage. Since this band, which contains $O^{rn}_{\varepsilon_{km_i}}$, and is contained in $\hat{Q}^{rn}_{\varepsilon_{km_i}}$, does not intersect the small squares, one can choose it for the band B in the geometric lemma, while for the band A of that lemma, we can take ${}^{x_i}O^{rn}_{11}$. Applying the lemma, we obtain a polygon S which 1) intersects the band ${}^0O^{r\,n+1}_{\varepsilon_{km_i}}$ in an interval, 2) separates ${}^1O^{r\,n+1}_{11}$ from ${}^0O^{r\,n+1}_{11}$, 3) lies inside ${}^{x_i}O^{rn}_{11}$, and 4) winds among the small squares without touching them. Dilating S to the closed polygonal band d, which has the properties 2), 3), and 4) and which is such that ${}^{x_i}O^{r\,n+1}_{11} = d \cup \hat{Q}^{r\,n+1}_{\varepsilon_{km_i}}$ is also a closed polygonal band (that this is possible is obvious), we obtain the following approximation ${}^{x_i}O^{r\,n+1}_{11}$ to the set of the level $u^r_{11} = x_i$.

${}^{x_i}\Theta^{r\,n+1}_{11} = {}^{x_i}O^{r\,n+1}_{11} \setminus Q^{r\,n+1}_{\varepsilon_{km_i}}$ is the next approximation to the boundary of this level set.

Having completed the indicated operation for all the bands ${}^{x_i}O^{rn}_{11}$, we will have the set of all closed polygonal nonintersecting bands ${}^xO^{r\,n+1}_{11}$ that separate ${}^1O^{r\,n+1}_{11}$ from infinity. These bands will be referred to as finished bands.

Let us begin to construct first approximations to the level sets of the function u^r_{11} that contain the sets $g^r_{n+1\,m}$ of rank $n + 1$.

The boundaries of the finished bands ${}^{x_i}O^{r\,n+1}_{11}$ do not intersect the small squares. Let us consider the numbers m that correspond to those small squares that lie in $Q^{r\,n+1}_{11}$, and do not lie in any of the finished bands. The set of all such m, we denote by $M^{r\,n+1}_{11}$. The small squares $\hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m}$ $(m \in M^{r\,n+1}_{11})$ must be included in the first approximations to the level sets. The finished bands divide the $Q^{r\,n+1}_{11}$ into open polygonal bands which contain the small squares. In each of these bands we proceed exactly as it was described in the performance at the second stage. The only difference is that we now have more finished bands. As a result, we obtain the bands ${}^{x_i}O^{r\,n+1}_{11}$ and the Θ-type bands ${}^{x_i}\Theta^{r\,n+1}_{11}$ which are approximations to the level sets and their boundaries. The values x_i in each open band between two finished bands are distributed uniformly.

In this manner one can accomplish the construction by building at each stage objects that have the properties 1°, 2°, 3°, 4°.

Suppose that all stages have been completed.

We define g^r_{km} as $\bigcup\limits_{i = k} Q^{ri}_{km}$.

The level sets of the function u^r_{km} which contain the sets $g^r_{k'm'}$ $(k' > k)$ are defined as the intersections of the corresponding polygonal bands, the approximations. The values of the function on these levels are determined at the k'th stage.

On all regions g^r_{km} the functions u^r_{km} are extended by continuation. Below it is proved that this can be done, and that the obtained functions will satisfy all the requirements of the fundamental lemma.

It is obvious that $\varepsilon_k \to 0$ when $k \to \infty$. Recalling how the squares P were constructed, we see that $\bigcup\limits_{\substack{ij \\ P^r_{k_i m_j} \subset g^r_{km}\ (k_i > k)}} P^r_{k_i m_j}$ is an everywhere dense set on g^r_{km}. Because of this, the sum Σ^r_{km} of all level sets on which we determined u^r_{km} is everywhere dense in g^r_{km}. We shall show that the function u^r_{km} is uniformly continuous on the set Σ^r_{km}.

Without restricting the argument, we will set $k = m = 1$, and will give the proof only for $u^r_{11} = u$.

Let $\varepsilon > 0$ be given. At each $(n + 1)$st stage one can find between any two bands ${}^{x}O^{r\,n+1}_{11}$, ${}^{y}O^{r\,n+1}_{11}$ at least one square $\hat{Q}^r_{\varepsilon_{n+1}m}$, if the construction of the levels $u = x$ and $u = y$ began before the $(n + 1)$st stage. Indeed, the width of the open band O^n between ${}^{x}O^{rn}_{11}$ and ${}^{y}O^{rn}_{11}$ is greater than d, while the squares $P^r_{\varepsilon_{n+1}}$ have diameters less than d and enter into

Lebesgue covering in such a way that one of them $P^r_{\varepsilon_{n+1}\bar{m}}$ has points in O^n. This square, and with it $\hat{Q}^{r\,n+1}_{\varepsilon_{n+1}\bar{m}}$ will, obviously, fall into the open band O^{n+1} between ${}^xO^{r\,n+1}_{11}$ and ${}^yO^{r\,n+1}_{11}$. But at the $(n+1)$st stage $(n > 1)$ the values between the newly constructed band were distributed uniformly. Therefore, the largest interval between the values of u on two level sets, whose approximations are neighboring bands of the nth stage, will decrease by two at each stage. Hence, there exists a stage k such that if ${}^xO^{rk}_{11}$ and ${}^yO^{rk}_{11}$ are neighboring bands, then $|x - y| < \varepsilon/2$.

Let us select $\delta = \varepsilon_{k+1}$. Suppose that $\rho(a,b) < \delta$. Then the points a and b are separated by one band ${}^zO^{rk}_{11}$ only, since the distance between the polygons that bound the bands is greater than $\varepsilon_{k+1} = \delta$. Hence, there exists a band ${}^zO^{rk}_{11}$ which is not separated from a and b by any other band. But it is obvious that at such points the function u differs from z by less than $\varepsilon/2$ (the rank k is chosen in this way). Therefore, $|u(a) - u(b)| < \varepsilon$, and the function u is thus uniformly continuous on the everywhere dense set of the compact $\bar{g}$.

This function can be extended (and in a unique manner) over the set $\bar{g}$.

We set $u = 0$ outside of g. Such a continuation of the functions u^r_{km} will satisfy the requirements 1) to 7) of the fundamental lemma.

Indeed, the fulfillment of the requirements 1), 2), 3), and 7) is obvious.

The condition 4) is satisfied with the constants $c = 1/2$ and $C = 3$, because for any k each point of E^2 is covered by at least one, and by not more than three squares $P^r_{\varepsilon_k m}$ for some m and r. But on these squares $1/2 \leqslant u^r_{km} \leqslant 1$. The level sets $u^r_{km} = 1/2$ were constructed especially for this purpose.

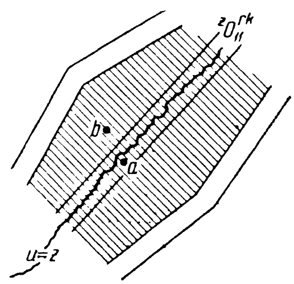


Figure 7. The bands are constructed at the nth stage. In the shaded area u differs from the value on the level $u = z$ (whose approximation is the middle band ${}^zO^{rk}_{11}$) by less than $\varepsilon/2$.

The condition 5) will be fulfilled if $g^r_{k'm'} \subset g^r_{km}$ because

$$g^r_{k'm'} = \bigcup_{i=k'}^{\infty} Q^{ri}_{\varepsilon_{k'}m'} \subseteq \bigcap_{i=k'+1}^{\infty} \hat{Q}^{ri}_{\varepsilon_{k'}m'} \subseteq \bigcap_{i=k'+1}^{\infty} {}^{x_{m'}}O^{ri}_{km},$$

that is, the set $g^r_{k'm'}$ is contained entirely in the level set of u^r_{km}. If $g^r_{k'm'} \subset R^2 \backslash g^r_{km}$, then $u_{k'm'} = 0$ on $g^r_{k'm'}$. The boundaries of g^r_{km} and $g^r_{k'm'}$ do not intersect, by their definition. Each of these sets is a region, and hence there can occur no other cases.

The condition 6) is also satisfied. This is obvious for the sets of the levels 0 and 1. (It is easy to see that each of the remaining level sets of the function u^r_{km} is obtained as the intersection of a sequence of closed polygonal bands, and it is, therefore, connected, and divides the plane into two parts, one containing the set where $u^r_{km} = 0$, the other one where $u^r_{km} = 1$.) The boundaries of the level sets of u^r_{km} that contain $g^r_{k'm'}$, divide the plane into not more than three parts because they are obtained as the intersection of sequences of closed polygonal Θ-type bands. The boundaries of the remaining level sets of u^r_{km} (with the exception of the 0 level set for which 6) is trivial) coincide with exactly these sets because none of such level sets contains points of the open set

$$\bigcup_{\substack{i,j \\ g^r_{k_i m_j} \subset g^r_{km}}} g^r_{k_i m_j},$$

which is everywhere dense in g^r_{km}, and consists of all points of g^r_{km} that belong to the sets of higher rank of the same system.

This completes the proof of the fundamental lemma.

§2. Proof of Theorem 2

Let u^r_{km} be functions that satisfy the conditions of the fundamental lemma, g^r_{km} be sets on which the functions are positive, and let d_k and $0 \leqslant c \leqslant C$ be the constants occurring in that lemma. For the purpose of constructing the representation of a function of three variables in the form indicated in Theorem 2, we first decompose a function of two variables into an absolutely and uniformly convergent series of the functions u^r_{km}.

Lemma 1. *Suppose that we are given on the square E^2 a family F of continuous functions which form a compact in the uniform metric (i.e. the family consists of uniformly bounded and equi-continuous functions u, and is closed with respect to uniform convergence). Then every function $f \in F$ can be represented in the form*

$$f(x) = \sum_{k=1}^{\infty} \sum_{r=1}^{3} \sum_{m=1}^{m_k} a^r_{km}(f)\, u^r_{km}(x), \tag{1}$$

where the coefficients a^r_{km} are independent of x, depend continuously (in the sense of the uniform metric) on the $f \in F$ and are such that

$$|a^r_{km}(f)| \leqslant a_k, \quad \sum_{k=1}^{\infty} a_k < \infty,$$

where the a_k *depend only on the family* F.

For the proof of this theorem we need the following proposition.

Lemma on the approximation by means of a linear combination of functions of rank k. *Let* $f(x)$ *be a continuous real function on* E^2, *and let*

$$\max_{x\in E^2} |f(x)| \leqslant M.$$

Let k *be a positive integer, and*

$$\max_{\rho(x,y)\leqslant d_k} |f(x)-f(y)| \leqslant \delta_k.$$

Then one can determine coefficients b_m^r, *independent of* x, *such that*

$$f(x) = S(x) + R(x), \tag{2}$$

where

$$S(x) = \sum_{r=1}^{3} \sum_{m=1}^{m_k} b_m^r u_{km}^r(x), \tag{3}$$

$$|R(x)| \leqslant \left(1-\frac{c}{C}\right)M + \delta_k. \tag{4}$$

Hereby one can select the b_m^n *so that they depend continuously (in the sense of the uniform metric) on* $f(x)$, *and satisfy the inequality* $|b_m^r| \leqslant M/C$.

Proof. We pick a point x_{km}^r in each one of the sets g_{km}^r, and set $b_m^r = f(x_{km}^r)/C$. Obviously, the b_m^r depend continuously on f and $|b_m^r| \leqslant M/C$. Next we will show that the inequality (4) is fulfilled at each point $x \in E^2$. The $R(x)$ is determined by means of (2) and (3) for the given choice of b_m^r. Let us keep the arbitrary point $x \in E^2$ fixed. From the properties 2) and 3) of the functions u_{km}^r (see the fundamental lemma) it follows that at most three of the functions u_{km}^r, for a given k, will be different from zero at each point x, and these will correspond to different r. Suppose that for the given point x these functions are $u_{km_r}^r$ $(r = 1,2,3)$. Then, for the given point x, we have

$$S(x) = \sum_{r=1}^{3} b_{m_r}^r u_{km_r}^r(x) = \frac{1}{C}\sum_{r=1}^{3} f(x_{km_r}^r)\, u_{km_r}^r(x).$$

Let us suppose at first that $x_{km_r}^r$ $(r = 1,2,3)$ and x were selected so fortunately that they coincided: $x_{km_r}^r = x$ $(r = 1,2,3)$. Then $s(x)$ would be

$$S'(x) = \frac{1}{C}\sum_{r=1}^{3} f(x)\, u_{km_r}^r(x) = \frac{f(x)}{C}\sum_{r=1}^{3} u_{km_r}^r(x) \tag{5}$$

and $R(x)$ would be, correspondingly,

$$R'(x) = f(x) - S'(x). \tag{6}$$

But from the requirement 4) of the fundamental lemma it follows that

$$0 < c \leqslant \sum_{r=1}^{3} u^r_{km_r}(x) \leqslant C.$$

Therefore, we have the following estimate for $R(x)$,

$$|R'(x)| = |f(x) - S'(x)| = |f(x)|\left(1 - \frac{c}{C}\right) \leqslant M\left(1 - \frac{c}{C}\right). \tag{7}$$

The same estimate for $R'(x)$, defined by the equations (5) and (6) holds, obviously, also without the hypothesis that $x^r_{km_r} = x$ $(r = 1, 2, 3)$. In order to appraise $R(x)$ in the general case, we consider

$$|R(x) - R'(x)| = |S(x) - S'(x)| =$$

$$= \frac{1}{C} \left| \sum_{r=1}^{3} [f(x^r_{km_r}) - f(x)] u^r_{km_r}(x) \right| \leqslant \frac{1}{C} \sum_{r=1}^{3} f(x^r_{km_r}) - f(x) | u^r_{km_r}(x).$$

Since (see condition 2) of the fundamental lemma) the diameter of the region g^r_{km} is less than d_k, we have that

$$|R(x) - R'(x)| < \frac{1}{C} \delta_k \sum_{r=1} u^r_{km_r}(x)$$

or, on the basis of property 4) of the fundamental lemma, that

$$|R(x) - R'(x)| < \delta_k.$$

This, in combination with (7), establishes the lemma.

Proof of Lemma 1. Let $f \in F$ be a real function continuous on E^2, and let

$$\sup_{x \in E^2,\, f \in F} |f(x)| \leqslant M = M_0, \qquad \sup_{\substack{x \in E^2,\, y \in E^2,\, f \in F \\ \rho(x, y) < d_k}} |f(x) - f(y)| = \delta_k.$$

As $k \to \infty$, $\delta_k \to 0$. Therefore, one can select a $k_1 = k_1(F)$ so large that $\delta_{k_1} < cM_0/2C$. Applying the lemma on the approximation, with $k = k_1$, and assuming that $a^r_{k_1 m} = b^r_m$, we obtain

$$f(x) = \sum_{r=1}^{3} \sum_{m=1}^{m_{k_1}} a^r_{k_1 m}(f) u^r_{k_1 m}(x) + R_1(x);$$

moreover

$$\sup_{x\in E^2,\, f\in F} |R_1(x)| \leqslant M_0\left(1-\frac{c}{C}\right)+\delta_{k_1} < M_0\left(1-\frac{c}{2C}\right),$$

where the $a^r_{k_1m}$ depend continuously on $f \in F$, and

$$|a^r_{k_1m}| < \frac{M_0}{C} = \frac{M}{C}.$$

Setting $1 - c/2C = \theta$, and $\theta M_0 = M_1$, we obtain

$$\sup_{x\in E^2,\, f\in F} |R_1(x)| < M_1.$$

It is obvious that the $R_1(x)$ that correspond to all possible $f \in F$, form a compact F_1, as a continuous image of a compact. In particular, these $R_1(x)$ are uniformly bounded and equi-continuous. Furthermore, each function $R \in F_1$ depends continuously on the corresponding function $f \in F_0$. Let us introduce the notation

$$\sup_{\substack{x\in E^2,\, y\in E^2,\, R_1\in F_1\\ \rho(x,\, y)\leqslant d_k}} |R_1(x)-R_1(y)| = \delta'_k.$$

We can repeat the preceding argument, and in conclusion obtain a $k_2 = k_2(F)$ such that

$$R_1(x) = \sum_{r=1}^{3}\sum_{m=1}^{m_{k_2}} a^r_{k_2m}(R_1)\, u^r_{k_2m}(x) + R_2(x),$$

where

$$\sup_{x\in E^2,\, f\in F} |R_2(x)| < \theta M_1 = \theta^2 M$$

and the $a^r_{k_2m}$ depend continuously on $R_1 \in F_1$, and, hence, on $f \in F_0$. Furthermore,

$$|a^r_{k_2m}| < \frac{M_1}{C} = \frac{M}{C}\,\theta.$$

Continuing in the same way, we obtain the sequence

$$R_n(x) = \sum_{r=1}^{3}\sum_{m=1}^{m_{k_{n+1}}} a^r_{k_{n+1}m}(R_n)\, u^r_{k_{n+1}m}(x) + R_{n+1}(x);$$

moreover

$$\sup_{x\in E^2,\, f\in F} |R_{n+1}(x)| < \theta M_n = \theta^{n+1}M, \tag{8}$$

where the $a^r_{k_n m}$ depend continuously on $f \in F$, and

$$|a^r_{k_{n+1}m}| < \frac{M_n}{C} = \frac{M}{C}\,\theta^n \tag{9}$$

($n = 0, 1, 2, \ldots,$ if we use the notation $R_0(x) = f(x)$).

Let us introduce the notation

$$S_n(x) = \sum_{i=0}^{n-1} (R_i(x) - R_{i+1}(x)) = \sum_{i=0}^{n-1} \sum_{r=1}^{3} \sum_{m=1}^{m_{k_{i+1}}} a^r_{k_{i+1}m} u^r_{k_{i+1}m}(x). \tag{10}$$

Then it is obvious that

$$f(x) = S_n(x) + R_n(x).$$

From the inequalities (8) and (9) it can be seen that the sequence $S_n(x)$ ($n = 1, 2, \ldots$) converges to $f(x)$ absolutely and uniformly, and that $|a^r_{k_i m}| < a_{k_i} = M\theta^{i-1}/C$ $(i = 1, 2, \ldots)$.

This proves the lemma, since one may set $a^r_{km} = 0$ when $k \neq k_i$ $(i = 1, 2, \ldots)$ and then obtain (1) from (10).

In the proof of Theorem 2 use is made of the following result.

Lemma 2. *The space of the components of the level sets of the function*

$$F_r(x, y) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{m_k} u^r_{km}(x, y)$$

is a tree with a branch point index not greater than three.

Every function

$$f_r(x, y) = \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} a^r_{km} u^r_{km} \tag{*}$$

is constant on each component of a level set of the function F_r if the a^r_{km} are such constants that the series (*) converges uniformly and absolutely.

Proof of Lemma 2. Let r be fixed. First, let us prove that all the components of the level sets of the function $F_r(x,y)$ are 1) components of the level sets of the function u^r_{km} $(k = 1, 2, \ldots;\ m = 1, \ldots, m_k)$, 2) boundaries of such components, 3) separate points which are intersections of sequences of the sets $g^r_{km'_k}$ $(k \to \infty)$.

Let us pick a point a in the plane. The point a belongs either to an infinite number of the sets g^r_{km}, or there exists a "last rank" $k_0 \geqslant 0$ after which the point a does not belong to any g^r_{km} $(k > k_0)$.

Let us consider the first case. We will prove that such a point is a component of the level sets of the function F_r. Suppose that the point a belongs to an infinite sequence $\{g^r_{k_i m_i}\}$. From the condition 3) of the fundamental lemma it follows that the k_i are all distinct. We shall assume that $k_{i+1} > k_i$. One can easily deduce from the fundamental lemma (requirements 2) and 5)), that if the sets g^r_{km} and $g^r_{k'm'}$ intersect, and if $k' > k$, then $g^r_{k'm'} \subset g^r_{km}$. In the proof of the fundamental lemma given above, this result is obtained automatically (see the proof of the fact that the requirement 5) is fulfilled). Therefore, we have a sequence of inscribed sets $g^r_{k_i m_i} \supset g^r_{k_{i+1} m_{i+1}} \supset \ldots \ni a$. In this connection, $\bigcap_{i=1}^{\infty} g^r_{k_i m_i} = a$, since the diameters of the sets $g^r_{k_i m_i}$ tend to zero as $i \to \infty$ (requirement 2) of the fundamental lemma).

On the boundary M_i of each set $g^r_{k_i m_i}$, the value of the function F_r is less than that at the point a. Indeed, all the functions u^r_{km} $(k \geqslant k_i)$ are zero on M_i (this is a direct consequence of the requirements 2) and 5) of the fundamental lemma), while all the functions u^r_{km} $(k < k_i)$ take on the same values as at the point a (requirement 5)). At the point a, however, all the functions $u^r_{k_j m_j}$ $(j \geqslant i)$ are positive, and, therefore, $F_r(a) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{m_k} u^r_{km}(a)$ is greater than F_r on M_i.

But each continuum that contains a, intersects some set of the M_i because $\bigcup_{i=1}^{\infty} M_i$ separates a from all the points of $R^2 \setminus a$ (Figure 8). This means that on each continuum that contains a one can find a point b where $F_r(b) \neq F_r(a)$, but this indicates precisely that a is a component of the level sets of the function F_r.

Now, let us consider the second case. Suppose the point $a \in g^r_{k_0 m_0}$ does not belong to any g^r_{km} $(k > k_0)$. Then a will belong to a continuum K, the set of a nonzero level z of the function $u^r_{k_0 m_0}$.

Let us assume at first that K does not contain the regions g^r_{km} $(k > k_0)$. Then $0 < z < 1$. We will prove that K is a component of a level set of the function F_r.

Let us select two sequences z^+_i and z^-_i $(i = 1, 2, \ldots)$ which converge to z from above and from below, and which are such that the sets M^-_i and M^+_i of the levels z^-_i and z^+_i $(i = 1, 2, \ldots)$ of the function $u^r_{k_0 m_0}$ do not contain the regions g^r_{km} and $0 < z^-_i < z < z^+_i < 1$. This can be done because

$0 < z < 1$ and the regions g^r_{km} constitute a denumerable set. The continua M^+_i and M^-_i, obviously, separate K from the points where $u^r_{k_0m_0}$ is greater than z^+_i and less than z^-_i, and all of them together, i.e. $\bigcup_{i=1}^{\infty} (M^+_i \cup M^-_i)$, separate K entirely from all points of $R^2\setminus K$, since at every point of $R^2\setminus K$, $u^r_{k_0m_0}$ is greater than some z^+_i or less than some z^-_i. On K, as well as on the sets M^+_i and M^-_i, the function F_r does not change since all the terms with u^r_{km} $(k > k_0)$ are zero in view of the assumption on the absence on K, M^+_i and M^-_i of the sets g^r_{km}. But the values of F_r on K, on M^+_i and on M^-_i are different, because all terms u^r_{km} with $k < k_0$ are the same on these continua (requirement 5)), all the terms u^r_{km} with $k > k_0$ are equal to zero, while the function $u^r_{k_0m_0}$ is equal to z on K, to z^+_i on M^+_i, and to z^-_i on M^-_i.

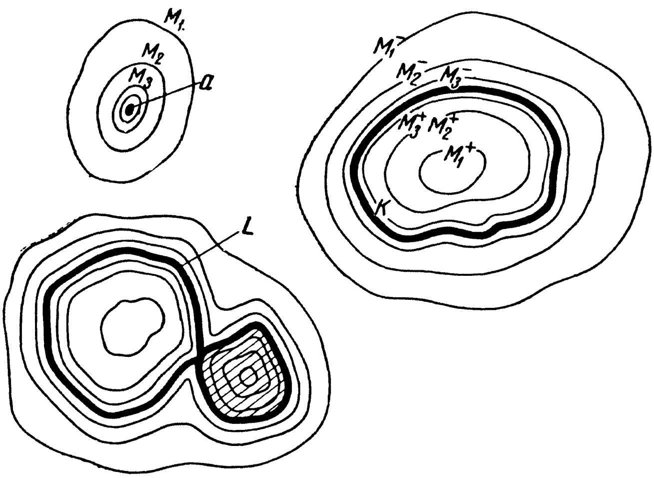


Figure 8. Representation of all types of components. In the third case $z \neq 1$. The $g^r_{k'm'}$ are lined. The case $z = 1$ is left to the reader.

Each continuum $M \neq K$, but containing K, intersects $\bigcup_{i=1}^{\infty} (M^+_i \cup M^-_i)$ (Figure 8). Therefore it has points where F_r differs from the values of F_r at the points of K. This means that K is a component of the set of levels of the function F_r.

In the remaining case the proof is analogous to the one given above, and we will only indicate it. If the set $K \ni a$ of the level $u^r_{k_0m_0} = z$ contains $g^r_{k'm'}$, then the component of the level set of the function F_r that contains a will be L, the boundary of K (Fig. 8). Actually, the point a does not belong to $g^r_{k'm'}$ (since k_0 is the "last rank"). The boundary of K divides the plane into no more than three parts (requirement 6)). First, suppose that $z \neq 1$. Then in two of these parts $u^r_{k_0m_0}$ will take on values greater and less than z, while in the third part $g^r_{k'm'}$, $u^r_{k'm'}$ is positive. The point a cannot lie in any of these parts but lies on the boundary of K. On the continuum L, the function F_r is constant, because all the functions u^r_{km} are constant (requirements 5) and 6)). In order to prove that L is a component of the level set of the function F_r, it is necessary to separate it by means of continua, with values of F_r, from all points of $R^2\setminus L$. For this it is necessary to use sets of levels near zero of the function $u^r_{k'm'}$ and sets of levels close to z of the function $u^r_{k_0m_0}$ (Figure 8).

The remaining case, $z = 1$, is even simpler because the boundary of the set $u^r_{k_0m_0} = 1$ divides the plane into two parts only (this is a direct consequence of the construction of the functions u^r_{km}, but it can also be deduced from requirements 2) and 6) of the fundamental lemma).

The structure of the components of a level set for the function F_r has thus been explained. Not a single one of them divides the plane into more than three parts. It follows (Appendix, Theorem 3) that the tree of the function F_r consists of points whose branching index does not exceed 3.

In order to complete the proof of Lemma 2, we note that all the functions u^r_{km} are constant on each component of the level sets of F_r. This implies the truth of the second assertion of the lemma.

Theorem 2. *Every real function $f(x_1, x_2, x_3)$ that is continuous on E^3 can be represented in the form*

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} h_i[\varphi_i(x_1, x_2), x_3],$$

where h_i and φ_i are continuous functions, the functions h_i are defined on the product $\Xi \times E^1$ of the tree by the interval E^1, while the $\varphi_i(x_1, x_2)$ are defined on the square E^2, and have for their values points of Ξ. Here Ξ is a tree whose points have branching indices not greater than 3.

Proof. A function $f(x_1, x_2, x_3)$ of three variables can be considered as a family of functions of two variables that depends on the third variable as a parameter: $f_{x_3}(x_1, x_2)$, where the function $f_{x_3}(x_1, x_2)$ is defined for each x_3 on a single square $0 \leqslant x_1, x_2 \leqslant 1$, and at a point (a, b) is equal to $f(a, b, x_3)$. Obviously, each of the functions $f_{x_3}(x_1, x_2)$ is continuous and depends continuously (in the sense of the uniform metric) on the parameter x_3 $(0 \leqslant x_3 \leqslant 1)$. Therefore, the family of functions $f_{x_3}(x_1, x_2)$ forms a compact. Hence, we can apply the Lemma 1 and obtain

$$f_{x_3}(x_1, x_2) = \sum_{k=1}^{\infty} \sum_{r=1}^{3} \sum_{m=1}^{m_k} a_{km}^r(x_3)\, u_{km}^r(x_1, x_2).$$

Since $|a_{km}^r(x_3)| \leqslant a_k$, and $\sum_{k=1}^{\infty} a_k < \infty$, it follows that each of the series

$$f_{x_3}^r(x_1, x_2) = \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} a_{km}^r(x_3)\, u_{km}^r(x_1, x_2) \quad (r = 1, 2, 3)$$

converges absolutely and uniformly. (But by the fundamental lemma only one of the u_{km}^r $(m = 1, \dots, m_k)$ is different from zero at any given point.) We shall show that $f_{x_3}^r(x_1, x_2)$ depends on x_3 continuously (in the same sense).

Indeed, suppose $\varepsilon > 0$. We can select N so large that $\sum_{k=N}^{\infty} a_k < \varepsilon/4$. Since the $a_{km}^r(x_3) u_{km}^r(x_1, x_2)$ depend continuously on x_3, the same thing must be true for the finite sum. Hence there exists a $\delta > 0$ such that if $|y - z| < \delta$ then

$$\sup_{x_1, x_2 \in E^2} \left| \sum_{k=1}^{N-1} \sum_{m=1}^{m_k} a_{km}^r(y)\, u_{km}^r(x_1, x_2) - \sum_{k=1}^{N-1} \sum_{m=1}^{m_k} a_{km}^r(z)\, u_{km}^r(x_1, x_2) \right| < \frac{\varepsilon}{4}$$

$$(r=1, 2, 3).$$

But since

$$\sup_{x_1, x_2 \in E^2} \left| \sum_{k=N}^{\infty} \sum_{m=1}^{m_k} a_{km}^r(y)\, u_{km}^r(x_1, x_2) - \sum_{k=N}^{\infty} \sum_{m=1}^{m_k} a_{km}^r(z)\, u_{km}^r(x_1, x_2) \right| \leqslant 2 \sum_{k=N}^{\infty} a_k < \frac{\varepsilon}{2},$$

we find that for $|y - z| < \delta$, it is true that

$$\sup_{x_1, x_2 \in E^2} |f_y^r(x_1, x_2) - f_z^r(x_1, x_2)| < \varepsilon \quad (r = 1, 2, 3).$$

Now we apply Lemma 2 and see that for any given x_3, each of the functions $x_3 \in [0,1]$ is constant on each component of the level set of one of the constructed functions $F_r(x_1, x_2) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{m_k} u_{km}^r(x_1, x_2)$ which does not depend on $f(x_1, x_2, x_3)$.

Let us consider (see Appendix) the tree of components of the level sets of the function $F_r(x_1, x_2)$. The mapping $t(a) = \varphi_r(x_1, x_2)$ associates with each point x of the square $E^2 = A$ a point φ_r of the tree T^r which represents the component t of the level set of $F_r(x_1, x_2)$ that contains (x_1, x_2). We can consider this mapping as a function $\varphi_r(x_1, x_2)$ defined on the square and with values from the tree. If one wishes, one can realize the tree on a plane. This mapping can then be written with the aid of two real functions defined on the square. The mapping $t(a)$ is continuous. The functions $f^r_{x_3}(x_1, x_2)$ generate on T^r functions $f^r_{x_0}(\varphi_r)$ which are equal to the values of $f^r_{x_3}(x_1, x_2)$ at any point of the component t of the counterimage of φ_r on E^2. Because of Lemma 2, this value is the same at all points of this component. It is obvious that the obtained functions $f^r_{x_3}(\varphi_r)$ are continuous on T^r and depend continuously on x_3. Therefore, one may consider the family $f^r_{x_3}(\varphi_r)$ $(x_3 \in [0, 1])$ as a continuous real function $f^r(x_3, \varphi_r)$ on the product of the tree by the interval of variation of x_3:

$$f^r_{x_3}(x_1, x_2) = f^r(x_3, \varphi_r(x_1, x_2)).$$

From the three trees T^r $(r = 1, 2, 3)$ we can compose a single tree Ξ. By Lemma 2, each of the three trees consists of points whose branching indices are 1, 2 or 3. The tree Ξ, obviously, can be constructed so that it has the same property. Each of the functions $f^r(x_3, \varphi_r)$ $(r = 1, 2, 3)$ can be extended continuously over the product of the entire tree Ξ by the interval (it does not matter in what way this is done). Let us denote this extension by $h_r(\varphi_r, x_3)$ $(r = 1, 2, 3)$. From the relation (1), Lemma 1, we obtain in this notation

$$f(x_1, x_2, x_3) = \sum_{r=1}^{3} h_r[\varphi_r(x_1, x_2), x_3].$$

This completes the proof of Theorem 1.

PART II

Proof of Theorem 3

We shall now construct the tree $X \subset E^3$ mentioned in Theorem 3. This tree is to be homeomorphic to the universal tree Ξ which does not have points whose branching index is greater than 3. The latter tree, as is well known (see Appendix, Theorem 5), can be obtained by attaching branches. More precisely, Ξ can be represented in the form

$$\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n}, \quad \Delta_n \subset \Delta_{n+1},$$

where Δ_n is a finite tree (curved complexes), Δ_1 is a simple arc and Δ_{n+1} is obtained from Δ_n by attaching at the point ρ_n (which is not a branch point) simple arcs σ_n (Figure 9). We note that the set of points ρ_n that are now branch points of Ξ, is at most denumerable. Everything that pertains to the abstract tree Ξ will be denoted by Greek letters, while the corresponding items of its realization X will de designated by Latin letters. The realization of X will be constructed in the form

$$X = \overline{\bigcup_{n=1}^{\infty} D_n}, \quad D_n \subset D_{n+1},$$

where the D_n are segment complexes in a three-dimensional space; the homeomorphism between Ξ and X will be constructed as a continuation of the homeomorphisms Δ_n and D_n.

It is known that in order for X to be a realization of Ξ (and, hence, to be a tree), it is sufficient that the following conditions be fulfilled (see Appendix, Lemmas 10 and 11):

α) Each newly constructed branch s_n, except for its base, must lie entirely inside the open, still empty, simplex T_n. Furthermore, for all twigs s_m attached to s_n $(p_m \in s_n)$ later $(m > n)$ $\overline{T_m} \subset T_n$ (Figure 9).

On Figure 9, and in Menger's work ([3], Chapter X), where the tree X lies in a plane, the simplexes T are triangles. In our case they are tetrahedra. This makes no essential difference.

β) The simplexes T_n must be sufficiently small: the diameters of the T_n tend to zero when $n \to \infty$.

γ) The points p_n at which the new branches are attached may not have been earlier branch points or endpoints for D_n.

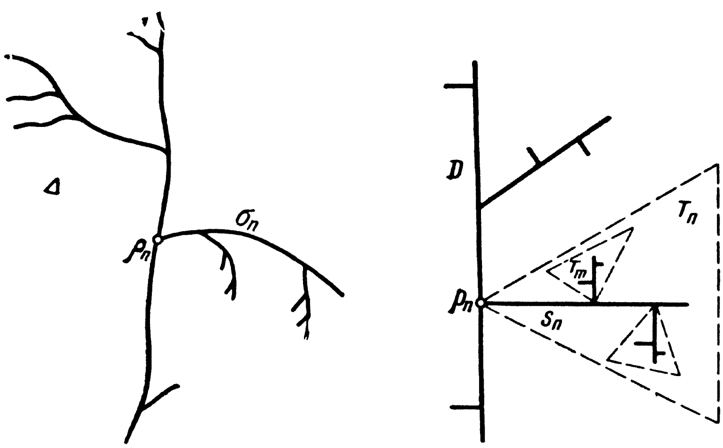


Figure 9. Finite trees: "abstract", curved tree Δ, and its realization as a complex D.

In the sequel (§§ 3-7), in the construction of D_n and X, we will always be able to choose the points p_n and the direction of the segments s_n with sufficient freedom: each time the forbidden points or directions will have an everywhere dense complement. The length of the s_n will always be chosen sufficiently small. The conditions α), β), and γ) can, therefore, be assumed to have been satisfied at each step. In order not to complicate the future presentation, we will not mention this in the sequel. We assume that by attaching each branch s_n we construct the corresponding tetrahedron T_n, and will not worry about the fulfillment of the conditions α), β), and γ).

In order that the obtained tree X may satisfy the conditions of Theorem 3, i.e. in order that each continuous function of the given family may be represented as the sum of functions of coordinates, it is necessary to select the p_n and s_n with certain restrictions. For the precise formulation of these restrictions, we need several new concepts which are presented in the next section.

§3. Fundamental definitions. Inductive properties 1-4

In a three-dimensional space* let K be a finite set of segments or straight lines. These segments (straight lines) are not to be parallel to the coordinate planes.

Definition 1 (Figure 10). A *zigzag* (certain type of broken line) is a system of points $a_0 \neq a_1 \neq \ldots \neq a_{n-1} \neq a_n$ of K, such that the segments $a_{i-1}a_i$ $(i = 1, \ldots, n-1)$ are perpendicular to the coordinate axes x_{α_i} and $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \ldots \neq \alpha_n$. The segments $a_{i-1}a_i$ are called *links* of the zigzag. If $a_0 = a_n$, the zigzag is said to be closed.

One should visualize the zigzag in the following way. The beginning a_0 is a point of K. We choose the first direction α_1. The plane that passes through a_0 and is perpendicular to the axis x_{α_1} (we shall refer to it as the " plane of the coordinate direction α_1") intersects K at a point a_1. We shall say that it leads from a_0 to a_1. In exactly the same way the link a_1a_2 lies in the plane of the direction α_2 $(\neq \alpha_1)$ so that at a_1 there occurs a break. At the point a_2, the direction again changes to α_3 $(\neq \alpha_2)$ and we arrive at the point a_3, and so on until we get to a_n, the end of the zigzag.

By somewhat modifying the described process we obtain the generating

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* In § 3-7 the number of dimensions could be $\geqslant 2$. The graphs correspond to the two-dimensional case.

scheme that was defined in note [1]. A more descriptive definition will be given here.

Definition 2 (Figure 11). The beginning of the generating scheme is the point $a_0 \in K$.

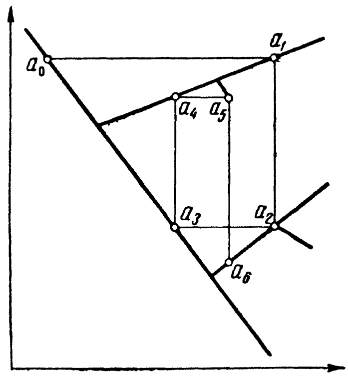


Figure 10. The zigzag $a_0a_1a_2a_3a_4a_5a_6$.

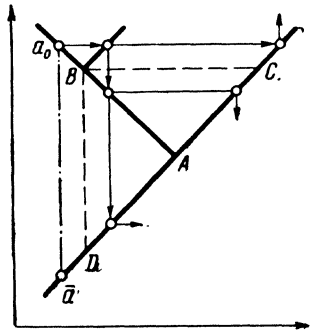


Figure 11. A generating scheme from the point a_0. If one includes the point $\bar{a}$, then the obtained double generating scheme will be of the class of the point a_0.

The beginning is also called the end of rank 0. We choose a coordinate direction α_0 and draw through the point a_0 the plane of this direction. In general, it will have several points of intersection with K in addition to the point a_0. We shall call this plane a plane of rank 1, and these points, ends of rank 1. The plane of rank 1 *leads* from a_0 into each of the ends of rank 1.

Next, this process is continued. At each end a of rank n we select a coordinate direction α different from the one along which we arrived at this end.* Through a we pass the plane of this direction. If this plane does not pass through any other point of K besides a, we do no more to this point a; it is called a free end. If, however, a is not a free end, then we obtain points of intersection of the plane with K, which are called ends of rank $n+1$. In this manner the constructed plane of rank $n+1$ leads away from the nonfree end of rank n and leads to ends of rank $n+1$.

If this process terminates, i.e. if all the ends of some rank N are free ends, and if all ends of all ranks as pairwise distinct,** then the entire

* That is, different from the direction of the plane of rank n at whose intersection with K the point a lies.

** This means that in the construction we do not arrive at the same point twice.

structure is called a generating scheme which leads from the point a_0 in the direction α_0. N is called the rank of the scheme.

In this manner, a *generating scheme* (or system) consists of a beginning a_0, of "supporting" planes of different ranks, and of ends of different ranks.

We will need a certain generalization of generating systems, a double generating system. It differs from the simple one defined above only in that from some of its points (ends) one draws two planes, and not just one. In this way, all three directional coordinate planes that pass through such a point, can be supporting planes in a double scheme if one of these planes leads into, and the other two away from the point. Double systems can be obtained, for example, by combining simple ones which have only the beginning in common, or by connecting to some nonfree end a of a simple scheme A a generating system B, for which a is the beginning, and which has no common points with A except a.

Every free end a of a double (or simple) generating system can be connected to the beginning a_0 by a unique zigzag all of whose points are ends of a scheme, and all of whose links lie in the supporting planes of the scheme. If there were several such zigzags, then the ends of the scheme could not be pairwise distinct. The indicated zigzags are called zigzags of the scheme. They are finite, not closed, and do not contain closed parts.

Definitions of stability. We shall say that two zigzags (generating schemes) on K are of the same type if their points can be put into a one-to-one reciprocal correspondence in such a way that corresponding points lie on the same segments (straight lines) of K,* while the corresponding links are perpendicular to the same coordinate axis.

We shall say that the zigzag $a_0 \dots a_n$ is not longer than the zigzag $b_0 \dots b_m$ $(m \geqslant n)$ if it is of the same type as a part $b_0 \dots b_n$ of the second one.

A generating scheme A is not longer than a generating scheme B if one can set up a correspondence between their zigzags under which all zigzags of A are not longer than the corresponding zigzags of B. The types of the generating schemes which are not longer than a given one form a finite set.

We shall say that a generating scheme A that begins at a_0 is stable if a_0 has such a neighborhood that the generating schemes of the points of K that lie in this neighborhood are not longer than A. For example, the complex K of Figure 11 admits a generating scheme, beginning at any point

* No branch points can lie within a segment of a segment-like complex K. The complex of Figure 11 consists of 5 segments. This remark does not apply to the set of straight lines of K.

with an arbitrary first direction. Here, for any point, except for the branch points A, B, and the end points C, D of the zigzag that issue from B, the scheme is stable.

The zigzags of the same type produce a mapping of the set of all their beginnings (initial points) into the set of their ends. This mapping is linear and nondegenerate (because the segments of K are not perpendicular to the coordinate axes). We will make frequent use of these facts in what follows.

The set of all points of K which are vertices of zigzags that issue from the point a_0 are called the class of points accessible from a_0, or simply the class of the point a_0 on K. The class of a set of points is defined in an analogous way. We call attention to the fact that the class of a point, and hence the class of a denumerable set is a denumerable set. All generating schemes of a point a_0, and of points belonging to the class of a_0 lie in the same class.

Now we can formulate the inductive lemma which will be proved in §§4-9.

Let us return to the function f on the tree Ξ.

Suppose that ω_n is the upper boundary of the variation of the functions $f \in F$ on the component difference $\Xi \setminus \Delta_n$. As $n \to \infty$, $\omega_n \to 0$.

Indeed, if Ξ' is a realization of Ξ constructed (see Appendix, Theorem 5) on the plane, then F will give rise to a family F' of equi-continuous functions defined on the planar continuum Ξ'. Since the diameter (see condition β, and Figure 9) of the triangles T_n tends to zero when $n \to \infty$, and since every component $\Xi' \setminus \Delta'_n$ lies in the triangle T_m $(m > n)$, it follows that for large enough n the diameter of the component $\Xi' \setminus \Delta'_n$ will be so small that the oscillation of any function $f' \in F'$ will be arbitrarily small on every component. Therefore one can pick a sequence

$$n_1 < n_2 < \ldots < n_r < \ldots,$$

so that $\omega_n \leqslant 1/r^2$ when $n > n_r$.

We shall next list the *inductive properties* of the tree D_n, of the homeomorphism of Δ_n on D, and of the functions $f_k^m(x_k)$ $(m \leqslant n;\ k = 1, 2, 3)$. Here the tree D is a realization of Δ_n. D lies in a three-dimensional cube of a segment-like complex whose segments are not perpendicular to the coordinate axes.

1. Let A be the set of points of D_n which are images of the branch points* of Ξ that lie on Δ_n. Let K_n be the set of straight lines whose segments form D_n, and let C_n be the class of the set of vertices of the

* More precisely, one should say of the "points ρ_m" because Theorem 4 is not proved in the Appendix. In the sequel, branch points can be taken as the points ρ_m and p_m.

closed zigzags on K_n.

Then:

a) C_n is at most denumerable,

b) C_n does not intersect A_n (and hence not the class A_n on K_n),

c) no two points of A_n belong to the same class on K_n.

2. On D_n there is a finite number of simple generating schemes such that from any point $a_0 \in D_n$ one can start in any direction to generate the scheme of one of the " canonic " type.

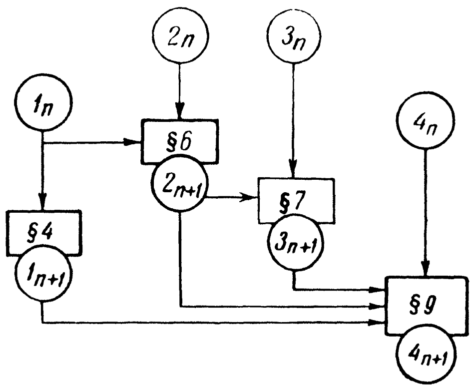
3. Every function $f \in F$ is representable on D_n in the form $f(x) = \sum_{k=1}^{3} f_k^n(x_k)$, where the x_k are the coordinates of the point $x \in D_n$, and the $f_k^n(x_k)$ are continuous functions which depend continuously on $f(x)$.

4. If $n_r < n \leqslant n_{r+1}$, then

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| < \left(3 + \frac{n - n_r}{n_{r+1} - n_r}\right)\frac{1}{r^2}.$$

Inductive lemma. *If the tree D_n, the homeomorphism of Δ_n on D_n, and the functions $f_k^m(x_k)$ $(k = 1,2,3;\ m \leqslant n)$ have the properties 1 to 4, then one can construct a tree D_{n+1}, a mapping of Δ_{n+1} on D_{n+1}, and functions $f_k^{n+1}(x_n)$, with the same properties, by attaching to the point p_n a branch-segment s_n that is not perpendicular to the coordinate axes.*

Scheme of proof:



i_n indicates the property i of the tree D_n, of the homeomorphism Δ_n onto D_n, or of the function $f_k^n(x_k)$. In the section that appears in any rectangle, the property i_{n+1} is derived from the properties that are connected with this section by means of arrows.

§4. Inductive preservation of property 1

We will assume that D_n has the property 1, and we will show what conditions have to be imposed on s_n in order that this property may be preserved on D_{n+1}. The conditions that one finds are not very restrictive: the direction may be chosen from an everywhere dense set of the second category;* the length can be arbitrarily small.

Let us now assume that on K_n, that is on D_n, to which there have been added rays which extend the segments D_n, the following conditions hold:

a) the class C_n of the points of closed zigzags is at most denumerable;

b) the points of closed zigzags of K_n are not accessible on K_n from the points of A_n which are the images on D_n of the branch points of Ξ;

c) no two points of A_n are such that one is accessible from the other on K_n.

Let us first restrict the selection of the direction of s_n in such a way that the condition a) is guaranteed on D_{n+1}. The number of the types of zigzags is at most denumerable for every choice of s_n, because the type is determined by the initial and successive straight lines of K_n and by the direction of the path, i.e. by a finite sequence of elements of a finite set. For each type there either is no closed zigzag, or there is one, or else all zigzags of the given type are closed. This follows from the linearity of the corresponding type of mapping of the initial straight line onto a finite one. In case that all zigzags of a type are closed, we say that a closed zigzag is stable. Obviously, it is sufficient that there be no closed zigzags on K_{n+1} in order that condition a) be satisfied on D_{n+1}.

Suppose that D_{n+1} has been constructed, and that the segment s_n is not perpendicular to the axes. The stable closed zigzags can occur only among zigzags which have a common point with the straight line l that supports s_n.

Let M be such a point. It can be taken for the beginning of a closed zigzag. Suppose that the equations of the straight line l in the system of coordinates in which the origin 0 has been translated to p_n are given as

$$x_2 = bx_1, \quad x_3 = cx_1,$$

where neither b nor c are zero, because the segment s_n is not parallel to the coordinate planes. For the sake of definiteness, let us assume that a closed zigzag issues from the point $M(x_0, bx_0, cx_0)$ in the direction x_1 and falls on l for the first time again exactly at the point M where it arrives from the direction x_2. Let the straight line at which the zigzag arrives at

* That is, from the complement of the sum of a denumerable number of nowhere dense sets.

the ith step, have the equations $x_2 = b_i x_1 + \beta_i$, $x_3 = c_i x_i + \gamma_i$. Neither one of the coefficients b_i, and c_i can be zero. The second point of the zigzag has the coordinates $x_0, b_1 x_0 + \beta_1$, $c_1 x_0 + \gamma_1$. If the second direction is, for example, x_2, then the coordinates of the third point will be

$$\frac{b_1 x_0 + \beta_1 - \beta_2}{b_2}, \quad b_1 x_0 + \beta_1, \quad c_2 \frac{b_1 x_0 + \beta_1 - \beta_2}{b_2} + \gamma_2.$$

In general the coordinates of each point depend linearly on x_0, and the coefficients are determined by the intermediate straight lines. We assume that the zigzag does not intersect l before it is closed. Then the last point will have the coordinates

$$l_1 x_0 + \lambda_1, \quad l_2 x_0 + \lambda_2, \quad \lambda_3 x_0 + \lambda_3$$

because the direction x_2 will lead to the point x_0, bx_0, cx_0, and one obtains $bx_0 = l_2 x_0 + \lambda_2$. For stable closure it is necessary that the equation be satisfied for all x_0, i.e. $b = l_2$ and $\lambda_2 = 0$. Hence, such a closed zigzag will be stable only if l lies in the plane $x_2 = l_2 x_1$. The corresponding directions l will be called forbidden directions.

If the zigzag closes after it has been on l several times, a necessary condition for stability is $b^i c^j = l_0$, where l_0 is some constant depending on K_n and on the type of the zigzag. Here i is the difference between the number of arrivals of the zigzag on l from the direction x_2 and the number of departures from l in this direction; j has a similar meaning for the direction x_3. If the direction of l is not a forbidden one (i.e. $b^i c^j \neq l_0$), then there can exist no closed zigzags of the considered type. Suppose that $(l_0 - 1)^2 + i^2 + j^2 \neq 0$. Then the directions l for which $b^i c^j = l_0$ form nowhere a nondense set (a curve) in the space of directions. Therefore all directions which are forbidden by some types of zigzag for which $(l_0 - 1)^2 + i^2 + j^2 \neq 0$, lie on a denumerable set of smooth curves and constitute an everywhere dense set of the second category of forbidden directions.

If, however, $i = 0$, $j = 0$, and $l_0 = 1$, then the closed zigzag will be stable for b and c arbitrary, and, in particular, if we direct l along the straight line q on whose segment $q_n \in D_n$ the point p_n lies, where the branch s_n is attached. It is true here that some, but not all, points of the zigzag (namely those lying on l and q) will run together. But it is easy to see that on K_n there is defined a stable zigzag and that the points of q will belong to its class. But on q there are points A_n. This yields a contradiction with the condition c) which is satisfied by D.

The final result is as follows: one can choose the direction from an everywhere dense set of the second category so that D_{n+1} will satisfy the condition a).

Let us now go over to the condition c). We consider two branching points of Ξ whose images lie on D_n (in A_n). The s_n must be chosen so that it will be impossible to connect the points from A_n in D_{n+1} by means of a zigzag. For zigzags which do not contain points of s_n, this is already so, because the condition c) is satisfied on D_n. The set of pairs of points A_n is denumerable. So is the set of all types of zigzags. Let us consider one of these types and one of the pairs. The requirement that a zigzag of this type connects these points leads to forbidden directions l for which such a connection can occur, and, just as in the preceding proof, all forbidden directions lie on a denumerable set of smooth curves. The condition b) leads to the same type of requirement for the points A_n and closed zigzags K_{n+1}.

We must now concern ourselves with the fulfillment of condition b) for the points of $A_{n+1}\setminus A_n$ (lying on s_n) and with condition c) for pairs of such points A_{n+1} of which at least one lies on s_n. Having selected in the manner described the direction l from the everywhere dense set of the second category (from the complement of the forbidden directions), we map σ_n on s_n.

Thus we have constructed K_{n+1}. Let us put on l the points of the class A_n. This denumerable set must not intersect the images of the branching points of Ξ on s_n. The same prohibition applies to the set of points of the closed zigzags on K_{n+1} and the classes (on K_{n+1}) of these points. The set of forbidden points or, as we shall say, the "taboo set" is at most denumerable because of the way in which l is chosen.

The requirements a) and b) will be fulfilled on D_{n+1}, while the requirement c) will hold on D_n if we do not map the branching points of Ξ into the forbidden points s_n. Such a mapping of s_n on l for which the requirement c) on D_{n+1} is also satisfied, will now be described. Here the segment s_n may be arbitrarily small. This fact will be used later.

Let us now assume that we have chosen s_n and its size. On s_n there is a taboo set (at most denumerable) which cannot contain the images a of branching points α of Ξ that lie on σ_n. The mapping must be homeomorphic, and we must take care that the points a are inaccessible to each other on K_{n+1}.

Let us arrange the branching points of Ξ on σ_n into a sequence $\alpha_1, \alpha_2, \ldots$. (The point ρ_n is not included in this sequence.) The denumerable set is everywhere dense on σ_n.* Therefore, a similar** mapping of this set on

* If this is not the case then we add to the points α some points σ_n.

** That is, an order preserving.

an everywhere dense set s_n can be extended to the homeomorphism σ_n on s_n. We still have to map the points α on s_n. Since there are no ends σ_n among the α_i, the images a_i of the α_i are distributed in the interval s_n' whose closure is s_n.

Let us consider on s_n a denumerable system of intervals δ_i^k, $1 \leqslant k < \infty$, $1 \leqslant i \leqslant i_k$, such that

1) for every k $\bigcup_{i=1}^{i_k} \delta_i^k = s_n'$,

2) each of the intervals $\delta_1^k, \ldots, \delta_{i_k}^k$ is less than ε_k; $\varepsilon_k \to 0$ when $k \to \infty$.

If each of the intervals contains at least one point α_j, then the points a_j $(j = 1, 2, \ldots)$ form an everywhere dense set s_n. Let us arrange all these intervals into one sequence δ_l $(l = 1, 2, \ldots)$.

Let us assume that the directions on σ_n and s_n have been selected so that ρ_n and p_n are the left endpoints.

First step. We pick a nonforbidden point a_1 on δ_1. It will be the image of a point α_1. The points of the class of a_1 form on s_n a denumerable set. We add this set to the taboo set.

Second step. The point α_1 divides σ_n into a left and a right part. Let α_{i_l} be a point α with smallest subscript in the left part, while α_{i_r} has the same meaning for the right part. The point a_1 divides the intervals δ into those that lie to the left of a_1, those that lie to the right of a_1, and those that contain a_1. Among the intervals that do not lie to the right of a_1, with a subscript greater than 1, we select the one with the smallest subscript. On it we pick a nonforbidden point to the left of a_1. This will be a_{i_l}, the image of α_{i_l}. We add to the taboo set all points of the class of a_{i_l}. We select from the remaining intervals δ which are not to the left of a_1, the one with the smallest subscript. In this interval we pick a nonforbidden point a_{i_r} to the right of a_1. We add to the taboo set the points of the class of a_{i_r}.

After the nth step, σ_n will be divided into 2^n intervals by the $2^n - 1$ points $\alpha_1, \alpha_{i_l}, \alpha_{i_r}, \alpha_{i_{ll}}, \alpha_{i_{lr}}, \ldots, \alpha_{i_{\underbrace{rr\ldots r}_{n-1 \text{ times}}}}$.

The $(n+1)$st step. In each one of the 2^n intervals we pick a point with the smallest subscript and denote this subscript in the left-most interval by $i_{\underbrace{ll\ldots l}_{n \text{ times}}}$, then by $i_{\underbrace{ll\ldots lr}_{n-1 \text{ times}}}, \ldots,$ in the right-most one by $i_{\underbrace{rr\ldots r}_{n-1 \text{ times}}}$.

The mapping of these 2^n points $\alpha_{i_{ll\ldots l}}, \alpha_{i_{rr\ldots r}}$ on s_n takes place in exactly the same way as described in the second step. The image of α, the point a, is always picked in the interval δ which is not to the left of α_i, the left end of the interval (α_i, α_j) from which α was picked, and not to the right of α_j. Hereby one picks the interval with the smallest subscript from all the intervals δ having the given property. In the interval δ, the point a is picked between a_i and a_j from the nonforbidden points. Then one adds to the taboo set all the points of the class of the point a. After this one constructs the image of the next point α until the $(n + 1)$st step is completed.

The proofs that the mapping of the points α on a is defined after the performance of all steps for all α, that this is a similar (order preserving) mapping, and that by the thus generated homeomorphism σ_n and s_n retain the properties a), b), and c), can be accomplished without difficulty.

§5. Lemma on generating schemes

Before we start the proof of the possibility of preserving the inductive properties 2, 3, and 4, let us investigate more closely generating schemes of segment-like complexes K. It is immaterial whether these schemes are simple or double.

If one omits the beginning in a generating scheme, then the remainder can be considered as the set of intersecting generating schemes starting with the ends of the first rank (of the shortened system).

Lemma 1 (Figure 12). *If the shortened systems A_i of a given system A are stable, and the initial point a_0 is not a branch point of K, then A is stable.*

Proof. Let $\varepsilon_1 > 0$ be such a number that an ε_1-shift* of the initial points a_i of the shortened schemes will not lengthen these systems (see definition of stability in §3). Furthermore, from the stability of A_i it follows that the a_i are not branching points of K. Since the complex K is a closed set, there exists an $\varepsilon_2 > 0$ such that the plane, which is parallel to the first plane of the scheme A and which is at a distance of at least ε_2 from it, intersects only those segments of K which contain a_0 and the points a_i.

By taking $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$, we obtain an ε-neighborhood of the point a_0. The existence of this neighborhood proves the stability of the scheme A.

* We recall that the distance between the points (x_1, x_2, x_3) and (x_1', x_2', x_3') is $\max_{1 \leqslant i \leqslant 3} (|x_i - x_i'|)$.

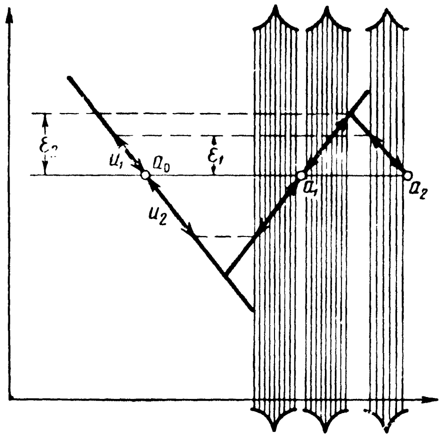


Figure 12. The generating scheme $a_0a_1a_2$ has rank 1. The layers of generating schemes of semineighborhoods are shaded.

Lemma 2. *If no vertex of a generating scheme is a branching point, then the generating scheme is stable.*

Proof. The proof is accomplished by induction. If the rank of the scheme is zero and the point a_0 is a free end, and not a branching point of K, then, obviously, there exists in K a neighborhood of the point a_0, which is composed of points with the same property (see Figure 12, where the points a_1 and a_2 are shown with the mentioned neighborhoods of stability). If the assertion of the Lemma 2 has been proved for a scheme of rank n, then its truth for a scheme of the next higher rank follows from Lemma 1.

Lemma 3. *Suppose that the generating scheme A with initial point a is stable. Then for every positive ε there exists a positive number δ such that every supporting plane that corresponds to the scheme A and leads away from the point b of B is at a distance not greater than ε from the corresponding plane of the scheme A, provided that the initial point b is nearer than δ to the initial point a.*

Proof. The generating scheme A has a finite number of supporting planes $\Pi_i^r(a)$ of each direction $r = 1, 2, 3$.

For the scheme B which leads away from the point b in the interval of stability of the scheme A, there are defined planes, points, and zigzags of the scheme B that correspond to planes, points, and zigzags of the scheme A. (The converse is in general not true, because the zigzags of the scheme B can terminate earlier.)

Let us consider the planes $\Pi_i^r(b)$. (This is the notation for the plane which corresponds to the plane $\Pi_i^r(a)$ in the scheme A.)

The coordinate x_r is the same for all the points $\Pi_i^r(b)$; it depends linearly on any coordinate of the initial point b. It follows from this and from the finiteness of the number of supporting planes of the scheme A, that for every point b in a sufficiently small neighborhood of the point a all planes of the scheme B are nearer than ε to the corresponding planes of the scheme A.

We note that the segments of a complex are always assumed to be non-perpendicular to the coordinate axes. From the finiteness of the number of the segments it follows that their inclination to the coordinate planes is bounded from below. Hence, Lemma 3 implies that a sufficiently small change of the beginning of a scheme will produce an arbitrarily small shift of the vertices of the scheme. These vertices will not disappear. These properties will be referred to as the continuous dependence of a stable generating system on its beginning. A finite number of stable generating schemes A_i depend in a uniformly continuous manner on the beginning, in the sense that for every $\varepsilon > 0$ there exists a $\delta > 0$ which is the same for all these schemes.

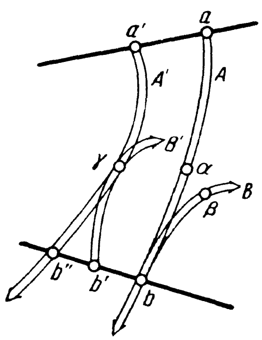


Figure 13. Relative to Lemma 4. The thick arrows represent the generating systems.

Lemma 4 (Figure 13). *Let us assume that the class of the point b does not contain a closed zigzag. Let A be a stable generating scheme which starts at a, and let B be a stable generating scheme with beginning at b, whose first direction is the same as that along which the scheme A leads into b. Then the points a and b have neighborhoods u_a and u_b such that if the scheme A' (this is a scheme that corresponds to A but its beginning is at $a' \in u_a$) passes through the point $b'' \in u_b$, then the scheme B' (which corresponds to B but leads away from $b' \in u_b$) has no points in common with A', provided $b' \neq b''$.*

Proof. Let us consider the set of points of the schemes A and B. Suppose that the shortest distance between two points is $\eta > 0$. We will pick for the points a and b neighborhoods u_a and u_b such that if $a' \in u_a$, $b' \in u_b$, then the points of the schemes A' and B' will be at a distance less than $\eta/3$ from the corresponding points of the schemes A and B. Such neighborhoods can be found in view of the remark relative to the Lemma 3. These are the neighborhoods sought.

Indeed, let a', b' and b'' be the points mentioned in the hypothesis of the lemma. Suppose that the point γ lies on A' and B' (Figure 13). As a point of A', it has a mate α on A. As a point of B', it has a mate β on B. We will prove that α and β coincide. Indeed, in the opposite case they would be at some distance not less than η from each other. But the point γ is at a distance less than $\eta/3$ from its mate β, and at a distance less than $\eta/3$ from α for the same reason. The obtained contradiction proves that $\alpha = \beta$. But this implies that the zigzag that connects b with β in B lies entirely in A; in the opposite case one could connect b with β $(= \alpha)$ by means of a zigzag through A in a different way. But the class of the point b, by the hypothesis of the lemma, contains no closed zigzags. The scheme A' is not longer than A. It contains γ, which corresponds to α, and it contains b'', which corresponds to b. This implies that A' contains a zigzag connecting b and γ of the same type as the zigzag $(b\alpha) \in A$. On the basis of similar arguments, the zigzags $(b\beta) = (b\alpha)$ and $(b'\gamma)$ are of the same type. The zigzags $b'\gamma$ and $b''\gamma$ must, therefore, also be of the same type. This, however, contradicts the nondegeneracy of the corresponding type of linear transformation because the points b' and b'' had been assumed to be distinct. This contradiction establishes Lemma 4.

In §8 we will make use of still another property of stable systems. We shall call it the property N. A scheme A which leads away from the point $a_0 \in K$, has the property N if the point α_0 lies on the segment $\Delta \subset K$, where it possesses neighborhoods* u_1 and u_2 (in the case when a_0 is an endpoint of K, α_0 has a one-sided neighborhood) such that for all points $a_0' \in u_1$ there exists a generating scheme $A'(a_0')$ of the same type and not longer than A, and for all points $a_0'' \in u_2$ there exists a generating scheme $A''(a_0'')$ of the same type not longer than A.

Examining Figure 12 one can understand that these types do not necessarily coincide, but may all three (type A, type A', and type A'') be different.

The following lemma is true:

Lemma 5. *Every stable generating scheme has the property* N.

Thus Lemma 5 can be deduced from Lemma 6 just as Lemma 2 can be deduced from Lemma 1.

Lemma 6. *Let* A *be a generating scheme that starts at* a_0 *in* K. *If each of the shortened schemes* A_i *of the scheme* A *has the property* N, *and the point* a_0 *is not a branching point of* K, *then the scheme* A *has the property* N.

The proof of Lemma 6 is analogous to the proof of Lemma 1.

* That is, intervals which lie on Δ and have a_0 for a limit point.

We introduce now the concept of a generating scheme (system) of intervals.

For this purpose we consider a generating scheme of points of the interval u of the complex K. Suppose they are all of the same type (as, for example, those of the scheme A' of the points of the interval u_a in the definition of the property N). The set of corresponding points of these schemes form intervals in which the zigzags of one type map the interval u. The corresponding planes of these systems form layers. If the parallel layers do not intersect then we have a generating scheme of the interval u. It consists of the intervals of a scheme analogous to the ends which lie in the intersection of K with the layers of the scheme that are analogous to the planes. The interval of a scheme of rank 0 is u; the set of all planes of the first direction of the schemes of the points u is a layer of rank 1. It will lead from u and will lead to the intervals of rank 1. And so on. From the combinatorial viewpoint, a generating scheme of intervals is constructed in the same way as a generating scheme of points. In place of free ends we have here free intervals.

The following concept was not introduced for the schemes (systems) of points. An interval of a layer is the intersection of the layer with the coordinate axis that is perpendicular to the layer. The generating schemes of points u associate with each point u a point in each interval of the scheme and a plane in each layer. This defines a linear mapping u on each interval of the layer.

Applying the Lemmas 2, 3, and 5 to the tree D_n which has the inductive properties 1 and 2, we can establish that D_n has a generating scheme that leads from the point p_n, and from each point of the class p_n. This scheme (system) is stable, has the property N and depends continuously on its beginning. The scheme exists because D_n has the inductive property 2, and the class of the point p_n does not contain branching points in view of property 2. Thus, the lemmas are applicable to this scheme.

§6. Inductive preservation of generating schemes

In §4 it was shown how one can add to D_n a branch s_n, as small as we please, in such a way that D_{n+1} would have property 1. In order that D_{n+1} may have the inductive properties 2, 3, and 4, it is necessary that s_n be small enough. Having chosen the direction of the straight line l in accordance with §4, having selected $\varepsilon > 0$ sufficiently small, and then s_n in the ε-neighborhood p_n, as described in §4, we find that all four inductive properties hold on the constructed tree D_{n+1}.

In this section it will be proved that if D_n has the properties 1 and 2. and if the direction l has been chosen correctly, then there exists an $\varepsilon > 0$ such that if s_n is placed in the ε-neighborhood of p_n, then D_{n+1} possesses the inductive property 2.

In accordance with the property 2, the tree D_n has a finite set of types of canonical generating schemes. We shall transform these types somewhat. We shall try to obtain a finite number of generating schemes A_i which pass through p_n and which are such that for every $\delta > 0$ there exists an $\varepsilon > 0$ such that the planes of the canonic schemes of points lying outside a δ-neighborhood of the beginning a_i of the scheme A_1 will not intersect the ε-neighborhood of p_n.

Suppose that the existing canonic types do not possess this property. Since the number of types is finite, one of them must be nonregular. By this we mean that this type contains generating canonic schemes which have planes arbitrarily close to p_n if there is no scheme that passes through p_n. We select from the sequence of initial points of the indicated schemes, a sequence that converges to a, and we consider the set of limit points of the set of points of all these schemes. This set of limit points cannot be a generating scheme. But it contains p_n, and by adding to some of its points (their number is, obviously, finite) their generating schemes, we obtain the generating scheme of the point a. The added points are all distinct from each other and from those that existed before, because in the class of p_n there are no points of closed zigzags.

By Lemma 2, the obtained system is stable. Therefore, there must exist a neighborhood of the point a such that the generating schemes which start in this neighborhood must be schemes that correspond to this point, because of stability. Let us replace (in this neighborhood of a) the nonregular type of generating schemes by the schemes that correspond to a. The obtained types will be considered to be canonic. It is clear that the remainder of the canonic nonregular type is regular. This can be proved easily by making use of the linearity of the corresponding mappings.

Having performed this operation with all the old nonregular types, we obtain new canonic types; we shall call them simply canonic types. A finite number of the points a_i have canonic schemes passing through p_n. All non-regular types are now in the intervals of stability of these schemes A_i. From the linearity of the mapping of the neighborhood of a_i into the neighborhood of p_n with the aid of the corresponding zigzags of the canonic schemes, there follows the following assertion.

For every $\delta > 0$ *there exists an* $\varepsilon > 0$ *such that the* ε*-neighborhoods* *of the point p_n can be intersected by the planes of only those new canonic schemes whose beginnings lie in a δ-neighborhood of the points a_i, and which correspond to the A_i in the sense of stability.*

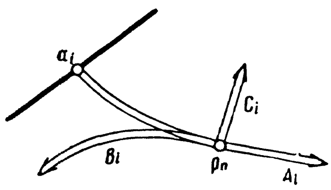


Figure 14. The thick arrows represent generating schemes.

Let A_i be a canonic generating scheme that leads away from a_i and passes through p_n. We shall consider the following generating schemes (Figure 14):

B_i, the scheme that leads away from p_n in the same direction along which A_i arrived at p_n.

C_i, the generating scheme, leading away from p_n, whose first direction is different from the ones along which A_i leaves p_n and arrives at p_n, and from the first direction of B_i. (In case $p_n = a_i$, the scheme B_i is not defined, and we do not consider it.)

All these schemes pass through p_n, and they are, therefore, stable.

According to the inductive condition 1, the constructed generating schemes have no points in common besides p_n, and the B_i and A_i satisfy condition 4.

Let us consider the set of all the points of all three schemes. Let the positive number η be the least distance between any two points of this set. Applying Lemma 4 to A_i and B_i, we find a δ-neighborhood of the point a_i, and an ε-neighborhood of the point p_n such that A_i and B_i' will not intersect if their beginnings lie in the indicated neighborhoods (for the definitions of A' and B' see Lemma 4). Applying Lemma 3 to the schemes A_i, B_i, C_i, we find a $\delta_2 > 0$ and an $\varepsilon_2 > 0$ such that all the points of A_i', B_i', C_i' will be at a distance greater than $\eta/3$ from their corresponding points of A_i, B_i, C_i provided that the beginning of A_i' lies in a δ_2-neighborhood of a_i, and the beginnings of the remaining schemes in an ε_2-neighborhood of p_n. Here C_i' is a scheme of the same type as C_i but shorter.

Let us choose a positive number δ less than δ_1 and δ_2. For this we find an $\varepsilon_3 > 0$ such that the ε_3-neighborhood of p_n is intersected by the planes of only those canonic generating schemes whose beginnings lie in δ-neighborhoods of the points a_i and whose first directions are the same as those of A_i. We choose a positive number ε less than ε_1, ε_2, and ε_3. From the finiteness of the number of types A_i it follows that all the numbers ε and δ can be chosen uniformly for all i. Consequently, we can obtain a system of δ-neighborhoods of the points a_i, and ε-neighborhoods of the p_n such that the following statements are true.

A) The ε-neighborhood of p_n is intersected by the planes of only those canonic schemes whose beginnings lie in δ-neighborhoods of the points a_i, and which correspond to A_i.

B) The schemes A_i' and B_i' do not intersect if their beginnings lie in the indicated δ- and ε-neighborhoods.

C) In the transition from A_i, B_i, C_i to A_i', B_i', C_i' the points of these schemes will be shifted over distances less than $\eta/3$ provided the beginnings remain within the indicated neighborhoods.

Recalling the meaning of the positive number η, we see that if the beginnings lie in the indicated neighborhoods, then the schemes B_i', C_i' cannot have any points in common besides the beginning. The same thing is true for C_i' and A_i', B_i' and A_i'.

Let us inscribe a parallelepiped P in the obtained neighborhood of p_n. The edges of the parallelepiped are to be parallel to the coordinate axes, its center is to be at p_n, and one of its diagonals is to lie on q_n. Inside P we attach to q_n at p_n a segment $2s_n$ in the direction l (see §3) (Figure 15).

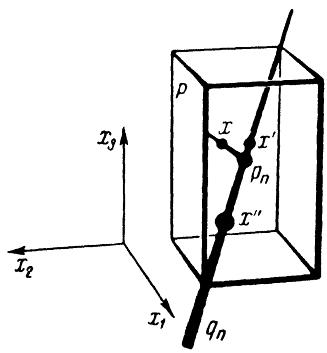


Figure 15. The attaching of s_n.

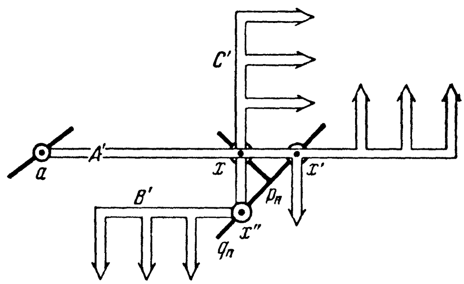


Figure 16. Generating scheme leading away from the point a on D_{n+1}.

The length of the segment s_n will be restricted from above. In order to preserve property 2 on D_{n+1}, it is sufficient that $2s_n \subset P$. We shall prove this.

If the planes of the canonic generating scheme D_n, which leads away from some point of D_n, do not intersect $2s_n$, then the scheme will remain to be a generating scheme also on D_{n+1}. In particular, this is known to be the case for all points that lie outside the δ-neighborhoods of the points a_i. From the property A) it follows that it is sufficient to construct a generating scheme leading away from each point of the δ-neighborhood of a_i in the first direction of A_i. Let A be such a point. The canonic scheme A'_i which leads away from a on D_n in the first direction of the scheme A_i, corresponds to A_i because the δ-neighborhood is smaller than the interval of stability.

Suppose that A intersects $2s_n$ at the point x. Then A'_i intersects q_n in some point x'; $x \neq x'$ if $a = a_i$.* In the sequel we will assume that $a \neq a_i$. Let us pass a plane through x'. The first direction of this plane is the same as that of C_i. Suppose that x'' is the point of intersection of this plane with q_n. From the choice of the direction of $2s_n$ it follows that $x'' \neq x'$ and x.

Let us construct (Figure 16) generating schemes C'_i and B'_i leading away from the points x''. From the properties B) and C) of the ε- and δ-neighborhoods it follows that the schemes A'_i and B'_i, as well as the schemes A'_i and C'_i have no points in common, while B'_i and C'_i have only the beginning in common. It is easy to see that the planes A'_i, B'_i, C'_i, that do not pass through x, x', x'', cannot intersect P.

All the thus far considered generating schemes led away from D_n. With their aid one can construct, however, schemes which will lead away from an a on D_{n+1}. The scheme A'_i does not lead to D_{n+1} only because the point x is not free on it. Let us select a direction at this point for the first plane of the scheme C_i. The obtained plane intersects D_{n+1} (in addition to the point x) also at the point x'' and at points of the first rank of the scheme C'_i. From the points of the first rank we leave along directions, along which we pass to C'_i. B'_i leads away from the point x'' on D. Since these schemes do not intersect, except at the point x'', because they cannot have any points in common with A_i, and since the planes of the schemes A'_i, B'_i, C'_i do not intersect P (except for the four planes which are here being considered and pass through x, x', x''), we obtain a generating scheme that leads away from a to D_{n+1}. In case $a_i = p_n$, and $a \in 2s_n$, one does not have to construct x''; the scheme C'_i is constructed at the point x' (Figure 17). The proof is analogous to the preceding one.

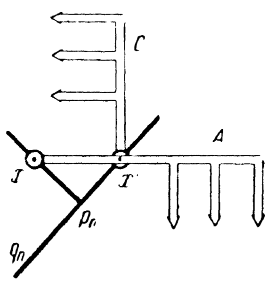


Figure 17. Generating scheme that leads from x on D_{n+1}.

The proof of the inductive fulfillment of the property 2 on D_{n+1} under the conditions of the fulfillment of the properties 1 and 2 on D_n, will be finished as soon as we give the finite number of types of generating schemes. But we have actually done

* In case $a = a_i$, the scheme A_i remains a generating scheme on D_{n+1}.

this in the construction of the generating schemes leading away from a. Indeed, it is easy to see that the set of types of schemes, which are here used (schemes A_i', B_i', and C_i') is finite, because they are not longer than the schemes A_i, B_i, C_i which are the schemes of the canonic type on D_n.

§7. Inductive preservation of the decomposition of functions

This section contains the construction of the representation of a function (defined on a finite tree D_n) in the form of the sum of functions of the coordinates.

Lemma 7. *Let A be a scheme that leads away from a point a_0 of a complex K, and let $f(x)$ be a function, defined on K, which differs from zero at the point a_0 only. Then there exists functions of the coordinates x_k of the point x such that for every point $x \in K$*

$$f(x) = \sum_{k=1}^{3} f_k(x_k), \qquad (*)$$

where the functions $f_k(x)$ differ from zero only at those points of the kth axis which are the intersections of this axis with the planes of the scheme A.

Proof. Let us assume that $f_k^0(x_k) \equiv 0$. If we substitute f_k^0 into the right-hand side of equation (*) then this equation will fail to hold only at the point of rank 0 of the scheme A. We will call the function $f_k^0(x_n)$ the zeroth approximation to $f_k(x_k)$. The function of the nth approximation, $f_k^n(x_k)$, will be constructed in such a way that the following conditions hold.

1) If the function $f_k^n(x_k)$ is substituted for $f_k(x_k)$ in the equation (*), then this equation will fail to hold only at the points of rank n of the scheme A.

2) $f_k^n(x_k) = f_k^{n-1}(x_k)$ $(n = 1, 2, \ldots)$, if the point x_k of the kth axis does not lie on planes of rank n of the scheme A.

The functions of the zeroth approximation possess the property 1), and if the rank of the scheme A is N, then the $f_k^{N+1}(x_k)$ will satisfy, obviously, all the requirements of Lemma 7. If the $f_k^{n-1}(x_k)$ are constructed so that the conditions 1) and 2) hold, then we set

$$f^{n-1}(x) = \sum_{k=1}^{3} f_k^{n-1}(x_k).$$

The expression $f(x) - f^{n-1}(x) = \Delta^n(x)$, the nth disjoint, is different from zero at the points of rank $n - 1$ of the scheme A. Let a be such a point, and suppose that the plane π, which leads away from this point, intersects the kth axis at the point $x_k(a)$. From the definition of the generating system it follows that all the $x_k(a)$, that correspond to different a and n, are distinct. Introducing corrections $\Delta_k^n(x_k) = \Delta^n(a)$ for $f_k^{n-1}(x_k)$ at the points $x_k(a)$, we set $f_k^n(x_k) = f_k^{n-1}(x_k) + \Delta_k^n(x_k)$. It is obvious that $f_k^n(x_k)$ has the properties 1), 2). Hence, one can construct all the $f_k^{n+1}(x_k)$. This completes the proof of Lemma 7.

Lemma 8. *Let A be a scheme which leads away from the interval s of the complex K, and let $f(x)$ be a continuous function, defined on K, and differing from zero on s only. Then there exist continuous functions $f_k(x)$ of the coordinates of the point x such that for every point $x \in K$*

$$f(x) = \sum_{k=1}^{3} f_k(x_k),$$

where the functions $f_k(x_k)$ are different from zero only on the intervals of the layers of the scheme A.

The proof of this assertion is analogous to the proof of Lemma 7. All points and planes are replaced by intervals and layers, while the functions which differ from zero at separate points are replaced by continuous functions differing from zero only on separate nonintersecting intervals at whose ends they are zero. In particular, the disjoints and corrections will be such functions.

Lemma 9. *The assertions of Lemmas 7 and 8 are true for double schemes.*

Proof. The proof of this lemma is again accomplished with the aid of the distribution of corrections. At the points (intervals) from which two planes (layers) issue, one may ignore one of them, obtain a simple system and make use of the Lemma 7 (8). But then the corrections and disjoints of all ranks will be equal. One can decrease the size of the corrections if one makes use of both issuing planes (layers) for the "distribution of the corrections along two directions".

Suppose, for example, that the planes π_1 and π_2 of the directions x_1 and x_2, respectively, lead away from the point a. In order that the equation (*) may hold at the point a_0, one may set

$$\Delta_1^1(x_1) = \gamma_1\Delta^1(x), \quad f_1^1(x_1) = f_1^0(x_1) + \Delta_1^1(x_1),$$

$$\Delta_2^1(x_2) = \gamma_2\Delta^1(x), \quad f_2^1(x_1) = f_2^0(x_2) + \Delta_2^1(x_2),$$

where, as before $f_k^0(x_k) \equiv 0$, $\Delta^1(x) = f(x) - \sum_{k=1}^{3} f_k^0(x_k)$ and where $\gamma_1, \gamma_2 > 0$,

$\gamma_1 + \gamma_2 = 1$. Then the equation (*) will fail to be satisfied at all ends of rank 1, and one has to introduce the correction at a greater number of points than one would have had to if one had ignored π_2 by assuming that $\gamma_2 = 0$.

In the final construction of the functions $f_k(x_k)$ in §9, use will be made of the distribution along two directions.

Lemma 10. *Suppose that we are given on a segment-like complex K two continuous functions: an "old" one,*

$$\bar{f}(x) = \sum_{k=1}^{3} \bar{f}_k(x_k),$$

where the $\bar{f}_k(x_k)$ are continuous functions of the coordinates x_k of the point $x \in K$, and a "new" one $f(x)$, which differs from the old one only on the interval s that possesses on K a generating scheme A (simple or double). Then one can find "corrections for $\bar{f}_k$" which are continuous functions $g_k(x_k)$, differing from zero only on the intervals of the layers of the scheme A, and which are such that if one writes $f_k(x_k) = \bar{f}_k(x_k) + g_k(x_k)$, then on the entire complex K

$$f(x) = \sum_{k=1}^{3} f_k(x_k).$$

Lemma 10 is a direct consequence of the Lemmas 8 and 9 if one introduces the function $g(x) = f(x) - \bar{f}(x)$. The process of the distribution of the corrections along two directions, which leads to the construction of the $g_k(x_k)$ $(\sum_{k=1}^{3} g_k(x_k) = g(x))$, determines the disjoints $\Delta^n(x)$, the corrections $\Delta_k^n(x_k) = \gamma_k(x)\Delta^n(x)$, and the approximations $g_k^n(x_k) = g_k^{n-1}(x_k) + \Delta_k^n(x_k)$. It is clear that one may consider the functions $f_k^n(x_k) = \bar{f}_k(x_k) + g_k^n(x_k)$ as approximations to the $f_k(x_k)$; the disjoints $f(x) - \sum_{k=1}^{3} f_k^n(x_k)$ and corrections $f_i^n(x_i) - f_i^{n-1}(x_i)$ will hereby be the same. The construction of the $f_k^n(x_k)$ $(k = 1, 2, 3;\ n = 0, 1, 2, \ldots, N + 1)$, which was described above, will be called the distribution of corrections.

Lemma 11. *For the preservation on D_{n+1} of the inductive property 3, it is sufficient that the interval $2s_n$ have a generating scheme on $D_n \cup 2s_n$. The expansion of $f(x)$ as a sum of functions f_k^{n+1} of the coordinates can be accomplished through the introduction of corrections for f_k^n with the aid of the distribution of corrections along two directions determined, in general, by the double generating scheme of $2s_n$.*

Proof. Suppose that on $D_n \cup 2s_n$, the interval $2s_n$ has a generating scheme. On D_n, every continuous function can be expressed as the sum of functions of coordinates (inductive requirement 3_n). We select for the old function $f^n(x) = \sum_{k=1}^{3} f_k^n(x_k)$, and for the new function, $f(x)$ on D_{n+1}. On $D_n \cup 2s_n$ we define this function so that the difference between it and the old function on $2s_n$ is an even function relative to the midpoint of $2s_n$. Then we will have the conditions of Lemma 10, from which follows the possibility of the representation of $f(x)$ on D_{n+1} as the sum of functions of the coordinates by the method of the distribution of the corrections along two directions. If each correction depends continuously on the expanded function (and this can, obviously, be obtained from the conditions of Lemmas 7-11), then the expansion $f(x) = \sum_{k=1}^{3} f_k^{n+1}(x_k)$ depends continuously on f. In §7 every correction depends continuously on the expanded function.

If the branch s_n is constructed as indicated in §§3-5, then the requirements 1_{n+1} and 2_{n+1} will be satisfied on D_{n+1}. The last requirement means that there exists on D_{n+1} a finite number of canonic generating schemes of intervals. For this it is only necessary that (§5) the direction of s_n be chosen correctly and that the branch s_n lie in a sufficiently small neighborhood P of the point p_n.

Lemma 12. *Suppose that the conditions* 1_n *and* 2_n *are fulfilled on* D_n. *If* s_n *lies in a small enough neighborhood* $P' \subset P$ *of the point* p_n, *then* $2s_n$ *has a generating scheme on* $D_n \cup 2s_n$.

Proof. Let us consider the above constructed canonic generating schemes of the points $2s_n'$ on D_{n+1} with a given first direction (Figure 17). When x changes on $2s_n'$, then x' runs through a one-sided neighborhood u of the point p_n on q_n. Because of the stability of the schemes A and C, there exists a semineighborhood u on the same side of p_n for whose points all schemes A' and all schemes C' will be of the same type (Lemma 5).

The points and parallel planes A and C do not intersect in pairs (excepting at the point p_n). From the continuous dependence of A' and C' on the initial point x' it follows that if x' changes in a sufficiently small neighborhood of the point p_n, then the points and planes of the schemes A' and C' will be as close as we please to the corresponding elements of the schemes A and C. Let $\eta > 0$ be the least of the distances between any plane π of one of the schemes A, C and a point (not on π) of one of these schemes. Let $\varepsilon > 0$ be the radius of a neighborhood of the point p_n such that the planes and points of the schemes A and C are shifted by not more than $\eta/3$ when the point x' varies over the ε-neighborhood of p_n. Then the intersection $\tilde{u}$ with the ε-neighborhood of p_n will yield a semineighborhood of the point p_n which is an interval u having on D_{n+1} nonintersecting generating schemes (see §5) A^* and C^*.

This follows from the fact that all schemes A' and C' are of the same type; any two parallel layers of these schemes will, obviously, not be separated by a distance greater than $\eta/3$. If we now place the segment $2s_n$ in a small enough neighborhood P_1 of the point p_n (namely such that x' falls into $\hat{u}$) then the interval $2s_n'$ will have a generating scheme on D_{n+1} whose first direction will coincide with the first direction of A.

From the Lemmas 11 and 12 it follows that *for the preservation of the inductive property* 3 *on* D_{n+1} *it is sufficient that the segment* s_n *be small and have a properly chosen direction* (§§3-5).

In §7 use is made of a generalization of the Lemma 12.

By the N-characteristic χ_N of a generating scheme A_u of the interval u on K we shall mean the set of directions of the generating layers of the intervals of rank less than N referred to these intervals. The N-characteristic χ_N and K determine uniquely the elements of the scheme A_u whose rank does not exceed N.*

Lemma 13. *Suppose that the conditions* 1_n *and* 2_n *are fulfilled on* D_n. *For every* $N > 0$, *there exist a neighborhood* $P(N)$ *of the point* p_n *and generating schemes of intervals* $u \subset P(N)$ *such that*

1) *Among them there exist schemes* $A_u^{\chi_N}$ *with any* N-*characteristic*.

2) *The intervals of the schemes* $A_u^{\overset{1}{\chi}_N}$ *and* $A_u^{\overset{2}{\chi}_N}$, *different from* u, *do not intersect if the first directions of these schemes are distinct.*

3) *If the intervals* $u_1 \in P(N)$, $u_2 \subseteq P(N)$ *do not intersect, then none of the intervals of the schemes* $A_{u_1}^{\chi_N}$ *and* $A_{u_2}^{\chi_N}$ *will intersect.*

The proof of Lemma 13 is analogous to the proof of Lemma 12, and is left to the reader.

Up to now our constructions have not depended on whether the function f belongs to the class F which is mentioned in the inductive lemma. In §9 the expansion constructed here will depend on F. This will not destroy the possibility of expanding any function on D_n into the sum of functions of the coordinates. We can, obviously, without loss of generality, assume that F is a compact. It is easy to see that within the limits of F, the continuous dependence of f_k^n on f is uniform.

* In the N-characteristic one can indicate the directions of the layers that lead away from the intervals which are not in the scheme A_u, because this scheme may terminate earlier with a free end.

§8. Arithmetic lemma

In this section there are proved two lemmas with whose aid there will be obtained, in the next section, corrections along two different directions.

Lemma 14. *Let*

$$a + b + c = d, \tag{1}$$

where

$$|a|, \quad |b|, \quad |c| < 3 + \theta, \tag{2}$$

$$|d| \leqslant 1. \tag{3}$$

Let

$$a' = a + \Delta a, \tag{4}$$

where

$$|\Delta a| < 1 + \varepsilon, \tag{5}$$

$$0 < \varepsilon < \theta < 1. \tag{6}$$

Then one can determine numbers $\Delta b(a, b, c, \Delta a)$ *and* $\Delta c(a, b, c, \Delta a)$ *such that if*

$$b' = b + \Delta b, \quad c' = c + \Delta c \tag{7}$$

then

$$|b'|, \quad |c'| < 3 + \theta + \varepsilon, \tag{8}$$

$$a' + b' + c' = d \tag{9}$$

$$|\Delta b|, \quad |\Delta c| \leqslant \max\left(\left| \, |\Delta a| - \frac{\varepsilon^2}{30} \right|, \ \varepsilon\right) \tag{10}$$

and, such that

the dependence of Δb *and* Δc *on* $a, b, c, \Delta a$, *which vary within the restrictions* (1) *to* (6), *be continuous and that as* $\Delta a \to 0$, Δb *and* Δc *will tend to zero.* (11)

Proof. We shall prove first that under the conditions of the lemma

$$|b + c| < 4 + \theta. \tag{12}$$

Indeed, from (1) it follows that $b + c = d - a$. Therefore, $|b + c| \leqslant |d| + |a|$. But since according to (2) and (3), $|a| < 3 + \theta$, $|d| < 1$, it follows that $|b + c| < 4 + \theta$. From (12) and (2) we obtain

$$2(3 + \theta) \pm (b + c) > 2 + \theta. \tag{13}$$

In order to satisfy the requirements of the lemma, we define Δb and Δc as

$$\begin{aligned} \Delta b &= -\gamma_b \Delta a, \quad 0 < \gamma_b < 1, \\ \Delta c &= -\gamma_c \Delta a, \quad 0 < \gamma_c < 1. \end{aligned} \tag{14}$$

If $\gamma_b + \gamma_c = 1$, then (9) will be fulfilled.

If here γ_b and γ_c depend continuously on $a, b, c, \Delta a \neq 0$, then (11) is satisfied. In order to select γ_b and γ_c so that the inequalities (8) will not be violated, we introduce

$$\begin{aligned} \lambda_b^- &= 3 + \theta - b + \frac{\varepsilon}{2}, \quad \lambda_c^- = 3 + \theta - c + \frac{\varepsilon}{2}, \\ \lambda_b^+ &= 3 + \theta + b + \frac{\varepsilon}{2}, \quad \lambda_c^+ = 3 + \theta + c + \frac{\varepsilon}{2}. \end{aligned} \tag{15}$$

These numbers, which are positive because of (2), give the leeway which one has for the introduction of the corrections Δb and Δc; thus, for example, λ_b^- shows how much one may add to b in order that the sum b' may not exceed $3 + \theta + \varepsilon/2$ (see (8)).

The inequality (13) shows the correction Δa, which does not exceed 2 in absolute value, can be made to satisfy (8) by selecting γ_b and γ_c in (14) between 0 and 1. Namely, if $\Delta a > 0$, we set

$$\gamma_b = \frac{\lambda_b^+}{\lambda_b^+ + \lambda_c^+}, \quad \gamma_c = \frac{\lambda_c^+}{\lambda_b^+ + \lambda_c^+}, \tag{16a}$$

and if $\Delta a < 0$, we let

$$\gamma_b = \frac{\lambda_b^-}{\lambda_b^- + \lambda_c^-}, \quad \gamma_c = \frac{\lambda_c^-}{\lambda_b^- + \lambda_c^-}. \tag{16b}$$

We shall prove that (1)-(7), (14), (15), (16a), and (16b) imply (8), (9), (10), (11). Indeed, (9) is satisfied because of the obvious equation $\gamma_b + \gamma_c = 1$. From (12), (13), and (15) we obtain

$$2 < \lambda_b^\pm + \lambda_c^\pm = 2\left(3 + \theta + \frac{\varepsilon}{2}\right) \pm (b + c) < 15, \tag{17}$$

and therefore, γ_b and γ_c will depend continuously on $a, b, c, \Delta a$ when $\Delta a \neq 0$. Since $0 < \gamma_b, \ \gamma_c < 1$, the condition (11) is satisfied. From (5), (6), and (7) it follows that

$$\frac{|\Delta a|}{\lambda_b^\pm + \lambda_c^\pm} < \frac{1 + \varepsilon}{2} < 1.$$

Therefore, $|\Delta b| < \lambda_b^\pm$, $|\Delta c| < \lambda_c^\pm$. But from (15) it follows that

$$|b \mp \lambda_b^\pm| < 3 + \theta + \varepsilon, \qquad |c \mp \lambda_c^\pm| < 3 + \theta + \varepsilon,$$

and because of (7), (14), (16a), and (16b), $|b'| < 3 + \theta + \varepsilon$, $|c'| < 3 + \theta + \varepsilon$, i.e. (8) is fulfilled. It remains to prove that (10) holds. In case $|\Delta a| \leqslant \varepsilon$, (10) is, obviously, a consequence of the relations $0 < \gamma_b < 1$, $0 < \gamma_c < 1$. From (15) and (2) it follows that $\lambda_{b,c}^\pm > \varepsilon/2$. Hence, in view of (17),

$\gamma_{b,c} > \varepsilon/30$. From this we have in accordance with (14) that $|\Delta b|, |\Delta c| > |\Delta a| \varepsilon/30$. Therefore, in case $|\Delta a| \geqslant \varepsilon$ it follows that $|\Delta b| > \varepsilon^2/30$, $|\Delta c| > \varepsilon^2/30$. But since (see (14) and (16)) $|\Delta b| + |\Delta c| = |\Delta a|$, it now follows that $|\Delta b| < |\Delta a| - \varepsilon^2/30$, $|\Delta c| < |\Delta a| - \varepsilon^2/30$, namely the condition (10) and Lemma 14 have been proved.

Lemma 15. *Let*

$$a + b + c = d \tag{1}$$

$$|a|, \quad |b|, \quad |c| < 3 + \theta, \tag{2}$$

$$|d| < 1 + \varepsilon. \tag{3}$$

Let

$$d' = d + \Delta d, \tag{4}$$

where

$$|\Delta d| < 1 + \varepsilon, \tag{5}$$

$$0 < \theta \leqslant 1, \quad 0 < \varepsilon < 1. \tag{6}$$

Then one can determine the numbers $\Delta a(a, b, c, \Delta d)$ *and* $\Delta b(a, b, c, \Delta d)$ *so that if*

$$a' = a + \Delta a, \quad b' = b + \Delta b \tag{7}$$

then

$$a' + b' + c = d', \tag{8}$$

$$|a'| < 3 + \theta + \varepsilon, \quad |b'| < 3 + \theta + \varepsilon, \tag{9}$$

$$|a - \Delta b| < 3 + \theta + \frac{\varepsilon}{2} \tag{10}$$

and that

> *the dependence of* Δa *and* Δb *on* $a, b, c,$ *and* Δd, *which vary within the given (see (1)-(6)) limits, will be continuous, and if* $\Delta d \to 0$ *then* Δa *and* Δb *will go to zero.* (11)

Proof. For the fulfillment of the inequalities (9) it is sufficient that

$$\begin{aligned} &0 \leqslant \Delta a < \lambda_a^+ \quad \text{or} \quad -\lambda_a^- < \Delta a \leqslant 0, \\ &0 \leqslant \Delta b < \lambda_{b1}^+ \quad \text{or} \quad -\lambda_{b1}^- < \Delta b \leqslant 0, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\lambda_a^+ = 3 + \theta + \varepsilon - a, \quad \lambda_a^- = 3 + \theta + \varepsilon + a, \\ &\lambda_{b1}^+ = 3 + \theta + \varepsilon - b, \quad \lambda_{b1}^- = 3 + \theta + \varepsilon + b, \end{aligned} \tag{13}$$

since a and b satisfy relation (2).

In order that (10) be satisfied, it is sufficient that

$$0 \leqslant \Delta b < \lambda_{b2}^+ \quad \text{or} \quad -\lambda_{b2}^- < \Delta b \leqslant 0, \tag{14}$$

where

$$\lambda_{b2}^{+}=3+\theta+\frac{\varepsilon}{2}+a, \quad \lambda_{b2}^{-}=3+\theta+\frac{\varepsilon}{2}-a, \tag{15}$$

again because of (2).

Setting now

$$\lambda_{b}^{+}=\min(\lambda_{b1}^{+}, \lambda_{b2}^{+}), \quad \lambda_{b}^{-}=\min(\lambda_{b1}^{-}, \lambda_{b2}^{-}), \tag{16}$$

we find that

$$\lambda_{b}^{+}+\lambda_{a}^{+}>2, \quad \lambda_{b}^{-}+\lambda_{a}^{-}>2. \tag{17}$$

Indeed, by (1) we have, $a+d=d-c$. Therefore, $|a+b| \leqslant |d|+|c|$ and, by (2) and (3),

$$|a+b|<4+\theta+\varepsilon. \tag{18}$$

But because of (13), $\lambda_a^+ + \lambda_{b1}^+ = 6+2\theta+2\varepsilon-(a+b)$. Hence, it follows from (18) that $\lambda_a^+ + \lambda_{b1}^+ > 2$. At the same time we have, in view of (13) and (15), that $\lambda_a^+ + \lambda_{b2}^+ = 6+2\theta+2\varepsilon > 2$. In accordance with (16), the first inequality of (17) has been established; the second one can be proved to be valid in a similar way.

Now we set

$$\Delta a=\gamma_a \Delta d, \quad \Delta b=\gamma_b \Delta d, \tag{19}$$

where if $\Delta d > 0$,

$$\gamma_a=\frac{\lambda_a^+}{\lambda_a^+ + \lambda_b^+}, \quad \gamma_b=\frac{\lambda_b^+}{\lambda_a^+ + \lambda_b^+} \tag{20a}$$

and if $\Delta d < 0$,

$$\gamma_a=\frac{\lambda_a^-}{\lambda_a^- + \lambda_b^-}, \quad \gamma_b=\frac{\lambda_b^-}{\lambda_a^- + \lambda_b^-}. \tag{20b}$$

We shall prove that (1)-(7), (13), (15), (16), (19), (20a), and (20b) imply (8), (9), (10), and (11).

Indeed, from (20) we obviously obtain $\gamma_a+\gamma_b=1$, which implies (8) in view of (19), (1), (4), and (7). From (2), (13), (15), and (16) it follows that every λ is positive, and, hence, that $0<\gamma_a<1$, $0<\gamma_b<1$. Since, if $\Delta d \neq 0$, the γ_a and γ_b depend continuously on a, b, c, and Δd (see (20a) and (20b)), it now follows that (11) must be fulfilled because of (19).

Finally, from (17), (3), and (16) we obtain

$$\frac{\Delta d}{\lambda_a^+ + \lambda_b^+}<\frac{1+\varepsilon}{2}<1, \quad \frac{-\Delta d}{\lambda_a^- + \lambda_b^-}<\frac{1+\varepsilon}{2}<1. \tag{21}$$

Taking into account the fact that λ is positive, we obtain with the aid of (20a), (20b), (19) and (21) the inequalities

$$0 \leqslant \Delta a < \lambda_a^+ \quad \text{or} \quad -\lambda_a^- < \Delta a \leqslant 0,$$
$$0 \leqslant \Delta b < \lambda_b^+ \quad \text{or} \quad -\lambda_b^- < \Delta b \leqslant 0,$$

These inequalities and (16) imply the relations (12) and (14). From (12) follows (9), and (14) implies the inequality (10). This completes the proof of Lemma 15.

§9. Inductive preservation of property 4

In this section it will be shown how one must distribute the corrections in the method of §7 in order to fulfil the inductive requirement 4_{n+1}.

In §3 we introduced the numbers n_r. The oscillation of any function f of the considered class F on any component of the complement of Δ_n in Ξ does not exceed $1/r^2$ provided $n \geqslant n_r$. In particular, this will be the case on each branch σ_n if $n \geqslant n_r$.

We will denote by $f^n(\xi)$ the function defined on Δ_n which coincides on Δ_n with $f \in F$, and also its continuous extension (over any Δ_m $(m > n)$ and on the entire Ξ) which is constant on each component of the complement of Δ_n in Ξ. That such an extension exists, and is unique, follows directly from the fact that the intersection of Δ_n with the closure of each component $\Xi \setminus \Delta_n$ consists of one point. The function which corresponds to $f^n(\xi)$ on D_m we will denote by $f^n(x)$ on X. Let us introduce the function

$$g^m(x) = f^m(x) - f^{n_r}(x) \tag{1}$$

$(n_r < m \leqslant n_{r+1})$. On D_{n_r} this function is zero, depends continuously on $f \in F$, and does not exceed $1/r^2$ anywhere in view of the definition of r and $f^m(x)$.

Let $n_r \leqslant n < n_{r+1}$. Suppose that D_n and $f_k^n(x_k)$ are determined so that the requirements 1_n, 2_n, 3_n, and 4_n are satisfied. Then (for $n = n_r$ this is trivial)

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| < \left(3 + \frac{n - n_r}{n_{r+1} - n_r}\right)\frac{1}{r^2}. \tag{2}$$

Our problem consists of selecting s_n and $f_k^{n+1}(x_k)$ so that the requirements 3_{n+1} and 4_{n+1} will be fulfilled.

From here on, till the end of this section, r will be kept fixed. In order to shorten the formulas in all estimates, the factor $1/r^2$ will be omitted. Thus, the inequality (2) will be written now in the form

$$| f_k^n(x_k) - f_k^{n_r}(x_k) | < 3 + \frac{n - n_r}{n_{r+1} - n_r}. \tag{2'}$$

This can be considered as a temporary change of the scale of the f-axis, or one can suppose that we are confining ourselves to the case $r = 1$, $1/r^2 = 1$, because the remaining cases can be treated in an analogous manner.

Thus, let us assume that on D_n the requirements 1_n, 2_n, 3_n, and 4_n are satisfied. Then on D_n

$$g^n(x) = \sum_{k=1}^{3} g_k^n(x_k), \tag{3}$$

where $g_k^n(x_k) = f_k^n(x_k) - f_k^{n_r}(x_k)$ when $n > n_r$, and when $n = n_r$, $g_k^n(x_k) = 0$, $g^n(x) = 0$. As usual, the x_k are the coordinates of the point x. In (3) $x \in D_n$. The fulfillment of the requirement 4_n on D_n means that

$$| g_k^n(x_k) | < 3 + \theta_n, \tag{4}$$

where we have introduced the notation

$$\theta_n = \frac{n - n_r}{n_{r+1} - n_r}. \tag{5}$$

We will construct D_{n+1} in accordance with §7, and will select functions $g_k^{n+1}(x_k)$, which depend continuously on $f \in F$, in such a way that if $x \in D_{n+1}$

$$\sum_{k=1}^{3} g_k^{n+1}(x_k) = g^{n+1}(x), \tag{6}$$

$$| g_k^{n+1}(x_k) | < 3 + \theta_{n+1}. \tag{7}$$

Here, $n_r < n + 1 \leqslant n_{r+1}$, and one has to assume that

$$f_k^{n+1}(x_k) = f_k^{n_r}(x_k) + g_k^{n+1}(x_k), \tag{8}$$

in order to prove 3_{n+1} and 4_{n+1}.

When n increases from n_r to n_{r+1}, then θ_n increases from 0 to 1, and when n increases by 1, θ_n increases each time by $1/(n_{r+1} - n_r)$. We choose ε, $0 < \varepsilon < 1/(n_{r+1} - n_r)$. Then $\theta_n + \varepsilon < \theta_{n+1}$. This will be kept fixed in the remainder of this section.

Construction of $2s_n$. On D_n there exists a point p_n where s_n is to be attached.

Let us consider the rays l' and l'' (Figure 18), into which the point

p_n divides the line containing q_n. When the direction s_n has been chosen, then the three coordinates which pass through $2s_n$ will intersect these rays. Let us now select the direction $2s_n$ so that one of these rays l' (it will be called the principal ray) will intersect the planes of one direction; this direction will be called the principal direction. The planes of the remaining two directions will intersect the ray l'' (it will be called the minor direction). One of these planes is chosen arbitrarily and is called the minor plane. Finally, this entire operation can be performed by not picking s_n from the forbidden directions of §4, which is now assumed. The direction s_n has been chosen.

The following assertions are true.

A. From every sufficiently small semineighborhood u_{pr} of the point p_n on the principal ray, one can start on D_n a double scheme A of the interval u_{pr} so that two layers will lead away from the intervals of ranks $1, 2, \ldots, N$, where N is taken equal to $[30/\varepsilon^2] + 1$ (in order to have $N\varepsilon^2/30 > 1$), and such that among the directions of the layers of rank 1 there is no principal direction.

From every sufficiently small semineighborhood u_{m} of the point p_n on a minor ray one can start on D_n a double scheme B of the interval u_{m} so that two layers will lead away from the intervals of ranks $1, 2, \ldots, N$, and that the first direction is the principal one. The symbol N has the same meaning here as in the preceding paragraph. The scheme C with the same N-characteristic can be started from the semineighborhood u_{pr} if this neighborhood is small enough. Finally, if the interval u_{m} is sufficiently small, then, on D_n, one can start from this neighborhood a double generating scheme D whose first direction is a minor direction and for which the splitting takes place in the intervals of ranks $1, 2, \ldots, N$.

B. If the mentioned semineighborhoods u_{pr} and u_{m} are small enough, then the intervals in the construction of A will not intersect except for those which coincide by construction (on l' and l'').

These assertions are consequences of Lemma 13 of §7.

The segment $2s_n$ of the direction selected above, is attached to p_n in the neighborhood P of p_n which is now chosen in such a way that the following three requirements are satisfied:

1) *The oscillation of each function* $g_k^n(x_k)$, *which corresponds to* $f \in F$, *in* P *must be less than* $\varepsilon/4$.

2) *The neighborhood* P *must be so small that under the condition that* $s_n \subset P$ *it is possible to map* σ_n *on* s_n *(see §4), and to satisfy the*

requirements 1_{n+1}, 2_{n+1} (§§4, 6).

3) *The projection of* $2s_n$ *on* l' *and* l'' *along the principal and minor directions must fall within the above constructed (see assertions* A *and* B) *semineighborhoods* u_{pr} *and* u_m *of the point* p_n *on* q_n *if* $2s_n \subset P$.

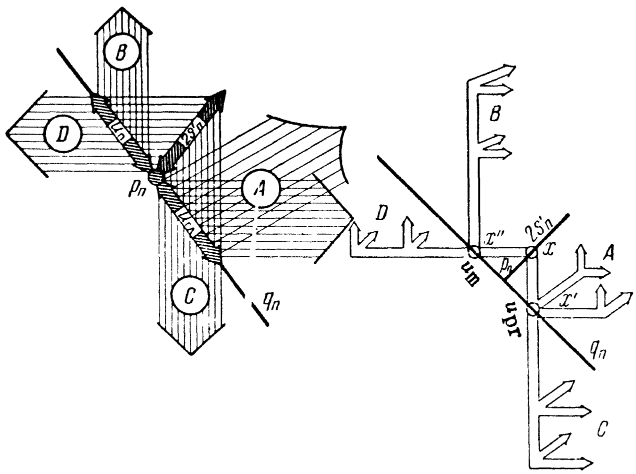


Figure 18. A double generating scheme of the interval $2s_n'$ on $D_{n+1} \cup 2s_n$. On the left, the first layers are shaded; on the right the representation is more schematic.

A sufficiently small neighborhood P of the point p_n will satisfy the requirement 1) because of the equicontinuity of the functions $f \in F$, the continuous dependence of $g_k^n(x_k)$ on $f \in F$, and the possibility of applying the Arzela-Ascoli lemma to the functions $g_k^n(x_k)$ and $f \in F$. Earlier (§§4, 6) it was established that for a sufficiently small P the requirement 2) is satisfied. Finally, the possibility of fulfilling the requirement 3) for small enough neighborhoods P is a consequence of the assertions A and B.

Now we select a neighborhood P that satisfies the requirements 1), 2), and 3). In P we pick $2s_n$ with the above chosen direction. We construct the mapping σ_n on s_n as in §4. On $D_{n+1} = D_n \cup s_n$ the conditions 1_{n+1} and 2_{n+1} are fulfilled because of 2).

Let us now construct on $D_n \cup 2s_n$ (Figure 18) a double generating scheme of the interval $2s_n$ of the following structure:

1. The initial interval $2s_n$ has two generating layers whose directions are the principal and the minor ones.

2. From the interval of the first rank, which lies on the principal direction, there starts a scheme A (see assertion A). From the remaining

intervals of the first rank to which the layer of the first direction leads, there issues the scheme C (see assertion A).

3. From the intervals of the first rank to which the layer of the minor direction leads, there start in the same way the schemes B (from u_m), and D (from the rest).

This construction is actually a generating scheme (double one). Indeed, the schemes A, B, C, D, and D_n do not intersect (except in the general intervals on u_{pr} and u_m). Since (except for the initial intervals) these schemes do not have intervals on u_{pr} and u_m, their layers of rank greater than 1 do not intersect u_{pr} and u_m, and hence not $2s_n$. The layers of the first rank do not intersect $2s_n$ because of the definitions of the principal and minor directions.

We will call the obtained scheme the large scheme.

Each zigzag of the large scheme which leads away from $2s_n$ either passes through at least N intervals distinct from $2s_n$, where the large scheme splits, or else terminates with a free end of lower rank. In any case, *from all the intervals of rank* $1, 2, \ldots, N$ *in the large scheme, which enter into the schemes* A *and* C, *and from the intervals of ranks* $2, 3, \ldots, N+1$ *in the large scheme, which enter into the schemes* B *and* D, *there issue two layers.* This follows from the assertions A and B.

Construction of the functions $g_k^{n+1}(x_k)$. We have seen (see (3)), that on D

$$g^n(x) = \sum_{k=1}^{3} g_k^n(x_k).$$

This formula can be considered to be the definition of $g(x)$ in the coordinate parallelepiped, stretched out over D_n in the product of the regions of definition of the functions $g_k^n(x_k)$ $(k = 1, 2, 3)$. On D_{n+1}, there is defined the function $g^{n+1}(x) = f^{n+1}(x) - f^{n_r}(x)$. The function $g_k^{n+1}(x_k)$ is to be found so that on D_{n+1} we would have

$$g^{n+1}(x) = \sum_{k=1}^{3} g_k^{n+1}(x_k). \tag{*}$$

In this manner, when $x \in D$, and, in particular, at the point p_n,

$$\sum_{k=1}^{3} g_k^{n+1}(x_k) = g^n(x).$$

We determine $g^{n+1}(x)$ on $2s_n$ so that the function

$$g^{n+1}(x) - g^n(x) = \Delta_0(x) \tag{9}$$

on $2s_n$ be even relative to the middle of this interval. It is obvious that the function $\Delta_0(x)$ is defined and continuous on $D_n \cup \overline{2s}_n$ and is different from zero only on $2s_n$.

We shall determine the functions $g_k^{n+1}(x_k)$ so that the equation (*) is fulfilled everywhere on $D_n \cup \overline{2s}_n$. We can do this by distributing the corrections along two directions that correspond to the larger scheme.

For the zeroth approximation to $g_k^{n+1}(x_k)$ we take ${}^0g_k^{n+1}(x_k) = g_k^n(x_k)$. If one substitutes the zeroth approximation in equation (*) for $g_k^{n+1}(x_k)$, the equation will be destroyed only on $2s_n$. We obtain the first approximation from the zeroth one by making corrections on the intervals of the layers of rank 1 of the large scheme. If $x \in 2s_n$, and if, for example, x_1 and x_2 are points (of these intervals of layers) that correspond to x, we obtain

$$\Delta_1^1(x) = \gamma_1\Delta_0(x),$$

$$\Delta_2^1(x) = \gamma_2\Delta_0(x).$$

But then if $\gamma_1 + \gamma_2 = 1$, and if

$${}^1g_1^{n+1}(x_1) = {}^0g_1^{n+1}(x_1) + \Delta_1^1(x_1),$$

$${}^1g_2^{n+1}(x_2) = {}^0g_2^{n+1}(x_2) + \Delta_2^1(x_2),$$

$${}^1g_3^{n+1}(x_3) = {}^0g_3^{n+1}(x_3),$$

the equation (*) will be vitiated on the intervals of the first rank only. In general, for the $(i-1)$st approximation the equation (*) will be destroyed on $D_n \cup 2s_n$ only on the intervals of the large scheme of rank $i - 1$. The ith approximation is then obtained from the $(i - 1)$st one by making corrections on the intervals of layers of rank i of the larger scheme. If x belongs to the layer u of rank $i - 1$ of the large scheme, and if, for example, u_2 and u_3 are intervals of layers that issue from u, while $x_2(x) \in u_2$, and $x_3(x) \in u_3$ correspond to x, and if the $(i - 1)$st disjoint at the point x is

$$\Delta_{i-1}(x) = g^{n+1}(x) - \sum_{k=1}^{3} {}^{i-1}g_k^{n+1}(x_k), \tag{10}$$

then we set

$$\begin{aligned} \Delta_2^i(x_2(x)) &= \gamma_2\Delta_{i-1}(x), \\ \Delta_3^i(x_3(x)) &= \gamma_3\Delta_{i-1}(x), \end{aligned} \tag{11}$$

where $\gamma_2 + \gamma_3 = 1$. (We do not assume that γ_2 and γ_3 are constants. They are functions of x, and will be determined later.) Now we suppose that

$$ {}^{i}g_2^{n+1}(x_2) = {}^{i-1}g_2^{n+1}(x_2) + \Delta_2^i(x_2) \text{ and so on} \tag{12}$$

and that the ith approximation is constructed so that the equation (*) is vitiated only on the intervals of rank i of the large scheme. The process described in §7 is called the distribution of corrections. Thanks to the construction of the large scheme, it proceeds in two directions when $1 \leqslant i \leqslant N$ or $2 \leqslant i \leqslant N+1$, and later terminates as in the case of a simple generating scheme when all intervals of some rank remain free.

We still have to take care of γ_1 and γ_2, for every distribution of the corrections, so that (see (7))

$$ | {}^{i}g_k^{n+1} | < 3 + \theta_{n+1} $$

and all corrections $\Delta_k^l(x_k)$ will be continuous, will vanish at the ends of the intervals of the layers of the large scheme, and will depend continuously on x and $f \in F$. Under these conditions the equation (*), i.e. (6), will be satisfied because of the results of the lemmas of §7; and, in view of (5), (6), (7), and (8), the fulfillment of the conditions 3_{n+1} and 4_{n+1} will have been established.

Lemma 16. *Suppose that the layer of the direction x_1 leads to the interval u of rank $i \geqslant 1$ of the large scheme, and that the layers of the directions x_2 and x_3 lead away from it. Let $x \in u$. Then*

$$ | {}^{i-1}g_k^{n+1}(x_k) | < 3 + \theta_n \quad (k = 1, 2, 3), \tag{13}$$

$$ \left| \sum_{k=1}^{3} {}^{i-1}g_k^{n+1}(x_k) \right| \leqslant 1. \tag{14}$$

Proof. Since u is an interval of rank i, it has not been touched previously in the distribution of the corrections: ${}^{i-1}g_k^n(x_k) = g_k^n(x_k)$. Hence (13) follows from (4), while (14) follows from the estimate of $g^n(x)$ (see definition $g^n(x)$).

Lemma 17. *In the hypotheses of Lemma 16, let $\Delta_{i-1}(x)$ be continuous on $\bar{u}$, vanishing at the ends of the disjoint u (see (10)), and depend continuously on $f \in F$. Furthermore, suppose that*

$$ | \Delta_{i-1}(x) | \leqslant 1 + \varepsilon. $$

Under these conditions one can find corrections $\Delta_2^i(x)$, $\Delta_3^i(x)$ so that

1) $|\Delta_2^i(x)|$, $|\Delta_3^i(x)| < \max\left(\left||\Delta_{i-1}(x)| - \frac{\varepsilon^2}{30}\right|, \varepsilon\right)$,

2) $|{}^i g_2^{n+1}(x_2)|$, $|{}^i g_3^{n+1}(x_3)| < 3 + \theta_{n+1}$,

3) $\Delta_2^i(x) + \Delta_3^i(x) = -\Delta_{i-1}(x)$,

4) $\Delta_2^i(x)$ *and* $\Delta_3^i(x)$ *will depend continuously on* $f \in F$, *and, when* $\Delta_{i-1}(x) \to 0$, $\Delta_2^i(x) \to 0$ *and* $\Delta_3^i(x) \to 0$. *(Here it is assumed in accordance with* (12), *that* ${}^i g_k^{n+1}(x_k) = {}^{i-1}g_k^{n+1}(x_k) + \Delta_i^k(x_k(x))$.*)*

Proof. The numbers $a = {}^{i-1}g_1^{n+1}(x_1)$, $b = {}^{i-1}g_2^{n+1}(x_2)$, $c = {}^{i-1}g_3^{n+1}(x_3)$, $d = g^n(x)$ (by Lemma 16), and $s = \Delta_{i-1}(x)$, $\theta = \theta_n$, and ε satisfy the conditions of the arithmetic Lemma 14. The conclusions of that lemma coincide in these notations with the conditions of the present lemma if one sets

$$\Delta_2^i(x) = \Delta b, \quad \Delta_3^i(x) = \Delta c.$$

Remark. It is obvious that Lemmas 16 and 17 remain valid if one makes a permutation of x_1, x_2, x_3 in their hypotheses and conclusions.

Lemma 18. *If the first disjoints* $\Delta_0(x)$, $\Delta_1(x)$, $\Delta_2(x)$ *do not exceed* $1 + \varepsilon$:

$$|\Delta_0(x)| \leqslant 1 + \varepsilon, \quad |\Delta_1(x)| \leqslant 1 + \varepsilon, \quad |\Delta_2(x)| \leqslant 1 + \varepsilon,$$

and if the functions of the first and second approximations ${}^1g_k^{n+1}(x_k)$, ${}^2g_k^{n+1}(x_k)$ *are less than* $3 + \theta_{n+1}$:

$$|{}^1g_k^{n+1}(x_k)| < 3 + \theta_{n+1}, \quad |{}^2g_k^{n+1}(x_k)| < 3 + \theta_{n+1},$$

then one can find $g_k^{n+1}(x_k)$,

$$|g_k^{n+1}(x_k)| < 3 + \theta_{n+1},$$

so that the equation (*) *will be satisfied. If* $\Delta_0(x)$ *and* $\Delta_1(x)$, ${}^1g_k^{n+1}$ *and* ${}^2g_k^{n+1}$, *depend continuously on* x *and* $f \in F$, *then* $g_k^{n+1}(x_k)$ *can be selected to be continuously dependent on* x *and* $f \in F$.

Proof. The Lemma 17 is in this case applicable to all intervals of the large scheme whose rank is greater than zero and from which issue (lead away) two layers. This is true, because in the use of Lemma 17 for the distribution of corrections the Δ_i decreases only when i increases. Making use of the conclusion 1) of Lemma 17, we see that if from the beginning of the large scheme up to a given one of its intervals there have been N intervals from which issued two layers, then in this distribution of corrections the quantity Δ_i is less than $\max(|1 + \varepsilon - N\varepsilon^2/30|, \varepsilon)$. But in the large scheme each zigzag with a free end either has at least N first intervals from which two layers issue, not counting the beginning, or all intervals of the zigzag up to the free one, included, have two issuing layers. Bearing in mind that $N\varepsilon^2/30 > 1$, we see that in both cases all corrections Δ_{n+1} are in absolute value less than ε. In the further distribution of the corrections with the aid of simple generating schemes of intervals of rank $N + 1$, as in Lemma 8 within §7, the functions $g_k^n(x_k)$ will receive corrections whose absolute value is less than ε, on the new intervals. But on these intervals

$$| g_k^{n+1}(x_k) | < | g_k^n(x_k) | + \varepsilon < 3 + \theta_n + \varepsilon < 3 + \theta_{n+1},$$

and since on the intervals of lower rank the inequality follows from Lemma 17 (rank > 1) and from the hypothesis of Lemma 18 (rank 0 and 1), the latter lemma is proved.

If one now determines Δ_0, Δ_1, Δ_2, ${}^1g_k^{n+1}$, ${}^2g_k^{n+1}$ so that they satisfy the conditions of Lemma 18, then, obviously, the construction of the function g_k^{n+1} under the requirements 3_{n+1} and 4_{n+1} will have been accomplished. Let us first consider the distribution and corrections from the interval of the zeroth rank $2s_n$. Here $\Delta_0(x) = g^{n+1}(x) - g^n(x)$, ${}^0g_k^{n+1}(x_k) = g_k^n(x_k)$, $\Delta_0(x)$ depends continuously on x and f, and vanishes at the ends $2s_n$ of the disjoint. For the sake of definiteness, let us assume that the coordinates of the principal and minor directions of the point $x \in 2s_n$ are x_1 and x_2. Let u_1 and u_2 be the corresponding intervals of the first rank of the large scheme, and let $x' \in u_1$, $x'' \in u_2$ be points which correspond to x (Figure 18). We will write also $x_1(x)$, $x_2(x)$, $x(x_1)$, $x(x_2)$, $x(x')$, $x_2(x_1')$, etc. to indicate this correspondence.

Lemma 19. *If the point* x *lies in the above-defined neighborhood* P *of the point* p_n, *then*

$$\left|\sum_{k=1}^{3} g_k^n(x_k)\right| < 1 + \varepsilon;$$

if $x \in 2s_n$, *then*

$$| \Delta_0(x) | = | g^{n+1}(x) - g^n(x) | < 1 + \varepsilon.$$

Proof. At the point $p_n = (p_{n_1}, p_{n_2}, p_{n_3}) \in D_n$

$$\sum_{k=1}^{3} g_k^n(p_{n_k}) = g^n(p_n)$$

(see definition $g^n(x)$),

$$| g^n(p_n) | \leqslant 1.$$

Because of the conditions on the neighborhood P, we find that in it

$$| g_k^n(x_k) - g_k^n(p_{n_k}) | < \frac{\varepsilon}{4}.$$

Using this and the preceding inequality, we obtain the first conclusion of the lemma.

The function $| g^{n+1}(x) - g^n(x) |$ is even (see the definition of $g^n(x)$) on $2s_n$, and it vanishes at the endpoints of this segment. Therefore, it will be sufficient to establish the second conclusion of the lemma on s_n.

By the definition of g^n we have

$$g^{n+1}(p_n) - g^n(p_n) = 0$$

and

$$| g^{n+1}(x) - g^{n+1}(p_n) | < 1.$$

The first requirement on P guarantees the fulfillment of the inequality

$$| g^n(x) - g^n(p_n) | < \frac{3}{4}\varepsilon,$$

which together with the preceding inequality proves Lemma 19.

Lemma 20. *For every $x \in 2s_n$ one can find $\Delta_1^1(x)$ and $\Delta_2^1(x)$ [we will write also $\Delta_1^1(x_1)$ and $\Delta_2^1(x_2)$ for $\Delta_1^1(x(x_1))$ and $\Delta_2^1(x(x_2))$ respectively] such that*

1) $\Delta_1^1(x) + \Delta_2^1(x) = \Delta_0(x)$,
2) $| {}^1g_k^{n+1}(x_k) | = | {}^0g_k^{n+1}(x_k) + \Delta_k^1(x_k) | < 3 + \theta_{n+1}$,
3) $| {}^0g_1^{n+1}(x_1) - \Delta_2^1(x(x_1)) | < 3 + \theta_n + \frac{\varepsilon}{2}$,

4) *$\Delta_1^1(x)$ and $\Delta_2^1(x)$ depend continuously on x and $\Delta_0(x)$, and when $\Delta_0(x) \to 0$ so does $\Delta_k^1(x) \to 0$.*

Proof. *The numbers*

$$a = {}^0g_1^{n+1}(x_1), \quad b = {}^0g_2^{n+1}(x_2), \quad c = {}^0g_3^{n+1}(x_3),$$

$$s = g^{n+1}(x), \quad \theta = \theta_n \text{ and } \varepsilon$$

satisfy (because of the fulfillment of condition 4_n and by the definition of θ_n and ε in Lemma 19) all the requirements of the arithmetic Lemma 15. Applying it, we obtain the conclusion of Lemma 20 if we set

$$\Delta_1^1(x) = \Delta a, \quad \Delta_2^1(x) = \Delta b.$$

In particular, for this definition of Δ_k^1 and ${}^1g_k^{n+1}$, we have

$$| \Delta_1(x) | < 1 + \varepsilon \text{ and } | {}^1g_k^{n+1}(x_k) | < 3 + \theta_{n+1}.$$

In order that the condition of Lemma 18 be satisfied, it is still necessary to determine Δ_k^2 and ${}^2g_k^{n+1}$ so that $| \Delta_3(x) | \leqslant 1 + \varepsilon$ and

$|\,{}^2g_k^{n+1}(x_k)\,| < 3 + \theta_{n+1}$. For those intervals of the large scheme where it splits, i.e. for all, except $u_2 \subseteq u_m$, this can be done with the aid of Lemma 17.

We introduced the point $x''(x)$ with the coordinates x_k'', whereby the point x'' and its coordinates are functions (linear) of the point x, or of any of its coordinates, and conversely. We thus have

$$^1g_2^{n+1}(x_2'') = {}^0g_2^{n+1}(x_2'') + \Delta_2^1(x_2'') \quad (x_2'' = x_2).$$

The remaining functions of the first approximation coincide with the functions of the zeroth approximation. Let us suppose that in accord with the distribution of the corrections along the directions of the large scheme,

$$^2g_1^{n+1}(x_1'') = {}^0g_1^{n+1}(x_1'') + \Delta_1^2(x_1''), \text{ where } \Delta_1^2(x_1'') = -\Delta_2^1(x(x_1'')).$$

Because of the choice of $\Delta_2^1(x)$ (see Lemma 20),

$$|\,\Delta_1^2(x_1'')\,| \leqslant 1 + \varepsilon.$$

Lemma 21. *In terms of the above notation*

$$|\,{}^2g_1^{n+1}(x_1'')\,| < 3 + \theta_{n+1}.$$

Proof. According to conclusion 3) of Lemma 20,

$$|\,{}^0g_1^{n+1}(x_1) - \Delta_2^1(x(x_1))\,| < 3 + \theta_n + \frac{\varepsilon}{2},$$

where x_1 is the coordinate of an arbitrary point $x \in 2s_n$, in particular $x(x'')$. In view of the first requirement on P (and u'', obviously, lies in P),

$$|\,{}^0g_1^{n+1}(x_1) - {}^0g_1^{n+1}(x_1'')\,| < \frac{\varepsilon}{4}.$$

Whence,

$$|\,{}^0g_1^{n+1}(x_1'') - \Delta_2^1(x(x_1))\,| < 3 + \theta_n + \frac{3}{4}\varepsilon < 3 + \theta_{n+1},$$

which was to be proved, because $\Delta_1^2(x_1'') = -\Delta_2^1(x(x_1))$.

Since each successive correction does not exceed, in the above described process, the preceding disjoints, we obtain from the mentioned fact that $|\,\Delta_1^2(x_1'')| \leqslant 1 + \varepsilon$, the result that $|\Delta_3(x)| \leqslant 1 + \varepsilon$. Bearing in mind Lemma 21, we can convince ourselves that our chosen $\Delta_k^2(x_k)$ does, indeed, satisfy the conditions of Lemma 18. This lemma has been proved, and we obtain functions $g_k^{n+1}(x_k)$ that fulfil all the requirements that were stated in the beginning of this section, and the inequalities (6) and (7). If we suppose (see (8)) that

$$f_k^{n+1}(x_k) = f_k^{n_r}(x_k) + g_k^{n+1}(x_k),$$

then we obtain a decomposition which has the properties 3_{n+1}, 4_{n+1}.

This completes the proof of the inductive lemma, because for $n = 1$, it is trivial.

Thus, the tree $X = \overline{\bigcup_{n=1}^{\infty} D_n}$, the homeomorphism of X on Ξ, and the decomposition of a function from F into the sum of functions of the coordinates on D_n have been constructed under the requirements of the inductive lemma.

§10. Proof of Theorem 3

As a result of the application of the processes described in the preceding section, one obtains trees D_n that are realizations of Δ_n, where $X = \overline{\bigcup_{n=1}^{\infty} D_n}$ realizes Ξ in the form of a subset of the three-dimensional space.

On each tree, every function $f \in F$ can be represented as

$$f(x) = \sum_{k=1}^{3} f_k^n(x_k),$$

where the continuous functions f_k^n of the coordinates x_k of the point $x \in D$ depend continuously on F. The sequence $f_k^n(x_k)$ converges uniformly as $n \to \infty$. This follows from the fact that $| f_k^n(x_k) - f_k^{n_r}(x_k) |$ is not greater than $4/r^2$ when $n_r < n \leqslant n_{r+1}$ and, hence,

$$| f_k^n(x_k) - f_k^{n_r}(x_k) | < \sum_{l=r}^{\infty} \frac{1}{l^2} \quad (n > n_r).$$

Let us denote by $f_k(x_k)$ the limits of these sequences. The sum of these three functions is a continuous function $f(x_1, x_2, x_3)$. For the point $(x_1, x_2, x_3) \in D_n$,

$$\sum_{k=1}^{3} f_k^m(x_k) = f(x) \text{ for all } m \geqslant n.$$

Therefore we have also for the limit the result

$$\sum_{k=1}^{3} f_k(x_k) = f(x) \quad \text{at each point} \quad x \in D_n \quad \text{for any} \quad n.$$

But $\bigcup_{n=1}^{\infty} D_n$ is an everywhere dense subset of its closure in X. The continuous functions $f(x)$ and $\sum_{k=1}^{3} f_k(x_k)$ coincide, therefore, on the entire tree X.

The proof will be complete if we can establish the continuous dependence of $f_k(x_k)$ on f.

Let $\varepsilon > 0$ be given. Let us consider an N so large that $|f_k^n(x_k) - f_k(x_k)| < \varepsilon/3$ for all $n \geqslant N$ and for all f_k^n, f_k which correspond to any function $f \in F$.

In view of the requirement 3_n, the functions $f_k^n(x_k)$, with a fixed $n = N$, depend continuously on $f \in F$. Therefore, f has a neighborhood of radius δ such that for $f' \in F$ and $|f' - f| < \delta$ it is true that $|f_k'^N(x_k) - f_k^N(x_k)| < \varepsilon/3$ for all x_k. From this it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|f' - f| < \delta$, then $|f_k'(x_k) - f_k(x_k)| < \varepsilon$, which was to be proved.

In this manner, *for every family F of real equi-continuous functions $f(\xi)$ defined on a tree Ξ, each of whose points has a branching index less than or equal to 3, one can realize the tree in the form of a subset X of the three-dimensional cube E^3 in such a way that every function of the family F can be represented in the form*

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1, x_2, x_3)$ is the image of $\xi \in \Xi$ in the tree X, the $f_k(x_k)$ are continuous real functions of a single variable, and f_k depends continuously on f in the sense of uniform convergence.

This is Theorem 3.

It implies Theorem 1, as was indicated in the Introduction.

APPENDIX

The space of the components of the level sets of a continuous function

That the set of the components of the level sets of a continuous function, defined on a square, is a tree is clear from Figure 19. Here we will assign an exact meaning to these words by following A.C. Kronrod [4] who introduced the concept of the space of the components of level sets, and K. Menger [3] who has made a study of trees. The theorems proved below are the main tools in both parts of the work. At the end of the Appendix there is placed (for the nonspecialists) a list of the basic concepts of point-set topology.

A. Construction of the metric space T_f

Let a continuous real function $f(a)$ be given on a continuum A (Figure 19). The set of a level, or a level set, is the set of all points a for which $f(a)$ has the same value. The set of a level is thus a closed set; the level sets do not intersect, and constitute all of A. Each set of a given level consists of components, continua that do not intersect each other.

Let us consider the entire set T_f of all components of all level sets of the continuous function $f(a)$. T_f will be called the space of components of the level sets of $f(a)$. We define a metric on this space so that T_f becomes a metric space. The components of the level sets of $f(a)$ are subsets of A and are points in T_f. Any given component will be denoted, the first time, by a capital letter, and after that by the same small letter.

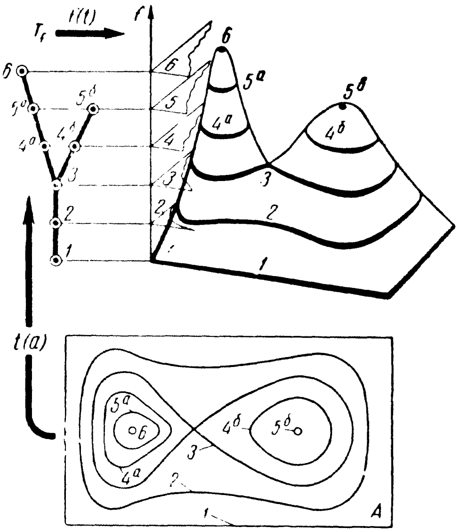


Figure 19. The set of levels, the space of components, and the graph of the function. Some components are denoted by numbers. The branching index of the points 1, 5^{σ}, $6 \in T_f$ is 1, of the points 2, 4^{a}, 4^{σ}, 5^{a} is 2, of the point 3 is 3. The corresponding components thus do not divide A, divide A into 2 parts, or into three parts, respectively.

As is known, the oscillation of a function on a set is the difference between its upper boundary and its lower boundary on the given set. The oscillation of a continuous function on a compact is finite and non-negative.

Let K_1 and K_2 be components of a level set of a continuous function $f(a)$ on a continuum A. By $\mathrm{P}(K_1, K_2)$, we denote the lower boundary of the oscillation $f(a)$ on all continua $F \subseteq A$ that contain K_1 and K_2:

$$\mathrm{P}(K_1, K_2) = \inf_{K_1 \cup K_2 \subset F \subset A} [\max_{a \in F} f(a) - \min_{a \in F} f(a)].$$

If one now defines the distance between points k_1 and k_2 of the space of components as $\rho(k_1, k_2) = \mathrm{P}(K_1, K_2)$, then T_f becomes a metric space. It is, indeed, obvious that

$$0 \leqslant \rho(k_1, k_2) = \rho(k_2, k_1) \leqslant \rho(k_1, k_3) + \rho(k_3, k_2).$$

In order to prove that $\rho(k_1, k_2) = 0$ implies $k_1 = k_2$, we have to make use of the next lemma.

Lemma 1. *For every open set* E $(E \subseteq A)$ *which contains a component* K *of a level set of a function* $f(a)$ *that is continuous on the continuum* A, *there exists a* $\delta > 0$ *such that if* $\rho(k, k_1) < \delta$, *then the component* K_1 *is contained in* E.

Proof. If the lemma were not true (Figure 20), there would exist a sequence of components K_n such that $\rho(k, k_n) < 1/n$ even though, for every n, K_n would contain a point b_n exterior to E. But by the definition of $\rho(k, k_n)$ there exist for $n = 1, 2, \ldots$ continua $F_n \subseteq A$, each of which contains K and K_n with the same n, and such that the oscillation of $f(a)$ on F_n is less than $2/n$. Therefore, the values of f on F_n must differ from the values of $f(a)$ at the points of K by less than $2/n$. The sequence of the points b_n $(n = 1, 2, \ldots)$ that are exterior to E have, because of the compactness of $A \setminus E$, a limit point $b \in A \setminus E$. The lower topological limit $\underline{\mathrm{lt}}\, F_n$ of the connected subsets F_n of the compact A is not empty, since it contains K. Thus the upper topological limit $\overline{\mathrm{lt}}\, F_n$ is connected. At the points of the upper limit, $f(a)$ takes on the same value as on K, because in every neighborhood of such a point there are points of F_n for every n (no matter how large), but these $f(a)$ will differ from $f(a)$ $(a \in K)$ by less than $2/n$.

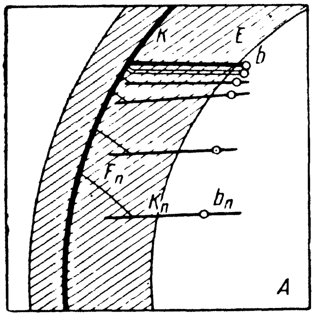


Figure 20. For Lemma 1. If for every $\rho(k, k_n)$ the components K_n have points b_n exterior to E, then K will have a point b exterior to E. The heavy curve is $\overline{\mathrm{lt}}F_n$.

The upper limit, obviously, contains also $K \subset E$ and $b \in A \setminus E$. This contradicts the fact that K is a component contained in E, because the upper limit, a connected set where $f(a)$ is constant, must lie entirely in one component. This establishes the lemma.

On the basis of Lemma 1, it follows from $\rho(k_1, k_2) = 0$ that K_1 and K_2 both lie in any given open set if this set contains either K_1 or K_2. But this can happen only if $K_1 = K_2$ because otherwise the distance between K_1 and K_2 in A would be positive.

The metric in T_f has thus been defined. The topology induced by this metric in T_f coincides with that of the work [4] if A is locally connected. A.S. Kronrod introduces a topology in T_f with the aid of neighborhoods which are defined as sets K that intersect with some open sets $E \subseteq A$. It can be easily seen that the topology on T_f depends only on the decomposition of A into components.

B. Two representations connected with a continuous function

Let us consider two representations, or mappings (Figure 19):

1. *$t(a)$ maps A on T_f and mates any point a of the continuum A with the point $t \in T_f$, where t is the component $T \subseteq A$ which contains a.*

2. *$f(t)$ maps T_f into the real axis f and mates any point $t \in T_f$ with a number f, the value of $f(a)$ at the points of the component $T \subseteq A$ that corresponds to $t \in T_f$.*

The use of the same letter f for $f(a)$ and $f(t)$ should not lead to any misunderstanding because these functions have entirely different definitions. We will say that the function $f(a)$ defined on A generates the function $f(t)$ on T_f.

If A is locally connected, then each of these mappings is continuous.

1. Since $f(a)$ is continuous, it is true that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the oscillation of $f(a)$ on any set of diameter less than δ is less than ε. Because of the local connectedness of A, any δ-neighborhood of a point $a \in A$ has a connected subneighborhood $u_\delta(a)$. Obviously, if b is contained in $u_\delta(a)$, the components K_a and K_b of the level sets that contain a and b are such that $\rho(k_a, k_b) < \varepsilon$.

2. If k_1, k_2 are two points of T_f that correspond to K_1, K_2, and if $\rho(k_1, k_2) < \varepsilon$, $a_1 \in K_1$, $a_2 \in K_2$, then $|f(a_1) - f(a_2)| < \varepsilon$, because the oscillation of a function is not less than its increment. Thus, $|f(k_1) - f(k_2)| < \varepsilon$.

The continuity of $t(a)$ and $f(t)$ has thus been proved.

If on A there is given a continuous function $g(a)$ which is constant on each component of every level set of the function $f(a)$, then $g(a)$ also generates a *continuous* function $g(t)$ on T_f (namely one which is equal to $g(a)$ at each point of the corresponding component), and we have $g(t(a)) = g(a)$. Indeed, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the oscillation of $g(a)$ on any set of diameter less than δ is less than ε. Let $E_\delta(T)$ be a δ-neighborhood of the component $T \subseteq A$, i.e. the set of points of A all of whose points are nearer than a distance δ from T. By Lemma 1, $t(T)$ has in T_f a neighborhood all of whose components lie in the interior of $E_\delta(T)$. Hence, we have found, for the given $\varepsilon > 0$, a neighborhood of the point $t \in T_f$ in which $|g(t) - g(t_1)| < \varepsilon$. This establishes the continuity of $g(t)$.

Let us now consider the counterimages of points for the mappings $t(a)$ and $f(t)$. The counterimage $t \in T_f$ is a component $T \subseteq A$, i.e., a connected set.

Definition [7]. A continuous mapping is said to be *monotone* if the counterimage of every point is connected.

By means of a monotone mapping one can transform a square with its boundary into a sphere, but not into a torus as we will see later. A monotone transformation is, so to speak, a contraction without "gluing together". Under monotone mappings there are preserved certain topological properties of sets. It is for this reason that the monotoneness of $t(a)$ yields some information on the space T_f.

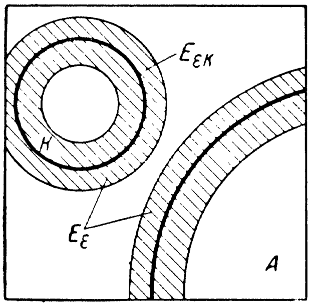


Figure 21. To Lemma 2. The construction of the neighborhood $E_{\varepsilon T}$ of the component T.

In the mapping $f(t)$, the counterimage of a point is the set of all points T_f where $f(t)$ takes on one value, i.e. the set of all components of a set of one level of $f(a)$.

From here on, A will be assumed to be locally connected, so that the functions $t(a)$ and $f(t)$ are continuous.

Lemma 2. *Every point $t \in T_f$ has a neighborhood $u(t)$ as small as we please (i.e. for every open subset $E \subset T_f$ that contains t, there exists an open set $u(t)$, $t \in u(t) \subset E$) such that its boundary consists of some points of two level sets of $f(t)$.*

Proof. Let T be the component that corresponds to t, and let α be the value of $f(a)$ at the points of T. Let us consider (see Figure 21) the open set E_ε of all points $a \in A$, where $|f(a) - \alpha| < \varepsilon$. E_ε contains T, and let $E_{\varepsilon T}$ denote the component of E_ε that contains T ($E_{\varepsilon T}$ is a region because A is locally connected. If a point lies in $E_{\varepsilon T}$, then the entire component containing this point of the level set $f(a)$ will, obviously, lie in $E_{\varepsilon T}$. It is clear that on the boundary of $E_{\varepsilon T}$, $f(a) = \alpha \pm \varepsilon$. We shall show that the image $u_\varepsilon(t)$ of the region $E_{\varepsilon T}$ under the mapping $t(a)$ satisfies the requirements of Lemma 2 for a small enough positive ε.

1. *$u_\varepsilon(t)$ is an open set in T_f that contains $t \in T_f$.*

This assertion is established by the application of Lemma 1 to $E_{\varepsilon T}$ and to the components contained in this region.

2. Suppose that K is a component which under the mapping $t(a)$ is transformed into one of the boundary points of $u_\varepsilon(t)$; then *K is contained in the boundary of $E_{\varepsilon T}$.*

The truth of this assertion can be proved by the application of Lemma 1 to the regions containing K.

3. For a sufficiently small positive ε, the oscillation of the function

$f(a)$ on E_ε, and on the continuum $\overline{E_{\varepsilon T}}$ is as small as we please. This implies that for a positive ε, small enough, $u_\varepsilon(t)$ *is an arbitrarily small neighborhood of* t.

This proves Lemma 2.

It follows from Lemma 2 that a level set of the function $f(t)$ is a zero-dimensional subset of T_f, since each of its points has an arbitrarily small neighborhood whose boundary is not intersected by the level set.

We have thus proved the next theorem.

Theorem 1. *The real continuous function* $f(a)$, *defined on a locally connected continuum* A *is the product of two continuous mappings: a monotone mapping* $t(a)$ *of the continuum* A *on the space* T_f *of the components of the level sets of the functions* $f(a)$, *and a mapping* $f(t)$ *of the space* T_f *on the real axis, under which the counterimage of every point* f *is of zero dimension. The function* $g(a)$, *which is continuous on* A *and constant on each component of the set of the level* $f(a)$, *generates a function* $g(t)$ *continuous on* T_f *such that* $g(a) = g(t(a))$.

C. Singly connected sets

Definition. A locally connected continuum M is said to be *singly connected* [7] if it cannot be represented as the sum of two continua whose intersection is not connected.

For example, the circle and the torus are not singly connected.

Remark. This definition is equivalent to the following ones.

A locally connected continuum is singly connected if every compact subset of it that divides it has a component that divides it.

A locally connected continuum is singly connected if every continuous mapping of it on a circle is homotopic to a mapping on a point.

It does not follow from singly connectedness that every simple closed curve on M can be contracted, without breaking it, into a single point.

Lemma 3 [7]. *The monotone image* F_2 *of a locally connected continuum* F_1 *is a singly connected, locally connected continuum.*

Lemma 4 [7]. *Under a monotone mapping of a compact, the complete counterimage of a continuum is a continuum.*

Proof of Lemma 4. In the opposite case, this complete counterimage could be divided into two nonintersecting closed sets A and B, whose images A' and B' would intersect. If C' were a point of intersection of the images, then its counterimage would intersect A and B, while at the same time it would lie in $A \cup B$, and hence would not be connected. Therefore, the mapping would not be monotone.

Proof of Lemma 3. F_2, the continuous image of a locally connected continuum, is a locally connected continuum. Let A_2 and B_2 be continua in F_2, $A_2 \cup B_2 = F_2$. In view of Lemma 4, the counterimages of A_2 and B_2, the sets A_1 and B_1, are continua. Obviously, $A_1 \cup B_1 = F_1$. Therefore, $A_1 \cap B_1$ is connected in view of the singly connectedness of F_1. But $A_2 \cap B_2$ is the image of $A_1 \cap B_1$ and hence is a connected set. This completes the proof of Lemma 3.

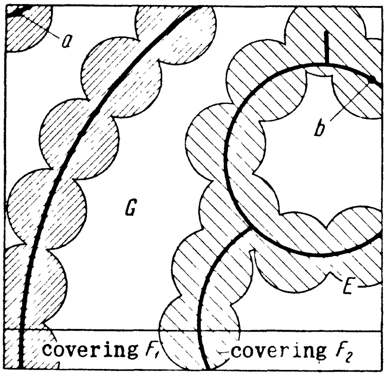


Figure 22. To Lemma 5. If it were true that $A \cup B = E$, $A \cap B = F_1 \cup F_2$, then the region G, which separates the point $a \in F_1$ from the point $b \in F_2$, would intersect the sets A, B that connect a and b. This would contradict that G is connected because $A \cap B = F_1 \cup F_2$ lies in the exterior of G.

Lemma 5 [7]. *The Euclidean cubes of any dimension, and the spheres of dimensions* 2 *and higher, are singly connected.*

Proof. Let us assume the opposite, and suppose, for the sake of definiteness, that the square $E = A \cup B$, where A and B are continua whose intersection $A \cap B$ consists of two nonintersecting compacts, i.e. $A \cap B = F_1 \cup F_2$. Let the distance between F_1 and F_2 be greater than $h > 0$. We will consider spherical neighborhoods with radius $h/3$ of all points of F_1 and F_2. These neighborhoods cover $F_1 \cup F_2$. It is possible to select from them a finite number of neighborhoods, and it is clear that they can be so chosen that F_1 and F_2 are covered, but their coverings do not intersect (Figure 22). It is obvious that the square is broken up by a finite number of curves each of which consists of a finite number of circular arcs, into parts of three types: those which are part of the covering of F_1, those which belong to the covering of F_2 and remaining ones. The coverings of F_1 and F_2 are at a distance greater than $h/3$ from each other. Therefore the remaining regions separate them. Let $a \in F_1$ and $b \in F_2$. Every broken line* that intersects a and b must intersect one of the regions of the remaining points. We consider it an obvious fact for E (a cube or sphere) that among the considered regions there is one G which separates a and b. We note only that this assertion is not true for a torus and other nonsingly connected sets. The continua A and B both contain a and b. Hence G contains

* And, hence, every continuum.

some points of A (which are not in B) and points of B (which are not in A, because $A \cap B = F_1 \cup F_2$). Both sets $A \cap \overline{G}$, $B \cap \overline{G}$ are closed and do not intersect, and their sum is $\overline{G}$, because $A \cup B = E$. This contradicts the connectedness of $\overline{G}$. This contradiction shows that the hypothesis on the incorrectness of Lemma 5 was false. Hence Lemma 5 is true.

By combining Theorem 1 and Lemmas 3 and 5, we obtain the following important property of T_f.

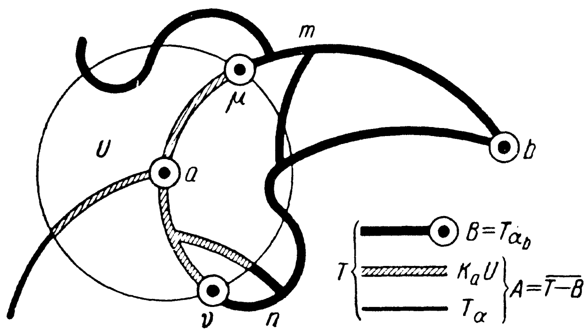


Figure 23. To Lemma 6. The locally connected one-dimensional continuum T that contains the cycle *ambna* can be broken into two connected parts (B is the heavy curve, $A = \overline{T \setminus B}$) by means of a nonconnected intersection.

Theorem 2. *The space of the components of the level sets of a continuous function defined on a singly connected locally connected continuum is a singly connected locally connected continuum. In particular, the space of the components of the level sets of a function that is continuous on a cube of any dimension and on a sphere of dimension greater than* 1 *is such a continuum.*

D. Trees

Definition. A *tree* is a locally connected continuum that does not contain homeomorphic images of a circle [3].

Since a tree is a locally connected continuum, any two points of it can be connected by a closed arc, and since the tree does not contain a homeomorphism of a circle, the arc is unique.

Lemma 6 [7]. *A one-dimensional singly connected continuum is a tree.*

Proof. Let us assume that such a continuum has two points a and b [Figure 23], which can be connected by nonintersecting arcs amb and anb. In view of the one-dimensionality of T, the point a has a neighborhood U, whose closure does not contain b, and whose boundary is of zero dimension.

Let K_aU be the component of the point a in this neighborhood. Because of the local connectedness of T, K_aU is an open set in T. Let us consider $T\setminus K_aU$. This closed set consists of the components, continua T_α, so that $T = (K_aU) \bigcup (\bigcup_\alpha T_\alpha)$. In particular, among these continua there is a component $T_{\alpha_b} \ni b$. Let us suppose that $B = T_{\alpha_b}$ and $A = \overline{T\setminus B}$. Obviously, $A \bigcup B = T$, B is a continuum, and A is a compact. We will show that A is connected. Indeed, from the fact that $T = (K_aU) \bigcup (\bigcup_\alpha T_\alpha)$, it follows that $T\setminus B = (K_aU) \bigcup (\bigcup_{\alpha\neq\alpha_b} T_\alpha) = \bigcup_{\alpha\neq\alpha_b} ((K_aU) \bigcup T_\alpha)$. It is easy to see that each set $(K_aU) \bigcup T_\alpha$ is connected. This implies that $T\setminus B$, and hence A, is connected.

Let us show also that $A \bigcap B$ contains the boundary of U. Indeed $A \bigcap B = \overline{B} \bigcap \overline{T\setminus B}$, i.e. $A \bigcap B$ is the boundary of $B = T_{\alpha_b}$ and, hence, is contained in the boundary K_aU, which is contained in the boundary of U. Each of the arcs amb and anb intersects the boundary of U, since a is in the interior of U, and b is in its exterior. Suppose that μ and ν are the first points of intersection of these arcs with the boundary of U starting from a. $A \bigcap B$ contains μ and ν, since it is obvious that these points are not contained in K_aU, but do lie in B, namely in the boundary of B. From the zero-dimensionality of the boundary of U it follows that $A \bigcap B$ is not connected, because a zero-dimensional connected set cannot have two distinct points. Thus, we have obtained a decomposition of T into the sum of two continua A and B whose intersection is not connected. This means that T is not a singly connected, locally connected continuum. This contradiction to the hypothesis of the lemma proves that T cannot contain homeomorphisms of a circle. Hence, T is a tree, which was to be proved.

Lemma 7. *The space of the components of the level sets of a real continuous function defined on a compact is at most one-dimensional.*

Proof. From Lemma 2 it follows that each point $t \in T_f$ has an arbitrarily small neighborhood whose boundary is contained in the sum of two level sets of $f(t)$ and is, therefore, either empty or zero-dimensional. Therefore, the space T_f is at most one-dimensional.

It is obvious that the space T_f can be zero-dimensional only in the case that the function f is a constant. Eliminating this case, when T_f is a single point, we can draw the following conclusion from Theorem 2, and from the Lemmas 6 and 7.

Theorem 3 [4]. *The space of the components of the level sets of a real continuous function defined on a locally connected, singly connected continuum. is a tree.*

The space of the components of the level sets of a real continuous function defined on an n-dimensional cube or on a sphere of dimension $n \geqslant 2$ *is a tree.*

The *branching index of a point of a tree* is the number* of parts (components) into which the tree falls after the given point is removed from the tree.

If the tree T is the space of the components of the level sets of a continuous function, then the branching index of a point of the tree is related to the structure of the component to which this point belongs.

Theorem 3[a] [4]. *The number of parts into which a component of a level set of a continuous function divides the region of definition of this function is equal to the branching index of the corresponding point of the space of the components.*

Proof. Indeed, the mapping $f(a)$ sets up a single-valued correspondence between the region of definition of the function f and the space of the components (Figure 19).

E. Structure of trees

We have seen that any two points of a tree can be connected by means of a simple arc, and by just one exactly. With the aid of this property one can obtain, following Menger [3], a convenient representation of trees, and can study their structure by reducing the investigation to finite trees, i.e. to trees with a finite number of branching points. We will confine ourselves to the consideration of trees which do not have any points with a branching index greater than three, since we use only this type of tree in Parts I and II of the present work.

Let Ξ be a tree whose points have branching indices not greater than 3. From the compact Ξ we pick a denumerable everywhere dense set $A : a_1, a_2, \ldots$ The pair of points a_1, a_2 determines in Ξ a unique simple arc $a_1 a_2$, which we denote by σ_0. From the remaining points $a_3, a_4, \ldots$ we pick the first point that is not contained in σ_0, and we denote it by $\bar{a}_3$. There is a unique simple arc $a_1\bar{a}_3$ in Ξ. We denote by ρ_1 the point nearest to $\bar{a}_3$ on the arc σ_0. (This point may happen to be a_1 or a_2.) Next, we denote the arc $\bar{a}_3\rho_1$ by σ_1, and, setting $\sigma_0 = \Delta_1$, $\Delta_1 \cup \sigma_1 = \Delta_2$, we see that when $i = 1$, the simple arc σ_i, the point ρ_1 and the finite trees Δ_i, Δ_{i+1} have the following properties:

1_i) $\Delta_{i+1} = \Delta_i \cup \sigma_i$,

2_i) $\sigma_i \cap \Delta_i = \rho_i$,

* Or the power, or cardinal number of the set of parts, if this set is infinite.

3_i) Δ_i contains all points a_k $(k \leqslant i+1)$.

If the finite trees Δ_i $(i = 1, \ldots, n)$ are constructed, and all Δ_{i+1}, Δ_i, σ_i, ρ_i $(i = 1, \ldots, n-1)$ satisfy the conditions 1_i), 2_i), 3_i), then it is easy to construct Δ_{n+1}. For this purpose we select, from the points of A that have not been included in Δ_n, the point with the smallest subscript. Let it be $\bar{a}_{n+2}$. In view of 3_{n+1}) the subscript of this point is greater than n. Hence, if we include it in Δ_{n+1} we guarantee the fulfillment of condition 3_n). The simple arc $a_1\bar{a}_{n+2} \subset \Xi$ that connects these points is uniquely determined. Suppose that ρ_n is the first point from $\bar{a}_{n+2}$ on $a_1\bar{a}_{n+2}$. We denote the simple arc $a_{n+2}\rho_n$ by σ_n. Then the conditions 1_n) and 2_n) are satisfied. In this manner we can determine Δ_n, σ_n, ρ_n for all $n \geqslant 1$, and the conditions 1_n), 2_n), 3_n) are all satisfied.

Each finite tree Δ_n has no point whose branching index is greater than 3. Indeed, in the opposite case there would be four simple arcs ad_r $(r = 1, \ldots, 4)$ that would intersect at a. Let us denote by B_r the set of those points of the tree that can be connected with a by means of simple arcs that intersect the arc ad_r (excluding, obviously, the point a). Such sets, for different r, will intersect each other, because the simple arc that connects two points of Δ_n is unique. The components of the set $\Xi \setminus a$ (which is open in the locally connected continuum Ξ) are open. Hence, any two points of such a component can be connected by a simple arc. This shows that every set B_r constitutes an entire component of $\Xi \setminus a$. Therefore, there should be at least four such components. But this is impossible, because the branching index of every point of the tree is less than 4.

Because of condition 3_i), and of the fact that A is everywhere dense

$$\overline{\bigcup_{n=1}^{\infty} \Delta_n} = \Xi.$$

The subsets $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ do not divide Ξ, because $\bigcup_{n=1}^{\infty} \Delta_n$ is connected, and through the addition of some limit points to a connected set, its connectedness is not destroyed. In particular, the points of the set $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ do not divide the tree Ξ into separate parts. The points of a tree which do not divide the tree are called ends of the tree.

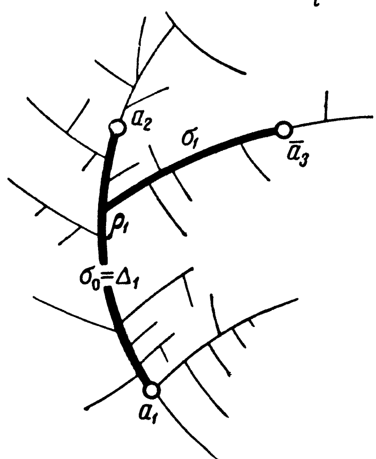


Figure 24. The heavy curve is Δ_2; Δ_1 and Δ_2 do not satisfy the requirement 4) of Lemma 8.

Before we give the conclusions of the study of the structure of a tree, we will change the construction of Δ_n so that the points ρ_n will not be ends of Δ_n. Suppose, for example, that σ_0 has for one of its ends a_2 the point ρ_{n_1}. We join σ_{n_1} to σ_0, and obtain a simple arc which we denote by σ_0^1. If one of the ends of σ_0^1 is ρ_{n_2}, then we join σ_{n_2} to σ_0^1, and obtain the simple arc $\sigma_0^2 = \sigma_0^1 \cup \sigma_{n_2}$, and so on, until either the end σ_0^N is not a ρ_m point for any m, or ad infinitum. In the first case we set $\sigma_0^N = \sigma_0^{\text{new}}$. In the second case, let l be a limit point of the ends σ_0^N. It will not divide Ξ, because it it did, then l would separate a_1 from some point $a_n \in A$,* and l would then belong to one of the sets Δ_n. By the construction of σ_n, l could not be a limit point of ends of Δ_n. It follows that $l \neq \rho_m$ for any m, and we have obtained for this second case that $\sigma_0^{\text{new}} = a_1 l$. After such a treatment of both ends of σ_0, we pick from the arcs σ_n the first one which is not contained entirely in σ_0^{new}, and repeat the same treatment of its ends. Hereby we will not touch the completed arcs; and, continuing this process, we will obtain a new system Δ_n^{new}, ρ_n^{new}, σ_n^{new}, whose elements we will denote simply by Δ_n, ρ_n, σ_n. This system will have, in addition to the properties 1), 2), 3), also the property

4) $\rho_m \neq \rho_n$ if $m \neq n$.**

We have thus proved the following lemma.

Lemma 8. *Every tree Ξ whose points have no branching index greater than 3 can be represented in the form*

$$\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n},$$

where the Δ_n are finite trees composed of arcs σ_n attached at the points ρ_n so that:

1) $\Delta_1 = \sigma_0$,

2) $\Delta_{n+1} = \Delta_n \cup \sigma_n$,

3) $\sigma_n \cap \Delta_n = \rho_n$,

4) $\rho_m \neq \rho_n$ *if* $m \neq n$, *and the points* ρ_n *are not ends of* Δ_n.

One can show that only the points ρ_n have a branching index greater than two, and that Lemma 8 without the condition 4) is true for every tree. This implies the next theorem.

Theorem 4 [3]. *Every tree Ξ consists of a set that is everywhere dense*

* Because the components of $\Xi \setminus l$ are regions.

** The old ρ's could coincide (Figure 24) if one connected successively two branches to ρ, the end of Δ. The new construction prevents this, and since Δ has no points with branching index greater than 3, property 4) is satisfied.

in Ξ and is composed of the points of an at most denumerable set of simple arcs which do not intersect pair-wise in more than one point, and of a set consisting of the ends of Ξ (which can be everywhere dense in Ξ and have the power of the continuum). The branching index of the points of Ξ is at most denumerable, and greater than two only in a denumerable set of points (namely, at the points of intersection of simple arcs indicated above).

Figure 25. To Theorems 4, 6, 7.

It is obvious that the representation of the tree in the form of Lemma 8 is not unique. The proof of Theorem 4 will not be given here, because this theorem is not being used in the present work.

Let us also consider the structure of the components of the remainder $\Xi \setminus \Delta_N$. This set is open in Ξ; its components are regions, and in each of them any two points can be connected by means of a simple arc, without passing outside the component.

Lemma 9. *Let Ξ, σ_n, ρ_n, Δ_n $(n = 1, 2, \ldots)$ be the objects defined in Lemma 8. Then the following statements are true.*

1. *The boundary of every component K of the set $\Xi \setminus \Delta_N$ consists of one point, namely of the point ρ_m $(m = m(N, K) \geqslant N)$.*

2. *Any two points of $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ lie in different components of $\Xi \setminus \Delta_N$ for N sufficiently large.*

Proof. 1. Let us suppose that this boundary has two distinct points a, $b \in \Delta_n \cap \overline{K}$ (Figure 26, 1). The points a and b have nonintersecting connected neighborhoods because Ξ is locally connected. Suppose that $a' \in u_a \cap K$ is a point of the first of these neighborhoods u_a, and $b' \in u_b \cap K$ one of the second neighborhood. The points a' and b' can be connected by means of a simple arc which lies entirely in K, while the points a and b belong to Δ_N as points of the boundary of K and can, therefore, be connected by a simple arc ab in Δ_N. The arcs ab and $a'b'$ do not intersect. From the fact that it is possible to connect a and a' by a simple arc in u_a, and b and b' by a simple arc in u_b, we conclude that in Ξ there is a curve $aa'b'ba$ that contains a homeomorph of the circle. Thus, the boundary of K must be a single point.

Since $\bigcup_{n=0}^{\infty} \sigma_n$ is everywhere dense in Ξ (by Lemma 8), there exists an arc σ_n that intersects the region K. Among such arcs, let σ_m be the one with least subscript. Obviously, $m > N$. Since Δ_{m+1} contains this arc (condition 2), Lemma 8), and since Δ_m does not intersect K, σ_m intersects the boundary of K. But this boundary is a single point that belongs to Δ_N and, hence, to Δ_m. Therefore (condition 3), Lemma 8) the truth of the first statement has been established.

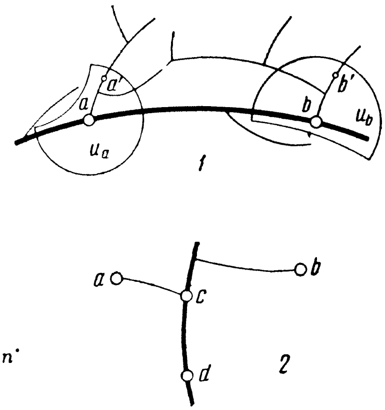


Figure 26. To Lemma 9. The heavy line is the tree Δ_N.

1. If the boundary of a component of the complement of Δ_N had two distinct points a and b, then Ξ would contain a homeomorph of a circle.
2. For sufficiently large N, Δ_N will separate any two points $a, b \in \Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$.

2. Suppose that a and b are two points of $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$, acd and bcd are simple arcs connecting a and b with the point $d \in \Delta_1$, c is the last point away from d that lies on both these arcs (Figure 26, 2). This point can coincide with only one of the points a, b, d, and we can, therefore, assume that $a \neq c$. In this case c separates a from d, for if a and d should belong to the same component of the open set $\Xi \setminus c$, one would be able to connect them by a simple arc not passing through c, and Ξ would contain a homeomorph of the circle, because this arc would not coincide with the simple arc acb. Therefore, $c \in \Delta_N$ for some N because it can be seen from Lemma 8 that the points $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ do not divide Ξ. This Δ_N separates a from b, for the points a and b can be connected by a simple arc acb, and hence by no other one. This establishes Lemma 9.

F. Realization of trees

All trees can be imbedded homeomorphically in a plane. We construct a planar set that is homeomorphic to a given tree Ξ whose points have branching indices not greater than three. In this we follow Menger [3].

Let $\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n}$ be the representation given in Lemma 8. We will select in the plane a straight line segment and an open triangle T_0 containing s_0. Let us map σ_0 on s_0 homeomorphically with the aid of the homeomorphism f_1. Then there will be on s_0 a point p_1 which is the image of ρ_1. We can construct an open triangle T_1, of diameter less than d_1 (this positive number will be defined later) with vertex at p_1, which does not intersect $D_1 = s_0$, except at the point p_1, and whose closure lies in T_0.

We select within T_1 a point and connect it with p_1. Then we obtain a segment s_1. We map σ_1 homeomorphically on s_1. We have constructed a homeomorphism f_2 of Δ_2 on $D_2 = s_0 \bigcup s_1$.

Suppose that we have constructed on R^2 complexes of segments (segment-like complexes) D_i from the segments s_i with the aid of the triangles T_i and the points p_i, and also let f_{i+1} be the homeomorphism Δ_{i+1}, on D_{i+1}, where $i, j = 1, 2, \ldots, n-1$ (see Figure 9) and

1_i) $D_1 = s_0$,

2_i) $D_{i+1} = D_i \bigcup s_i$,

3_i) $D_i \cap T_i = p_i$,

4_i) $(R^2 \setminus T_i) \cap s_i = p_i$,

5_i) if $i > j$, $\overline{T}_i \cap \overline{T}_j = 0$ or else $\overline{T}_i \subset T_j$,

6_i) the diameter T_i is less than $d_i > 0$,

7_i) f_i maps Δ_{i-1} the same way as f_{i-1} $(i > 1)$.

Let the arbitrary positive number d_n be given. On Δ_n there exists, in general, a point $p_n \in \sigma_k$ $(k \leqslant n)$ (if there is no such point, then Δ_n is the resulting tree). The homeomorphism f_n determines, on D_n, a point $p_n \in s_k$, the image of ρ_n. It is easy to select in the triangle T_k a small, open triangle T_n so that the following conditions hold:

1) one of this triangle's vertices is p_n,

2) $\overline{T}_n \subset T_k$,

3) T_n does not intersect s_k,

4) $\overline{T}_n$ does not intersect $\overline{T}_i$ $(i < n)$ if T_k does not lie in T_i,

5) the diameter T_n is smaller than d_n.

Having picked in T_n a point, and connected it to p_n, we obtain a segment which we denote by s_n. Obviously, by mapping σ_n homeomorphically on s_n, we determine the required homeomorphism f_{n+1} on Δ_{n+1} so that the conditions 1_i) to 7_i) will be satisfied. We have thus proved the truth of the following lemma.

Lemma 10. *Let* $\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n}$ *be the representation given in Lemma* 8. *Let* d_n *be a positive number. In the plane* R^2 *one can construct (with the aid of the segments* s_n, *the points* p_n, *and the triangles* T_n) *complexes* D_n *and homeomorphisms* $f_n : \Delta_n \to D_n$ *such that the conditions* 1_n)-7_n) *are satisfied for any* $n = 2, 3, \ldots$.

Now, let Ξ, Δ_n, D_n, σ_n, s_n, ρ_n, p_n, T_n, f_n $(n = 1, 2, \ldots)$ be such a system of objects, and suppose that $d_n > 0$, $d_n \to 0$ as $n \to \infty$.

Lemma 11. *In the notation given above,* $X = \bigcup_{n=1}^{\infty} D_n$ *is a tree that is homeomorphic to* Ξ, *and the homeomorphism can be constructed so that it coincides with* f_n *on* Δ_n *if* $n = 1, 2, \ldots$.

Proof. We define a sequence of mappings f'_n $(n = 1, 2, \ldots)$ of Ξ in X, namely on D_n, so that on Δ_n, f'_n coincides with f_n. We obtain f'_n on Ξ as $f_n(\varphi_n(\xi))$: the product of a continuous mapping φ_n of all of Ξ on Δ_n, and f_n which transfers Δ_n on D_n homeomorphically. Such a mapping will coincide with f_n on Δ_n if φ_n keeps every point of Δ_n unchanged. We have, therefore, defined a mapping φ_n on Δ_n so that $\varphi_n(\xi) = \xi$ $(\xi \in \Delta_n)$. Every component $K \subset \Xi \setminus \Delta_n$ has a unique boundary point ρ_m $(m = m(K, n) \geqslant n)$ in accordance with assertion 1 of Lemma 9. Let us set $\varphi_n(\xi) = \rho_{m(K,n)}$ $(\xi \in K)$. Now, $\varphi_n(\xi)$ is everywhere defined; we will show that this mapping is continuous. The point $\xi \in \Xi \setminus \Delta_n$ has a neighborhood K which is transformed into the same point as ξ. We still have to prove the continuity at the points of Δ_n. We will point out a neighborhood for such a point ξ, which will be transformed into an arbitrarily previously given neighborhooa u_ξ. A connected neighborhood $v \subseteq u_\xi$ of the point ξ will do. (This neighborhood exists because of the local connectedness of Ξ.) The points η of this neighborhood of ξ will go into its interior by the transformation φ_n. Indeed, this is obvious for the points $\eta \in \Delta_n$. Let $\eta \in \Xi \setminus \Delta_n$. Then η will be contained in some component K of the set $\Xi \setminus \Delta_n$. Let $\rho = \rho(K, n)$, be the boundary of K. Firstly, $\rho \in v$, because the points η and ξ of the region v can be connected by a simple arc lying in v. On this arc one can find a point of the boundary K because the initial point η of this arc belongs to K while the end ξ does not belong to K; this is the point ρ (Lemma 9). Secondly, the image of ρ under the mapping φ_n is ρ by the definition of φ_n. The continuity of φ_n has thus been proved, and it implies the continuity of $f'_n(\xi) = f_n(\varphi_n(\xi))$.

The sequence of mappings f'_n $(n = 1, 2, \ldots)$ converges uniformly on Ξ. Let a positive ε be given. From the fact that $d_n \to 0$, it follows that for $n \geqslant N(\varepsilon)$ $d_n < \varepsilon$. We will show that at every point $\xi \in \Xi$, and for any $n > N(\varepsilon)$, $\rho(f'_n(\xi), f'_{N(\varepsilon)}(\xi)) < \varepsilon$. This follows from the fact that the image of ξ under the mapping f'_n lies in the triangle $\overline{T}_m$ $(m > N(\varepsilon))$, when $p_n \in D_{N(\varepsilon)}$, or on D_n in accordance with the conditions 1) to 7) of Lemma 10.

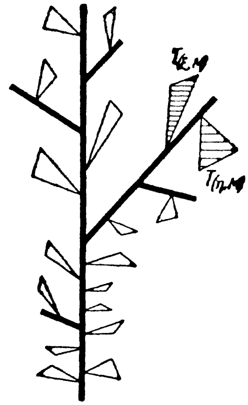


Figure 27. To Lemma 11. The heavy line tree D_6 is homeomorphic to Δ_6 that separates ξ and η. Some of the triangles T_m $(m \geqslant 6)$ have been drawn, for which $p_m \in D_6$. Among them $T_{(\xi,M)}$ and $T_{(\eta,M)}$ $(M > 6)$ have been shaded. They contain the images of ξ and η under all mappings f'_m $(m > M)$.

Thus, $f = \lim_{n\to\infty} f'_n$ is a continuous mapping. Obviously, it coincides with f_n on Δ_n. We shall prove that to distinct points of Ξ there correspond distinct images in X. This is obvious for the points $\xi \in \bigcup_{n=1}^{\infty} \Delta_n$. The points ξ and η of $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ lie, for sufficiently large N, in different components, K_1, K_2 of the complement of Δ_N (Lemma 9). From this, and from the definition of f'_n with the aid of properties 3) and 4) of Lemma 8, it follows that from some M on $(M > N)$ the images ξ, η under f'_m $(m \geqslant M)$ lie in different triangles $T_{(\xi,M)}$, $T_{(\eta,M)}$, whose closures intersect D_N (Figure 27). From the condition 5) of Lemma 10 we now see that $\overline{T}_{(\xi,M)} \cap \overline{T}_{(\eta,M)} = 0$, which shows that $f(\xi) \neq f(\eta)$. In exactly the same way, one can consider the case when $\xi \in \bigcup_{n=1}^{\infty} \Delta_n$, $\eta \in \Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$. The image of the entire tree Ξ under the mapping f contains all of D_n, and hence it is X. Therefore, f is a reciprocal one-to-one continuous mapping of the compact Ξ on X, i.e. it is a homeomorphism. This implies that X is a tree. Lemma 11 has thus been established.

The process used in the proofs of Lemmas 10 and 11 for the construction of the tree X, and of the mapping f in accord with the conditions 1) to 4) of Lemma 8, for Ξ, Δ_n, σ_n, ρ_n $(n = 1, 2, \dots)$ and $d_n \to 0$, can be called the *method of attaching branches*. Our result can then be formulated as follows.

Theorem 5 [3]. *Let there be given a tree* Ξ *whose points have no branching indices greater than three; then one can construct in the plane, by the method of attaching branches, a tree* X, *homeomorphic to* Ξ, *and a homeomorphism* f *between* Ξ *and* X.

The next more general theorem can be proved in an analogous way.

Theorem 6 [3]. *Every tree* Ξ *has a homeomorphic image in the plane.*

A set M is said to be universal for a class A_α if each set A_α has a homeomorphic image in M.

Theorem 7 [3]. *If in the representation of Theorem 4 the set of points of intersection of the simple arcs is everywhere dense, and if the branching index of* Ξ *at every one of its points is* n *(respectively, denumerably infinite), then the tree is universal for the class of all trees whose branching index does not exceed* n *(respectively, for all trees). The trees which are described above do actually exist.*

Theorems 6 and 7 are not used in this work. The reader can provide the proofs himself, or he can find them in the work [3]. We note without proof that the space of the components of the level sets of a continuous function defined on a square can be a universal tree. An example (for the case $n = 3$) is the function $F(x, y)$ constructed in Part I (§ 2) of this work.

Concepts and theorems of point-set topology used without further comment

1. **Concepts** ([6], Chapters VII and VIII; [7]; [8]; [9]).

Metric space. Topological space. Open and closed sets, boundary. Continuous mapping and homeomorphism. Everywhere dense set. Connectedness.

A compact is a metric space in which one can select from every infinite sequence a convergent subsequence. A continuum is a connected compact. The component of a point of a set (or simply a component of a set) is the largest connected subset that contains the given point.

A set is locally connected if every neighborhood* of any point contains a subneighborhood of this point.

A set is zero-dimensional if in any neighborhood of each of its points there is a neighborhood of the same point whose boundary is empty.

A set is one-dimensional if in any neighborhood of each of its points there lies a subneighborhood of the same point whose boundary is zero-dimensional.

* Here and in the sequel, a neighborhood of a point is any open set containing this point.

A region is an open connected set. A simple arc is a set that is homeomorphic to a segment of a straight line. The set A separates B from C if every continuum that contains B and C contains A. If A separates $b \in B \subset M$ from $c \in C \subset M$, then one says that A divides M.

The point x belongs to the upper topological limit $\overline{\mathrm{lt}}\, M_i$ of the sets M_i $(i = 1, 2, \ldots)$ if in every one of its neighborhoods there lie points of an infinite number of the sets M_i. The point belongs to the lower topological limit $\underline{\mathrm{lt}}\, M_i$ if in every one of its neighborhoods there are points of all but a finite number of the sets M_i.

2. **Theorems.**

A metric space which is a continuous image of a compact is a compact, of a continuum is a continuum, of a locally connected continuum is a locally connected continuum [6].

A reciprocal one-to-one continuous mapping of a compact is a homeomorphism [6]. A continuous mapping of a compact is uniformly continuous.

The components of a compact are continua; the components of an open set in a connected space are regions [6].

In a region of a locally connected continuum any two points can be connected by means of a closed arc ([3]; [7]; [9]).

The intersection of a decreasing sequence of continua $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ is a continuum [6].

If the sets $B \subset M$ and $C \subset M$ lie in different components of $M \setminus A$, then A separates B from C. If the closed set A of a locally connected continuum M separates B from C, then B and C lie in different components of $M \setminus A$.

A set that consists of two noncoinciding simple arcs with common ends contains a simple closed arc (homeomorph of a circle). The sum of four simple arcs aa', $a'b'$, $b'b$, ba have the same property if $a'b' \cap ba = 0$ and $aa' \cap bb' = 0$.

In a compact, the upper topological limit of a sequence of connected sets is connected, provided the lower topological limit is not empty [6].

A connected zero-dimensional set consists of one point [8].

A uniformly continuous function defined on a set that is everywhere dense in a compact, can be extended to a function over the entire compact. This extension is unique.

A reciprocal one-to-one, and similar (order preserving) correspondence between two sets s_1 and s_2, where s_1 is a denumerable everywhere dense subset of a segment I, and s_2 is a denumerable everywhere dense subset of a segment I_2, can be extended to a homeomorphism between the segments. Such an extension is unique.

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SOME QUESTIONS OF APPROXIMATION AND REPRESENTATION OF FUNCTIONS*

V. I. ARNOL'D

1. Statement of the problem. Let f and g be functions of two variables. Then

$$F(x, y, z) = f[x, g(y, z)]$$

is a function of the three variables x, y and z. This is an example of a superposition constituted of the functions f and g.

In general, a superposition constituted of given function, or a superposition of given functions, means a function which is obtained by substitution of some of the functions in place of the arguments in other functions of the set.

Superposition is a fundamental idea in analysis. For instance, the elementary functions are, by their definition, those functions which are superpositions of the functions $a(x, y) = x + y$, $b(x, y) = xy$, $c(x, y) = x^y$ and the known functions of one variable, $\log x$, $\sin x$, etc.

Obviously a superposition constituted of functions of two variables can be a function of an arbitrary number of variables. Here we shall consider the converse problem: which functions of several variables are superpositions of functions of a smaller number of variables.

The problem is due to D. Hilbert. The roots of the equations of 5th and 6th degree are superpositions of functions of two variables, when considered as functions of the coefficients. For equations of 7th degree it has not been possible to find such a representation; the representation reduces to functions of three variables. This led Hilbert to pose the following problem (the 13th of his "Mathematical Problems" [1]):

Is every analytic function of three variables a superposition of continuous functions of two variables? Is the root $x(a, b, c)$ of the equation

$$x^7 + ax^3 + bx^2 + cx + 1 = 0$$

a superposition of continuous functions of two variables?

It is necessary to consider the class of functions of which the superposition is constituted. It is easy to see [2] that any function of three variables can be represented as a superposition of functions of two variables by a proper choice of the latter functions. On the other hand, we have:

Theorem 1 (D. Hilbert). *There is an analytic function of three variables*

* Editor's note: translation into English published in Amer. Math. Soc. Transl. (2) 53 (1966), 192–201
Translation of V.I.Arnol'd: Some questions of approximation and representation of functions. Proc. Internat. Congress Math. (Edinburgh, 1958), Cambridge Univ. Press, New York, 1960, pp. 339–348

which is not a superposition of infinitely differentiable functions of two variables.

This can be explained in the following way: the number of independent coefficients of the Taylor series up to order n for functions of three variables is of order n^3, and for functions of two variables is of order n^2. Consequently, if a function of three variables is a superposition of a particular form (for example, $f[x, g(y, z)]$) of infinitely differentiable functions of two variables, then the coefficients of a sufficiently high order in its Taylor series must obey some relation depending on the form of the superposition (for the simplest form given above it is enough to take coefficients for terms up to fourth order). The different types of superposition form a countable set; there is an analytic function of three variables which does not obey any of the corresponding relations. It cannot be a superposition of analytic functions of two variables of any form whatever.

This explains the posing of the problem of the possibility of reduction to superpositions of continuous functions. Hilbert expected that a reduction of this type would also be not always possible.

For superpositions of the simplest forms this is actually the case [2,3,4].

2. Superpositions of smooth functions. Vituškin [5,4] showed that if the resolution of an arbitrary smooth function (having p derivatives) of three variables into a superposition of functions of two variables is possible, then this requires a lowering of the order of smoothness one and a half times.

We consider the class of functions given on the n-dimensional unit cube E^n which have all parital derivatives up to order p inclusive and are such that their pth order partial derivatives all satisfy a Hölder condition of degree $0 < \alpha \leq 1$. [1)]

Definition. The class of all such functions will be denoted by $F^n_{p,\alpha}$; here n is the *dimension*, $p + \alpha$ the *smoothness*, and $(p + \alpha)/n$ the *quality* of the functions of the class.

We note that, for example, functions which obey a Lipschitz condition (Hölder condition of degree 1) have smoothness 2.

Theorem 2 (Vituškin). *There is a function of class $F^n_{p,\alpha}$ which cannot be represented as a superposition of functions of better quality and of smoothness ≥ 1, i.e. functions of classes $F^m_{q,\beta}$ with $(q + \beta)/m > (p + \alpha)/n$, $q \geq 1$.*

For example, a function of three variables of smoothness 3 cannot always be represented as a superposition of functions of two variables of smoothness greater than 2.

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1) A function $f(x_1, x_2, \cdots, x_n)$ obeys a Hölder condition of degree α with constant M if $|f(x) - f(y)| \leq M \|x - y\|^{\alpha}$.

This result, which was obtained in a very complicated way by means of the theory of multidimensional variation [4], created by Vituškin on the basis of work of Kronrod, Adel'son-Vel'skiĭ, Landis and others [6] has since then been related by Kolmogorov [7] to the concepts of Shannon's information theory. The considerations which arise in this connection have independent interest, going far beyond the limits of the Hilbert problem which stimulated their development (see also the report of Kolmogorov "Linear dimension of topological vector spaces").

3. Entropy of function classes. Three decimal figures are needed to give the position of a point of the interval (0, 1) with an accuracy of 0.001. To give a point of a segment with accuracy ϵ the number of decimal places needed is, $\log 1/\epsilon$ and for a point of an n-dimensional cube $n \log 1/\epsilon$ figures are needed. It therefore makes sense to talk of "the number of signs which are necessary to specify a function $f \in F$ with accuracy ϵ" where F is some class of functions. This number is the "minimal capacity of a table of functions". It is known how much the capacity of a table of functions increases with an increase in the number of variables, and how it decreases when the smoothness of the functions increases (interpolation of high order!). It turns out that the minimal capacity of a table of functions of class $F^n_{p,\alpha}(C)$ to accuracy ϵ has an order of growth $(1/\epsilon)^{n/(p+\alpha)}$ as $\epsilon \to 0$. Here $F^n_{p,\alpha}(C)$ means the class of functions $f \in F^n_{p,\alpha}$ for which $|f|$ and the absolute values of the derivatives of order $\leq p$ do not exceed C and which obey the Hölder condition with constant C. In information theory it is convenient to consider not decimals to the base ten but binary decimals. For this reason logarithms to the base 2 occur in the exact formulations below.

Let X be a totally bounded* set in the metric space R. By definition for any $\epsilon > 0$ there is a finite set of points R such that X is covered by the balls of radius ϵ with centers at these points (ϵ-net in R for X).

Definitions. Let

$N^R_\epsilon(X)$ be the minimal number of points of an ϵ-net in R for X;

$N_\epsilon(X)$ be the minimal number of sets of diameter 2ϵ covering X;

$M_\epsilon(X)$ be the maximal number of ϵ-different signals in X, i.e. the maximal number of points of X such that balls of radius ϵ with centers in these points do not intersect.

Then

$$H^R_\epsilon(X) = \log_2 N^R_\epsilon(X), \quad H_\epsilon(X) = \log_2 N_\epsilon(X), \quad C_\epsilon(X) = \log_2 M_\epsilon(X)$$

are called respectively the ϵ-entropy of X relative to R, the ϵ-entropy of X and the ϵ-capacity of X.

*Translator's note. I.e. precompact.

It is easy to show that

$$H^R_{2\epsilon}(X) \le C_\epsilon(X) \le H_\epsilon(X) \le H^R_\epsilon(X).$$

By a *table to accuracy* ϵ for a function of a given class we mean a choice of 0's and 1's by means of which this function is determined with accuracy ϵ at every point. The number of tables formed of n signs is 2^n. Consequently, the result of Vituškin which shows that $H_\epsilon(X)$ is the minimal capacity of a table of functions of class X with accuracy ϵ in the case when X is compact in the space C[1] is natural.

Estimates of H_ϵ have been given for a series of important classes [8,9,11]:

(A) for the class of functions of n complex variables analytic in a region $G = \Pi^n_{i=1} G_i$ (where G_i is a region in the complex plane) uniformly bounded by some constant C and considered on $K = \Pi^n_{i=1} K_i$ (where K_i is a continuum contained in G_i)

$$\lim_{\epsilon \to 0} \frac{H_\epsilon}{C(K, G)\,(\log 1/\epsilon)^{n+1}} = 1;$$

here $C(G, K)$ is the geometrical characteristic of G and K calculated in the regular manner defined in [9].

(B) In [8,11] it was proved that

Theorem 3 (Vituškin and Kolmogorov).

$$k\left[\frac{C}{\epsilon}\right]^{n/(p+\alpha)} \le H_\epsilon(F^n_{p,\alpha}(C) \le K\left[\frac{C}{\epsilon}\right]^{n/(p+\alpha)},$$

where $0 < k \le K$ *are constants not depending on* ϵ.

It is easy to show by means of these formulas for $q \ge 1$ that the superposition of any fixed type of functions of quality $(q + \beta)/m$ of a given compact family $F^m_{q,\beta}(C_1)$ cannot give all the functions of a compact family $F^n_{p,\alpha}(C)$ if the quality of the latter is worse than $((p + \alpha)/n < (q + \beta)/m)$.

Roughly speaking the point is that the minimal capacity of a table of functions of class $F^n_{p,\alpha}(C)$ is of order $(1/\epsilon)^{n/(p+\alpha)}$ and the capacity of the table of all functions of class $F^m_{q,\beta}(C_1)$ which enter into a superposition of a concrete form is of order $(1/\epsilon)^{m/(q+\beta)}$. If the representation of all functions of class $F^n_{p,\alpha}(C)$ by such superpositions of functions of class $F^m_{q,\beta}(C_1)$ is possible, then a sufficiently full table of all the functions entering into the superposition could replace a table of the superposition functions to accuracy ϵ; consequently

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1) The continuous functions on a compact with metric $\rho(f, g) = \max|f - g|$.

$(1/\epsilon)^{n/(p+\alpha)} \leq K(1/\epsilon)^{m/(q+\beta)}$ where $0 < K < \infty$ is independent of ϵ. This means that $n/(p+\alpha) \leq m/(q+\beta)$: if the representation is possible, then it has to be in terms of functions of quality not better than the functions which are represented. By the well-known method of condensation of singularities we now construct a function of the class $F^n_{p,\alpha}(C)$ which cannot be represented by any superposition of functions of the classes $F^m_{q,\beta}(C_1)$ for any possible C_1 and $(q+\beta)/m > (p+\alpha)/n$. This is the theorem of Vituškin.

4. Superpositions of continuous functions. Hilbert's problem, however, is concerned not with smooth but with continuous functions. In this domain the results have turned out contrary to his hypothesis.

In 1956 Kolmogorov showed [12] that any continuous function given on the n-dimensional cube E^n for $n \geq 3$ has a representation

$$f(x_1, \cdots, x_n) = \sum_{r=1}^{n} h^r[x_n, g_1^r(x_1, \cdots, x_{n-1}), g_2^r(x_1, \cdots, x_{n-1})]$$

where the functions g of $n-1$ variables and the functions h of 3 variables are real and continuous.

Applying this representation several times one can see that any continuous function of $n \geq 4$ variables is a superposition of continuous functions of 3 variables.

The proof is very complicated. The basic apparatus used is the tree of components of the level sets of a function, introduced by Kronrod.

A level set of a function $f(x)$ is a set of all the values of x in its domain of definition for which, for some c, $f(x) = c$.

A *component* of a level set is any connected piece of the level set. On Diagram 1 the level sets for $0 \leq c \leq 1/2$ consist of one component those for $1/2 < c \leq 2/3$ of two components, and for $2/3 < c \leq 1$ of one.

A function is a map of its domain of definition onto its range of values. This map can be represented as the product of the following two maps:

(1) A map of the domain of definition onto the set of components of the level sets, each point being put into correspondence with the component to which it belongs.

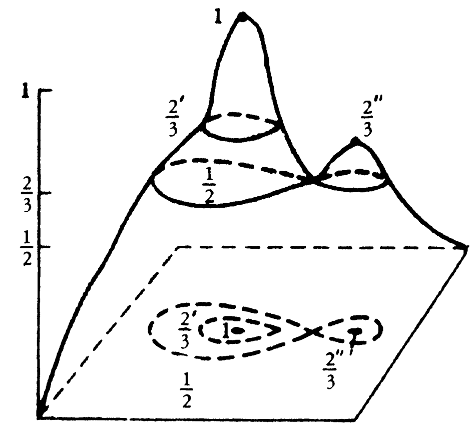

Figure 1. The level sets for 0, 1/2, 2/3 and 1 are designated, by the values to which they correspond. $2/3'$ and $2/3''$ are the two components of the level set for 2/3.

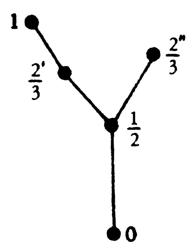

Figure 2. The tree of components of the level sets of the function of Figure 1. The components are designated as in Figure 1.

(2) A map of the set of components onto the set of values: each component goes into the value which the function takes on that component.

Let the domain of definition be a compact set F.

If the function is continuous then a "natural topology" is introduced for the set of components. Let A be a component and U any open set in F which contains A. Then the set of all components of level sets which meet U form a neighborhood U_A of the component A.

Now the first map is continuously montonic [1)] and the second continuous with zero-dimensional inverse images. It follows that the space of components is a locally connected simply connected continuum, i.e. a tree [13,14]. It is called the tree of the function.

The connection of this with the function is very simple. For instance, the function of Figure 1 has a tree which is homeomorphic to the horn shown in Figure 2. The number of segments into which a point of the tree divides the tree is equal to the number of parts into which the corresponding component of the level sets divides the domain of definition.

One of numerous noteworthy properties of trees [12] makes use of the result that there is a universal tree in the plane (one which contains a homeomorphic image of any tree). The functions which appear in the superposition indicated above and in later superpositions have trees which are universal, or almost universal, and this is an indication of the serrated nature of their graphs.

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1) I.e. the inverse image of each point is connected.

The Hilbert problem was formulated for functions of three variables, and the Kolmogorov theorem does not answer it. However, it turned out that, after a further complication in the construction and on arranging a tree in three-space in such a way that any function on it is represented as a sum of functions of the coordinates, one can represent any continuous function given on the three dimensional cube in the form

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij}[\phi_{ij}(x_1, x_2), x_3]$$

where h and ϕ are real continuous functions of two variables.

This proves, contrary to Hilbert's hypothesis, that any continuous function of $n \geq 3$ variables can be represented as the superposition of continuous functions of two variables.

Finally, soon after this Kolmogorov succeeded in proving the following theorem.

Theorem 4. *Any function continuous on the n-dimensional cube can be represented in the form*

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{2n+1} \chi_i\left[\sum_{j=1}^{n} \phi_{ij}(x_j)\right],$$

where χ and ϕ are real continuous functions of one variable.

Every continuous function, therefore, is a superposition of continuous functions of a single variable and of a single particular function of two variables, namely addition.

The method of the present article is more elementary than that of [12] and [15] and does not use the concept of a tree. The proof of Theorem 4 can easily be understood in the note [16].

The functions ϕ_{ij} are standard and independent of $f(x_1, \cdots, x_n)$. The construction in [16] can be regarded, for this reason, as a method of using the standard functions

$$\phi_i(x_1, \cdots, x_n) = \sum_{j=1}^{n} \phi_{ij}(x_j) \quad (i = 1, \cdots, 2n+1)$$

to imbed a special n-dimensional cube E^n homeomorphic to F in $(2n+1)$-dimensional space. The function $f(x_1, \cdots, x_n)$ induces a continuous function $f(\phi_1, \cdots, \phi_{2n+1})$ on F. F has the noteworthy property that any continuous function $f(\phi_1, \cdots, \phi_{2n+1})$ on F can be represented as the sum of functions $\chi_i(\phi_i)$ of the coordinates of a point in F.

Problems. The results obtained can be reduced to the following table:

Functions used / Functions represented	C^m	$q+\beta\to\infty$ $F^m_{q,\beta}$	F^m_∞	A^m
C^n	+	–	–	–
$p+\alpha \downarrow \infty$ $F^n_{p,\alpha}$	+	$q+\beta>(p+\alpha)m/n,\ q\geq 1$: – ; $q<1$, or $q+\beta\leq(p+\alpha)m/n$: ?	–	–
F^n_∞	+	?	–	–
A^n	+	?	–	–

Here C^n is the class of all continuous functions on the n-dimensional cube, F^n_∞ all the infinitely differentiable functions, A^n the analytic functions; the sign + means that all the functions of the class on the left are superpositions of functions of the class above $(m < n)$. From this the following problem arises naturally:

Problem. *Can every function of the class* $F^n_{p,\alpha}$ *be represented as a superposition of functions of the class* $F^m_{q,\beta}$ *for* $(q+\beta)/m = (p+\alpha)/n$? *for*

$$(q+\beta)/m > (p+\alpha)/n - \epsilon \quad (\epsilon > 0) \quad (m < n)?$$

Can every function of the class F^n_∞ *or of* A^n *be represented as a superposition of functions of* $F^m_{q,\beta}$? *of* F^m_∞?

The study of particular forms of superposition reveals very remarkable properties of classes of functions which can be represented as superpositions of a given form [17]. We derive the following problem.

Problem 2. *Find the simplest superposition of functions of* $m < n$ *variables in terms of which one can represent* (a) *a given function of* n *variables*; (b) *a given class of continuous functions of* n *variables*; (c) *all continuous functions of* n *variables. Investigate analogous problems for approximation with arbitrary degree of accuracy.*

For a more practical approach to such problems see [18].

The estimate of H_ϵ in Theorem 3 is coarse in so far as the constants k and K are not determined. It is not clear how they depend on C, n, p and α; the asymptotic behavior of H_ϵ is unknown, i.e. for which function $\phi(\epsilon)$ (supposedly, for constants) is

$$\lim_{\epsilon\to 0} \frac{H_\epsilon}{\phi(\epsilon)\,(1/\epsilon)^{n/(p+\alpha)}} = 1.$$

The difficulty of this question is clear if we note that in the case of the Euclidean metric, which is by far the easiest case, it corresponds to the problem of the densest packing of balls and the most economical covering of space by balls.

Problem 3. *Improve the estimate of* H *given in Theorem* 3. *Establish the asymptotic behavior of* $H_\epsilon(F^n_{p,\alpha}(C))$ *for* $\epsilon \to 0$.

Since the ϵ-entropy $H_\epsilon(F)$ characterizes the minimal capacity of a table of functions of class F to accuracy ϵ, a knowledge of the behavior of H_ϵ is essential for an estimate of different methods of approximating given functions, of introducing them to machines and of storing them in the memory of the machine. However, what is important here are the values of H_ϵ for small but finite ϵ.

Problem 4. *For various classes* ($F^n_{p,\alpha}(C)$ *etc.*) *give accurate estimates of* H_ϵ *for finite* ϵ. *Investigate methods of tabulation by means of which the capacity of the table approximates to the minimal value. Estimate the increase in the difficulty of using a tabulation with decrease of its capacity.*

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Translated by J. L. B. Cooper

KOLMOGOROV SEMINAR ON SELECTED QUESTIONS OF ANALYSIS*

V.I. Arnol'd and L.D. Meshchalkin
translated by Gerald Gould

The seminar was concerned with two groups of questions.

I. *Ill-posed problems* in analysis and mechanics, that is, problems whose solutions are everywhere discontinuously dependent on the parameter, in the main, 'problems with a small denominator'.

The simplest example of a problem with small denominators is the equation

$$g(z + 2\pi\mu) - g(z) = f(z)\,, \tag{1}$$

where $f(z)$ is given and $g(z)$ is an unknown function of period 2π. The solution is formally given by the Fourier series

$$g(z) = \sum_{n \neq 0} \frac{f_n}{\exp(2\pi i n\mu) - 1} \exp(inz)\,,$$

where f_n is the Fourier coefficient of $f(z)$.

If μ is rational, then some of the denominators will be zero, while if μ is irrational, then some of the denominators will be arbitrarily small. The convergence of the series is determined by the arithmetical properties of μ; for sufficiently small $f(z)$ the solution exists for almost all μ (in the sense of Lebesgue measure). The dependence of the solution on μ is everywhere discontinuous in general. Nevertheless, series of this kind occur both in analysis and in applications. They were first encountered in astronomy. "The difficulties encountered in celestial mechanics due to the existence of small denominators and the approximate commensurability of the mean motions relate to the very nature of things and cannot be avoided." (Poincaré).

The lectures by Vakhaniya on boundary-value problems with data on the entire boundary for the vibrating-string equation, by Boyarskiĭ and Vakhaniya on singularities of shells of negative curvature, by Arnol'd on maps of the circle onto itself, and by Kolmogorov on certain problems of classical mechanics

* Uspekhi Mat. Nauk **15**, No. 1, 247–250 (1960)

contained a survey of material put together here ([1] –[14]). Apart from problems that, like (1), are formally immediately soluble, certain problems have been considered that require perturbation theory even for their formal solution. It has not been possible to establish convergence of power series of a small parameter, which is usually required here (see, for example, [10]). To get round these difficulties Kolmogorov ([13]) applied Newton's rapidly convergent method. Of the new results along these lines we note the following:

1. An analytical investigation of maps of the circle onto itself enabling one, for example, to carry over results in the study of the vibrating-string equation on an ellipse to curves that are analytically close to an ellipse (V.I. Arnol'd).

2. A presentation of the results of [13] in terms of maps of the annulus, enabling one, for example, to solve D. Birkhoff's celebrated problem on the motion of a point on a billiard table in the case when the table is analytically close to an ellipse (Kolmogorov).

3. Establishing the analytic independence of solutions obtained by Newton's method on a small parameter and the *monogenic* (*in the sense of Borel*) dependence ([15]) of solutions of certain problems on a parameter playing the role of μ in (1) (Arnold). The conjecture that the first dependence was non-analytic, and the second was monogenic was expressed by Kolmogorov ([14]).

The lecture by Artsimovich and Leontovich was devoted to questions of "magnetic traps", which are necessary for the realization of controlled thermonuclear reactions. A rigorous qualitative study of the behaviour of a charged particle in magnetic fields of certain configurations is, it would seem, reminiscent of the analysis of problems of billiard type [16].

The consideration of an ill-posed problem shouldconclude with a discussion of the actual meaning of the mathematical result. In applications ill-posedness turns up after passing to the limit (for example, as certain parameters tend to 0 or ∞: instances or such parameters are time in qualitative theory, viscosity in hydrodynamics, and thickness in the theory of shells). The interpretation of the mathematical model (in which the passage to the limit is carried out *just before* the solution of the problem: the parameters are set *equal to* 0 or ∞) must consist in indicating which properties of the model correspond to which properties of the *prelimit model*. Such an interpretation can be produced only in a very small number of cases. However, such properties apparently can be interpreted in this way and have a real meaning, and not merely for "structurally stable" properties in the sense of Andronov and Pontriyagin [17].

II. Some sessions were devoted to *the study of mathematical models of the turbulent motion of an incompressible viscous liquid*. In his opening remarks on this theme Kolmogorov pointed out the following two main conclusions from experimental material which must lie at the basis of one's considerations:

1. As the viscosity ν decreases the laminar solutions of stationary problems usually become either unstable, or the stability is so slight that it is not actually observed.

2. As $\nu \to 0$ the decrease in smoothness of the solutions observed in practice is so strong that the order of dissipation of the energy per unit mass depends only on the typical veleocity of a typical path, and not on ν.

Kolmogorov put forward for consideratikon the solution of the system

$$\begin{aligned} \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \nu \Delta u + \gamma \sin y \,, \\ \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \nu \Delta v \,, \\ \frac{\partial u}{\partial x} &+ \frac{\partial u}{\partial y} = 0, \end{aligned} \tag{2}$$

where

$$\begin{aligned} \frac{D}{Dt} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \,, \\ \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \,, \end{aligned}$$

with periods $2a$ and 2π with respect to x and y, respectively, and

$$\int_{-a}^{a} v(x,y)dx = \int_{-\pi}^{\pi} u(x,y)dy = 0 \,. \tag{3}$$

He expressed the conjecture that for small ν there must be a structurally stable solution (that is, there is a non-trivial invariant measure μ_ν in the (u, v) space) such that $\mu_\nu \to 0$ as $\nu \to 0$, where the limit measure is concentrated on the continuous functions. Among other models we note the following, proposed by G.I. Barenblatt:

$$\frac{Du}{Dt} = v\frac{\partial^2 u}{\partial y^2} + f \,.$$

However, none of the models confirms Kolmogorov's conjecture, which remains open. Yaglom and Volevich have made a survey of papers by Burgess and Hopf closely related to this topic [18]–[20].

Meshchalkin and Sinaĭ have, with the help of straightforward calculations with continued fractions, managed to investigate completely the question of the stability of the laminar solution

$$u = \frac{\gamma}{\nu}\sin y, \quad v = 0$$

of equations (2), (3). Thus it turns out, in particular, that for $a < \pi$ the laminar soltion is always stable, while for $a > \pi$ (for a fixed velocity profile, that is, for $\frac{\gamma}{\nu} = \text{const}$) one can always find a γ_0† such that for $\nu < \nu_0$ the laminar solution is unstable. These arguments involving artificial models are

† *Translator's note*: This should be ν_0 .

justified by the fact that already the proof given by Wasow and Lin [21]–[23] of the stability of a Couette flow and the proof of the instability at high speeds of a Poiseuille planar flow based on computer calculations is not in total compliance with the mathematics. The results of Lin were even declared to be erroneous, because Petrov [24] presented a proof of the stability of Poiseuille planar flow at all speeds. However, at the seminar V.B. Lidskiĭ indicated a flaw in Petrov's proof (see [25]).

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ON ANALYTIC MAPS OF THE CIRCLE ONTO ITSELF*

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translated by Gerald Gould

After the appearance of the papers by Siegel [1], [2] and Kolmogorov [3], [4] problems with small denominators lost their unapproachability.

One problem of this kind is the following. Let $t(\phi)$ be a continuous function of period 2π with $\phi_1 + t(\phi_1) > \phi_2 + t(\phi_2)$ for $\phi_1 > \phi_2$. Then the function $T(\phi) = \phi + t(\phi)$ determines in natural fashion a *map* T of the circle of points $\exp(i\phi)$ onto itself. If $t(\phi) \equiv \alpha$, then T is a *rotation* through an angle α. The map $\Psi(\phi) = \phi + \psi(\phi)$ can also be regarded as a *change of variable* on the circle.; the new variable at the point ϕ is $\Psi(\phi)$, and the new variable at the point $T(\phi)$ is $\Psi(T(\phi))$. We say that the change of variable $\Psi(\phi)$ *converts* T *into* T' if $T'[\psi(T(\phi)] = \Psi[T(\phi)]$: T' is T written in terms of the parameter $\Psi(\phi)$. Clearly, T' is a map of the circle onto itself, namely, $T' = \Psi T \Psi^{-1}$.

Poincaré's Problem. *Under what conditions can T be converted into a rotation under a suitable change of variable?*

Poincaré [5] introduced the *rotation number* of T. It is necessary that the rotation number of T be equal to α. Denjoy [6] proved that *if the rotation number α of T is incommensurable with 2π and $\frac{dt}{d\phi}$ is of bounded variation. then the required change of variable exists.*

Many authors have considered the Dirichlet problem for the string equation in a rectangle and in an ellipse. In order to reduce the general case to these two cases one has to solve Poincaré's problem. It is therefore natural to restrict the admissible changes of variable by insisting on smoothness and solving Poincaré's problem for smooth T.

Until now this has only been achieved for analytic transformations that 'are slightly different from a rotation'. The idea is as follows.

Let $\mathbf{t}(\phi) = t_0 + \tilde{\mathbf{t}}(\phi) = t_0 + \sum_{n\neq 0} t_n \exp(in\phi)$ be the Fourier expansion of the given function. Poincaré's problem is equivalent to the functional equation $\Psi[\phi + t(\phi)] = \Psi(\phi) + \alpha$ or, which is the same,

$$\psi(\phi + t(\phi) - \psi(\phi) = \alpha - t(\phi). \tag{1}$$

* Uspekhi Mat. Nauk **15**, No. 2, 212–214 (1960) (Summary of reports announced to the Moscow Math. Soc.)

Suppose that $\alpha - t(\phi)$ is small. We select a change of variable $\overline{\Psi}(\phi)$ such that $\overline{T} = \overline{\Psi} T \overline{\Psi}^{-1}$ (that is, the T in the parameter $\overline{\Psi}(\phi)$) differs from a rotation by an angle much smaller than T. For this we define $\overline{\psi}$ from the 'equation of first approximation' (cf. (1)

$$\overline{\psi}(\phi + \alpha) - \overline{\psi}(\phi) = -\bar{t}(\phi)\,. \tag{2}$$

The Fourier coefficients of $\overline{\psi}(\phi)$ (if they exist) are

$$\overline{\psi}_n = \frac{t_n}{1 - \exp(in\alpha)} \qquad (n = \pm 1, \pm 2, \ldots)\,. \tag{3}$$

If α is commensurable with 2π, then there will be zeros in some of the denominators in the above equation, while if it is incommensurable, then there will be arbitrarily small denominators in the above equation. By a well-known theorem in the metric theory of Diophantine approximation, for almost all x there exists $c > 0$ such that

$$\left| x - \frac{m}{n} \right| > \frac{c}{|n|^3} \tag{4}$$

for all integers m and n. It is easy to see that the Fourier series with coefficients (3) converges to the smooth solution (2) if $t(\phi)$ is a sufficiently smooth function, and that $x = \frac{\alpha}{2\pi}$ satisfies (4). Making the change of variable $\overline{\Psi}(\phi) = \phi + \overline{\psi}(\phi)$ we find that $\overline{T} = \overline{\Psi} T \overline{\Psi}^{-1}$: $\overline{\phi} \to \overline{\phi} + t^1(\overline{\phi})$, where $t^1(\phi) = t_0 + \overline{\psi}(\phi + t(\phi)) - \overline{\psi}(\phi + \alpha)$, and $t^1(\overline{\phi})$ is obtained on substituting $\overline{\Psi}^{-1}(\phi)$ in place of ϕ. We now estimate the difference between $\overline{T}$ and a rotation by an angle α:

$$\begin{aligned} t^1(\phi) - \alpha &= t_0 - \alpha + \overline{\psi}(\phi + t(\phi)) - \overline{\psi}(\phi + \alpha) \\ &= t_0 - \alpha + \frac{d\overline{\psi}}{d\phi}\Big|_{\xi} (t(\phi) - \alpha)\,. \end{aligned}$$

One naturally expects that $|t(\phi) - \alpha| \sim \epsilon$ implies that $|\overline{\psi}| \sim \epsilon$ and $\left|\frac{d\overline{\psi}}{d\phi}\right| \sim \epsilon$; it can be shown that $|t_0 - \alpha| \sim \epsilon^2$, so that $|t(\phi) - \alpha| \sim \epsilon^2$. We proceed with the transformation $\overline{T}$ in the same way as with T and for each approximation the 'error' $|t^{(n)}(\phi) - \alpha|$ is squared. The rapid convergence typical of Newton's method paralyses the effect of small denominators.[2]

A transformation T is said to be *analytic* if the function $t(\phi)$ is analytic in a strip enclosing the real axixs.

Theorem *For $c > 0$, $R > 0$ there exist $M > 0$ and $\varrho > 0$ such that if an analytic transformation T of the circle satisfies the conditions*

$1°.\quad |t(\phi) - \alpha| < M \quad$ *for* $|\operatorname{Im} \phi| < R$,

$2°.\quad \left| \frac{\alpha}{2\pi} - \frac{m}{n} \right| > \frac{c}{n^3} \quad (m = 1, 2, \ldots;\ n = \pm 1, \pm 2, \ldots)$,

[2] Newton's method was first used for a similar purpose by Kolmogorov [4].

where α is the rotation number of T, then there exists a change of variable $\overline{\Psi}$ that is analytic in the strip $|\mathrm{Im}\,\phi| < \varrho$ and converts T into a rotation through an angle α.

In [8] more complicated theorems of this type are proved. It turns out, in particular, that the power series in a small parameter proposed by Poincaré converge. The space of maps of the circle onto itself is studied, and we consider certain approaches to the question of typicality and exceptionality of various cases.

It is interesting to note that the requirement 2° is essential: the change of variable in Denjoy's theorem (mentioned earlier) can even turn out to be not absolutely continuous in spite of the fact that T is analytic and $\frac{\alpha}{2\pi}$ is irrational.

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SMALL DENOMINATORS. I
MAPPINGS OF THE CIRCUMFERENCE ONTO ITSELF*

V. I. ARNOL'D

In the first part of the paper it is shown that analytic mappings of the circumference, differing little from a rotation, whose rotation number is irrational and satisfies certain arithmetical requirements, may be carried into a rotation by an analytic substitution of variables. In the second part we consider the space of mappings of the circumference onto itself and the place occupied in this space by mappings of various types. We indicate applications to the investigation of trajectories on the torus and to the Dirichlet problem for the equation of the string.

Introduction

Continuous mappings of the circumference onto itself were studied by Poincaré (see [1], Chapter XV, pp. 165–191) in connection with the qualitative investigation of trajectories on the torus. The problem of Dirichlet for the equation of the string can be reduced to such mappings, but the topological investigation turns out here to be insufficient (see [5]). In the first portion of the present paper we attempt an analytic refinement of the Denjoy theorem completing the theory of Poincaré [2].

Suppose that $F(z)$ is periodic, $F(z + 2\pi) = F(z)$, real on the real axis and analytic in its neighborhood, with $F'(z) \neq -1$ for $\operatorname{Im} z = 0$. Then to the mapping of a strip of the complex plane defined by $z \longrightarrow Az \equiv z + F(z)$ there corresponds an orientation-preserving homeomorphism B of the neighborhood of the points $w(z) = e^{iz}$:

$$w = w(z) \longrightarrow w(Az) \equiv Bw.$$

In this sense we say that A is an analytic mapping of the circumference onto itself.

Suppose that the rotation number * of A is equal to $2\pi\mu$. From Denjoy's theorem it follows that for irrational μ there exists a continuous inversible real function $\phi(z)$ of the real variable z, periodic in the sense that

$$\phi(z + 2\pi) = \phi(z) + 2\pi$$

and such that

$$\phi(Az) = \phi(z) + 2\pi\mu. \tag{1}$$

*We assume that the reader is acquainted with the results of the papers [1] (pp. 165–191, 322–335) and [2], which appear in the textbooks [3] (pp. 65–76) and [4] (pp. 442–456).

* Editor's note: translation into English published in Am. Math. Soc. Transl. (2) 46 (1965), 213–284
Translation of V.I. Arnol'd: Small denomnators. I. Mapping the circle onto itself. Izv. Akad. Nauk SSSR Ser. Mat. 25:1 (1961). Corrections in Izv. Akad. Nauk SSSR Ser. Mat. 28:2 (1964), 479–480

We shall say that ϕ is a new parameter and that when expressed in the parameter ϕ the transformation A becomes a rotation by the angle $2\pi\mu$. Such a function ϕ must be unique up to an additive constant.

In §1 it is shown that for certain irrational μ, in spite of the analyticity of $F(z)$, the function ϕ in (1) may turn out not to be absolutely continuous. The idea of this example consists of the following. Since under rotations of the circumference length is preserved, the reduction of a transformation to a rotation by an appropriate choice of parameter amounts to the determination of the invariant measure of the transformation. In the case of a rational rotation number the invariant measure is concentrated, as a rule, at separate points, the points of the cycles of the transformation. However, if the rotation number is irrational, but can be approximated extremely well by rationals, then the invariant measure retains its singular character, though it is distributed everywhere densely on the circumference.

The following conjecture appears to be plausible:

There exists a set $M \subseteq [0, 1]$ of measure 1 such that for each $\mu \in M$ the solution of the equation (1) *for any analytic transformation A with rotation number $2\pi\mu$ is analytic.*

At present this is proved only for analytic transformations sufficiently close to a rotation by the angle $2\pi\mu$ (§4, Theorem 2).* The proof consists in the construction of the solution of equation (1) by means of the solution of equations of the form

$$g(z + 2\pi\mu) - g(z) = f(z). \tag{2}$$

In the solution of this equation by the use of Fourier series, there appear small denominators, making the convergence difficult. The calculation of the successive corrections, adapting the solution of the equation (2) to the equation (1), is carried out by a method of the type of Newton's method, and the rapid convergence of this method guarantees the possibility of realizing not only all the approximations of the theory of perturbations, but also the passage to the limit.

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*Note added in proof. As this paper was going to press the author learned of the work of A. Finzi [38], [39]. From the results of [38] it follows that if the rotation number of a sufficiently smooth mapping of the circumference onto itself satisfies certain arithmetical requirements, then the transformation may be converted into a rotation by a *continuously differentiable* change of variables. Thus the method of A. Finzi does not require that the transformation be close to a rotation. This partly confirms the conjecture stated above. A. Finzi notes, however, that he does not see how to extend his method to the case when a higher smoothness of the substitution of variables is required. The present paper contains a partial answer to some of the questions posed by Finzi. For a partial answer to some of the questions posed here the reader is referred to the Finzi papers.

Newton's method was applied for a similar purpose by A. N. Kolmogorov [6]. Theorem 2 of the present paper is in a way a discrete analogue of his theorem on the preservation of conditionally periodic motions under small changes of the Hamilton's function. In distinction to [6] we have no analytic integral invariants at our disposal, but rather we seek them. Moreover, we prove (in Theorem 2) the analyticity of the dependence on a small parameter ϵ, from which there follows the convergence of all the series in powers of ϵ that are usual in the theory of perturbations.

A direct proof of the convergence of these series has not been achieved, and A. N. Kolmogorov has even conjectured* (before studying the paper [7] of K. L. Siegel) that they might diverge.

Another conjecture of Kolmogorov, stated by him in the report [8], turned out to be true: questions in which small denominators play a role are connected with the monogenic functions of Borel [9]. For our case this is established in §§7,8 and used in §11.

Certain important problems with small denominators were solved by K. L. Siegel (see [7], [33], [34], [35]). There is a direct connection between mappings of the circumference and the problem of the center for the Schroeder equation: *is it possible to make an analytic substitution of variables* $\phi(z) = z + b_2 z^2 + \cdots$ *which will convert a mapping of the neighborhood of the origin of the complex plane, given by the analytic function* $f(z) = e^{2\pi i \mu} z + a_2 z^2 + \cdots$, *into a rotation by the angle* $2\pi\mu$?

The result of Siegel in [7] is analogous to our Theorem 2 and may be obtained by the same method. The problem of the center is a singular case of the problem of the mapping of a circumference whose radius, in the singular case, is equal to zero. In comparison with the general case the position here is simpler, since the solution (the Schroeder series) may be formally written down directly. The application of Newton's method also gives the Schroeder series; in distinction to Theorem 2, each coefficient of the solution will be exactly defined after a finite number of approximations.

In the second part of the paper we cite the classical mappings of the circumference onto itself and discuss the question of the typicality of various cases. In §9 we introduce the function $\mu(T)$ (rotation number) on the space of mappings of the circumference. Further we study, for rational (§10) and irrational (§11) μ, the level sets $\mu(T) = \mu$ from the point of view of their structure (Theorems 6 and 7) and density (Theorems 5 and 8). Of greatest importance from the topological

*In a report to the Moscow Mathematical Society on January 13, 1959.

point of view are the rough mappings (the word "rough" being taken in the sense of Andronov and Pontrjagin [10]) with normal cycles and rational rotation numbers; these mappings form an open everywhere dense set.* From the point of view of measure in finite-dimensional subspaces the ergodic case also is typical. In §12 we consider the two-dimensional subspace of mappings $x \to x + a + \epsilon \cos x$.

In §§13 and 14 the preceding results are applied to the qualitative investigation of trajectories on the torus and to the Dirichlet problem for the equation of the string.

I wish to express my thanks to A. N. Kolmogorov for his valuable advice and assistance.

Part I
On analytic mappings of the circumference onto itself

The basic content of the first part of this paper is contained in §§4–6 (Theorem 2). For an understanding of the proof of Theorem 2 (§§5,6) it is necessary to study subsections 2.1 and 2.3 of §2 and subsection 3.3 of §3. For the lemmas on implicit functions and on finite increments contained in §3 one may turn at need to the references. Each of §§1, 2, 7 may be read independently of all the rest. In §8 we prove a generalization of Theorem 2 (Theorem 3), used in the second portion of the paper.

§1. The case when the new parameter is not an absolutely continuous function of the old parameter

1.1. In this section we construct an analytic mapping A of the circumference C, subsets G_n $(n = 1, 2, \cdots)$ of the circumference and integers N_n $(n = 1, 2, \cdots)$ such that:

1. $\text{mes}\, G_n \to 0$ as $n \to \infty$.
2. $A^{N_n} (C \setminus G_n) \subset G_n$.
3. The rotation number μ of the transformation A is irrational.

This transformation A cannot be converted into a rotation by an absolutely continuous change of variables. Indeed, let ϕ be a continuous parameter in which the transformation becomes a rotation by the angle $2\pi\mu$ (ϕ exists from Denjoy's theorem). Suppose that $G \subset C$. The measure of the set $\phi(G)$ of values $\phi(x)$, $x \in G$, coincides with the measure of $\phi(A^N G)$, since these sets superpose under a rotation. Therefore it follows from condition 2 that:

$$2\pi - \text{mes}\, \varphi(G_n) \leqslant \text{mes}\, \varphi(G_n)$$

*Note added in proof. This result was also obtained by V. A. Pliss in the paper [43], published while this paper was being printed.

and

$$\operatorname{mes} \varphi (G_n) \geqslant \pi.$$

In view of condition 1, ϕ is not an absolutely continuous function on C.

1.2. For the construction we use the following lemmas.

Lemma α. *Let the transformation A of the circumference be semistable forward* and analytic in the neighborhood of the real axis, and suppose that the points $z_0, z_k = A(z_{k-1})$ $(0 < k < n)$ form a cycle, i.e., $A(z_{n-1}) = z_0$. Then for any $\epsilon > 0$ there is in the indicated neighborhood of the real axis a transformation A' differing from A by less than ϵ and having exactly one cycle, in fact $z_0, z_1, \cdots, z_{n-1}$.*

Proof. We construct a correction $\Delta(z)$ analytic in the strip in question, vanishing at the points $z_0, z_1, \cdots, z_{n-1}$ and positive on the remainder of the real points.

Put

$$A'(z) = A(z) + \varepsilon' \Delta(z);$$

for sufficiently small $\epsilon' > 0, |\epsilon' \Delta(z)| < \epsilon$ in the indicated strip and $A'(z)$ is a transformation of the circumference. Evidently the transformation $(A')^n$ moves forward all the points z not less than the transformation A^n; furthermore the points $z_0, \cdots, z_{n-1}$ move by $2\pi m$, and the remaining points by not less than $2\pi m$. Lemma α is proved.

Definition. Suppose that A is a transformation of the circumference C and that G is a set on C. We shall say that the transformation A has property 2 relative to G and N if $A^N (C \setminus G) \subset G$.

Lemma β. *Given a transformation A with the single cycle $z_0, \cdots, z_{n-1}$ and any $\epsilon > 0$, then A possesses property 2 relative to the set G_ϵ of points of the ϵ-neighborhood of the cycle and any N exceeding some $N_0(\epsilon)$.*

Proof. Suppose that $z_i < x < z_j$, where $x_i x_j$ is one of the arcs into which the cycle divides the circumference. The points $A^{kn}(x)$ $(n = 1, 2, \cdots)$ lie on the arc $z_i z_j$ and form a monotone sequence (for more details see §10). Therefore it follows that in the case when the transformation A is semistable forward (the case of backward semistability is completely analogous),

$$A^{kn}(x) \underset{k \to +\infty}{\longrightarrow} z_j.$$

Indeed, suppose that λ is the limit of the monotone sequence $A^{kn}(x)$. Then λ is invariant with respect to A^n and belongs to a cycle satisfying the inequalities

$$z_i < \lambda \leqslant z_j.$$

*This means that for some integers m, n and any real z, $A^n(z) \geq z + 2\pi m$, with equality attained.

Thus

$$\lim_{k\to\infty} A^{kn+l}(x) = A^l(z_j).$$

The same is true for the other intervals into which the cycle divides the circumference.

Consider the points $x_i = z_i + \epsilon$. By what has been proved, beginning with some $N_0(\epsilon)$, all the points $A^N x_i$ lie in an ϵ-neighborhood of the cycle. Evidently that N_0 is the one desired.

Lemma γ. *Suppose that the transformation has property 2 relative to G and N, and suppose that $\epsilon > 0$. Then there exists a $\delta > 0$ such that each transformation B differing from A by less than δ has property 2 relative to N and the ϵ-neighborhood of G.*

Proof. The lemma follows in an obvious way from the continuous dependence of A^N on A.

Lemma δ. *Suppose that A is a semistable forward transformation, $B(z) = A(z) + h$, $h > 0$. Then the rotation number μ of the transformation B is strictly larger than the rotation number m/n of the transformation A.*

Proof. Evidently $\mu \geq m/n$. In addition $B^n(z) > A^n(z)$ and therefore B does not have a cycle of order n. Hence $\mu > m/n$.

Lemma ϵ (degenerate case of Liouville's theorem). *If the inequality $|\alpha - m/n| < c/|n|$ for any $c > 0$ has an infinite set of irreducible solutions m/n, then the number α is irrational.*

Proof. If $\alpha = p/q$, then for $n > q$

$$\left|\frac{p}{q} - \frac{m}{n}\right| > \frac{\frac{1}{|q|}}{|n|},$$

since the quotient m/n is irreducible, so that $|pn - qm| \neq 0$ for $q < n$.

1.3. The transformation A is formed as a limit of a sequence of transformations A_n with rational rotation numbers. Beginning with the transformation $z \to A_1(z)$, we shall suppose that it has the following properties:

1_1. A_1 is analytic in the strip $|\operatorname{Im} z| < R$, and in this strip $|A_1(z)| < C/2$.

2_1. The rotation number of A_1 is rational: $\mu_1 = p_1/q_1$.

3_{1a}. A_1 is semistable forward.

3_{1b}. A_1 has exactly one cycle.

The existence of such an A_1 is evident: from each A_1'' with property 1_1 one may obtain, with an appropriate choice of $h > 0$, $A_1' = A_1'' + h$ with properties 1_1, 2_1 and 3_1, and then one may correct A_1' to A_1 using Lemma α. The subsequent transformations A_n are obtained from the preceding ones by using a process based on the following Induction Lemma.

Induction Lemma. *Suppose that* $\delta_n > 0$ *and suppose given transformations* A_k $(k = 1, 2, \cdots, n)$ *and* $R > 0$, $C > 0$ *such that*

1_n. *For* $|\mathrm{Im}\, z| < R$ *the* A_k *are analytic and satisfy the inequalities*

$$|A_k(z) - A_{k-1}(z)| < \frac{C}{2^k} \quad (A_0(z) \equiv 0).$$

2_n. *The rotation numbers of the* A_k *are rational and for* $k > 1$

$$\left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| < \frac{1}{(k-1)^2 (\max_{l<k} q_l)^2}$$

3_n. A_k *is semistable forward and has a unique cycle.*

Then one may construct a transformation A_{n+1} *such that the sequence* A_k $(k = 1, 2, \cdots, n+1)$ *will have properties* $1_{n+1}, 2_{n+1}, 3_{n+1}$ *and*

4_{n+1}. $\quad |A_{n+1}(z) - A_n(z)| < \delta_n$ *for* $\mathrm{Im}\, z = 0$.

Proof. Consider the transformation A_λ: $z \to A_n(z) + \lambda$, $\lambda > 0$. Evidently there exists a $\lambda_0 > 0$ such that for $\lambda < \lambda_0$

$$|A_\lambda(x) - A_n(z)| < \frac{C}{2^{n+2}} \quad (|\mathrm{Im}\, z| < R),$$

$$|A_\lambda(z) - A_n(z)| < \frac{\delta_n}{2} \quad (\mathrm{Im}\, z = 0)$$

and the rotation number of A_λ is strictly larger than p_n/q_n (Lemma δ) and less than

$$\frac{p_n}{q_n} + \frac{1}{n^2 (\max_{l \leqslant n} q_j)^2}$$

(continuity of the rotation number, see §9). Suppose that the rotation number of A_{λ_0} is μ. We select a rational number p_{n+1}/q_{n+1},

$$\frac{p_n}{q_n} < \frac{p_{n+1}}{q_{n+1}} < \mu.$$

Among all the λ for which the rotation number of A_λ is p_{n+1}/q_{n+1} we select the largest. Suppose that it is λ_1. The transformation A_{λ_1} has the properties $1_{n+1}, 2_{n+1}, 4_{n+1}$, and, as is easily seen, is semistable forward. We apply Lemma α to it. Then we obtain a transformation A_{n+1} satisfying all the requirements of the Induction Lemma.

1.4. The transformation A_1 satisfies requirements $1_1, 2_1, 3_1$ of the Induction Lemma for the same C, R. We shall describe the choice of δ_n in carrying out the induction from A_n to A_{n+1}. We denote by G_n^* the ϵ-neighborhood of the single cycle A_n, where $\epsilon > 0$ is chosen so that the measure of G_n^* is less than 2^{-n-2}. By Lemma β there is an N_n such that A_n has property 2 relative to G_n^* and N_n. By Lemma γ there exists a $\delta_n^* > 0$ for which the transformation A has property 2 relative to N_n and to a G_n-neighborhood of G_n^* of measure 2^{-n-1} if on the real axis

$$|A(z) - A_n(z)| < \delta_n^*.$$

Choose

$$\delta_{n+1} = \min\left(\frac{\delta_n}{2}, \frac{\delta_n^*}{2}\right)$$

(we formally take $\delta_0 = 0$). Applying the Induction Lemma, we obtain A_{n+1}.

If the transformations A_n, $n = 1, 2, \cdots$, are constructed in the way described, then, in view of property 1_n this sequence converges uniformly in the strip $|\mathrm{Im}\, z| < R$, so that the limit A is an analytic transformation. Evidently

$$|A(z) - A_n(z)| \leqslant \sum_{k=n}^{\infty} |A_{k+1}(z) - A_k(z)| \leqslant \sum_{k=n}^{\infty} \delta_n \left(\frac{1}{2}\right)^{n+1} \leqslant \delta_n \quad (\mathrm{Im}\, z = 0)$$

for any n and therefore A has property 2 relative to G_n and N_n, $n = 1, 2, \cdots$. From property 2_n and the continuity of the rotation number, we conclude on the basis of Lemma ϵ that the rotation number of A is irrational. Indeed, for any n

$$\left|\mu - \frac{p_n}{q_n}\right| \leqslant \sum_{k=n}^{\infty} \frac{1}{k^2 (\max_{l \leqslant k} q_l)^2} \leqslant \sum_{k=n}^{\infty} \frac{1}{k^2 q_n^2} < \frac{2}{q_n^2}.$$

Thus all three properties of subsection 1.1 are satisfied, so that A is the desired transformation.

1.5. **Remark.** Considering the example just constructed, it is not difficult to see that a transformation A with the indicated properties may be found in any family of analytic transformations

$$z \to A_\Delta z \equiv z + \Delta + F(z)$$

and therefore in any neighborhood of any transformation with an irrational rotation number, given only that the family has the following property: among the transformations A_Δ^n there are no rotations. Probably the family $z \to z + \Delta + \frac{1}{2}\cos z$ has this property; in this case an example may be given by a simple analytic formula.

§2. On the functional* equation $g(z + 2\pi\mu) - g(z) = f(z)$

* Hilbert [12] gave this equation as an example of an analytic problem with a nonanalytic solution. It is encountered in investigations on the metric theory of dynamical systems (see [13], [14]), and is the simplest example of a problem with small denominators.

Added in proof. This paper was already in press when the author became acquainted with the paper [40] of A. Wintner in which this equation was apparently first studied from a modern point of view.

2.1. Suppose that $f(z)$ is a function of period 2π, μ a real number. It is required to define from the equation

$$g(z+2\pi\mu)-g(z)=f(z) \tag{1}$$

a function $g(z)$ having period 2π.

In case the equation (1) is solvable, evidently

$$\int_0^{2\pi} f(z)\,dz=0.$$

Furthermore, if $g(z)$ is a solution, then $g(z)+C$ is also a solution. Therefore we shall consider only right sides which are in the mean equal to zero and seek only solutions in the mean equal to zero. In each function $\phi(z)$ on $[0, 2\pi]$ we single out the constant part

$$\bar{\varphi}=\frac{1}{2\pi}\int_0^{2\pi}\varphi(z)\,dz$$

and the variable part

$$\tilde{\varphi}(z)=\varphi(z)-\bar{\varphi}.$$

The equation $\bar{f}=0$ is thus a necessary condition for the solvability of equation (1). By a solution of (1) we shall from now on always understand the variable part $g(z)$.

If $\mu=m/n$, i.e., is rational, then for the existence of a solution it is necessary that

$$\sum_{k=1}^{n} f\left(z+2\pi\frac{k}{n}\right)=0,$$

since this sum may be expressed in terms of the solution in the form

$$\sum_{k=1}^{n} g\left(z+2\pi\frac{m}{n}+2\pi\frac{k}{n}\right)-\sum_{k=1}^{n} g\left(z+2\pi\frac{k}{n}\right),$$

and in these two sums the terms are identical. If such a condition is satisfied, then a solution exists but it is defined only up to an arbitrary function of period $2\pi/n$, since such a function satisfies the homogeneous equation

$$g\left(z+2\pi\frac{m}{n}\right)-g(z)=0.$$

Now if μ is irrational, then the equation has a unique solution; in fact,

1) *For irrational* μ *equation* (1) *cannot have two distinct continuous solutions.*

Proof. The difference of two continuous solutions of equation (1) satisfies the equations

$$g(z+2\pi)-g(z)=0,$$
$$g(z+2\pi\mu)-g(z)=0;$$

i.e., this continuous function has two incommensurable periods. Such a function is a constant (see [15], pp. 55–56); it takes on one and the same value at all points $2\pi k + 2\pi\mu l$, which form an everywhere dense set. Since

$$\int_0^{2\pi} g(z)\,dz = 0,$$

then the constant in question is zero.

2) *For an irrational μ equation* (1) *cannot have two measurable solutions not coinciding almost everywhere.*

Proof. Again we consider the difference of two solutions of (1) and denote it by $g(z)$. It can be considered as a function on the circumference, since it has period 2π. By condition 1

$$g(z + 2\pi\mu) - g(z) = 0;$$

i.e., $g(z)$ does not change under a rotation through the angle $2\pi\mu$. Therefore the set E_a of points of the circumference where $g(z) > a$ is invariant under a rotation through the angle $2\pi\mu$. If the function $g(z)$ is constant almost everywhere, then this constant, as in case 1), is zero. If $g(z)$ is not constant, then for some a the set E_a has a measure satisfying $0 < \text{meas}\, E_a < 2\pi$. But it is well known that a set invariant with respect to rotations by an angle noncommensurable with 2π has measure zero or a complete measure (see, for example, [3]; for the proof it is sufficient to use the theorem on points of density). Thus $g(z) = 0$ almost everywhere.

If the function $f(z)$ is expanded into the Fourier series

$$f(z) = \sum_{n \neq 0} f_n e^{inz},$$

then for the Fourier coefficients of $g(z)$ we have

$$g_n e^{2\pi i\mu n} - g_n = f_n,$$

i.e.,

$$g_n = \frac{f_n}{e^{2\pi i\mu n} - 1}, \quad g(z) = \sum_{n \neq 0} g_n e^{inz}. \tag{2}$$

For rational μ some of the denominators vanish. For irrational μ there are arbitrarily small denominators. We note that

$$|e^{2\pi i\mu n} - 1| > |\mu n - m| \tag{3}$$

for any integer n and some integer m. Therefore the smallness of the denominators in (2) depends on the approximation of μ by rational numbers.

Lemma 1 (see [16]). *Suppose that $\epsilon > 0$. For almost every (in the sense of Lebesgue measure) μ with $0 \leq \mu \leq 1$ there exists a $K > 0$ such that*

$$|\mu n - m| \geqslant \frac{K}{n^{1+\varepsilon}} \tag{4}$$

for any integers m *and* $n > 0$.

Proof. We select any $K > 0$ and estimate the measure of the set E_K of points μ, $0 < \mu < 1$, not satisfying the inequality (4), which we rewrite in the form

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^{2+\varepsilon}}.$$

This set contains all the points m/n with circumferences of radius $K/n^{2+\epsilon}$. For a fixed n the number of these points will be equal to $n + 1$, and the common length of the circumferences (on [0, 1]) will be equal to $K/n^{1+\epsilon}$. Therefore

$$\operatorname{mes} E_K \leqslant \sum_{n=1}^{\infty} \frac{K}{n^{1+\varepsilon}} = c(\varepsilon) K.$$

The set of points μ, for which the number K required in the lemma does not exist, is contained in E_K for any $K > 0$, so that this measure is less than $c(\epsilon) K$ for any K; i.e., it is equal to zero.

2.2. We shall show that for almost all μ small denominators worsen the convergence of the series (2) only a little.

Lemma 2 (see [17]). *The series*

$$S = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \frac{1}{|n\mu - m_n|} \tag{5}$$

converges for any $\epsilon > 0$ *and any integers* m_n, *if* μ *is such that*

$$|\mu n - m| \geqslant \frac{K}{n^{1+\varepsilon-\delta}} \quad (K > 0, \quad 0 < \delta < \varepsilon) \tag{6}$$

for all integers m *and* $n > 0$.

Proof. Without loss of generality we may suppose that $|\mu n - m_n| < 1$. We consider series S_i of the same type as S, but in which the summation is extended only over those indices $n = n_k^{(i)}$ for which

$$\frac{1}{2^{i+1}} \leqslant |\mu n_k^{(i)} - m_{n_k^{(i)}}| < \frac{1}{2^i} \quad (i = 0, 1, 2, \ldots; \; n_{k+1}^{(i)} > n_k^{(i)}). \tag{7}$$

The series S_i taken together contain all the terms of S, so that it is sufficient to prove that

$$\sum_{i=0}^{\infty} S_i < \infty.$$

To estimate S_i we note that from (6) the successive indices $n_k^{(i)}, n_{k+1}^{(i)}$ of terms of the series S_i are significantly far apart: since from (7) there follows the inequality

$$|\mu (n_k^{(i)} - n_{k+1}^{(i)}) - m| < \frac{1}{2^{i-1}},$$

from (6) we deduce

$$\frac{1}{2^{i-1}} > \frac{K}{N_i^{1+\varepsilon-\delta}},$$

where

$$N_i = \min_{0<k<\infty} (n_{k+1}^{(i)} - n_k^{(i)}).$$

Therefore we obtain

$$N_i > (2^{i-1}K)^{\frac{1}{1+\varepsilon-\delta}}. \tag{8}$$

Evidently $n_1^{(i)} > N_i$, and more generally $n_k^{(i)} > kN_i$, so that in view of (5), (7), (8) we have

$$S_i < \sum_{k=1}^{\infty} \frac{2^{i+1}}{(kN_i)^{1+\varepsilon}} = \frac{2^{i+1}}{N_i^{1+\varepsilon}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} = \frac{2^{i+1}}{2^{(i-1)\frac{1+\varepsilon}{1+\varepsilon-\delta}}} L(\varepsilon, K) \quad (L(\varepsilon, K) > 0),$$

$$S_i < 2^{1+\frac{1+\varepsilon}{1+\varepsilon-\delta}} L 2^{i\left(1-\frac{1+\varepsilon}{1+\varepsilon-\delta}\right)} = L'(\varepsilon, \delta, K)\theta^i.$$

Here

$$\theta = 2^{1-\frac{1+\varepsilon}{1+\varepsilon-\delta}} < 1,$$

so that

$$\sum_{i=0}^{\infty} S_i < \infty,$$

as was required to be proved.

As is well known, if $f(x)$ is a function $p + \epsilon$ times differentiable,* then its Fourier coefficients have an order of decrease

$$f_n = O\left(\frac{1}{n}\right)^{p+\varepsilon},$$

and if

$$f_n = O\left(\frac{1}{n}\right)^{p+1+\varepsilon}$$

then $f(x)$ is differentiable $p + \epsilon$ times. From this and from inequality (3) and Lemmas 1 and 2, applied to the series (2), we obtain the following result:

If the function $f(z)$ is $p + 1 + \epsilon + \delta$ times differentiable, then for almost all μ equation (1) *has a $p + \epsilon$ times differentiable solution.*

On the other hand, it is not hard to construct examples for which the number

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*I.e., a function whose pth derivative satisfies a Hölder condition of degree ϵ: $|f^{(p)}(x+h) - f^{(p)}(x)| < Ch^{\varepsilon}$.

μ can be approximated by rationals so well that in spite of the rapid decrease of the numerators f_n the series (2) converges slowly or not at all. So even if $f(z)$ is analytic there may appear cases where $g(z)$ is not analytic but is infinitely differentiable, or even only differentiable finitely many times, or only continuous, or even discontinuous, or the solution is not measurable (see [14], [17]).*

2.3. Consider the equation (1) in the class of analytic functions. To investigate this case we recall two lemmas concerning the Fourier coefficients of analytic functions.

Lemma 3. *If the function $f(z)$ of period 2π in the strip $|\mathrm{Im}\ z| \le R$ is analytic and in this strip $|f(z)| \le C$, then its Fourier coefficients satisfy the inequalities*

$$|f_n| \leqslant Ce^{-|n|R}.$$

Proof. By definition,

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(z)\, e^{-inz}\, dz.$$

From the periodicity of $f(z)e^{-inz}$,

$$\int_0^{i\tau} f(z)\, e^{-inz}\, dz = \int_{2\pi}^{2\pi+i\tau} f(z)\, e^{-inz}\, dz,$$

so that

$$f_n = \frac{1}{2\pi}\int_{0+i\tau}^{2\pi+i\tau} f(z)\, e^{-inz}\, dz$$

for any $\tau \in [-R, R]$. Integrating in the case $n > 0$ along the line $\tau = -R$ and for $n < 0$ along $\tau = R$, we obtain

$$|f_n| \leqslant \frac{1}{2\pi}\int_0^{2\pi} Ce^{-|n|R}\, dz,$$

as was required to be proved.

Lemma 4. *Suppose that the Fourier coefficients of $f(z)$ satisfy the inequalities $|f_n| \le Ce^{-|n|R}$. Then $f(z)$ is analytic and satisfies for $|\mathrm{Im}\ z| \le R - \delta$, $0 < \delta < R$, the inequality*

$$|f(z)| \leqslant \frac{2C}{1 - e^{-\delta}},$$

and its derivative satisfies the inequality

$$|f'(z)| \leqslant \frac{2C}{(1 - e^{-\delta})^2}.$$

*A. N. Kolmogorov has conjectured that this last case is realized whenever the series $\sum_{n\neq 0} |f_n^2|/|e^{2\pi i\mu n} - 1|^2$ diverges.

Proof. For $|\operatorname{Im} z| \leq R - \delta$, $0 < \delta < R$ it is evident that

$$|e^{inz}| \leqslant e^{|n|(R-\delta)}.$$

Therefore

$$|f_n e^{inz}| \leqslant Ce^{-|n|\delta}$$

and

$$\sum_{n=-\infty}^{\infty} |f_n e^{inz}| \leqslant 2 \sum_{n=0}^{\infty} Ce^{-n\delta} \leqslant \frac{2C}{1-e^{-\delta}} .$$

In the same way

$$\sum_{n=-\infty}^{\infty} |f_n ine^{inz}| \leqslant 2C \sum_{n=0}^{\infty} ne^{-n\delta} \leqslant \frac{2C}{(1-e^{-\delta})^2} .$$

In the strip $|\operatorname{Im} z| \leq R - \delta$ the series converge absolutely uniformly. The lemma is proved.

Now it is not difficult to investigate the analytic solutions of equation (1).

Theorem 1. *Suppose that* $f(z) = \tilde{f}(z)$ *is an analytic function of period* 2π *and that, for* $|\operatorname{Im} z| \leq R$, $|f(z)| \leq C$. *Let* μ *be irrational,* $K > 0$ *and*

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^3} \tag{9}$$

for any integers m *and* $n > 0$. *Then the equation*

$$g(z + 2\pi\mu) - g(z) = f(z)$$

has an analytic solution $g(z) = \tilde{g}(z)$, *and for* $|\operatorname{Im} z| \leq R - 2\delta$ *and any* $\delta < 1$, $0 < \delta < R/2$,

$$|g(z)| \leqslant \frac{4C}{K\delta^3}, \tag{10}$$

$$|g'(z)| \leqslant \frac{8C}{K\delta^4} . \tag{11}$$

Proof. Applying Lemma 3 for the estimate of the Fourier coefficients f_n of the function $f(z)$, and using inequalities (3) and (9), we obtain from (2)

$$|g_n| \leqslant \frac{C}{K} n^2 e^{-|n|R}. \tag{12}$$

We note the simple inequality

$$|n|^p \leqslant \left(\frac{p}{e}\right)^p \frac{e^{|n|\delta}}{\delta^p}, \tag{13}$$

valid for any $\delta > 0$. In fact $p \ln x < p \ln (p/e) + x$, since the function $p \ln x - x$ has its maximum at $x = p$. Putting $x = \delta|n|$, we obtain (13). Applying (13) to (12) (for $p = 2$), we have

$$|g_n| \leqslant \frac{Ce^{-|n|R} e^{|n|\delta}}{K\delta^2} = \frac{Ce^{-|n|(R-\delta)}}{K\delta^2},$$

so that from Lemma 4 we obtain in the strip $|\operatorname{Im} z| \leq R - 2\delta$:

$$|g(z)| \leqslant \frac{2C}{K\delta^2(1-e^{-\delta})}, \quad |g'(z)| \leqslant \frac{2C}{K\delta^2(1-e^{-\delta})^2}.$$

Since $|1-e^{-\delta}| > \delta/2$ for $\delta < 1$, we therefore obtain the inequalities (10) and (11). The theorem is proved.

Remark 1. Evidently the solution is real if $f(z)$ is real on the real axis.

Remark 2. If the function $f(z, \lambda)$ depends analytically on a parameter λ, then *the solution (under the conditions of Theorem 1) also depends analytically on that parameter.*

2.4. We consider equation (1) for complex μ. In this case the solution of the homogeneous equation

$$g(z+2\pi\mu) - g(z) = 0$$

is any doubly periodic function with periods 2π and $2\pi\mu$, so that the solution of the problem is certainly not unique. If we require that $g(z)$ be analytic in a strip *of width greater than $|\text{Im } 2\pi\mu|$, then the solution of (1) is defined uniquely up to a constant.* Indeed, a strip of that width contains a parallelogram of periods, and a solution of the homogeneous equation analytic in it is bounded in the entire plane; i.e., it is a constant. The condition $\bar{g} = 0$ singles out the unique solution which is given by the series (2). This series converges for any nonreal μ, but we are interested in estimates, and thus we must exclude neighborhoods of rational μ. We shall denote by M_K^r the set of points μ of the rectangle in the complex plane $0 \le \text{Re}\,\mu \le 1$, $|\text{Im}\,\mu| \le r$ such that for all integer m, n the inequality

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3}$$

is satisfied. It is evident that along with μ the points $\bar{\mu}$, $1-\mu$ and $1-\bar{\mu}$ are also contained in M_K^r.

Instead of inequalities (3) we have

$$|e^{2\pi i z} - 1| \geqslant \min\left(\frac{1}{2}, \pi|z-m|\right) \tag{14}$$

for any complex z with some integer n. We shall prove (14). If $|e^{2\pi i z} - 1| \ge \frac{1}{2}$, then (14) is proved. If $|e^{2\pi i z} - 1| < \frac{1}{2}$, then we join the points 1 and $e^{2\pi i z}$ by a segment and consider the integral

$$\frac{1}{2\pi i}\int_1^{e^{2\pi i z}} \frac{dw}{w} = \frac{1}{2\pi i}(\ln e^{2\pi i z} - \ln 1) = z - m,$$

where $\ln w$ is one of the branches of the logarithm and $\ln 1 = 2\pi i m$, m an integer. Since the segment of integration lies entirely in the circle

$$|w-1| < \frac{1}{2},$$

and in this circle $|w| > \frac{1}{2}$, we have

$$\left| \int_1^{e^{2\pi i z}} \frac{dw}{w} \right| \leqslant 2 \, | e^{2\pi i z} - 1 | .$$

Therefore

$$| z - m | \leqslant \frac{1}{\pi} | e^{2\pi i z} - 1 | ,$$

as was required to be proved.

If $\mu \in M_K^r$, then by applying (14) to $z = \mu n$ we find

$$| e^{2\pi i \mu n} - 1 | \geqslant \min\left(\frac{1}{2}, \frac{\pi K}{n^2}\right).$$

Thus, *if* $\mu \in M_K^r$, *where* $K < 1/2\pi$, *then*

$$| e^{2\pi i \mu n} - 1 | \geqslant \frac{\pi K}{n^2} .$$

Theorem 1′. *Suppose that* $f(z) = \tilde{f}(z)$ *is an analytic function of period* 2π *and that* $|f(z)| \le C$ *for* $|\mathrm{Im}\, z| \le R$, *and suppose that* $\mu \in M_K^r$, $K < 1/2\pi$. *Then the equation*

$$g(z + 2\pi\mu) - g(z) = f(z) \tag{1}$$

has an analytic solution $g(z) = \tilde{g}(z)$, *and for* $|\mathrm{Im}\,(z - 2\pi\mu)| < R - 2\delta$ *and any* $\delta < 1$, $0 < \delta < R/2$,

$$| g(z) | \leqslant \frac{4C}{\pi K \delta^3}, \quad | g'(z) | \leqslant \frac{8C}{\pi K \delta^4} . \tag{16}$$

Proof. From formula (2) and Lemma 3, we have

$$| g_n e^{inz} | \leqslant \frac{C e^{-|n| R}}{e^{2\pi i \mu n} - 1} e^{in(z - 2\pi\mu + 2\pi\mu)}. \tag{17}$$

But for $|\mathrm{Im}\,(z - 2\pi\mu)| < R - 2\delta$

$$| e^{in(z - 2\pi\mu)} | < e^{|n|(R - 2\delta)},$$

so that it follows from (17) that

$$| g_n e^{inz} | \leqslant \frac{C e^{-2\delta |n|}}{1 - e^{-2\pi i \mu n}} .$$

Since $1 - \mu \in M_K^r$, we have from (15),

$$| 1 - e^{-2\pi i \mu n} | \geqslant \frac{\pi K}{n^2} ,$$

which means that

$$| g_n e^{inz} | \leqslant \frac{C e^{-2\delta |n|} n^2}{\pi K} .$$

Hence from (13) it follows that the series $g(z)$ and $g'(z)$ converge, and accordingly the inequalities (16) are valid (see the proofs of Theorem 1 and Lemma 4).

Remark 1. Remark 2 to Theorem 1 applies also to Theorem 1′.

Remark 2. Let us fix the function f and the number z and consider the dependence of the solution just found on μ:

$$g(\mu) = \sum_{n \neq 0} \frac{f_n}{e^{2\pi i\mu n} - 1} e^{inz}. \qquad (2).$$

The function $g(\mu)$ is analytic in the upper and lower half-planes, but the axis $\operatorname{Im} \mu = 0$ is a cut. On it the series (2) converges almost everywhere, but to an everywhere discontinuous limit. That does not prevent us in §7 from differentiating the solution with respect to μ even for $\operatorname{Im} \mu = 0$ if we make use of the ideas of Borel [9]. For the time being we shall take the formula

$$\frac{\partial g}{\partial \mu} = - \sum_{n \neq 0} \frac{2\pi i n e^{2\pi i\mu n} f_n}{(e^{2\pi i\mu n} - 1)^2} e^{inz}$$

to have a meaning only in the upper and lower half-planes separately.

§3. Lemmas necessary for the proof of Theorem 2

3.1. Lemma 5. *If at each point of the segment $z_1 z_2$ the function $f(z)$ is analytic and $|df/dz| \leq L$, then $|f(z_2) - f(z_1)| \leq L\,|z_2 - z_1|$.*

Proof. Indeed,

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} \frac{df(z)}{dz} dz,$$

from which it follows that

$$|f(z_2) - f(z_1)| \leqslant \int_{z_1}^{z_2} \left| \frac{df(z)}{dz} \right| |dz| \leqslant L|z_2 - z_1|.$$

Remark. The example $f(z) = e^{iz}$, $z_1 = 0$, $z_2 = 2\pi$ shows that in the complex domain the theorem on the finite increment in the form

$$f(z_2) - f(z_1) = \frac{df(\xi)}{dz} (z_2 - z_1)$$

or

$$|f(z_2) - f(z_1)| = \left| \frac{df(\xi)}{dz} \right| |z_2 - z_1|$$

is invalid.

3.2. Lemma 6 (on implicit functions). *Suppose that the functions $F(\epsilon)$, $\Phi(\epsilon, \Delta)$ are analytic and that for $|\epsilon| \leq \epsilon_0$, $|\Delta| \leq \Delta_0$*

$$|F(\varepsilon)| \leqslant M_1, \quad |\Phi(\varepsilon, \Delta)| \leqslant M_2 |\Delta|,$$

where $M_1/(1 - M_2) < \Delta_0/3$ and $M_2 < 1/6$. Then

1. *The equation $\Delta + F(\epsilon) + \Phi(\epsilon, \Delta) = 0$ has analytic solution $\Delta^*(\epsilon)$, satisfying for $|\epsilon| < \epsilon_0$ the inequality $|\Delta^*(\epsilon)| \leq M_1/(1 - M_2)$.*

2. *The equation $\Delta + F(\epsilon) + \Phi(\epsilon, \Delta) = \Delta_1$ has a solution $\Delta = \Delta(\Delta_1, \epsilon)$,*

analytically depending on Δ_1 *and* ϵ, $|\Delta_1| < \Delta_0/6$, $|\epsilon| < \epsilon_0$, *where*

$$|\Delta(\Delta_1, \varepsilon) - \Delta^*(\varepsilon)| \leqslant 2|\Delta_1|$$

Proof. The disk $|\Delta| < M_1/(1-M_2)$ lies, when $M_1/(1-M_2) < \Delta_0$, $|\epsilon| \le \epsilon_0$, in the region where $|F(\epsilon)| \le M_1$, $|\Phi(\epsilon, \Delta)| < M_2|\Delta|$, and therefore under the transformation $\Delta \to -F(\epsilon) - \Phi(\epsilon, \Delta)$ is carried inside itself:

$$|F(\varepsilon) + \Phi(\varepsilon, \Delta)| \leqslant M_1 + \frac{M_1}{1-M_2} M_2 = \frac{M_1}{1-M_2}.$$

The fixed point of the transformation is the desired solution $\Delta^*(\epsilon)$. Analyticity follows from the usual theorem on implicit functions, since

$$\frac{\partial}{\partial\Delta}(\Delta + F(\varepsilon) + \Phi(\varepsilon, \Delta)) \neq 0,$$

which follows from the estimate of $\partial\Phi/\partial\Delta$ using Cauchy's integral formula: for $|\Delta| \le 2\Delta_0/3$, $|\epsilon| < \epsilon_0$

$$\left|\frac{\partial\Phi}{\partial\Delta}\right| \leqslant \frac{M_2\Delta_0}{\frac{\Delta_0}{3}} < \frac{1}{2}.$$

2. Under the transformation $w \to w + \Phi(w, \epsilon)$ the point $\Delta^*(\epsilon)$ goes into $-F(\epsilon)$, and the point w of the disk $|w - \Delta^*(\epsilon)| \le 2|\Delta_1|$ into the point

$$w + \Phi(\Delta^*(\varepsilon), \varepsilon) + [\Phi(w, \varepsilon) - \Phi(\Delta^*(\varepsilon), \varepsilon)].$$

Since under the conditions of the lemma

$$|\Phi(w, \varepsilon) - \Phi(\Delta^*(\varepsilon), \varepsilon)| \leqslant |\Delta_1|$$

for the points of this disk (Lemma 5), the image of the disk $|w - \Delta^*(\epsilon)| \le 2|\Delta_1|$ contains the entire disk $|w + F(\epsilon)| \le \Delta_1$ and has the point $\Delta(\Delta_1, \epsilon)$, going into $\Delta_1 - F(\epsilon)$. This point satisfies the inequality

$$|\Delta - \Delta^*| \leqslant 2|\Delta_1|$$

and the equation

$$\Delta = \Delta_1 - F(\varepsilon) - \Phi(\varepsilon, \Delta).$$

Uniqueness and analyticity follows from the inequality $|\partial\Phi/\partial\Delta| < \frac{1}{2}$.

Remark. It is easy to see that if under the conditions of Lemma 6 the functions $F(\epsilon)$ and $\Phi(\epsilon, \Delta)$ are real for real ϵ, Δ, then $\Delta^*(\epsilon)$ and $\Delta(\Delta_1, \epsilon)$ are real for real Δ_1, ϵ.

3.3. **Newton's method** (see [18], [19]). Suppose that we are seeking a solution of the equation $f(x) = 0$ (Figure 1). We determine x roughly as x_0 and find the point of intersection x_1 of the tangent at x_0 to the curve $y = f(x)$ with the x axis:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

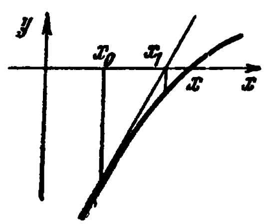

Figure 1

Further, we define successively

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

and estimate the rapidity of convergence of the process.* Suppose that x is the desired solution and $|x_0 - x| = \epsilon$. Then the deviation of the curve from its tangent at the point x_0 has order ϵ^2 at the point x, which means that $|x_1 - x|$ is a quantity of order ϵ^2. Thus after the nth step the error will be of order 2^n, which represents extraordinarily fast convergence.

We shall apply a similar method to the solution of a linear functional equation approximated by the equation considered in §2. The rapid convergence will paralyze the denominators appearing at each step.

§4. Theorem 2 and the Fundamental Lemma

4.1. Heuristic considerations. The transformation

$$z \to z + 2\pi\mu$$

is a rotation of the circumference. The transformation

$$z \to z + 2\pi\mu + \varepsilon F(z)$$

is a rotation perturbed by the term $\epsilon F(z)$, which is small along with ϵ. Its rotation number, even if $\bar{F} = 0$, may be different from $2\pi\mu$. However, we may seek $\Delta = \Delta(\epsilon)$ such that the transformation

$$z \to z + 2\pi\mu + \Delta + \varepsilon F(z)$$

will have a rotation number equal to $2\pi\mu$. We shall show that *for numbers μ that are normally approximable by rational numbers, and sufficiently small ϵ,*

1) *$\Delta(\epsilon)$ depends analytically on ϵ;*

2) *the transformation $z \to z + 2\pi\mu + \Delta + \epsilon F(z)$ may be converted into a rotation through the angle $2\pi\mu$ by an analytic substitution of variables $\phi(z) = z + g(z)$.*

Here $g(z)$ is a correction small with ϵ, and property 2) means that

$$\varphi(z + 2\pi\mu + \Delta(\varepsilon) + \varepsilon F(z), \varepsilon) = \varphi(z, \varepsilon) + 2\pi\mu,$$

or, what is the same thing (the dependence of g on ϵ is implied),

$$g(z + 2\pi\mu + \Delta + \varepsilon F(z)) - g(z) = -\Delta - \varepsilon F(z). \tag{1}$$

This equation differs from that considered in §2 only by small quantities of second order, and therefore it is natural in the first approximation to choose $\Delta = \Delta(\epsilon)$ so that the right side of equation (1) will be equal to zero in the mean:

$$\Delta_1 = -\varepsilon\bar{F}$$

*Here we cite no exact assumptions and estimates. They are given in the paper [18] in a very general form, which, however, does not include the arguments of the following sections.

and to seek $g_1(z)$ as the solution of the equation

$$g_1(z + 2\pi\mu) - g_1(z) = -\varepsilon \bar{F}(z).$$

The g_1 thus defined has order ϵ and in the variable $\phi_1 = z - g_1$ our transformation

$$z \to z + 2\pi\mu + \Delta_1(\varepsilon) + \varepsilon F(z)$$

has the form

$$\begin{aligned}\varphi_1(z + 2\pi\mu + \Delta_1(\varepsilon) + \varepsilon F(z)) &= z + 2\pi\mu + \Delta_1 + \varepsilon F \\ + g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) &= z + g_1(z) + 2\pi\mu \\ + [g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) &- g_1(z + 2\pi\mu)] \\ + [g_1(z + 2\pi\mu) - g_1(z) + \varepsilon \tilde{F}(z)] &+ (\Delta_1 + \varepsilon \bar{F}).\end{aligned}$$

The last two terms vanish because of the choice of Δ_1 and $g_1(z)$ and we obtain

$$\varphi_1(z) \to \varphi_1(z) + 2\pi\mu + F_2(z, \varepsilon).$$

Now the "perturbation" has the form

$$F_2(z, \varepsilon) = g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) - g_1(z + 2\pi\mu) = \frac{dg_1(\xi)}{dz}(\Delta_1 + \varepsilon F).$$

Here dg_1/dz, as also g_1, is a quantity of order ϵ, and, since the same relates to the second factor, the perturbation in the parameter ϕ_1 has order ϵ^2. With the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + F_2$$

one may proceed in the same way and define a "correction to the frequency" Δ_2 and a new parameter ϕ_2 such that the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2$$

in the parameter ϕ_2 goes into the transformation

$$\varphi_2 \to \varphi_2 + 2\pi\mu + F_3,$$

where $F_3 \sim \epsilon^4$. However, here in the parameter z the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2$$

will not have the form

$$z \to z + 2\pi\mu + \Delta + \varepsilon F.$$

Therefore we need to begin with the transformation

$$z \to z + 2\pi\mu + \Delta_1(\varepsilon) + \Delta_1'(\Delta_2) + \varepsilon F;$$

then with a proper choice of $\Delta_1'(\Delta_2)$ we may in the parameter ϕ_1 obtain the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2'(\varphi_1),$$

and in the parameter ϕ_2 the transformation

$$\varphi_2 \to \varphi_2 + 2\pi\mu + F_3',$$

and so forth. The rapid convergence of the method ($F_n \sim \epsilon^{2^{n-1}}$) makes it possible to carry out the limit transition and in the limit to obtain a new parameter

$\phi(z, \epsilon)$ and a final correction $\Delta(\epsilon)$ with the properties 1) and 2).

The usual method of solution of our problem in the theory of perturbations would consist in seeking $\Delta(\epsilon)$ and $\phi(z, \epsilon)$ in the form of series in powers of ϵ, while the coefficients of the series would be successively determined by equation (1) in the first approximation, in the second, and so forth. The proof of convergence of such series by direct estimates has not been achieved, though it results from the following fundamental theorem of this paper.

4.2. **Theorem 2.** *Suppose given a family of analytic transformations of the circumference, depending analytically on two parameters* ϵ, Δ;

$$z \to A(z, \varepsilon, \Delta) \equiv z + 2\pi\mu + \Delta + F(z, \varepsilon) \tag{2}$$

and numbers $R > 0$, $\epsilon_1 > 0$, $K > 0$, $L > 0$ *such that*

1) $F(z + 2\pi, \epsilon) = F(z, \epsilon)$;

2) *for* $\operatorname{Im} z = \operatorname{Im} \epsilon = 0$ *we always have* $\operatorname{Im} F(z, \epsilon) = 0$;

3) *for* $|\operatorname{Im} z| \le R$, $|\epsilon| \le \epsilon_0$

$$|F(z, \varepsilon)| \leqslant L|\varepsilon|; \tag{3}$$

4) *the irrational number* μ *for any integers* m *and* n *satisfies the inequality*

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3}. \tag{4}$$

Then there exist numbers ϵ' *and* R', $0 < \epsilon' \le \epsilon_0$, $0 < R' \le R$, *and functions* $\Delta(\epsilon)$, $\phi(z, \epsilon)$, *real for real* ϵ *and* z *and analytic for* $|\epsilon| < \epsilon'$, $|\operatorname{Im} z| < R'$, *such that*

$$\varphi(A(z, \varepsilon, \Delta(\varepsilon)), \varepsilon) = \varphi(z, \varepsilon) + 2\pi\mu. \tag{5}$$

This theorem is proved in §6 on the basis of the following lemma.

Fundamental Lemma. *Suppose given a family of analytic transformations of the circumference, depending analytically on the parameters* ϵ, Δ:

$$z \to A_0(z, \varepsilon, \Delta) \equiv z + 2\pi\mu + \Delta + F(z, \varepsilon) + \Phi(z, \varepsilon, \Delta) \tag{6}$$

and numbers $R_0 > 0$, $\epsilon_0 > 0$, $K > 0$, $\delta > 0$, $C > 0$, $0 < \Delta_0 < 1$ *such that*

1) $F(z + 2\pi, \epsilon) = F(z, \epsilon)$ $\Phi(z + 2\pi, \epsilon, \Delta) = \Phi(z, \epsilon, \Delta)$;

2) *for* $\operatorname{Im} z = \operatorname{Im} \epsilon = \operatorname{Im} \Delta = 0$ *always* $\operatorname{Im} F = \operatorname{Im} \Phi = 0$;

3) *for* $|\operatorname{Im} z| \le R_0$, $|\epsilon| \le \epsilon_0$, $|\Delta| \le \Delta_0$

$$|F(z, \varepsilon)| \leqslant C < \delta^8, \tag{7}$$

$$|\Phi(z, \varepsilon, \Delta)| < \delta|\Delta|; \tag{8}$$

4) *the irrational number* μ *for any integers* m *and* n *satisfies the inequality* (4);

5) *the number* δ *satisfies the inequalities*

$$\delta < \frac{K}{64}, \quad \delta < \frac{R_0}{8}, \tag{9}$$

$$\delta < \frac{1}{36}, \tag{10}$$

and moreover

$$C < \frac{\Delta_0}{6}. \tag{11}$$

Then there exist analytic functions $z(\phi, \epsilon)$, $\Delta(\Delta_1, \epsilon)$, $F_1(\phi, \epsilon)$, $\Phi_1(\phi, \epsilon, \Delta_1)$ *such that*

1. *Identically*

$$z[A_1(\varphi, \varepsilon, \Delta_1), \varepsilon] = A_0[z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)], \tag{12}$$

where

$$A_1(\varphi, \varepsilon, \Delta_1) \equiv \varphi + 2\pi\mu + \Delta_1 + F_1(\varphi, \varepsilon) + \Phi_1(\varphi, \varepsilon, \Delta_1). \tag{13}$$

2. $F_1(\varphi + 2\pi, \varepsilon) = F_1(\varphi, \varepsilon)$, $\Phi_1(\varphi + 2\pi, \varepsilon, \Delta_1) = \Phi_1(\varphi, \varepsilon, \Delta_1)$; $z(\varphi + 2\pi, \varepsilon) = z(\varphi, \varepsilon) + 2\pi$.

3. *For* $\operatorname{Im}\varphi = \operatorname{Im}\Delta_1 = \operatorname{Im}\varepsilon = 0$ *always* $\operatorname{Im} z = \operatorname{Im}\Delta = \operatorname{Im} F_1 = \operatorname{Im}\Phi_1 = 0$.

4. *For* $|\Delta_1| \leqslant C$, $|\operatorname{Im}\varphi| \leqslant R_0 - 7\delta$, $|\varepsilon| \leqslant \varepsilon_0$

$$|F_1(\varphi, \varepsilon)| \leqslant \frac{C^2}{\delta^6}, \tag{14}$$

$$|\Phi_1(\varphi, \varepsilon, \Delta_1)| \leqslant \delta^2 |\Delta_1|, \tag{15}$$

$$|z(\varphi, \varepsilon) - \varphi| \leqslant \frac{C}{\delta^4}, \quad \left|\frac{\partial z}{\partial \varphi}\right| < 2, \tag{16}$$

$$|\Delta(\Delta_1, \varepsilon)| \leqslant \Delta_0, \quad \left|\frac{\partial \Delta}{\partial \Delta_1}\right| < 2. \tag{17}$$

The Fundamental Lemma shows that small (of order C) perturbations of the rotation $z \to z + 2\pi\mu$ may be compensated by the change in the parameter $z \to \phi$ for $\Delta = \Delta(\Delta_1, \epsilon)$, so that in the new parameter the difference from a rotation will be of order C^2. The proof of the lemma is given in the next section.

4.3. In §11 we shall use the following assertion.

Corollary to Theorem 3. *Suppose that the irrational number* μ *satisfies inequality* (4) *of Theorem 2, and suppose that* $R > 0$. *Then there exists a* $C(R, K) > 0$ *such that if the transformation*

$$Az : z \to z + 2\pi\mu + F(z)$$

has a rotation number $2\pi\mu$ *and* $|F(z)| \leq C$ *for* $|\operatorname{Im} z| \leq R$, *then* Az *may be converted into a rotation by the angle* $2\pi\mu$ *by an analytic change of variables.*

Proof. Consider the function

$$F_1(z) = \frac{F(z)}{\max\limits_{|\operatorname{Im} z| \leqslant R} |F(z)|}$$

and the family of transformations

$$A_\varepsilon z : z \to z + 2\pi\mu + \varepsilon F_1(z),$$

satisfying the conditions of Theorem 2 for $L = 1$, since $|F_1(z)| \le 1$ for $|\operatorname{Im} z| \le R$. According to Theorem 2, there exists an $\epsilon'(R, K) > 0$ such that for $\epsilon < \epsilon'$ the transformation

$$z \to z + 2\pi\mu + \Delta(\varepsilon) + \varepsilon F_1(z)$$

can be converted into a rotation through the angle $2\pi\mu$. Choose $C(R, K) < \epsilon'$. Then, if $|F(z)| \ge C$ for $|\operatorname{Im} z| \le R$, there exists a Δ such that

$$z \to z + 2\pi\mu + \Delta + F(z)$$

can be turned by an analytic transformation of coordinates into a rotation through the angle $2\pi\mu$, since

$$F(z) = \max_{|\operatorname{Im} z| \leqslant R} |F(z)| F_1(z),$$

and

$$\max |F(z)| \leqslant C < \varepsilon'.$$

But the rotation number of Az is equal to $2\pi\mu$, from which it follows that $\Delta = 0$ (see item 2 in the proof of Theorem 4 in §10, where it is shown that for an arbitrarily small Δ the rotation number of the transformation $z \to z + 2\pi\mu + \Delta + F(z)$ is larger than $2\pi\mu$). The corollary is proved.

The assertion of the corollary may be obtained directly as well, using constructions analogous to those of Theorem 2. Because of the absence of the parameters ϵ and Δ, these constructions will be less clumsy.

4.4. **Remark on the multidimensional case.** All the constructions of §§2–8 may be considered to be multidimensional if we replace a point of the circumference by a point of a torus of k variables. Condition 4) of Theorem 2 is replaced by the following condition of "incommensurability" for the vector $\vec{\mu}$:

$$|n_0 + (\vec{\mu}, \vec{n})| \geqslant \frac{K}{|\vec{n}|^\omega} \tag{18}$$

for any integer vector $\vec{n} = (n_0, \cdots, n_k)$. Here $(\vec{\mu}, \vec{n})$ is the scalar product

$$\sum_{i=1}^{k} \mu_i n_i, \qquad |\vec{n}| = \sum_{i=0}^{k} |n_i|.$$

For sufficiently large ω condition (18) is satisfied for almost all vectors $\vec{\mu}$.

Without dwelling in detail on the formulations and proofs of all the inequalities, lemmas and theorems for the multidimensional case, we present only one result.

Multidimensional Theorem 2. *Suppose that $\vec{\mu} = (\mu_1, \cdots, \mu_k)$ is a vector with incommensurable components such that for any integer vector $\vec{n}$*

$$|n_0 + (\vec{\mu}, \vec{n})| > \frac{K}{|\vec{n}|^{k+1}} .$$

Then there exists an $\epsilon(R, C, k) > 0$ *such that for the vector field* $\vec{F}(\vec{z})$ *on the torus, analytic and sufficiently small,* $|\vec{F}(\vec{z})| < \epsilon$ *for* $|\mathrm{Im}\, \vec{z}| < R$, *there exists a vector* $\vec{a}$ *for which the transformation*

$$\vec{z} \to \vec{z} + \vec{a} + \vec{F}(\vec{z})$$

of the torus into itself is converted into

$$\vec{\varphi} \to \vec{\varphi} + 2\pi\vec{\mu}$$

by an analytic substitution of variables.

§5 Proof of the Fundamental Lemma

5.1. Construction of $z(\phi, \epsilon)$, $\Delta(\Delta_1, \epsilon)$, $F_1(\phi, \epsilon)$ **and** $\Phi_1(\phi, \epsilon, \Delta_1)$. **The function** $z(\phi, \epsilon)$ **is constructed as the inverse to**

$$\varphi(z, \varepsilon) = z + g(z, \varepsilon), \tag{1}$$

and the function $\Delta(\Delta_1, \epsilon)$ **as the inverse to** $\Delta_1(\Delta, \epsilon)$. **In subsection 4.1 we saw these functions had to be chosen so that the expression**

$$g(A_0(z, \varepsilon, \Delta), \varepsilon) - g(z, \varepsilon) + F(z, \varepsilon) + \Delta + \Phi(z, \varepsilon, \Delta)$$

would be small. Without defining $\Delta(\Delta_1, \epsilon)$ **for the time being (i.e., considering** Δ **as an independent variable) we define** $g^*(z, \epsilon, \Delta)$ **as the solution of the equation**

$$g^*(z + 2\pi\mu, \varepsilon, \Delta) - g^*(z, \varepsilon, \Delta) = -\tilde{F}(z, \varepsilon) - \tilde{\Phi}(z, \varepsilon, \Delta). \tag{2}$$

Expressing the transformation A_0 **(see §4, formula (6)) in terms of the parameter**

$$\varphi^*(z, \varepsilon, \Delta) = z + g^*(z, \varepsilon, \Delta),$$

we obtain

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = z + 2\pi\mu + \Delta + F(z, \varepsilon) + \Phi(z, \varepsilon, \Delta) + g^*(z + 2\pi\mu, \varepsilon, \Delta) + g^*(A_0(z, \varepsilon, \Delta)) - g^*(z + 2\pi\mu, \varepsilon, \Delta),$$

or, transforming the right side by means of (2),

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = z + g^*(z, \varepsilon, \Delta) + 2\pi\mu + \Delta + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta) + g^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] - g^*(z + 2\pi\mu, \varepsilon, \Delta).$$

Thus from (1) we obtain

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = \varphi^*(z, \varepsilon, \Delta) + 2\pi\mu + \Delta + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta) + g^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] - g^*(z + 2\pi\mu, \varepsilon, \Delta).$$

We define $\Delta_0^*(\epsilon)$ **as the solution of the equation**

$$\Delta_0^*(\varepsilon) + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta_0(\varepsilon)) = 0 \tag{4}$$

and put

$$g^*(z, \varepsilon, \Delta_0^*(\varepsilon)) = g(z, \varepsilon). \tag{5}$$

Now the new parameter $\phi(z, \epsilon)$ is defined by equations (5) and (1). We represent (3) in the form

$$\varphi[A_0(z, \varepsilon, \Delta), \varepsilon] = \varphi(z, \varepsilon) + 2\pi\mu + \Delta_1(\varepsilon, \Delta) + \hat{F}_1(z, \varepsilon) + \hat{\Phi}_1(z, \varepsilon, \Delta), \tag{6}$$

where

$$\hat{F}_1(z, \varepsilon) = g(z_{\mathrm{I}}, \varepsilon) - g(z_{\mathrm{II}}, \varepsilon), \tag{7}$$

$$\hat{\Phi}_1(z, \varepsilon, \Delta) = g(z_{\mathrm{III}}, \varepsilon) - g(z_{\mathrm{I}}, \varepsilon), \tag{8}$$

$$\Delta_1(\varepsilon, \Delta) = \Delta + \overline{F}(\varepsilon) + \overline{\Phi}(\varepsilon, \Delta), \tag{9}$$

$$z_{\mathrm{I}} = z + 2\pi\mu + \widetilde{F}(z, \varepsilon) + \hat{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon)), \tag{10}$$

$$z_{\mathrm{II}} = z + 2\pi\mu, \tag{11}$$

$$z_{\mathrm{III}} = z + 2\pi\mu + \widetilde{F}(z, \varepsilon) + \Delta_1(\varepsilon, \Delta) + \widetilde{\Phi}(z, \varepsilon, \Delta). \tag{12}$$

We define $z(\phi, \epsilon)$ from (1), $\Delta(\Delta_1, \epsilon)$ from (9), and write

$$F_1(\varphi, \varepsilon) = \hat{F}_1(z(\varphi, \varepsilon), \varepsilon), \tag{13}$$

$$\Phi_1(\varphi, \varepsilon, \Delta_1) = \hat{\Phi}_1(z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)), \tag{14}$$

$$A_1(\varphi, \varepsilon, \Delta_1) = \varphi[A_0(z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)), \varepsilon]. \tag{15}$$

5.2. We shall prove that the functions just constructed are those sought. Assertions 1, 2, and 3 of the Fundamental Lemma are satisfied in an obvious way. The proof of assertion 4 is based on the following estimates.

1°. Estimate of $\Delta_0^*(\epsilon)$. On the basis of the inequalities (10), (11) of §4, Lemma 6 of §3 is applicable to equation (4). Here $M_1 = C$, $M_2 = \delta$, and since

$$\frac{C}{1-\delta} < \frac{\Delta_0}{3}, \quad \delta < \frac{1}{2}$$

(see formulas (10), (11) of §4),

$$|\Delta_0^*(\varepsilon)| < \frac{C}{1-\delta}\cdot$$

Taking into account that $\delta < \frac{1}{2}$, we find for $|\epsilon| < \epsilon_0$ that

$$|\Delta_0^*(\varepsilon)| < 2C. \tag{16}$$

2°. Estimate of $g(z, \epsilon)$. Inequality (16) makes it possible to estimate the right side of equation (2). For $|\mathrm{Im}\, z| < R$, $|\epsilon| \le \epsilon_0$, $\Delta = \Delta_0^*(\epsilon)$, from (16) and inequalities (7), (8), (10) of §4 it follows that

$$|\widetilde{F}(z, \varepsilon) + \widetilde{\Phi}(z, \varepsilon, \Delta)| \leqslant 2C + 2\delta \cdot 2C < 4C. \tag{17}$$

Applying Theorem 1 of §2 to equation (2), we obtain on the basis of (5), (17) and condition 4) of the Fundamental Lemma that for $|\mathrm{Im}\, z| \le R_0 - 2\delta$, $|\epsilon| \le \epsilon_0$ and any $\delta < 1$, $0 < \delta < R_0/2$,

$$|g(z, \varepsilon)| < \frac{8\cdot 4C}{K\delta^3}, \quad \left|\frac{\partial g}{\partial z}\right| < \frac{16\cdot 4C}{K\delta^4},$$

from which, in view of inequality (9) of §4,

$$|g(z, \varepsilon)| < \frac{C}{\delta^4}, \quad \left|\frac{\partial g(z, \varepsilon)}{\partial z}\right| < \frac{C}{\delta^5}. \tag{18}$$

Since $C < \delta^8$ by inequality (7) of §4, it follows that

$$|g(z, \varepsilon)| < \delta.$$

Therefore under the mapping $z \to \phi(z, \epsilon) = z + g(z, \epsilon)$ the strip

$$|\mathrm{Im}\, z| \leqslant R_0 - 2\delta$$

goes into a region containing the strip

$$|\mathrm{Im}\, \varphi| \leqslant R_0 - 3\delta.$$

In the latter the inverse function is analytic, since $|\partial\phi/\partial z| > \frac{1}{2}$ for $|\mathrm{Im}\, z| < R_0 - 2\delta$. In the same way one proves inequality (16) of §4.

3°. **Estimate of** $F_1(\phi, \epsilon)$. Suppose that $|\mathrm{Im}\, z| < R_0 - 3\delta$, $|\epsilon| \le \epsilon_0$. Since, from inequality (16) and conditions 3) and 5) of the Fundamental Lemma,

$$F_1(\varphi, \varepsilon) = \hat{F}_1(z(\varphi, \varepsilon), \varepsilon),$$

the imaginary parts z_{I} and z_{II} (see (10) and (11)) do not exceed $R_0 - 2\delta$. Applying Lemma 5 of §3, we find on the basis of (17) and (18) that for $|\mathrm{Im}\, z| < R_0 - 3\delta$, $|\epsilon| \le \epsilon_0$

$$|\hat{F}(z, \varepsilon)| \leqslant \frac{4C^2}{\delta^5}. \tag{19}$$

We note that the appearance of C^2 in this inequality is the most essential feature of the proof of Theorem 2.

For $|\mathrm{Im}\, \phi| \le R_0 - 4\delta$ and $|\epsilon| \le \epsilon_0$ we have from 2°

$$|\mathrm{Im}\, z(\varphi, \varepsilon)| < R_0 - 3\delta,$$

and therefore estimate (14) of §4 follows from (19) in view of the definition of $F_1(\phi, \epsilon)$ and inequality (10) of §4.

4°. **Estimate of** $|\Delta(\Delta_1, \epsilon) - \Delta_0^*(\epsilon)|$. The equation

$$\Delta = \Delta_1 - \overline{F}(\varepsilon) - \overline{\Phi}(\varepsilon, \Delta),$$

defining $\Delta(\Delta_1, \epsilon)$, belongs to the type considered in Lemma 6 of §3. We have seen (see (16)) that $|\Delta_0^*(\epsilon)| < 2C$, from which, on the basis of formula (11) of §4, it results that

$$|\Delta_0^*(\varepsilon)| < \frac{\Delta_0}{3}. \tag{20}$$

Thus Lemma 6 is applicable, and for $|\Delta_1| \le C < \Delta_0/6$, $|\epsilon| \le \epsilon_0$

$$|\Delta(\Delta_1, \varepsilon) - \Delta_0^*(\varepsilon)| < 2|\Delta_1|. \tag{21}$$

Comparing (20) and (21), we find that for $|\epsilon| \le \epsilon_0$, $|\Delta_1| \le C$

$$|\Delta(\Delta_1, \varepsilon)| < \frac{2}{3}\Delta_0.$$

For $|\epsilon| < \epsilon_0$, $|\Delta| < (2/3)\Delta_0$, from Cauchy's formula we have

$$\left|\frac{\partial\Phi}{\partial\Delta}\right| < \frac{\delta\Delta_0}{\frac{\Delta_0}{3}} < \frac{1}{2}$$

(see inequalities (8), (10) of §4). Estimate (17) of §4 is proved since it is evident that

$$\left|\frac{\partial\Delta}{\partial\Delta_1}\right| = \left|\frac{1}{1 + \frac{\partial\Phi}{\partial\Delta}}\right| < 2.$$

5°. Estimate of $|\Phi_1(\phi, \epsilon, \Delta_1)|$. Let us set up the difference $z_{III} - z_I$. From formulas (12) and (10) it is equal to

$$\Delta_1 + \widetilde{\Phi}(z, \varepsilon, \Delta(\Delta_1, \varepsilon)) - \widetilde{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon)).$$

From Lemma 5 of §3, for $|\mathrm{Im}\, z| \le R_0$, $|\epsilon| \le \epsilon_0$, $|\Delta_1| < \Delta_0/6$

$$|\widetilde{\Phi}(z, \varepsilon, \Delta(\Delta_1, \varepsilon)) - \widetilde{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon))| < |\Delta - \Delta_0^*|,$$

since $|\partial\widetilde{\Phi}/\partial\Delta| < 1$. Comparing the inequality just obtained with inequality (21), we have

$$|z_{III} - z_I| < 3|\Delta_1|. \tag{22}$$

Applying Lemma 5 of §3 to the right side of (8), on the basis of (22), (18) and inequalities (7), (10) of §4 we find that

$$|\hat{\Phi}_1(z, \varepsilon, \Delta)| < \frac{C}{\delta^5} 3|\Delta_1| < \delta^2|\Delta_1| \tag{23}$$

under the condition that $|\epsilon| \le \epsilon_0$, $|\Delta_1| < \Delta_0/6$,

$$|\mathrm{Im}(z + \Delta_1 + \widetilde{F} + \widetilde{\Phi})| \leqslant R_0 - 2\delta.$$

This last inequality is satisfied if

$$|\mathrm{Im}\, z| < R_0 - 6\delta, \quad |\Delta_1| < C, \quad |\varepsilon| < \varepsilon_0.$$

Indeed, then

$$|\widetilde{F} + \widetilde{\Phi}| < \delta + 2\delta\Delta_0 < 3\delta$$

(see formulas (7), (8), (17) of §4 and inequality (20)) in both the terms z_{III} and z_I. For $|\mathrm{Im}\,\phi| \le R_0 - 7\delta$, $|\Delta_1| < C$ we have, from 2°,

$$|\operatorname{Im} z| < R_0 - 6\delta.$$

Therefore estimate (15) of §4 follows from (23).

The Fundamental Lemma is proved.

§6. Proof of Theorem 2

6.1. Construction of $z(\phi, \epsilon)$ and $\Delta(\epsilon)$. We put $\Phi = 0$ in the Fundamental Lemma, and as $F(z, \epsilon)$ we take the function $F(z, \epsilon)$ of Theorem 2. We choose $\delta_1 > 0$ so that

1) $\sum_{n=1}^{\infty} \delta_n < \frac{R_0}{8}$, where $\delta_n = \delta_{n-1}^{1\frac{1}{2}}$ $(n = 2, 3, \ldots)$;

2) $\delta_1 < \frac{K}{64}$, $\delta_1 < \frac{1}{36}$.

Let $6\delta_1^{12} < \Delta_0 < 1$, $R = R_0$, K be the same as in the condition of the theorem. Let $L\epsilon' < C_1 = \delta_1^{12}$, $0 < \epsilon' < \epsilon_0$, C_1 and δ_1 be respectively ϵ_0, C and δ of the Fundamental Lemma. Then all the hypotheses of that lemma are satisfied, and for $|\operatorname{Im} \phi_1| \leq R - 7\delta_1$, $|\epsilon| \leq \epsilon'$, $|\Delta_1| \leq C_1$, we find that

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_1 + F_1(\varphi_1, \varepsilon) + \Phi_1(\varphi_1, \varepsilon, \Delta_1),$$

where

$$|F_1(\varphi_1, \varepsilon)| \leqslant \delta_1^{18} = \delta_2^{12}, \tag{1}$$

$$|\Phi_1(\varphi_1, \varepsilon, \Delta_1)| \leqslant \delta_1^2 |\Delta_1| < \delta_2 |\Delta_1|, \tag{2}$$

$$|z(\varphi_1, \varepsilon) - \varphi_1| \leqslant \delta_1, \quad \left|\frac{\partial z}{\partial \varphi_1}\right| < 2, \tag{3}$$

$$|\Delta(\Delta_1, \varepsilon)| \leqslant \Delta_0, \tag{4}$$

$$\left|\frac{\partial \Delta}{\partial \Delta_1}\right| < 2. \tag{5}$$

More generally, if the functions*

$$\Delta_{k-1}(\Delta_k, \varepsilon), \quad F_k(\varphi_k, \varepsilon), \quad \Phi_k(\varphi_k, \varepsilon, \Delta_k), \quad \varphi_{k-1}(\varphi_k, \varepsilon),$$
$$A_k(\varphi_k, \varepsilon, \Delta_k),$$

are defined for $k = 1, 2, \cdots, n$ and satisfy the conclusion of the Fundamental Lemma with z replaced by ϕ_{k-1}^*, ϕ by ϕ_k, R_0 by R_{k-1}, $R_0 - 7\delta$ by $R_k = R_{k-1} - 7\delta_k$, Δ_0 by δ_{k-1}^*, A_0 by A_{k-1}, A_1 by A_k, δ by δ_k, C by $C_k = \delta_k^{12}$ for each $k = 1, 2, \cdots, n$, then we may introduce functions ϕ_{n+1} and Δ_{n+1} such that the conclusion of the Fundamental Lemma will be valid for them for $k = 1, 2, \cdots, n + 1$. Indeed, inequalities (9) and (10) are satisfied for δ_n from the definition of δ_1, (11) follows from the inequality $C_{k+1} = C_k^{3/2} < (1/6)\, C_k$, and all the other

* ϕ_0 denotes z, C_0 denotes Δ_0; $\Delta_{1-1}(\Delta_1, \varepsilon) = \Delta(\Delta_1, \varepsilon)$.

conditions of the lemma enter into the conclusion (of course, for the functions with the preceding index). Therefore we may consider all the functions indicated above as having been constructed. The functions $\phi_{n-1}(\phi_n, \epsilon)$, $\Delta_{n-1}(\Delta_n, \epsilon)$ $(n = N, N-1, \cdots, 1)$ define the functions

$$z^{(N)}(\varphi_N, \varepsilon) = z(\varphi_1(\ldots(\varphi_N, \varepsilon)\ldots), \varepsilon), \tag{6}$$

$$\Delta_0^{(N)}(\Delta_N, \varepsilon) = \Delta(\Delta_1(\ldots(\Delta_N, \varepsilon)\ldots), \varepsilon). \tag{7}$$

Put $\Delta_N = 0$, and suppose that $\Delta_0^{(N)}(0, \epsilon) = \Delta^{(N)}(\epsilon)$. Then

$$\Delta(\varepsilon) = \lim_{N\to\infty} \Delta^{(N)}(\varepsilon),$$

$$z(\varphi, \varepsilon) = \lim_{N\to\infty} z^{(N)}(\varphi, \varepsilon).$$

For the basis of the convergence of $\Delta^{(N)}(\epsilon)$ and $z^{(N)}(\phi, \epsilon)$ we note first of all that from the definition of δ_n, for $\omega > 0$

$$\lim_{N\to\infty} 2^N \delta_N^{\omega} = 0.$$

6.2. Convergence of $\Delta^{(N)}(\epsilon)$. The functions $\Delta_0^{(N)}(\Delta_N, \epsilon)$, as follows from formula (7) and from inequality (17) of §4, are defined for $|\epsilon| \le \epsilon_0$, $|\Delta_N| \le \delta_N^{12}$. Since

$$\frac{\partial \Delta_0^{(N)}}{\partial \Delta_N} = \frac{\partial \Delta}{\partial \Delta_1} \cdots \frac{\partial \Delta_{N-1}}{\partial \Delta_N},$$

in the indicated region, on the basis of (5), the inequality

$$\left| \frac{\partial \Delta_0^{(N)}}{\partial \Delta_N} \right| < 2^N$$

is satisfied, and since

$$|\Delta_N [\Delta_{N+1}(\ldots(\Delta_M, \varepsilon)\ldots\varepsilon), \varepsilon]| \leqslant \delta_N^{12}$$

if $|\Delta_M| \le \delta_M^{12}$ $(M \ge N)$, therefore from Lemma 5 of §3,

$$|\Delta_0^N [\Delta_N(\Delta_{N+1}\ldots(\Delta_M, \varepsilon)\ldots, \varepsilon), \varepsilon] - \Delta_0^{(N)}(0, \varepsilon)| < 2^N \delta_N^{12}.$$

Thus in view of (7) we deduce that

$$|\Delta^{(N)}(\varepsilon) - \Delta^{(M)}(\varepsilon)| < 2^N \delta_N^{12},$$

from which it immediately follows that $\Delta^{(N)}(\epsilon)$ converges for $|\epsilon| \le \epsilon_0$, and also that $\Delta(\epsilon)$ is analytic.

6.3. Convergence of $z^{(N)}(\phi, \epsilon)$. From the Fundamental Lemma, the functions $\phi_{n-1}(\phi_n, \epsilon)$ are defined for $|\mathrm{Im}\, \phi_n| \le R$, $|\epsilon| \le \epsilon_0$, and, in view of (3), differ from their arguments ϕ_n by less than δ_n, so that

$$|\mathrm{Im}\, \varphi_{n-1}(\varphi_n, \varepsilon)| < R_{n-1}.$$

Thus formula (6) defines $z^{(N)}(\phi, \epsilon)$ in the strip

$$|\operatorname{Im} \varphi| \leqslant R_n = R_0 - 7\sum_{k=1}^{n} \delta_k.$$

From condition 1) on the choice of δ_1, all these strips contain the strip $|\operatorname{Im} \phi| \leq R/8$, so that all the functions $z^{(N)}(\phi, \epsilon)$ are defined in the latter.

Since

$$|\varphi_N(\varphi_{N+1}\cdots(\varphi_M, \varepsilon), \ldots, \varepsilon) - \varphi_M| < \sum_{k=N}^{M} \delta_k,$$

and this sum, from the definition of δ_n, is not larger than $2\delta_N$, we find from (6) that

$$|z^{(N)}(\varphi, \varepsilon) - z^{(M)}(\varphi, \varepsilon)| < \left|\frac{\partial z^{(N)}}{\partial \varphi}\right| 2\delta_N.$$

On the basis of (3),

$$\left|\frac{\partial z^{(N)}}{\partial \varphi}\right| < 2^N,$$

so that

$$|z^{(N)}(\varphi, \varepsilon) - z^{(M)}(\varphi, \varepsilon)| < 2^{N+1}\delta_N,$$

which proves the uniform convergence of $z^{(N)}(\phi, \epsilon)$ for $|\operatorname{Im} \phi| \leq R/8$, $|\epsilon| \leq \epsilon_0$.

6.4. We shall define $\phi(z, \epsilon)$ as the inverse to $z(\phi, \epsilon)$. From inequalities (1) and (2) and from the fact that $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$ it results that

$$\varphi(z, \varepsilon) \to \varphi(z, \varepsilon) + 2\pi\mu$$

when $z \longrightarrow A(z, \epsilon, \Delta(\epsilon))$. Theorem 2 is proved.

§7. On monogenic functions

7.1. **The concept of monogeneity.** In the investigation of the dependence of the solutions of equation (1) of §2 on the parameter μ we encounter functions analytic in the upper and in the lower half-plane, and everywhere discontinuous on the real axis. All the functions, Δ_n, g_n, ϕ_n, F_n, Φ_n constructed in §6, considered as functions of μ, have these properties (see §8). These functions belong to the type called by Borel [9] monogenic.

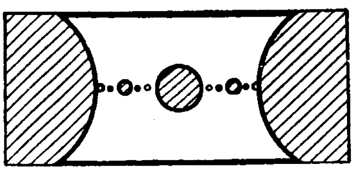

Figure 2

The monogenic functions of Borel are defined on the set $E = \bigcup_{k=1}^{\infty} E_k$, where $E_k \subseteq E_{k+1}$ are perfect compact subsets of the complex plane. In our case E_k is the set M_K^R of points μ of the rectangle $|\operatorname{Im} \mu| \leq R$, $0 \leq \operatorname{Re} \mu \leq 1$ of the complex plane, for which

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3} \quad \left(K = \frac{1}{k}\right),$$

i.e., the set formed by rejecting from the rectangle $|\mathrm{Im}\,\mu| \le R$, $0 \le \mathrm{Re}\,\mu \le 1$ the circles $C_{m/n,K}$, shaded in Figure 2, of radii $K/|n|^3$ with centers at rational points m/n.

Definition. A function $f(\mu)$ is said to be *uniformly differentiable* on a perfect compact set F of the complex plane, and the function $g(\mu)$ its derivative, if for any $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$\left|\frac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} - g(\mu_3)\right| < \varepsilon,$$

whenever $|\mu_1 - \mu_3| < \delta$, $|\mu_2 - \mu_3| < \delta$, $\mu_1, \mu_2, \mu_3 \in F$.

A function is *monogenic* on $E = \bigcup_{k=1}^{\infty} E_k$ if it is uniformly differentiable on each E_k.

In particular, a uniformly differentiable function on E is monogenic on $E = \bigcup_{k=1}^{1} E_k$, and conversely a function monogenic on $E = \bigcup_{k=1}^{1} E_k$ is uniformly differentiable on E. Such functions will be called monogenic on E, in distinction to those that are monogenic on $E = \bigcup_{k=1}^{\infty} E_k$.

The following properties of monogenic functions are evident.

1) From monogenicity on $E = \bigcup_{k=1}^{\infty} E_k$ follows continuity of the derivative on E_k.

2) If Γ is a rectifiable curve joining two points α and β in E_k, then

$$\int_{\Gamma} f'(\mu)\, d\mu = f(\beta) - f(\alpha).$$

3) If a function is analytic in a neighborhood of each point of a set, it is monogenic on the set.

4) If E_k contains a region, then a function in it which is monogenic on $E = \bigcup_{k=1}^{\infty} E_k$ is analytic.

An example of a nonanalytic monogenic function was constructed in §2, as is proved in subsection 7.4 (see Lemma 10; the fact that $g(\mu)$ is not analytic for $\mathrm{Im}\,\mu = 0$ is left to the reader to prove).

Properties of monogenicity of a function may essentially depend on its region of definition $E = \bigcup_{k=1}^{\infty} E_k$ and on the decomposition of E into the E_k. If the rapidity of decrease of the components of the complements to the E_k is sufficiently great, then, as Borel proved, monogenic functions on $E = \bigcup_{k=1}^{\infty} E_k$ have many properties of analytic functions (Cauchy integral, infinite differentiability, uniqueness of the monogenic prolongation). The question as to which of these properties are preserved in our case will be left aside, since in the sequel

(§§8 and 11) we use only the definition of uniform differentiability.

The class of functions monogenic on $E = \bigcup_{k=1}^{\infty} E_k$ depends not only on E but also on E_k. However, if E is obtained by using another system of sets, $E = \bigcup_{k=1}^{\infty} F_k$, $F_k \subseteq F_{k+1}$, such that

$$E_{\alpha k} \subseteq F_k \subseteq E_{\beta k} \quad (\alpha < 1 < \beta),$$

then the class of functions monogenic on $E = \bigcup_{k=1}^{\infty} E_k$ and on $E = \bigcup_{k=1}^{\infty} F_k$ coincide. The sets M_K^R (Figure 2) are not convenient for the investigation of monogenic functions because of the complex character of the intersections of the disks $C_{m/n,K}$. Making use of the above remark, we replace these sets by another system of sets N_K^R such that:

1. $M_{2K}^R \subseteq N_K^R \subseteq M_{\frac{K}{2}}^R$.

2. The set N_K^R is obtained from the rectangle $|\operatorname{Im} \mu| \leq R$, $\operatorname{Re} \mu \in [0, 1]$ by deleting nonintersecting open disks.

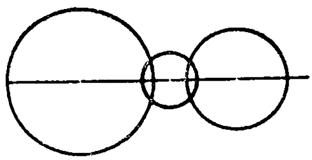

Figure 3

The construction of the N_K^R $(K < 1/9)$ is given in subsection 7.2; it is complex and may be omitted by the reader.

7.2. **Construction of the N_K^R.** The transformation of the M_K^R into the N_K^R consists of two operations. First the disks being deleted are diminished to disks $C'_{m/n,K}$ so that in the system $C'_{m/n,K}$ $(m = 0, 1, \cdots; n = 1, 2, \cdots)$ there are no "bridges" (see Figure 3), i.e., triples of disks of which the smallest intersects both the larger, while these latter do not intersect one another. Then the disks C' are increased to disks $C''_{m/n,K}$ so that two such disks either do not intersect or else one lies inside the other. Here it is necessary to order them so that

$$C_{\frac{m}{n}, K} \supseteq C'_{\frac{m}{n}, K} \supseteq C_{\frac{m}{n}, \frac{K}{2}},$$
$$C'_{\frac{m}{n}, K} \subseteq C''_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, 2K}$$

Then

$$C_{\frac{m}{n}, \frac{K}{2}} \subseteq C''_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, 2K}$$

and after deletion from the rectangle of the disks $C''_{m/n,K}$ there remains a set N_K^R, having both of the needed properties.

Lemma 7. *Suppose that the disks $C_{m/n,K}$ and $C_{p/q,K}$ $(n \geq q)$ intersect* *and* $K < 1/9$. *Then* $n > 2q^{4/3}$; *i.e., the smaller disk is much smaller than the larger.*

Proof. Indeed, the sum of the radii of the circles is larger than the distance between their centers, so that

$$\frac{K}{n^3} + \frac{K}{q^3} > \left| \frac{p}{q} - \frac{m}{n} \right|.$$

Since $pn - qm \neq 0$, then $\left| \frac{p}{q} - \frac{m}{n} \right| \geqslant \frac{1}{qn}$, and

$$K(n^3 + q^3) \geqslant q^2 n^2;$$

in view of the inequality $n \geq q$ we obtain

$$K(n^3 + q^3) \geqslant q^4,$$

or

$$n^3 > \frac{q^4}{K} - q^3.$$

Taking into account the fact that $K < 1/9$, we have

$$n^3 > 9q^4 - q^3 \geqslant 8q^4,$$

as was required to be proved.

Operation 1: Construction of the $C'_{m/n,K}$. This construction consists of an infinite number of successively realized stages such that after the nth stage disks $C'_{m/n,K}$ $(0 \leq m \leq n)$ have been constructed with the following properties:

A_n. No disk $C_{m_1/n_1,K}$ $(n_1 > n)$ can join a disk $C'_{m/n,K}$ to a disk $C'_{m_2/n_2,K}$ $(n_2 \leq n)$ if these disks $C'_{m/n,K}$ and $C'_{m_2/n_2,K}$ do not intersect each other.

B_n. $C_{\frac{m}{n},\frac{K}{2}} \subseteq C'_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},K}$.

We begin with the first stage. Suppose that $C'_{m/1,K} = C_{m/1,K}$. Then the property B_1 is satisfied. Property A_1 is also satisfied, since the diameter of the disk $C_{m_1/n_1,K}$ $(n_1 > 1)$ is less than

$$\frac{2K}{n_1^3} < \frac{2}{9 \cdot 8} \quad \left(K < \frac{1}{9}\right),$$

and the distance between the disks $C_{0/1,K}$ and $C_{1/1,K}$ is larger than

$$1 - 2K > \frac{2}{3}.$$

The first stage is done.

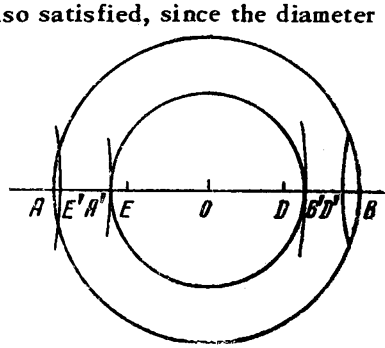

Figure 4

Now we suppose that we have successively performed $n-1$ stages. We consider any disk $C = C_{m/n,K}$ (Figure 4). Suppose that O is its center, AB the diameter lying on the real axis, E and D the means of AO and OB. The disk C can only intersect with those disks $C'_{m_2/n_2,K}$ $(n_2 < n)$ for which $C_{m_2/n_2,K}$ intersects with C (because of property B_k, $k \le n-1$). Further, all such disks $C'_{m_2/n_2,K}$ intersect also one with another (as a consequence of property A_k, $k \le n-1$).

Now we arrange the disks in the order of decrease of n_2 (i.e., of the growth of the disks):

$$C_i = C_{\frac{m_{2,i}}{n_{2,i}}, K} \qquad (n = n_{2,0} > n_{2,1} > \ldots > n_{2,l} \geqslant 1).$$

From Lemma 7, $n_{2,i} \ge 2n_{2,i+1}$ $(0 \le i \le l-1)$, so that $n > 2^l$ and $l < \log_2 n$. Thus, the circumferences of the disks $C_{m_2/n_2,K}$ yield in their intersection with the diameter AB not more than $2\log_2 n$ points. Therefore among the portions into which these points divide the segment BD and the segment AE, there is at least one length larger than $K/4n^3 \log_2 n$. Now the diameter of the circumference $C_{m_1/n_1,K}$ $(n_1 > n)$, which intersects with C, by Lemma 7 does not exceed

$$\frac{K}{8n^4} < \frac{K}{4n^3 \log_2 n}.$$

We take the ends B' and A' closest to O of the largest pieces of BD and AE, which we denote by $B'D'$ and $A'E'$, as the ends of the diameter of $C'_{m/n,K}$. Such a choice does not contradict property B_n. It is clear that if the circumference $C_1 = C_{m_1/n_1,K}$ $(n_1 > n)$ intersects $C'_{m/n,K}$, then it lies inside C, and among the disks $C_{m_2/n_2,K}$ $(n_2 \le n)$ can only intersect the C_i. But since the diameter of C_1 is less than the lengths of $B'D'$ and $A'E'$, therefore C_1 can only intersect those C_i which are intersected by $C'_{m/n,K}$. Therefore property A_n is also satisfied, and thus we have given the construction of the nth stage.

At the conclusion of all the stages one obtains a system of disks $C'_{m/n,K}$ with the following properties:

A. *No disk* $C_{\frac{m_1}{n_1}, K}$ *can join* $C'_{\frac{m_2}{n_2}, K}$ *and* $C'_{\frac{m_3}{n_3}, K}$ *if* $n_1 > n_2,\ n_1 > n_3$ *and* $C'_{\frac{m_2}{n_2}, K} \cap C'_{\frac{m_3}{n_3}, K} = 0.$

B. $C_{\frac{m}{n}, \frac{K}{2}} \subseteq C'_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, K}.$

Property B follows from B_n, and property A from A_{n_2} if $n_2 \ge n_3$, and from A_{n_3} if $n_3 \ge n_2$.

Operation 2: Construction of the $C''_{m/n,K}$. Now we shall enlarge the disks of the system $C'_{m/n,K}$.

By a *tail* $C = C'_{m/n,K}$ we shall mean the collection of all the $C'_{m_i/n_i,K}$ $(n_i > n)$ which may be joined to C by a monotone finite chain of pairwise intersecting disks $C'_{m_{j_k}/n_{j_k},K}$ $(0 \leq k \leq l_i)$:

$$\frac{m_{j_0}}{n_{j_0}} = \frac{m}{n}, \quad n_{j_k} < n_{j_{k+1}}, \quad C'_{\frac{m_{j_k}}{n_{j_k}}, K} \cap C'_{\frac{m_{j_{k+1}}}{n_{j_{k+1}}}, K} \neq 0, \quad \frac{m_{j_{l_i}}}{n_{j_{l_i}}} = \frac{m_i}{n_i}.$$

Obviously, if the disk C_1 enters into the tail of the disk C_2, then the tail of C_1 always enters into the tail of C_2. Moreover, if the tails of C_1 and C_2 intersect,* then one of the tails lies entirely in the other. We shall prove this fact. We suppose on the contrary that the disks C_1 and C_2 may be joined to a common disk of their tails, C_3, by monotone chains. Two such chains at the same time join C_1 and C_2. Of the chains joining C_1 and C_2 we select one consisting of the smallest number of disks. In this chain only successive disks intersect one another (see Figure 5; in the system of circles drawn there the shaded tail is the largest). If this chain is monotone, then our assertion is proved. If the chain is not monotone, then there is a disk in it which joins two preceding it, which contradicts property A of operation 1. Thus, if two tails intersect, then one of them contains the other.

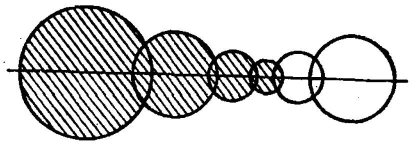

Figure 5

Suppose that α and β are the upper and lower bounds of the points of the real axis covered by the tail of the disk $C = C'_{m/n,K}$. The disk with diameter $\alpha\beta$ will also be a disk $C''_{m/n,K}$. From what has been stated above it follows that the circumferences of two such disks do not intersect.** Evidently $C''_{m/n,K} \supseteq C'_{m/n,K}$. We shall show that

$$C''_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, 2K}$$

Indeed, on the basis of Lemma 7, it is easy to estimate the measure of the tail of C. Suppose that the disk C_1 belongs to the tail of C and the monotone chain joining C_1 to C consists of N disks. Since each of those following them is not less than 8 times smaller than the preceding one, the sum of their diameters does not exceed the diameter of C for any N. Therefore it is evident

* It is easy to see that if two tails intersect as point sets, then they have a common disk.

** But they can touch.

that α and β are distant from $C'_{m/n,K}$ by not more than 1/7 of the diameter of $C_{m/n,K}$, and from the center m/n by not more than 9/7 of the radius of $C_{m/n,K}$. Hence it follows that

$$C''_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},2K}$$

The construction of the N_K^R is complete.

7.3. **Differentiation of sequences.** The passage to the complex plane of μ was undertaken largely for the sake of the following lemma, which is not valid if by the set N_K^R is meant its part lying on the real axis.

Lemma 8. *Suppose that the sequence $f_n(\mu)$ of functions, monogenic on the set N_K^R, converges there uniformly to $f(\mu)$, and that the derivatives converge uniformly to $g(\mu)$. Then $f(\mu)$ is monogenic on N_K^R and $f'(\mu) = g(\mu)$.*

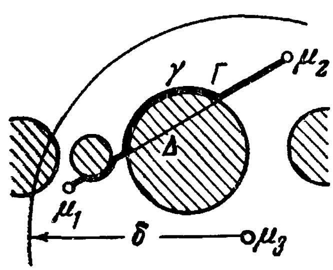

Figure 6

Proof. Suppose that $\epsilon > 0$. We may choose $\delta > 0$ so that

$$\left| \frac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} - g(\mu_3) \right| < \varepsilon$$

when $|\mu_1 - \mu_3| < \delta$, $|\mu_2 - \mu_3| < \delta$, $\mu_1, \mu_2, \mu_3 \in N_K^R$.

If $\delta > 0$ is sufficiently small, then all of these points lie in one component of N_K^R.

We shall show that in such a case the points μ_1, μ_2 may be joined in N_K^R by a rectifiable curve Γ such that the following conditions are satisfied:

1) for any point $\mu \in \Gamma$ $|\mu - \mu_3| < 2\delta$;

2) the length of Γ is less than $2|\mu_1 - \mu_2|$.

Indeed, let us join the points μ_1 and μ_2 by a segment $\mu_1\mu_2$ (see Figure 6). This segment may intersect certain disks C_i, by the deletion of which from the rectangle $|\mathrm{Im}\,\mu| \leq R$, $\mathrm{Re}\,\mu \in [0, 1]$ the set N_K^R was formed. These disks are disjoint and do not separate μ_1 from μ_2 in N_K^R, since the points μ_1 and μ_2 lie in one component. The disks C_i excise on $\mu_1\mu_2$ nonintersecting intervals Δ_i. We replace each such interval Δ_i by the smaller of the two arcs into which $\mu_1\mu_2$ divides the circumference C_i, and denote this arc by γ_i. The length of Δ_i is increased by such a substitution by not more than $\pi/2$ times, and therefore the length of Γ will be less than $2|\mu_1 - \mu_2|$. The distance $|\mu_1 - \mu_2|$, by hypothesis, does not exceed 2δ, so that all the points of γ_i are less than δ units distant from the midpoint of Δ_i. This last point, as well as all the points of the segment $\mu_1\mu_2$, lies in the disk $|\mu_1 - \mu_2| < \delta$, so that for any point $\mu \in \gamma_i$

$$|\mu - \mu_3| < 2\delta.$$

Thus the curve Γ is the one desired.

2. We have already noted that if $\phi(\mu)$ is monotone in N_K^R and Γ is a rectifiable curve with endpoints μ_1 and μ_2, then

$$\int_\Gamma \varphi'(\mu)\,d\mu = \varphi(\mu_2) - \varphi(\mu_1).$$

(For the proof it is only necessary to equate the integral to the integral sum.)

Applying this equation to the curve Γ constructed above and to the function $f_n(\mu)$, which is monogenic by hypothesis, we obtain

$$\int_\Gamma f_n'(\mu)\,d\mu = f_n(\mu_2) - f_n(\mu_1).$$

In view of the uniform convergence of the f_n to f and f_n' to g, we may pass to the limit on left and right:

$$\int_\Gamma g(\mu)\,d\mu = f(\mu_2) - f(\mu_1).$$

3. Now we shall estimate

$$\left|\frac{f(\mu_2)-f(\mu_1)}{\mu_2-\mu_1} - g(\mu_3)\right|.$$

To this end we consider the integral

$$\int_\Gamma (g(\mu) - g(\mu_3))\,d\mu = f(\mu_2) - f(\mu_1) - (\mu_2-\mu_1)\,g(\mu_3).$$

We have

$$\left|\int_\Gamma (g(\mu) - g(\mu_3))\,d\mu\right| \leqslant \int_\Gamma |g(\mu) - g(\mu_3)|\,|d\mu| \leqslant \max_{\mu\in\Gamma} |g(\mu) - g(\mu_3)|\cdot 2\,|\mu_2-\mu_1|,$$

since the length of Γ is less than $2|\mu_2 - \mu_1|$.

Thus,

$$\left|\frac{f(\mu_2)-f(\mu_1)}{\mu_2-\mu_1} - g(\mu_3)\right| \leqslant 2\max_{\mu\in\Gamma} |g(\mu) - g(\mu_3)|.$$

The right side of the last inequality, from property 1) of the curve Γ, is twice the increment of $g(\mu)$ on a segment of length less than 2δ, and, in view of the uniform continuity of the function $g(\mu)$ on the compactum N_K^R, tends to zero along with δ. Lemma 8 is thus proved.

7.4. **Functions of several variables and operations on them.** In what follows we shall need functions analytic in one variable and monogenic in the others.

Suppose that the variable z is angular (varies in the strip $\operatorname{Im} z \in (ab)$)* and has period 2π,** the variables ϵ and Δ vary in the neighborhood of zero, and

* The boundaries may depend on μ.

** I.e., as z increases by 2π, functions of z have increments of 0 or 2π.

$\mu \in N_K^R$.

Definition. The function $f(z, \epsilon, \Delta, \mu)$ is *analytic in* z, ϵ, *and* Δ, *and monogenic in* $\mu \in N_K^R$ if the series

$$f(z, \varepsilon, \Delta, \mu) = \sum f_{kmn}(\mu) e^{ikz} \varepsilon^m \Delta^n,$$

in which the coefficients are monogenic functions of $\mu \in N_K^R$, converges along with its derivative with respect to μ uniformly for $\mu \in N_K^R$ and z, ϵ, Δ, varying in the indicated regions.

Evidently such a function is continuous, while

a) for fixed μ it is analytic in z, ϵ, Δ and

b) for fixed z, ϵ, Δ it is monogenic in $\mu \in N_K^R$.

Property b) follows from Lemma 8.

Lemma 9. *Suppose that the functions* $h_i(z, \epsilon, \Delta, \mu)$ *are monogenic with respect to* $\mu \in E$ *and analytic in* z, ϵ, Δ. *Then the following functions have the same property in the corresponding regions:*

1) *the functions*

$$h_1(z, \varepsilon, \Delta, \mu) + h_2(z, \varepsilon, \Delta, \mu), \quad h_1(z, \varepsilon, \Delta, \mu) h_2(z, \varepsilon, \Delta, \mu),$$
$$h_1(h_2(z, \varepsilon, \Delta, \mu), \varepsilon, \Delta, \mu), \quad h_1(z, \varepsilon, h_2(z, \varepsilon, \Delta, \mu), \mu);$$

2) *the solution* $\phi(z, \epsilon, \Delta, \mu)$ *of the equation* $h(\phi, \epsilon, \Delta, \mu) = z$;

3) *the solution* $\gamma(z, \epsilon, \Delta, \mu)$ *of the equation* $h(z, \epsilon, \gamma, \mu) = \Delta$;

4) *the partial derivatives of* h *with respect to* z, ϵ, Δ;

5) *the integral with respect to a parameter* $\int_0^{2\pi} h(z, \epsilon, \Delta, \mu)\, dz$;

while in all these cases the usual rules of differentiation apply; for example, in case 2)

$$\frac{\partial \varphi}{\partial \mu} = -\frac{\frac{\partial h}{\partial \mu}}{\frac{\partial h}{\partial \varphi}}.$$

The proof repeats well-known arguments from standard analysis and will be omitted.

Lemma 10. *Suppose that the function* $f(z, \epsilon, \Delta, \mu) = \tilde{f}$ *is analytic with respect to* z *in the region* $|\operatorname{Im} z| \le R$; ϵ, $|\epsilon| \le \epsilon_0$; $|\Delta| \le \Delta_0$ *and is monogenic with respect to* $\mu \in N_K^R$, *and suppose that in the indicated region*

$$|f| \leqslant C, \quad \left|\frac{\partial f}{\partial \mu}\right| \leqslant L.$$

Then the solution of the equation

$$g(z + 2\pi\mu, \varepsilon, \Delta, \mu) - g(z, \varepsilon, \Delta, \mu) = f(z, \varepsilon, \Delta, \mu)$$

is monogenic with respect to $\mu \in N_K^R$ *and analytic with respect to* z *in the* *region* $|\operatorname{Im}(z-2\pi\mu)| \le R-2\delta$, ϵ, $|\epsilon| \le \epsilon_0$, Δ, $|\Delta| \le \Delta_0$, *while in this region*

$$|g| \leqslant \frac{4C}{K\delta^3}, \quad \left|\frac{\partial g}{\partial z}\right| \leqslant \frac{8C}{K\delta^4}, \quad \left|\frac{\partial^2 g}{\partial z^2}\right| \leqslant \frac{10C}{K\delta^5},$$

$$\left|\frac{\partial g}{\partial \mu}\right| \leqslant \frac{C+L}{K^2}\frac{10^3}{\delta^6}, \quad \left|\frac{\partial^2 g}{\partial z \partial \mu}\right| \leqslant \frac{C+L}{K^2}\frac{10^3}{\delta^7}.$$

Proof. The solution is given for fixed μ by the series (2) of §2:

$$\sum_{n \neq 0} \frac{f_n(\mu, \varepsilon, \Delta)}{e^{2\pi i n\mu}-1} e^{inz},$$

of which it is required to establish the uniform convergence for $|\operatorname{Im}(z-2\pi\mu)| \le R-2\delta$, since

$$f_n(\mu, \varepsilon, \Delta) = \sum f_{nkl}(\mu)\, \varepsilon^k \Delta^l.$$

But the uniform convergence of this series has been established in §2 along with the desired estimates of g and $\partial g/\partial z$ in the proof of Theorem 1′, since

$$N_K^{\frac{1}{2\pi}} \subseteq M_{\frac{K}{2}}^{\frac{1}{2\pi}}.$$

Estimates of the other derivatives are obtained by differentiation of the series using the usual formulas and taking account of inequality (13) of §2.

§8. On the dependence of the constructions of Theorem 2 on μ

8.1. We have seen, in subsection 7.4, that the solution of the linear equation (1) of §2 depends on μ monogenically. In the present section we shall prove the monogenicity with respect to μ of the functions Δ_n, F_n, Φ_n, g_n, $\Delta^{(n)}$ constructed in §6.

It turns out that the region of monotonicity contracts as n increases (by $|\operatorname{Im} 2\pi\mu|$ at each step) and the author has not been able to establish whether the solution of equation (1) of §4 depends monogenically on μ.

The monogenicity of $\Delta^{(n)}$ with respect to μ for real μ is used in §11. There we shall also make use of the smallness (uniformly with respect to n) of $\partial\Delta^{(n)}/\partial\mu$ for small ϵ.

In order to shorten complicated expressions in this section the argument ϵ will be dropped in all functions. This is similar to the way in which we earlier ignored the dependence on μ and took only z, ϕ, ϵ, Δ as arguments.

The construction of $\Delta^{(n)}(\mu)$ was carried out in the following way.

Step by step we constructed new parameters $\phi_n = \phi_n(\phi_{n-1}, \mu)$ and quantities $\Delta_{n-1} = \Delta_{n-1}(\Delta_n, \mu)$ such that the transformation

$$\varphi_{n-1} \to \varphi_{n-1} + 2\pi\mu + \Delta_{n-1}(\Delta_n, \mu) + F_{n-1}(\varphi_{n-1}, \mu) + \Phi_{n-1}(\varphi_{n-1}, \Delta_{n-1}(\Delta_n, \mu)\, \mu)$$

is converted into the transformation

$$\varphi_n \to \varphi_n + 2\pi\mu + \Delta_n + F_n(\varphi_n, \mu) + \Phi_n(\varphi_n, \Delta_n, \mu)$$

with significantly smaller F and Φ, where $\phi_0 = z$, $F_0 = F$, $\Phi_0 = 0$, $\Delta_0 = \Delta$.

Further, we constructed $\Delta^{(n)}(\mu)$ such that the transformation

$$z \to z + 2\pi\mu + \Delta^{(n)}(\mu) + F(z)$$

converts, in the variable ϕ_n, into the transformation

$$\varphi_n \to \varphi_n + 2\pi\mu + F_n(\varphi_n, \mu) + \Phi_n(\varphi_n, 0, \mu),$$

to which end we put

$$\begin{aligned} &\Delta_k^{(n)}(\mu) = \Delta_k(\Delta_{k+1}^{(n)}(\mu), \mu) \quad (k = 0, 1, \ldots, n-1), \\ &\Delta_n^{(n)}(\mu) = 0. \end{aligned} \tag{1}$$

Thus we obtained

$$\Delta_0^{(n)}(\mu) = \Delta^{(n)}(\mu).$$

Theorem 3. *Under the conditions of Theorem 2, for sufficiently small* $\epsilon > 0$, $0 < K < 1/9$

$$\Delta(\mu) = \lim_{n\to\infty} \Delta^{(n)}(\mu),$$

where the functions $\Delta^{(n)}(\mu)$ *are monogenic with respect to* $\mu \in N_K^{r_n}$ $(r_n > 0)$ *and under these conditions* $|\partial\Delta^{(n)}/\partial\mu| < 6L|\epsilon|$.

The proof of this theorem rests on the following lemma, which repeats the Fundamental Lemma (see §§4 and 5).

Lemma 11. *Suppose we are given a family of analytic mappings of the circumference, depending analytically on* Δ *and monogenically on* $\mu \in N_K^r$,

$$z \to A_0(z, \Delta, \mu) = z + 2\pi\mu + F(z, \mu) + \Delta + \Phi(z, \Delta, \mu)$$

and numbers $R_0 > 0$, $1/9 > K > 0$, $\delta > 0$, $C > 0$, $0 < \Delta_0 < 1$, $0 < r < 1/2\pi$, $2\pi r \le R_0 - 5\delta$ *such that*

1) $F(z + 2\pi, \mu) = F(z, \mu)$, $\Phi(z + 2\pi, \Delta, \mu) = \Phi(z, \Delta, \mu)$;

2) *for* $\operatorname{Im} z = \operatorname{Im} \mu = \operatorname{Im} \Delta = 0$ *always* $\operatorname{Im} F = \operatorname{Im} \Phi = 0$;

3) *for* $|\operatorname{Im} z| \leqslant R_0$, $\mu \in N_K^r$, $|\Delta| \leqslant \Delta_0$

$$|F(z, \mu)| \leqslant C, \tag{2}$$

$$\left|\frac{\partial F(z, \mu)}{\partial\mu}\right| \leqslant C, \tag{3}$$

$$|\Phi(z, \mu, \Delta)| \leqslant \delta^2 |\Delta|, \tag{4}$$

$$\left|\frac{\partial\Phi(z, \mu, \Delta)}{\partial\mu}\right| \leqslant \delta^2 |\Delta|; \tag{5}$$

4) *the number* δ *satisfies the inequality*

$$\delta < \frac{K^2}{5\cdot 10^4}; \tag{6}$$

5) $C = \delta^{27}$, $\Delta_0 = \delta^{26}$.

Then there exist functions $z(\phi, \mu)$, $\Delta(\Delta_1, \mu)$ *analytic in* ϕ, Δ_1 *and mono-*

genic in $\mu \in N^r_K$ *such that*

1. *Identically*

$$z(A_1(\varphi, \mu, \Delta_1), \mu) = A_0(z(\varphi, \mu), \Delta(\Delta_1, \mu), \mu),$$

where

$$A_1(\varphi, \mu, \Delta_1) \equiv \varphi + 2\pi\mu + \Delta_1 + F_1(\varphi, \mu) + \Phi_1(\varphi, \mu, \Delta_1).$$

2. $F_1(\varphi + 2\pi, \mu) = F_1(\varphi, \mu), \quad \Phi_1(\varphi + 2\pi, \mu, \Delta_1) = \Phi_1(\varphi, \mu, \Delta_1),$

$$z(\varphi + 2\pi, \mu) = z(\varphi, \mu) + 2\pi.$$

3. *For* $\operatorname{Im}\varphi = \operatorname{Im}\Delta_1 = \operatorname{Im}\mu = 0$ *always* $\operatorname{Im} z = \operatorname{Im}\Delta = \operatorname{Im} F_1 = \operatorname{Im}\Phi_1 = 0$

4. *For* $|\Delta_1| \leqslant \delta^{28}$, $|\operatorname{Im}\varphi| \leqslant R_0 - 7\delta - |\operatorname{Im} 2\pi\mu|$, $\mu \in N^r_K$ *the functions constructed above are analytic in* ϕ, Δ_1, *monogenic in* $\mu \in N^r_K$, *and the following relations hold*:

$$|F_1| \leqslant \frac{C^2}{\delta^6}, \tag{7}$$

$$|\Phi_1| \leqslant \frac{C}{\delta^6}|\Delta_1|, \tag{8}$$

$$\left|\frac{\partial F_1}{\partial \mu}\right| \leqslant \frac{C^2}{\delta^{13}}, \tag{9}$$

$$\left|\frac{\partial \Phi_1}{\partial \mu}\right| \leqslant \frac{C}{\delta^{13}}|\Delta_1|, \tag{10}$$

$$\left|\frac{\partial z}{\partial \mu}\right| \leqslant \frac{C}{\delta^7}, \tag{11}$$

$$\left|\frac{\partial \Delta}{\partial \mu}\right| \leqslant 4C, \tag{12}$$

$$|\Delta(\Delta_1, \mu)| \leqslant \Delta_0, \tag{13}$$

$$|z(\varphi, \mu) - \varphi| \leqslant \frac{C}{\delta^4}, \tag{14}$$

$$\left|\frac{\partial \Delta}{\partial \Delta_1}\right| \leqslant 2, \tag{15}$$

$$\left|\frac{\partial z}{\partial \varphi}\right| \leqslant 2. \tag{16}$$

8.2. **The proof of Lemma 11 is more complicated than the proof of the Fundamental Lemma. The construction repeats the considerations of subsections 5.1 with the difference that μ changes from a fixed real number to an independent complex variable. In the construction of $\Delta(\Delta_1)$, $z(\phi)$, g, F_1 and Δ_1, following subsections 5.1, one uses integration with respect to z, the solution of equation (1) of §2, the construction of an inverse function and the substitution of a function into a function. From the lemmas of subsection 7.4 all of these operations do not lead out of the class of functions monogenic in $\mu \in N^r_K$ and analytic with respect to z, Δ, ϕ, Δ_1 in the corresponding regions.**

Therefore special attention need be directed only to inequalities (9), (10), (11), and (12), which are not in the Fundamental Lemma. Their proof is based on the following estimates.

1°. **Estimate of $\partial g^*/\partial \mu$.** On the basis of subsections 5.1, 7.4, and in view of the conditions of the lemma, for $|\mathrm{Im}\, z| \leq R_0$, $\mu \in N_K^r$, $|\Delta| \leq \Delta_0$

$$\left|\frac{\partial \widetilde{F}}{\partial \mu}\right| \leqslant 2C, \quad \left|\frac{\partial \widetilde{\Phi}}{\partial \mu}\right| \leqslant 2\delta^2 |\Delta_0| \leqslant 2C.$$

Thus the right side of equation (2) of §5 has a derivative with respect to μ not exceeding $4C$. Applying Lemma 10, we find that

$$|g^*| \leqslant \frac{16C}{K\delta^3}, \tag{17}$$

$$\left|\frac{\partial g^*}{\partial z}\right| \leqslant \frac{32C}{K\delta^4}, \tag{18}$$

$$\left|\frac{\partial^2 g^*}{\partial z^2}\right| \leqslant \frac{40C}{K\delta^5}, \tag{19}$$

$$\left|\frac{\partial g^*}{\partial \mu}\right| \leqslant \frac{5\cdot 10^3 C}{K^2\delta^6}, \tag{20}$$

$$\left|\frac{\partial^2 g^*}{\partial z \partial \mu}\right| \leqslant \frac{5\cdot 10^3 C}{K^2\delta^7} \tag{21}$$

for $|\mathrm{Im}\,(z - 2\pi\mu)| \leqslant R - 2\delta$, $\mu \in N_K^r$, $|\Delta| \leqslant \Delta_0$.

2°. **Estimate of $\partial \Delta_0^*/\partial \mu$.** From equation (4) of §5 and subsection 7.4 it follows that

$$\frac{\partial \Delta_0^*(\mu)}{\partial \mu} = -\frac{\dfrac{\partial \overline{F}}{\partial \mu} + \dfrac{\partial \overline{\Phi}}{\partial \mu}}{1 + \dfrac{\partial \overline{\Phi}}{\partial \Delta}}.$$

Estimating Δ_0^* as in 1° of subsection 5.2, we find that

$$|\Delta_0^*| < 2C < \frac{\Delta_0}{2}.$$

For $|\Delta| \leq \Delta_0/2$, using Cauchy's integral formula, we find from (4) that

$$\left|\frac{\partial \Phi}{\partial \Delta}\right| \leqslant \frac{\delta^2 \Delta_0}{\frac{\Delta_0}{2}} = 2\delta < \frac{1}{2}, \quad \left|\frac{\partial \overline{\Phi}}{\partial \Delta}\right| < \frac{1}{2}, \quad \left|\frac{\partial \widetilde{\Phi}}{\partial \Delta}\right| < 1.$$

Accordingly $|1 + \partial\overline{\Phi}/\partial\Delta| > 1/2$ for $|\Delta| \leq \Delta_0/2$. Therefore, on the basis of (3), (5), and Lemma 9,

$$\left|\frac{\partial \Delta_0^*}{\partial \mu}\right| < 2(C + \delta^2 \Delta_0).$$

In view of (6), $\delta^2 \Delta_0 < C$, so that

$$\left|\frac{\partial \Delta_0^*}{\partial \mu}\right| < 4C \tag{22}$$

for $\mu \in N_N^r$.

3°. **Estimate of $\partial g/\partial \mu$.** From subsections 7.4 and 5.1,

$$\frac{\partial g}{\partial \mu} = \frac{\partial g^*}{\partial \mu} + \frac{\partial g^*}{\partial \Delta}\frac{\partial \Delta_0^*}{\partial \mu}, \tag{23}$$

$$\frac{\partial^2 g}{\partial z \partial \mu} = \frac{\partial^2 g^*}{\partial z \partial \mu} + \frac{\partial^2 g^*}{\partial z \partial \Delta} \frac{\partial \Delta_0^*}{\partial \mu} . \tag{24}$$

First we shall estimate $\partial g^*/\partial \Delta$ and $\partial^2 g^*/\partial z \partial \Delta$. We note that the equation

$$g^*(z + 2\pi\mu, \Delta, \mu) - g^*(z, \Delta, \mu) = -\widetilde{F}(z, \mu) - \widetilde{\Phi}(z, \Delta, \mu)$$

on differentiation with respect to Δ gives the equation

$$\frac{\partial g^*}{\partial \Delta}(z + 2\pi\mu, \Delta, \mu) - \frac{\partial g^*}{\partial \Delta}(z, \Delta, \mu) = -\frac{\partial \widetilde{\Phi}}{\partial \Delta}$$

of the same form with respect to $\partial g^*/\partial \Delta$, and we may use Lemma 10. To this end we estimate $\partial\widetilde{\Phi}/\partial \Delta$ using Cauchy's integral formula: for $|\operatorname{Im} z| \leq R, |\Delta| \leq \Delta_0/2$

$$\left| \frac{\partial \widetilde{\Phi}}{\partial \Delta} \right| \leqslant \frac{2\delta^2 \Delta_0}{\frac{\Delta_0}{2}} < 4\delta^2.$$

From Lemma 10, for $|\operatorname{Im}(z - 2\pi\mu)| \leq R_0 - 2\delta$, $|\Delta| \leq \Delta_0/2$, $\mu \in N_K^r$

$$\left| \frac{\partial g^*}{\partial \Delta} \right| < \frac{4}{K\delta^3} 4\delta^2,$$

$$\left| \frac{\partial^2 g^*}{\partial \Delta \partial z} \right| < \frac{8}{K\delta^4} 4\delta^2.$$

Substituting these estimates, and also estimates (20), (21) and the estimate of Δ_0^* from point 2°, into formulas (23) and (24), we find that

$$\left| \frac{\partial g}{\partial \mu} \right| < \frac{5C}{K^2} \frac{10^3}{\delta^6} + \frac{16}{K\delta} 4C < \frac{C10^4}{K^2\delta^6},$$

$$\left| \frac{\partial^2 g}{\partial z \partial \mu} \right| < \frac{5 \cdot 10^3 C}{K^3 \delta^7} + \frac{32}{K\delta^2} 4C < \frac{C10^4}{K^2\delta^7}$$

for $|\operatorname{Im}(z - 2\pi\mu)| \leq R - 2\delta$, $\mu \in N_K^r$.

4°. **Estimate of** $\partial \Delta(\Delta_1, \mu)/\partial \mu$. Analogously to subsection 2°, we have

$$\frac{\partial \Delta}{\partial \mu} = -\frac{\frac{\partial \overline{F}}{\partial \mu} + \frac{\partial \overline{\Phi}}{\partial \mu}}{1 + \frac{\partial \overline{\Phi}}{\partial \Delta}},$$

and if $|\Delta| \leq \Delta_0/2$, then, as in point 2°, we obtain

$$\left| \frac{\partial \Delta}{\partial \mu} \right| < 4C.$$

In order that the inequality $|\Delta| < \Delta_0/2$ should be satisfied it is sufficient that $|\Delta_1| \leq \delta^{27}$. For then, as was shown in §5, $|\Delta_0^*| \leq 2C$, $|\Delta - \Delta_0^*| \leq 2|\Delta_1|$, and since $C = \delta^{27}$, then for $|\Delta_1| \leq \delta^{27}$ we have

$$|\Delta(\Delta_1, \mu)| \leqslant 4\delta^{27} < \frac{\delta^{26}}{2} = \frac{\Delta_0}{2}.$$

Thus, for $|\Delta_1| \leq \delta^{27}$, $\mu \in N_K^r$,

$$\left| \frac{\partial \Delta(\Delta_1, \mu)}{\partial \mu} \right| < 4C. \tag{25}$$

At the same time we have shown that for $|\Delta_1| \leq \delta^{27}$ the estimates of point 1° are valid.

5°. Estimate of $\partial \hat{F}_1/\partial\mu$. From subsections 5.1 and 7.4 we have

$$\frac{\partial \hat{F}_1(z,\mu)}{\partial\mu} = \left[\frac{\partial g(z_{\mathrm{I}},\mu)}{\partial\mu} - \frac{\partial g(z_{\mathrm{II}},\mu)}{\partial\mu}\right] + \left[\frac{\partial g(z_{\mathrm{I}},\mu)}{\partial z} - \frac{\partial g(z_{\mathrm{II}},\mu)}{\partial z}\right] 2\pi$$
$$+ \frac{\partial g(z_{\mathrm{I}},\mu)}{\partial z}\left[\frac{\partial \widetilde{F}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\Delta}\frac{\partial\overset{*}{\Delta}_0}{\partial\mu}\right], \tag{26}$$

where

$$z_{\mathrm{I}} = z + 2\pi\mu + \widetilde{F}(z,\mu) + \widetilde{\Phi}(z,\mu,\overset{*}{\Delta}_0(\mu)), \tag{27}$$
$$z_{\mathrm{II}} = z + 2\pi\mu. \tag{28}$$

The first two brackets on the right side of (26) may be estimated by using the lemma on finite increments, Lemma 5 of §3. We have

$$\left|\frac{\partial g(z_{\mathrm{I}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{II}})}{\partial\mu}\right| \leqslant |z_{\mathrm{I}} - z_{\mathrm{II}}|\left|\frac{\partial^2 g}{\partial\mu\,\partial z}\right|;$$

putting their estimates in place of $z_{\mathrm{I}} - z_{\mathrm{II}}$ and $\partial^2 g/\partial\mu\partial z$, we obtain

$$\left|\frac{\partial g(z_{\mathrm{I}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{II}})}{\partial\mu}\right| \leqslant \frac{4\cdot 10^4 C^2}{K^2\delta^7}$$

and analogously

$$\left|\frac{\partial g(z_{\mathrm{I}})}{\partial z} - \frac{\partial g(z_{\mathrm{II}})}{\partial z}\right| \leqslant \left|\frac{\partial^2 g}{\partial z^2}\right| |z_{\mathrm{I}} - z_{\mathrm{II}}| \leqslant \frac{40\,C}{K\delta^5}4C = \frac{160\,C^2}{K\delta^5}.$$

The last term on the right side of (26) may be estimated using inequalities (3), (5), (22), (18) and does not exceed

$$\frac{32\,C}{K\delta^4}(4\,C + 2\,C) < \frac{200\,C^2}{K\delta^4}.$$

Thus

$$\left|\frac{\partial\hat{F}_1}{\partial\mu}\right| < C^2\left[\frac{4\cdot 10^4}{K^2\delta^7} + 2\pi\frac{160}{K\delta^5} + \frac{200}{K\delta^4}\right] < \frac{5\cdot 10^4}{K^2\delta^7}C^2.$$

All of these estimates are valid if the arguments z_{I} and z_{II} do not leave the region $|\mathrm{Im}\,(z - 2\pi\mu)| \leq R_0 - 2\delta$, where the estimates of g and its derivatives operate. To this end it suffices, for example, that $|\mathrm{Im}\, z| \leq R_0 - 3\delta$. Indeed, then

$$|\widetilde{F}(z,\mu) + \widetilde{\Phi}(z,\mu,\overset{*}{\Delta}_0(\mu))| \leqslant 2C < \delta,$$

i.e.,

$$|\mathrm{Im}\,(z_{\mathrm{I}} - 2\pi\mu)| < R_0 - 2\delta.$$

Thus, for $|\mathrm{Im}\, z| \leq R_0 - 3\delta$, $\mu \in N_K^r$, $|\Delta_1| < \delta^{27}$

$$\left|\frac{\partial \hat{F}_1}{\partial \mu}\right| < \frac{5\cdot 10^4}{K^2\delta^7}C^2. \tag{29}$$

6°. **Estimate of** $(\partial/\partial\mu)(\Delta - \Delta_0^*)$. **We have**

$$\frac{\partial}{\partial\mu}(\Delta(\Delta_1, \mu) - \Delta_0^*(\mu)) = \frac{\partial\Delta(\Delta_1, \mu)}{\partial\mu} - \frac{\partial\Delta(0, \mu)}{\partial\mu};$$

by the lemma on finite increments,

$$\left|\frac{\partial}{\partial\mu}(\Delta - \Delta_0^*)\right| \leqslant \left|\frac{\partial^2\Delta(\Delta_1, \mu)}{\partial\Delta_1\partial\mu}\right| |\Delta - \Delta_0^*|.$$

We estimate $|\partial^2\Delta(\Delta_1, \mu)/\partial\Delta_1\partial\mu$, using the Cauchy integral, as the derivative of $\partial\Delta/\partial\mu$. For $|\Delta_1| \leq \delta^{27}$, as follows from (25), $|\partial\Delta/\partial\mu| < 4C$. Therefore in the disk $|\Delta_1| \leq \delta^{27}/2$ always

$$\left|\frac{\partial^2\Delta}{\partial\Delta_1\partial\mu}\right| < \frac{4C}{\frac{\delta^{27}}{2}} = 8.$$

In particular, $|\partial^2\Delta/\partial\Delta_1\partial\mu| < 8$ when $|\Delta_1| \leq \delta^{28}$. Since

$$|\Delta - \Delta_0^*| \leqslant 2|\Delta_1|,$$

then for $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$

$$\frac{\partial}{\partial\mu}(\Delta(\Delta_1, \mu) - \Delta_0^*(\mu))| < 16|\Delta_1|. \tag{30}$$

7°. **Estimate of** $(\partial/\partial\mu)[\tilde{\Phi}(\dot{\Delta}(\Delta_1, \mu)) - \tilde{\Phi}(\Delta_0^*(\mu))]$. **This derivative is equal to**

$$\frac{\partial\tilde{\Phi}(\Delta)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\mu} + \frac{\partial\tilde{\Phi}}{\partial\Delta}\frac{\partial\Delta(\Delta_1)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}.$$

The first difference may be estimated using the lemma on finite increments: for $|\Delta| \leq \Delta_0/2$, $\mu \in N_K^r$, $|\text{Im } z| \leq R$

$$\left|\frac{\partial\tilde{\Phi}(\Delta)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta^*)}{\partial\mu}\right| \leqslant \left|\frac{\partial^2\tilde{\Phi}}{\partial\mu\partial\Delta}\right| |\Delta - \Delta_0^*| \leqslant 8\delta^2|\Delta_1|$$

(here $|\partial^2\tilde{\Phi}/\partial\mu\partial\Delta|$ is estimated using the Cauchy integral: $|\partial^2\tilde{\Phi}/\partial\mu\partial\Delta| < 2\delta^2|\Delta_0|/\frac{1}{2}|\Delta_0| = 4\delta^2$).

The second difference may be written in the form

$$\frac{\partial\tilde{\Phi}(\Delta)}{\partial\Delta}\left(\frac{\partial\Delta(\Delta_1)}{\partial\mu} - \frac{\partial\Delta_0^*}{\partial\mu}\right) + \frac{\partial\Delta_0^*}{\partial\mu}\left(\frac{\partial\tilde{\Phi}(\Delta)}{\partial\Delta} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\Delta}\right), \tag{31}$$

where the first term is estimated with the use of inequality (30) and does not exceed $16|\Delta_1|$, since $|\partial\tilde{\Phi}/\partial\Delta| < 1$ (see point 2°) and the second term, from the lemma on finite increments, does not exceed

$$\left|\frac{\partial\Delta^*}{\partial\mu}\right|\left|\frac{\partial^2\tilde{\Phi}}{\partial\Delta^2}\right| |\Delta(\Delta_1) - \Delta_0^*| \leqslant 4C\frac{16}{\delta^{24}}2|\Delta_1|.$$

Here the only new estimate is that of $\partial^2\widetilde{\Phi}/\partial\Delta^2$. To find it we employ the expression for the second derivative obtained from the Cauchy integral:

$$\left|\frac{\partial^2\widetilde{\Phi}}{\partial\Delta^2}\right| \leqslant 2\frac{2\delta^2\Delta_0}{\left(\frac{\Delta_0}{2}\right)^2} = \frac{16}{\delta^{24}}$$

for $|\Delta| \le \Delta_0/2$, for which, as we have seen in point 4°, it is sufficient that the inequality $|\Delta_1| \le \delta^{27}$ should be satisfied. Comparing all three estimates, we find that

$$\left|\frac{\partial}{\partial\mu}[\widetilde{\Phi}(\Delta) - \widetilde{\Phi}(\Delta_0^*)]\right| < 8\delta^2|\Delta_1| + 16|\Delta_1| + 128\,\delta^3|\Delta_1|.$$

Finally we have

$$\left|\frac{\partial}{\partial\mu}[\widetilde{\Phi}(\Delta(\Delta_1, \mu)) - \widetilde{\Phi}(\Delta_0^*(\mu))]\right| < 100|\Delta_1| \tag{32}$$

for $|\Delta_1| \le \delta^{28}$, $|\operatorname{Im} z| \le R_0$, $\mu \in N_K^r$.

8°. **Estimate of** $(\partial/\partial\mu)\,\hat{\Phi}_1(z, \mu, \Delta(\Delta_1, \mu))$. It is convenient for us first to consider the function of z, μ and Δ_1, and not of z, μ, and Δ. We have

$$\frac{\partial\hat{\Phi}_1}{\partial\mu} = \left[\frac{\partial g(z_{\mathrm{III}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{I}})}{\partial\mu}\right] + \left[\frac{\partial g(z_{\mathrm{III}})}{\partial z} - \frac{\partial g(z_{\mathrm{I}})}{\partial z}\right]\frac{\partial z_{\mathrm{I}}}{\partial\mu} + \frac{\partial g(z_{\mathrm{III}})}{\partial z}\left[\frac{\partial z_{\mathrm{III}}}{\partial\mu} - \frac{\partial z_{\mathrm{I}}}{\partial\mu}\right], \tag{33}$$

where

$$z_{\mathrm{I}} = z + 2\pi\mu + \overline{F}(z, \mu) + \widetilde{\Phi}(z, \mu, \Delta_0^*(\mu)), \tag{27}$$

$$z_{\mathrm{III}} = z + 2\pi\mu + \Delta_1 + \overline{F}(z, \mu) + \widetilde{\Phi}(z, \mu, \Delta(\Delta_1, \mu)) + \Delta_1. \tag{34}$$

The first two brackets on the right side of (33) may be estimated as in point 5°:

$$\left|\frac{\partial g(z_{\mathrm{III}})}{\partial\mu} - \frac{\partial g(z_{\mathrm{I}})}{\partial\mu}\right| \leqslant \left|\frac{\partial^2 g}{\partial\mu\,\partial z}\right| |z_{\mathrm{III}} - z_{\mathrm{I}}| \leqslant \frac{C\,10^4}{K^2\delta^7} 3|\Delta_1|,$$

since

$$z_{\mathrm{III}} - z_{\mathrm{I}} = \Delta_1 + \widetilde{\Phi}(z, \mu, \Delta) - \widetilde{\Phi}(z, \mu, \Delta^*(\mu))$$

and, using the estimate (22) of §5,

$$|z_{\mathrm{III}} - z_{\mathrm{I}}| \leqslant 3|\Delta_1|.$$

Analogously,

$$\left|\left(\frac{\partial g(z_{\mathrm{III}})}{\partial z} - \frac{\partial g(z_{\mathrm{I}})}{\partial z}\right)\frac{\partial z_{\mathrm{I}}}{\partial\mu}\right| \leqslant \left|\frac{\partial^2 g}{\partial z^2}\right| |z_{\mathrm{III}} - z_{\mathrm{I}}| \left|\frac{\partial z_{\mathrm{I}}}{\partial\mu}\right|$$

$$\frac{40\,C}{K\delta^5} 3|\Delta_1| \left|2\pi + \frac{\partial\overline{F}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}\right| \leqslant \frac{40\,C}{K\delta^5} 3|\Delta_1|(2\pi + 6\,C) \leqslant \frac{1600\,C}{K\delta^5}|\Delta_1|,$$

where the factor $|\partial z_{\mathrm{I}}/\partial\mu|$ is estimated using conditions 3) of Lemma 11 and estimate (22), taking account of the fact that $C < 1$. It remains for us to estimate

$(\partial/\partial\mu)(z_{III} - z_I)$. We have

$$z_{III} - z_I = \Delta_1 + \widetilde{\Phi}(z, \mu, \Delta(\Delta_1, \mu)) - \widetilde{\Phi}(z, \mu, \Delta_0^*(\mu)).$$

From estimate (32) we find that

$$\frac{\partial}{\partial\mu}(z_{III} - z_I) \leqslant 100 |\Delta_1|,$$

where $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$.

Thus,

$$\left|\frac{\partial g(z_{III})}{\partial z}\left(\frac{\partial z_{III}}{\partial\mu} - \frac{\partial z_I}{\partial\mu}\right)\right| \leqslant 100 |\Delta_1| \frac{32C}{K\delta^4} \leqslant \frac{10^4 C}{K\delta^4} |\Delta_1|.$$

Comparing the estimates of all three terms of the right side of equation (33), we find that

$$\frac{\partial}{\partial\mu}\hat{\Phi}_1(z, \mu, \Delta(\Delta_1, \mu))\Big| \leqslant \frac{C\,10^4}{K^2\delta^7} 3|\Delta_1| + \frac{1600\,C}{K\delta^5}|\Delta_1| + \frac{C\,10^4}{K\delta^4}|\Delta_1| \leqslant \frac{C\,10^5}{K^2\delta^7}|\Delta_1|.$$

All of these estimates have been derived under the hypothesis that $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$ and that z_I, z_{III} do not leave the strip $|\mathrm{Im}\,(z - 2\pi\mu)| \leq R_0 - 2\delta$, where Lemma 10 operates. For this it is sufficient, for example, that $|\mathrm{Im}\, z| \leq R_0 - 4\delta$, since then

$$|\Delta_1 + \widetilde{F}(z, \varepsilon) + \widetilde{\Phi}(z, \varepsilon, \Delta)| \leqslant \delta + 2C + 2C < 2\delta,$$
$$|\mathrm{Im}\,(z_{III} - 2\pi\mu)| \leqslant R_0 - 4\delta + 2\delta = R_0 - 2\delta.$$

6°. **Estimate of $\partial z/\partial\mu$.** The function $g(z, \mu)$ is defined for

$$|\mathrm{Im}\,(z - 2\pi\mu)| \leqslant R_0 - 2\delta.$$

Therefore the following function is also defined in that strip:

$$\varphi(z, \mu) = z + g(z, \mu).$$

Since in that strip $|g(z, \mu) < \delta|$ (see (6), (17)), then the image of this strip as $z \to \phi$ contains the strip

$$|\mathrm{Im}\,(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta,$$

which as $\phi \to z$ goes into a region containing the strip

$$|\mathrm{Im}\,(z - 2\pi\mu)| \leqslant R_0 - 4\delta.$$

From subsections 5.1 and 7.4 it follows that

$$\frac{\partial z}{\partial\mu} = -\frac{\dfrac{\partial g}{\partial\mu}}{1 + \dfrac{\partial g}{\partial z}}.$$

From inequality (18) and conditions 4), 5) of Lemma 11, $|\partial g/\partial z| < \frac{1}{2}$. Thus, applying estimate (23), we obtain

$$\left|\frac{\partial z}{\partial\mu}\right| \leqslant \frac{10^4 C}{K^2\delta^6}$$

for $|\operatorname{Im}(z-2\pi\mu)| \le R_0 - 2\delta$, $\mu \in N_K^r$ and, in particular, for

$$|\operatorname{Im}(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta.$$

10°. **Estimate of $(\partial/\partial\mu) F_1(\phi, \mu)$, $(\partial/\partial\mu)\Phi_1(\phi, \mu, \Delta_1)$.** From subsection 5.1,

$$F_1(\varphi, \mu) = \hat{F}_1(z(\varphi, \mu), \mu),$$
$$\Phi_1(\varphi, \mu, \Delta_1) = \hat{\Phi}_1(z(\varphi, \mu), \mu, \Delta(\Delta_1, \mu)).$$

The function $z(\phi, \mu)$ is defined for $|\operatorname{Im}(\phi - 2\pi\mu)| \le R_0 - 3\delta$, $\mu \in N_K^r$, and if

$$|\operatorname{Im}(z - 2\pi\mu)| \leqslant R_0 - 4\delta,$$

then for this z there exists a ϕ such that $z = z(\phi, \mu)$ and

$$|\operatorname{Im}(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta.$$

The functions $\hat{F}_1(z)$, $\hat{\Phi}_1(z)$ are defined for $|\operatorname{Im} z| \le R_0 - 4\delta$ and therefore the functions $F_1(\phi, \mu)$, $\Phi_1(\phi, \mu, \Delta_1)$ are defined for

$$|\operatorname{Im}\varphi| \leqslant R_0 - |\operatorname{Im} 2\pi\mu| - 5\delta$$

under the hypothesis that $|\operatorname{Im} 2\pi\mu| \le R_0 - 5\delta$, i.e., that $2\pi\mu \le R_0 - 5\delta$. In this region

$$\frac{\partial F_1}{\partial\mu} = \frac{\partial \hat{F}_1}{\partial\mu} + \frac{\partial \hat{F}_1}{\partial z}\frac{\partial z}{\partial\mu}, \qquad \frac{\partial \Phi_1}{\partial\mu} = \frac{\partial \hat{\Phi}_1}{\partial\mu} + \frac{\partial \hat{\Phi}_1}{\partial z}\frac{\partial z}{\partial\mu},$$

where in the calculation of $\partial\hat{\Phi}_1/\partial\mu$ the independent variables are taken to be z, μ and Δ_1, as in point 8°.

For the estimation of $\partial\hat{F}_1/\partial z$ and $\partial\hat{\Phi}/\partial z$ we use the Cauchy integral. Staying at a distance δ from the boundary of the strip, where the estimates of $\hat{F}_1$ and $\hat{\Phi}_1$ are known, we see from the estimates of 3° and 5° of §5 that

$$\left|\frac{\partial \hat{F}_1}{\partial z}\right| \leqslant \frac{4C^2}{\delta^6}, \quad \left|\frac{\partial \hat{\Phi}_1}{\partial z}\right| \leqslant \frac{3C\,|\Delta_1|}{\delta^6}$$

for $|\operatorname{Im} z| \le R_0 - 5\delta$. Applying estimates 5°, 8° and 9°, we find from (35) that

$$\left|\frac{\partial F_1}{\partial\mu}\right| \leqslant \frac{5\cdot 10^4 C^2}{K^2\delta^7} + \frac{10^4 C}{K^2\delta^6}\frac{4C^2}{\delta^6},$$
$$\left|\frac{\partial \Phi_1}{\partial\mu}\right| \leqslant \frac{C\cdot 10^5\,|\Delta_1|}{K^2\delta^7} + \frac{3C\,|\Delta_1|}{\delta^6}\frac{10^4 C}{K^2\delta^6}.$$

Thus, for

$$|\Delta_1| \leqslant \delta^{28}, \quad \mu \in N_K^r, \quad 2\pi r \leqslant R_0 - 5\delta, \quad |\operatorname{Im}\varphi| \leqslant R_0 - |\operatorname{Im} 2\pi\mu| - 6\delta,$$

we have

$$|F_1(\varphi, \mu)| \leqslant \frac{C^2}{\delta^6}, \qquad |\Phi_1(\varphi, \mu, \Delta_1)| < \frac{|\Delta_1|}{\delta^6}$$
$$\left|\frac{\partial F_1}{\partial\mu}\right| \leqslant \frac{C^2}{\delta^{13}}, \qquad \left|\frac{\partial \Phi_1}{\partial\mu}\right| \leqslant \frac{C\,|\Delta_1|}{\delta^{13}},$$

since

$$\frac{5\cdot 10^4}{K^2}\delta < 1. \tag{6}$$

In exactly the same way all the remaining estimates 1°–9°, in view of conditions 4) and 5) of Lemma 11, may be brought into the form (7)–(16).

Lemma 11 is proved.

8.3. **Proof of Theorem 3.** Theorem 3 is derived from Lemma 11 in the same way as Theorem 2 was derived from the Fundamental Lemma in §6.

We choose $\delta_1 > 0$ such that

1) $\sum_{n=1}^{\infty} \delta_n < \frac{R}{8}$, where $\delta_n = \delta_{n-1}^{1\frac{1}{2}}$ $(n = 2, 3, \ldots)$,

2) $\delta_1 < \frac{K^2}{5\cdot 10^4}$.

Let $R = R_0$, K be the same as in condition of Theorem 2, $\mu \in N_K^{\frac{K}{16\pi(n+1)}}$, $\Delta_0 = \delta_1^{26}$, $L\varepsilon_0 < C_1$, where

$$C_1 = \delta_1^{27}, \tag{35}$$

and C_1, δ_1 are respectively the C and δ of Lemma 11. Then from inequalities (7)–(16) we obtain

$$|F_1| < \frac{\delta_1^{54}}{\delta_1^{13}} < \delta_1^{40,5} = (\delta_1^{1\frac{1}{2}})^{27} = \delta_2^{27},$$

$$\left|\frac{\partial F_1}{\partial \mu}\right| < \delta_2^{27},$$

$$|\Phi_1| \leqslant \frac{\delta_1^{27}}{\delta_1^{13}} |\Delta_1| < \delta_1^3 |\Delta_1| = \delta_2^2 |\Delta_1|,$$

$$\left|\frac{\partial \Phi_1}{\partial \mu}\right| < \delta_2^2 |\Delta_1|$$

for

$$|\Delta_1| < \delta_2^{26} = \delta_1^{39} < \delta_1^{28}, \quad |\operatorname{Im} \varphi_1| \leqslant R_0 - 7\delta_1 - |\operatorname{Im} 2\pi\mu| = R_1, \quad \mu \in N_K^{\frac{R}{16\pi(n+1)}}$$

Thus, we again find ourselves in the conditions of Lemma 11, but with a decrease of $7\delta_1 + R/8(n+1)$ in the radius of R_1. Since

$$\sum_{n=1}^{\infty} \delta_n < \frac{R}{8},$$

then we may carry out n successive approximations, and the last will operate for

$$|\operatorname{Im} \varphi_n| \leqslant \frac{R}{8(n+1)}, \quad \mu \in N_K^{\frac{R}{16\pi(n+1)}}, \quad |\Delta_n| < \delta_{n+1}^{26}.$$

Omitting the usual (see §6) proof of the convergence of the approximations for real μ, we estimate $|\partial\Delta^{(n)}/\partial\mu|$.

From subsection 8.1 it follows that

$$\frac{\partial\Delta_k^{(n)}}{\partial\mu} = \frac{\partial\Delta_k}{\partial\mu} + \frac{\partial\Delta_k}{\partial\Delta_{k+1}}\frac{\partial\Delta_{k+1}^{(n)}}{\partial\mu}.$$

Putting $C_k = \delta_k^{27}$, on the basis of Lemma 11 we find that

$$\left|\frac{\partial\Delta_k^{(n)}}{\partial\mu}\right| \leqslant 4C_{k+1} + 2\left|\frac{\partial\Delta_{k+1}^{(n)}}{\partial\mu}\right|.$$

If

$$\left|\frac{\partial\Delta_{k+1}^{(n)}}{\partial\mu}\right| < C_{k+1},$$

then

$$\left|\frac{\partial\Delta_k^{(n)}}{\partial\mu}\right| < 6C_{k+1} < C_k.$$

Since

$$\left|\frac{\partial\Delta_n^{(n)}}{\partial\mu}\right| = 0,$$

then

$$\left|\frac{\partial\Delta_0^{(n)}}{\partial\mu}\right| < 6C_1.$$

Theorem 3 is proved.

Remark. In exactly the same way we may prove the monogenicity of the functions g_n, F_n, Φ_n, ϕ_n and obtain analogous estimates.

Part II

On the space of mappings of the circumference onto itself

The problem of studying the dependence of the rotation number on the coefficients of the equation was posed by Poincaré [1]. The consideration of the rotation number as a function on the space of mappings makes it possible to elucidate questions concerning typical and exceptional cases.

Angular coordinates of a point on a circumference will be denoted by small greek letters; ϕ and $\phi + 2\pi$ are one and the same point of the circumference. We shall use capital letters to denote transformations:

$$T : \phi \longrightarrow T\phi.$$

We shall consider only continuous one-to-one direct (orientation-preserving) transformations. As an example one may cite the rotation through the angle $\theta : \phi \longrightarrow \phi + \theta$. To each transformation we assign a "displacement," namely a function on the circumference showing how much each point is displaced. We shall denote a displacement by the same letter as the transformation, but in lower case:

$$T : \varphi \to T_\varphi = \varphi + t(\varphi).$$

Here $t(\phi)$ is the displacement. If T is a rotation through the angle θ, then $t(\phi) \equiv \theta$. Generally speaking, a shift, as also ϕ, is defined only up to a multiple of 2π. However, if we define $t(\phi)$ at one point, we may uniquely continue it by continuity.

If T is a smooth transformation, then $t(\phi)$ is a smooth periodic function:

$$t(\varphi + 2\pi) = t(\varphi).$$

We denote by

$$T^n\varphi = \varphi + t^{(n)}(\varphi)$$

the nth power of the transformation of T. In connection with this notation we suppose that branches of $t^{(n)}(\phi)$ are chosen to correspond to the branches of $t(\phi)$:

$$t^{(n)}(\varphi) = t^{(n-1)}(\varphi) + t(T^{n-1}(\varphi)) \quad (n = 2, 3, \ldots).$$

Under this condition $t^{(n)}(\phi)$ is called a displacement with n steps.

§9. The function $\mu(T)$ and its level sets

We consider the spaces

$$C \supset C^1 \supset C^2 \supset \ldots \supset C^n \supset \ldots \supset C^\infty \supset A$$

of one-to-one direct mappings of the circumference onto itself, continuous, continuously and infinitely differentiable, and analytic in the neighborhood of the real axis, with the topologies usual in these spaces. Each successive topology is stronger than the preceding one and each of the spaces is everywhere dense in the preceding one.*

Poincaré [1] defined for each transformation $T \in C$ the rotation number $2\pi\mu$. Thus on the space C there is given a function $\mu(T)$. The following theorem was stated by Poincaré without proof.

Theorem 4. *The function $\mu(T)$ is continuous at each point of C.*

Proof. We shall show that $\mu(T)$ is continuous at the point T_0.

Suppose given a point $\epsilon > 0$. We choose a number $n > 2/\epsilon$ such that

$$\frac{m}{n} < \mu(T_0) < \frac{m+1}{n}.$$

*If T lies in one of the spaces $C^1, C^2, \cdots, A$ without distinction as to which one, then we shall call T a *smooth* transformation.

Then under the transformation

$$T_0^n : \varphi \to \varphi + t_0^{(n)}(\varphi)$$

each point is shifted by more than $2\pi m$. Indeed, if some point were shifted less, and another point more, then there would be a point shifted by exactly $2\pi m$, i.e., a point which is fixed for T_0^n. Then, evidently, in spite of the choice of n, we would have

$$\mu = \frac{m}{n}.$$

If all the points were shifted by less than $2\pi m$, then we would have $\mu \leq m/n$, which again contradicts the choice of n.

Analogously one proves that each point is shifted in the course of n steps through less than $2\pi(m+1)$. Thus

$$2\pi m < t_0^{(n)}(\varphi) < 2\pi(m+1).$$

In view of the continuity of $t_0^{(n)}(\phi)$,

$$2\pi m + \eta < t_0^{(n)}(\varphi) < 2\pi(m+1) - \eta$$

for some $\eta > 0$, and in view of the continuous dependence of $T^{(n)}$ on T there exists a $\delta > 0$ such that

$$|t^{(n)}(\varphi) - t_0^{(n)}(\varphi)| < \eta,$$

as soon as the transformation T differs from T_0 by less than δ:

$$|t(\varphi) - t_0(\varphi)| < \delta.$$

For such T

$$2\pi m < t^{(n)}(\varphi) < 2\pi(m+1),$$

so that

$$\frac{m}{n} < \mu(T) < \frac{m+1}{n}.$$

Thus, $|\mu(T) - \mu(T_0)| < \epsilon$ for $|t(\phi) - t_0(\phi)| < \delta$. The theorem is proved.

Remark. Even in very nice cases the function $\mu(T)$ is only continuous. For example, consider the family of transformations

$$T_h : \varphi \to \varphi + h + 0{,}1 \sin^2\varphi,$$

where h is a parameter. By what has been proved, $\mu(T_h)$ is a continuous function of h. With increasing h the function $\mu(T_h)$ grows, but with a lag at each rational value of μ. To this value there corresponds a whole segment $[h_1 h_2]$ of values of h. On the other hand, for $h > h_2$ the function $\mu(T_h)$ increases with extreme rapidity. E. G. Belaga showed that, for example, in the neighborhood of the origin $\mu(T_h)$ grows at least as fast as $C\sqrt{h}/-\log h$.

A level set of $\mu(T)$ is a set of transformations with the same rotation number $2\pi\mu$. To such transformations there belong the rotation through the angle $2\pi\mu$, transformations convertible into rotations through the angle $2\pi\mu$ by an appropriate change of variables, and possibly other transformations.

The structure of the level set $\mu(T) = \mu$ essentially depends on whether μ is rational or irrational.

§10. The case of rational μ

10.1. If $\mu(T) = m/n$, then, as Poincaré showed, T^n has fixed points: $t^{(n)}(a) = 2\pi m$. The set of these points is invariant relative to T and closed, as the level set of the continuous function $t^{(n)}(a)$. The points $a, Ta, \cdots, T^{n-1}a$ are called a *cycle*. In the investigation of cycles it is convenient to consider the graph of the transformation T^n and the graph of the function $t^{(n)}(\phi)$ (see Figure 7; on this drawing we have shown the graph of $T(\phi) = \phi + \frac{1}{2}\cos\phi$ and we have indicated the image of 0 for several iterations of T). A cycle is called *isolated* if in some neighborhood of its points there are no points of other cycles. An isolated cycle is *stable* if one of its points, and thus all of its points, has arbitrarily small neighborhoods which are taken into their own interiors by the transformation T^n. It is easy to see that as $n \to +\infty$ the points of such a neighborhood tend to points of the cycle, which explains the usage. A stable cycle of the transformation T^{-1} is called an *unstable* cycle of T. An isolated cycle is *semistable forward* (*backward*) if all the points of some neighborhood of a point of the cycle (the point itself excluded) are moved forward (backward) by the transformation T^n, i.e., if in this neighborhood

$$t^{(n)}(\varphi) - 2\pi m > 0 \quad (< 0).$$

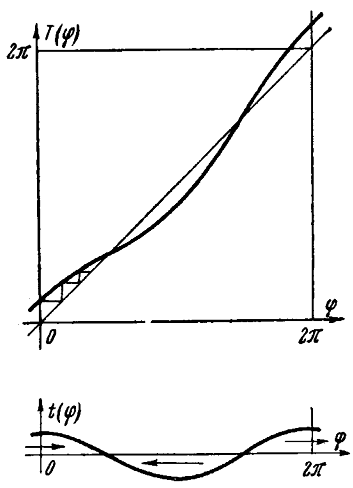

Figure 7

A transformation $T \in C^1$ is *normal* if at the points of its cycles

$$\frac{dt^{(n)}(\varphi)}{d\varphi} \neq 0.$$

Evidently, a normal transformation has a finite number of cycles, while all of these cycles are stable or unstable. Indeed, those roots of $t^{(n)}(\phi) - 2\pi m$, where $dt^{(n)}/d\phi < 0$, are points of stable cycles, and those where $dt^{(n)}/d\phi > 0$ are points of unstable cycles. Therefore it follows that the points of stable and unstable cycles of a normal transformation alternate.

10.2. **Theorem 5.** *Normal transformations form a set open in C^1 and everywhere dense in A.*

Proof. 1. The points of a cycle are the points where $t^{(n)}(\phi) = 2\pi m$. At these points $dt^{(n)}(\phi)/d\phi \neq 0$. Therefore for a small, along with the first derivative, variation of $t^{(n)}(\phi)$ the function $t^{(n)}(\phi) - 2\pi m$ does not acquire any new roots and the old ones do not disappear, but rather are displaced continuously, while the derivatives at the roots preserve sign. This means that the transformation T with such a variation of the function $t^{(n)}(\phi)$ becomes normal. In view of the continuous dependence of $t^{(n)}(\phi)$ on T, the first assertion of the theorem is proved.

2. We shall show that *arbitrarily close to any transformation there is an analytic transformation with a cycle.* Evidently it is sufficient to prove this for an analytic transformation and analytic proximity. Suppose that T is an analytic transformation with an irrational rotation number, and suppose that $\epsilon > 0$. Among the points $\phi_n = T^n\phi_0$ is one displaced from ϕ_0 by less than ϵ, for example, backward:

$$2\pi m - \varepsilon < t^{(n)}(\varphi_0) < 2\pi m$$

(Denjoy's theorem). We consider a family of analytic transformations T_λ ($\lambda \geq 0$, $T_0 = T$):

$$T_\lambda : \varphi \to \varphi + t(\varphi) + \lambda.$$

It is not hard to see that for $\lambda = \epsilon$ T^n_λ displaces ϕ_0 ahead:

$$t^{(n)}_\lambda(\varphi_0) \geqslant 2\pi m.$$

Hence, in view of the continuity of $t^{(n)}_\lambda(\phi_0)$ in λ, it follows that for some $\lambda_0 \leq \epsilon$ T_{λ_0} has a cycle $\phi_0, T_{\lambda_0}\phi_0, \cdots$:

$$t^{(n)}_{\lambda_0}(\varphi_0) = 2\pi m.$$

3. **An analytic transformation with a cycle can be converted into a normal transformation by an arbitrarily small variation.** Indeed, suppose that T is an analytic transformation, and among its cycles there is no stable cycle (and therefore also no unstable cycle). We choose a cycle $\phi_0, \phi_1, \cdots, \phi_{n-1}$ and introduce an analytic function $\Delta(\phi)$, vanishing at these points and having there negative derivatives. The transformation

$$T_\theta : t_\theta(\varphi) = t(\varphi) + \theta\Delta(\varphi)$$

for small θ is arbitrarily proximate to T and has at least one stable cycle $\phi_0, \phi_1, \cdots, \phi_{n-1}$. Therefore it is sufficient to consider the case when the desired transformation T has a stable cycle. We shall construct over T an analytic function $\delta(\phi)$ which

1) is equal to zero and has a negative (positive) derivatives at the points of the stable cycles of T;

2) is positive (negative) at the points of the cycles of T which are semistable forward (backward).

The existence of such a function is obvious, since the number of all cycles of T is finite, because the analytic function $t^{(n)}(\phi) - 2\pi m$ has an isolated root and therefore is not identically zero.

Consider the transformation $T_\theta : \phi \longrightarrow \phi + t(\phi) + \theta\delta(\phi)$. For small θ this transformation is normal. The formal proof of the fact that the stable cycles of T for small θ are only somewhat shifted, while the roots of $t^{(n)}(\phi) - 2\pi m$ become multiple, and the semistable cycles vanish, is left to the reader. For sufficiently small θ the transformation T_θ is the desired one.

Theorem 5 is proved.

10.3. The construction of a normal transformation may be easily perceived from the graph of the function $t^{(n)}(\phi) - 2\pi m$. Its roots are the points of the cycles of the transformation and divide the circumference into arcs. Each such arc $\alpha\beta$ is bounded at one end by a point α of a stable cycle and at the other end by a point β of an unstable cycle. For $n \longrightarrow +\infty$ the points of the arc wind around onto the stable cycle, and for $n \longrightarrow -\infty$ onto the unstable cycle, i.e.,

$$\lim_{k\to\infty} T^{kn}(\gamma) = \alpha \,(\mathrm{mod}\, 2\pi), \qquad \lim_{k\to-\infty} T^{kn}(\gamma) = \beta \,(\mathrm{mod}\, 2\pi),$$

where $\gamma \in (\alpha, \beta)$. Assertions of this type are well known in the qualitative theory of differential equations, and we omit the proof.

Thus a topologically normal transformation is characterized by three integers m, n, k, where m/n is the rotation number and k the number of stable (and therefore of unstable) cycles. Two transformations with the same m, n, k are similarly arranged in the sense that one of them can be converted into the other by a continuous change of variables on the circumference (i.e., $T_2 = \Phi T_1 \Phi^{-1}$, where $\Phi \in C$). In addition the derivative $dt^{(n)}(\phi)/d\phi$ at the points of the cycle, which characterizes the rapidity of winding around onto the cycle, is an invariant under a smooth change of variables. Probably there are no other invariants, but I have not been able to prove this.

Theorem 6. *The set $E_{m/n}$ at the level $\mu = m/n$ in any of the spaces $C^1, \cdots \cdots, A$ is connected and consists of*

1) a kernel $\bigcup_{k=1}^{\infty} E^k_{m/n}$ of normal transformations dense in $E_{m/n}$ and open in $C^p(A)$. The kernel consists of connected components $E^k_{m/n}$ of transformations with k stable and k unstable cycles. Two transformations of one component $E^k_{m/n}$ may be converted one into the other by a continuous change of variables;

2) *the boundaries of* $E_{m/n}$ *and* $E^k_{m/n}$. *The boundary of* $E_{m/n}$ *consists of transformations* T *for which* $t^{(n)}(\phi) - 2\pi m$ *does not change sign. Its parts* $F_+(t^{(n)}(\phi) - 2\pi m \geq 0)$ *and* $F_-(t^{(n)}(\phi) - 2\pi m \leq 0)$ *contain transformations semi-stable forward and backward, and are connected and intersect in a connected set* F_0. *Transformations from* F_0 *change under a smooth substitution of variables into rotations.* F_0 *lies in the boundary of each component of* $E^k_{m/n}$.

Proof. 1. The sets $E_{m/n}$, F_+, F_- are connected. For the proof we join, without leaving the set in question, any transformation $T \in E_{m/n}(F_+, F_-)$ with the rotation T_2 through the angle $2\pi m/n$ by an arc T_θ $(0 \leq \theta \leq 2,\ T_0 = T)$. Suppose that $\phi_0, \cdots, \phi_{n-1}$ is a cycle of T. Making the smooth substitution of variables

$$\varphi \to \Psi\varphi = \varphi + \psi(\varphi)$$

we transfer the points $\phi_0, \cdots, \phi_{n-1}$ into $2\pi ml/n$ $(0 \leq l \leq n-1)$. Put

$$\Psi_\theta\varphi = \varphi + \theta\psi(\varphi)$$

and consider

$$T_\theta\varphi = \Psi_\theta T \Psi_\theta^{-1}\varphi = \varphi + t_\theta(\varphi) \quad (0 \leqslant \theta \leqslant 1).$$

This transformation is the transformation T described in the variable Ψ_θ, and belongs to $E_{m/n}(F_+, F_-)$.

Now we consider the segment joining T_1 to T_2:

$$T_\theta\varphi = \varphi + (\theta - 1)2\pi\frac{m}{n} + (2 - \theta)t_1(\varphi) \quad (1 \leqslant \theta \leqslant 2).$$

The points $2\pi ml/n$ $(0 \leq l \leq n-1)$ form a cycle T_θ for all $1 \leq \theta \leq 2$ and therefore the curve T_θ lies entirely in $E_{m/n}$ (respectively F_+, F_-). The connectedness is proved.

2. The set $E^k_{m/n}$ **of normal transformations with given** m, n, k **is connected in any of the spaces** $C^1, \cdots, A$. For the proof we join in the space in question the transformations T_0, T_2 by the arc T_θ $(0 \leq \theta \leq 2)$. We carry out a smooth substitution of variables

$$\Psi\varphi = \varphi + \psi(\varphi),$$

taking the points of the cycles T_0 into the corresponding points of the cycles T_2 (which is not hard to do since the number of these points is the same and they follow in the same order). The transformation $T_1 = \Psi T_0 \Psi^{-1}$ operates on the points of the cycles of T_2 in the same way as the transformation T_2; it is easy to see that it does not have other cycles. Putting

$$\Psi_\theta(\varphi) = \varphi + \theta\psi(\varphi)$$

and

$$T_\theta = \Psi_\theta T_0 \Psi_\theta^{-1} \quad (0 \leqslant \theta \leqslant 1),$$

we join T_0 to T_1 by a curve lying in $E^k_{m/n}$.

Consider the transformation

$$T_1(\varphi) = \varphi + t_1(\varphi), \quad T_2(\varphi) = \varphi + t_2(\varphi).$$

The functions $t_1(\phi)$ and $t_2(\phi)$ coincide at the points of the cycles, and therefore all the transformations

$$T_\theta(\varphi) = \varphi + (2-\theta)\, t_1(\varphi) + (\theta - 1)\, t_2(\varphi) \quad (1 \leqslant \theta \leqslant 2)$$

have the same cycles. Accordingly, the curve T_θ $(0 \le \theta \le 2)$, joining T_0 to T_2, lies entirely in $E^k_{m/n}$.

3. The proof of the fact that the set $E^k_{m/n}$ is open and that the set $\bigcup_k E^k_{m/n}$ of normal transformations with the rotation number m/n is everywhere dense in $E_{m/n}$ is analogous to the proof of Theorem 5 (subsections 1 and 3).

4. If $T_1, T_2 \in E^k_{m/n}$, then we may carry out a continuous change of variables $\Psi = \phi + \psi(\phi)$ such that T_1 goes into T_2: $T_2 = \Psi T_1 \Psi^{-1}$. Indeed, we denote the points of the stable cycles of T_1 by a^l_i $(1 \le l \le k,\ 1 \le i \le n,\ T_1 a_i = a_{i+1},\ a_{n+1} = a_1)$ and the points of the unstable cycles of T_1 by b^l_i (by l we denote the number of the cycle in the order in which it follows on the circumference). Here there are no points of the cycles on the arc $a^1_1 b^1_1$ (thus the same is true of each arc $a^l_i b^l_i$ and $b^l_i a^{l+1}_i$).*

Suppose further that c^l_i and d^l_i are the points of the stable and unstable cycles of T_2, enumerated in an analogous way. The substitution of variables Ψ carries the points a^l_i, b^l_i into c^l_i, d^l_i, and it remains for us to complete the definition of Ψ to the arcs $a^l_i b^l_i$, $b^l_i a^{l+1}_i$. We choose the points x and y inside the arcs $a^1_1 b^1_1$ and $c^1_1 d^1_1$. The points $T^n_1 x$ and $T^n_2 y$ lie in the same arcs closer to a^1_1 and c^1_1 respectively. We map the arc $(x, T^n_1 x)$ onto the arc $(y, T^n_2 y)$ homeomorphically and directly using Ψ: $x \longrightarrow y$, $T^n_1 x \longrightarrow T^n_2 y$. Evidently under the transformations T^p_1 the images of the arc $[x, T^n_1 x]$ (or of the arc $[y, T^n_2 y]$ under the transformations T^p_2) entirely cover the whole arc $a^1_i b^1_i$ $(1 \le i \le n)$ (the whole arc $c^1_i d^1_i$). Thus we define $\Psi(\phi)$ on the arc $T^p_1 x,\ T^{p+n}_1 x$ as $T^p_2 \Psi T^{-p}_1$. An analogous construction is possible on $a^l_i b^l_i$ and $b^l_i a^{l+1}_i$. The proof of the fact that the substitution of variables just found is the desired one is not complicated and we omit it.

5. **The structure of the boundary.** If $t^{(n)}(\phi) - 2\pi m$ changes sign, then T is an interior point of $E_{m/n}$ since under a small variation of T, $t^{(n)}(\phi) - 2\pi m$ will change sign as before, and T preserves the cycle. Therefore the boundary $E_{m/n}$ enters into the sum of F_+ ($T \in F_+$ if $t^{(n)}(\phi) - 2\pi m \ge 0$) and F_-. In order to convert the transformation $T \in F_0 = F_+ \cap F_-$ into a rotation, we need to carry the points of one cycle into $2\pi m l/n$ by a smooth substitution of variables and then to

*By $l+1$ for $l = k$ we understand 1.

redefine the parameters on all the arcs $[2\pi ml/n, 2\pi(ml+1)/n]$, except one ($l=0$), according to the formula

$$\Psi(\varphi)=2\pi\frac{ml}{n}+T^{-l}(\varphi).$$

By a small variation of a rotation through the angle $2\pi m/n$ one may convert it into a transformation in any $E^k_{m/n}$, roughly as was done in the proof of Theorem 5 of subsection 3. From the preceding considerations it follows that the same is true also for all transformations of F_0, which proves the last assertion of Theorem 6.

10.4. From Theorem 6 (point 4 of the proof) it follows that normal transformations are *rough* in the sense of Andronov-Pontrjagin [10]. Since, by Theorem 5, the set of all normal transformations is everywhere dense, no nonnormal transformation can be rough.

From the topological point of view normal transformations fill out a predominant part of the space of transformations, namely an everywhere dense open set. In the following section it will be proved that from the point of view of measure the typical case is also the ergodic case.

§11. The case of irrational μ

11.1. Consider now the set E_μ of irrational level μ. In the spaces $C^2, \cdots, A$, by Denjoy's theorem, each transformation $T\in E_\mu$ may be converted into a rotation through the angle $2\pi\mu$ by a *continuous* change of variables. We are also concerned with transformations which can be converted into a rotation by a *smooth* change of variables. The set of such transformations will be denoted by $E^{C^p}_\mu$ (respectively by E^A_μ; the common notation is E'_μ).

Theorem 7. 1°. *The set E^A_μ is everywhere dense in E_μ in the topology of C. All sets E'_μ are connected.*

2°. *If μ is such that $|\mu - m/n| > K/|n|^3$ for any integers m and n not equal to zero, then the set E^A_μ is open in E_μ in the topology of A.*

Proof. 1°. Suppose that T_0 denotes a rotation through the angle $2\pi\mu$, and suppose that $T_1\in E'_\mu$. Then there exists a smooth substitution of variables

$$\Psi(\varphi)=\varphi+\psi(\varphi)$$

such that $T_1=\Psi T_0\Psi^{-1}$. The substitution

$$\Psi_\theta(\varphi)=\varphi+\theta\psi(\varphi)\quad(0\leqslant\theta\leqslant1)$$

converts T_0 into $T_\theta=\psi_\theta T_0\psi_\theta^{-1}$; thus the curve T_θ joining T_0 to T_1 lies entirely in E'_μ. The connectedness of E'_μ is proved.

We shall construct in E^A_μ a transformation T^* in a given neighborhood of $T\in E_\mu$. By Denjoy's theorem there exists a continuous substitution of variables

$\Psi(\phi)$ such that $T = \Psi T_0 \Psi^{-1}$. We shall construct an analytic substitution $\Psi^*(\phi)$ of variables ϕ such that Ψ and Ψ^*, Ψ^{-1} and Ψ^{*-1} differ only slightly in the metric of C. Then $T^* = \Psi^* T_0 \Psi^{*-1}$ approximates T in the metric of C and lies in E_μ^A. Assertion 1° is completely proved.

2°. The fact that the set E_μ^A is open in $E_\mu \cap A$ follows from Theorem 2. Evidently it is sufficient to show that some neighborhood of the rotation T_0 in $E_\mu \cap A$ lies in E_μ^A. The transformation $T \in E_\mu \cap A$ may be written in the form

$$\varphi \to \varphi + 2\pi\mu + F(\varphi),$$

while the neighborhood $U_{R,C}$ of the transformation T_0 is given by the inequality $|F(\phi)| < C$ for $|\mathrm{Im}\,\phi| < R$. But by the Corollary to Theorem 3 (see subsection 4.3), for a given R there exists a C such that all the transformations $T \in U_{R,C} \cap E_\mu$ analytically reduce to rotations. Theorem 7 is proved.

11.2. In turning to the question of typicality from the point of view of measure (see [8]) we encounter the absence of a reasonable measure in functional spaces and therefore we are forced to restrict ourselves to finite-dimensional subspaces.

Consider the two-dimensional space of analytic transformations

$$A_{a,b}: z \to z + a + F(z, b),$$

where for $|\mathrm{Im}\, z| < R$, $|b| < b_0$ $F(z, b)$ is an analytic function satisfying the inequality $|F(z, b)| < L|b|$.

Theorem 8.

$$\lim_{\theta \to 0} \frac{\mathrm{mes}\, E_\theta}{2\pi\theta} = 1, \tag{1}$$

where E_θ is the set of points of the set (ab), $a \in [0, 2\pi]$, $b \in [0, \theta]$, such that the transformation A_{ab} converts into a rotation by an analytic substitution of the coordinate z.

Proof. 1. Consider the set M_K, namely the compact set of points $0 < \mu < 1$ satisfying the inequality

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^3}$$

for all $m, n > 0$. By Theorem 2, for any $\mu \in M_k$ there exist $C = C(K, R) > 0$ and a function $\Delta(b, \mu)$, analytic in b, such that the transformation $A_{2\pi\mu + \Delta(b,\mu), b}$ for $\mu \in M_K$, $|b| < C$ may be converted into a rotation by an analytic change of parameter: $(2\pi\mu + \Delta(b, \mu), b) \in E_\theta$. We denote by $M_K(b)$ the set of points $\mu + \Delta(b, \mu)/2\pi$, $\mu \in M_K$ for a fixed b. Then the transformation $D_b: \mu \to \mu + \Delta(b,\mu)/2\pi$ carries M_K into the set $M_K(b)$.

Put $\epsilon > 0$ and choose $K > 0$ so that $M_{2K} > 1 - \epsilon/3$ (from Lemma 1 of §2 this is possible). We shall show that for sufficiently small b the inequality

$$\mathrm{mes}\, M_{\frac{K}{2}}(b) > 1 - \varepsilon$$

is valid, from which Theorem 8 will follow immediately, since it is evident that

$$2\pi\theta \geqslant \operatorname{mes} E_0 \geqslant 2\pi \int_0^\theta \operatorname{mes} M_{\frac{K}{2}}(b)\, db.$$

2. In §7 we constructed a perfect set $N_K^0 = N_K$, $M_{2K} \subseteq N_K \subseteq M_{K/2}$. Evidently it is sufficient to show that for sufficiently small b

$$\operatorname{mes} N_K(b) > 1 - \varepsilon. \tag{2}$$

(Since $K > 0$ is fixed, we may now drop the index K: $N_K = N$.)

From Theorem 3, the mapping $D_b : N \longrightarrow N(b)$ is the limit of a uniformly converging sequence of monogenic mappings

$$D_b^n : \mu \rightarrow \mu + \frac{1}{2\pi} \Delta^n(b, \mu).$$

We shall show that for any $\epsilon > 0$ there exists a $b(\epsilon)$ such that for $b < b(\epsilon)$ and any n

$$\operatorname{mes} D_b^n(N) > 1 - \varepsilon. \tag{3}$$

From Theorem 3, there exists a $b(\epsilon)$ such that for n, $b < b(\epsilon)$, $\mu \in N$ the following inequality will hold:

$$\left| \frac{\partial \Delta^n}{\partial \mu} \right| < \frac{\varepsilon}{3};$$

i.e., under the mapping D_b^n, N maps almost without dilation.

We shall show that $b(\epsilon)$ has the desired property (the index n will be dropped everywhere, since the argument is always carried out for n fixed). Suppose $b < b(\epsilon)$. From the definition of monogenicity, for $\epsilon/3$ there exists a $\delta > 0$ such that

$$\left| \frac{\Delta(\mu_1) - \Delta(\mu_2)}{\mu_1 - \mu_2} - \frac{\partial \Delta(\mu_3)}{\partial \mu} \right| < \frac{\varepsilon}{3}$$

if $|\mu_1 - \mu_3| < \delta$, $|\mu_2 - \mu_3| < \delta$, $\mu_1, \mu_2, \mu_3 \in N$. Then under the same conditions

$$\left| \frac{\Delta(\mu_1) - \Delta(\mu_2)}{\mu_1 - \mu_2} \right| < \frac{2\varepsilon}{3}, \tag{4}$$

in view of the choice of $b(\epsilon)$.

3. We decompose N into nonintersecting (of course, measurable) parts N^i, $\bigcup_{i=1}^L N^i = N$, the diameter of each of which is less than δ, and suppose that $N^i(b)$ are their images under the transformation D_b^n. Since under this transformation the distance between two points of N^i cannot decrease, as follows from (4), by more than $1 - 2\epsilon/3$ times, therefore

$$\operatorname{mes} N^i(b) > \left(1 - \frac{2\varepsilon}{3}\right) \operatorname{mes} N^i,$$

from which it follows that

$$\sum_{i=1}^{L} \operatorname{mes} N^i(b) > \left(1 - \frac{2\varepsilon}{3}\right) \sum_{i=1}^{L} \operatorname{mes} N^i.$$

Thus

$$\operatorname{mes} N(b) > \left(1 - \frac{2\varepsilon}{3}\right) \operatorname{mes} N,$$

and since

$$\operatorname{mes} N > 1 - \frac{\varepsilon}{3},$$

we obtain

$$\operatorname{mes} N(b) > \left(1 - \frac{2\varepsilon}{3}\right)\left(1 - \frac{\varepsilon}{3}\right) > 1 - \varepsilon,$$

and inequality (3) is proved. Inequality (2) follows from this, since the following lemma is valid.

Lemma. *Suppose that* $E \subseteq [0, 1]$ *is a perfect set and that* f_n *is a sequence of continuous mappings of this set onto* $F_n \subseteq [0, 1]$, *uniformly converging to the mapping* $f\colon E \to F$, *and suppose* $0 \le \Delta < 1$. *If* $\operatorname{mes} F_n > 1 - \Delta$ *for all* n, *then* $\operatorname{mes} F \ge 1 - \Delta$.

Proof. Suppose that $\epsilon > 0$. We consider the set D_ϵ of contiguous intervals of F larger than ϵ. There will be a finite number of them, and for a sufficiently large n these intervals will be arbitrarily little different from the corresponding contiguous intervals of F_n. The sum of the length of the latter for any n is less than Δ, since $\operatorname{mes} F_n > 1 - \Delta$. Therefore the total length of D_ϵ does not exceed Δ. In view of the arbitrariness of the choice of $\epsilon > 0$, the measure of the entire complement to F is also not larger than Δ, as was required to be proved.

Putting $E = N$, $f_n = D_b^n$, $F_n = D_b^n(N)$, $\Delta = \epsilon$, we obtain inequality (2) from (3). Theorem 8 is proved.

§12. Example

We consider the two-dimensional space of mappings of the circumference onto itself of the form

$$\varphi \to \varphi + a + \varepsilon \cos \varphi \equiv T_{a,\varepsilon}(\varphi). \tag{1}$$

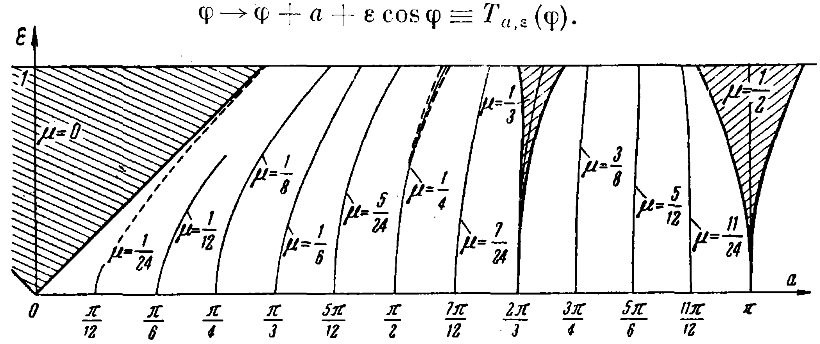

Figure 8

For $\epsilon = 0$ we obtain $T_{a,0}$, namely a rotation through the angle a. For $|\epsilon| < 1$ formula (1) defines a direct one-to-one continuous mapping of the circumference onto itself.

The level sets of the function

$$\mu(a, \varepsilon) = \mu(T_{a,\varepsilon})$$

continuous for $|\epsilon| \leq 1$ may be studied from two points of view. First, we may seek those points (a, ϵ) of the plane for which μ is rational; the boundaries of such regions are given by the conditions of semistability of the cycle. For example, the point (a, ϵ) enters into the level set $\mu = 0$ if the equation

$$\varphi = \varphi + a + \varepsilon \cos \varphi$$

has a real solution, i.e., the boundary of the region $\mu = 0$ is the straight line $a = \pm\epsilon$. In the same way we find the regions $\mu = m/n$. They approach the line $\epsilon = 0$ with ever narrowing tongues (Figure 8); two boundaries of the region $\mu = m/n$ have contact of $(n-1)$st order. For example, the regions $\mu = 1/2$ and $\mu = 1/3$ have bounding curves

$$a = \pi \pm \frac{\varepsilon^2}{4} + O(\varepsilon^4), \tag{2}$$

$$a = \frac{2\pi}{3} + \frac{\sqrt{3}}{12}\varepsilon^2 \pm \frac{\sqrt{7}}{24}\varepsilon^3 + O(\varepsilon^4). \tag{3}$$

Therefore one obtains approximate formulas, valid also for not very small ϵ: for $\epsilon = 1$ formula (2) gives $\pi \pm 0.25$ instead of $\pi \pm 0.23237\cdots$.

The second approach to the determination of the level sets $\mu(a, \epsilon)$ consists in using Newton's method for the approximate determation of the curves of irrational level μ. After two steps of Newton's method we obtain the following approximate equation for the level lines:

$$a = 2\pi\mu + \frac{\varepsilon^2}{4}\operatorname{ctg}\pi\mu - \frac{\varepsilon^4}{32}\operatorname{ctg}^3\pi\mu + \frac{\varepsilon^4}{32}\operatorname{ctg}2\pi\mu\,(1 + \operatorname{ctg}^2\pi\mu), \tag{4}$$

which works well when the cotangents are not large. Figure 9 gives an idea of the character of the convergence of the approximations and on the relation of this result to the preceding one. On this drawing we have shown the graph of the function $\mu(a) = \mu(a, 1)$. We have denoted the zeroth approximation of Newton's method by 0, the first by I, and the second by II. The horizontal segments for $\mu = 0, 1/2, 1/3$ are determined independently in accordance with formulas (2) and (3). For the number a given by

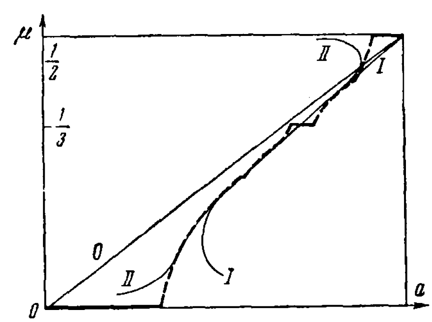

Figure 9

formula (4) the substitution of variables

$$\psi(\varphi) = \varphi - \frac{\varepsilon}{2}\frac{\sin(\varphi - \pi\mu)}{\sin \pi\mu} + \frac{\varepsilon^2}{4}\frac{\sin(2\varphi - \pi\mu)}{\sin \pi\mu \sin 2\pi\mu}$$

converts the transformation (1) into the transformation

$$\psi \to \psi + 2\pi\mu + F_2(\psi, \varepsilon, \mu),$$

where $F_2 \sim \epsilon^4$.

Remark. In the theory of oscillations the phenomenon of "locking in" is well known. This phenomenon corresponds to zones with rational rotation numbers.

Transformations of type (1) and diagrams of the type of Figure 8 describe a certain regime of the work of a generator of relaxation oscillations, synchronized by a sinusoidal impulse (see [25]). Another problem of a similar sort connected also with the mappings of a circumference onto itself is considered in the book [37] (pp. 221–231 of 2nd ed.).

§13. On trajectories on the torus*

13.1. Suppose that we are given on the torus $x, y \in [0, 2\pi]$ a differential equation

$$\frac{dy}{dx} = F(x, y) \quad (F(x + 2\pi k, y + 2\pi l) = F(x, y) > 0)$$

and that the usual conditions of existence and uniqueness theorems are satisfied. Through each point y_0 of the meridian $x = 0$ there passes a trajectory

$$y(x, y_0), \quad y(0, y_0) = y_0.$$

Following Poincare, we make correspond to the point y_0 the point $y(2\pi, y_0)$. Then we obtain a mapping of the circumference $x = 0$ onto itself, direct, one-to-one, continuous, and smooth or analytic for sufficiently smooth or analytic right side. If now the function $F(x, y)$ differs by little from a constant, then this mapping will be close to a rotation. All the properties of the transformation $y_1(y_0)$ reflect the corresponding properties of the solutions of equation (1), and we need only formulate the results of the preceding sections in the new terms.

If the mapping $y_1(y_0)$ is converted by the change of variables from y to $\phi(y)$ into a rotation through the angle $2\pi\mu$, then this substitution may be extended in a natural way to the whole torus if at the point $(x, y(x, y_0))$ we set

$$\varphi(x, y) = \varphi(y_0) + \mu x.$$

Evidently, if $\phi(y)$ is a smooth, or analytic, substitution, then the substitution $\phi(x, y)$ on the whole torus will also be smooth or analytic. In the x, ϕ coordinates the trajectories are written in the form

$$\varphi = \varphi_0 + \mu x$$

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*See [1] – [4], [14], [19] and [20].

and one therefore says that a substitution of this kind *straightens out*, or *rectifies*, *the trajectories*. An analytic rectification of trajectories was obtained by A. N. Kolmogorov [14] in the case of the presence of an analytic integral invariant. On the basis of Theorem 2 we may now assert that *if the function* $F(x, y)$ *is analytically close to a constant and if the rotation number* μ *satisfies the usual arithmetic conditions, then the trajectories may be analytically rectified.* Thus it follows that the dynamical system

$$\frac{dy}{dt} = F(x, y), \quad \frac{dx}{dt} = 1$$

has an analytic integral invariant with invariant measure equal to the area in the x, ϕ coordinates.

On the other hand, in the same way as in the example of §1 one may construct an analytic function $F(x, y)$ such that the invariant measure of the system is not absolutely continuous relative to the area $dxdy$, although the rotation number μ is irrational and the system ergodic.*

13.2. Suppose that on the torus we are given a system of differential equations

$$\frac{dx}{dt} = A(x, y), \quad \frac{dy}{dt} = B(x, y) \quad (A(x, y) > 0, \quad B(x, y) > 0) \tag{1}$$

with analytic right side. Consider the equation

$$\frac{dy}{dx} = \frac{B(x, y)}{A(x, y)},$$

which has the same integral curves as the system. If these may be rectified in accordance with subsection 13.1, then in the new coordinates the system has the form

$$\frac{dx}{dt} = A'(x, \varphi), \quad \frac{d\varphi}{dt} = \mu A'(x, \varphi),$$

where $A'(x, \phi) = A(x, y(x, \phi))$. This system has the analytic integral invariant $1/A'(x, \phi)$, and in the paper [14] it was shown, with the usual hypotheses on μ, how to convert it to the system

$$\frac{du}{dt} = 1, \quad \frac{dv}{dt} = \mu$$

by an analytic substitution of variables.

The contrary possibility, both in the case of an equation and in the case of a system, is the presence of limit cycles [20]. The decomposition of the space

*Added in proof. The contrary assertion in the review [41], which appeared while this paper was being printed, is mistaken.

of right sides of the system (1) into level sets for the rotation number, the characterization of rough systems and the consideration of the question as to the typicality, are analogous to the considerations of §§9–11. It results that

1. The case of normal cycles (it is still rough) is topologically predominant.* The corresponding set of right sides is open and everywhere dense; however, in systems with an integral invariant this case cannot happen at all.

2. The ergodic case (the case of irrational μ) is typical as well if one uses measures in finite-dimensional spaces as the point of departure for judging typicality. For systems with an analytic integral invariant this case is predominant.

In the multidimensional case, in the absence of an integral invariant, the rotation number is not defined. Nevertheless, by making use of the remark of subsection 4.4, we may obtain the following assertion.

Theorem 9. *Suppose that* $\vec{\mu} = (\mu_1, \cdots, \mu_n)$ *is a vector with noncommensurable components such that for any integer* $\vec{k}$

$$|(\vec{\mu}, \vec{k})| > \frac{C}{|\vec{k}|^n}.$$

Then there exists an $\epsilon(R, C, n) > 0$ *such that for any analytic vector field* $\vec{F}(\vec{x})$ *on the torus, i.e., a field with* $\vec{F}(\vec{x} + 2\pi\vec{k}) = \vec{F}(\vec{x})$), *which is sufficiently small,* $|\vec{F}(\vec{x})| < \epsilon$ *for* $|\mathrm{Im}\,\vec{x}| < R$, *there exists a vector* $\vec{a}$ *for which the system of differential equations*

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}) + \vec{a}$$

converts into

$$\frac{d\vec{u}}{dt} = 2\pi\vec{\mu}$$

by an analytic change of variables.

§14. Dirichlet's problem for the equation of the string

14.1. Suppose that D is a region on the plane, convex in the coordinate directions; i.e., its boundary Γ intersects each line $x = c$, $y = c$ at not more than two points.

The *Dirichlet problem* for the equation $\partial^2 u/\partial x \partial y = 0$ on D consists in *finding on* D *a function* $u(x, y) = \phi(x) + \psi(y)$ *which on* Γ *is transformed into a given function* $f(a)$ $(a \in \Gamma)$: $u|_\Gamma = f$.

Here one may impose various requirements of smoothness, analyticity and so

* In the paper [19], to judge from the review [21], it is asserted that a necessary and sufficient condition for roughness is the presence of *one* stable cycle. This is not true.

forth on f, ϕ, ψ, Γ.

In the case when D is the rectangle $0 \leq x + y \leq l$, $0 \leq y - x \leq t$, it is convenient to refer to the coordinates $\xi = x + y$, $\tau = y - x$. Then our equation becomes the equation of the string, and the problem may be interpreted as the problem of finding the motion of the string with respect to two instantaneous photographs and the motion of the ends. From physical considerations (standing waves) it is clear that with commensurable l and t the problem is not always solvable, and if it is solvable, not always uniquely. This problem has been the object of a series of papers, e.g., [22], [23], [5], [24], [17], [28]. There are difficulties of an analogous order in the solution of certain other problems, e.g., [25]–[27].

14.2. **Uniqueness theorems** (see [5]). We shall associate with the boundary certain of its mappings onto itself (see Figure 10). Suppose that P is a transformation carrying the point $a \in \Gamma$ into the point $Pa \in \Gamma$ with the same coordinate x, and that Q is a transformation carrying the point $a \in \Gamma$ into the point $Qa \in \Gamma$ with the same coordinate y. These transformations are continuous, one-to-one, and change the orientation of the contour Γ. We write $QP = T$. Evidently

$$P^2 = Q^2 = E, \quad PQ = T^{-1}.$$

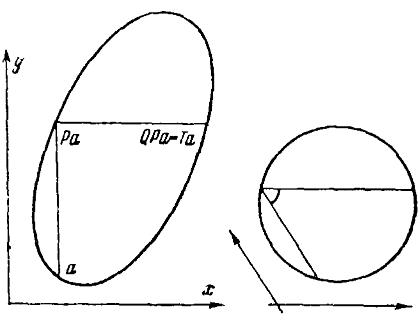

Figure 10

T is a direct homeomorphic mapping.

Theorem 10 (see [5]). *If the contour* Γ *is such that for some point* $a_0 \in \Gamma$ *the set* $T^n a_0$ $(n = 0, 1, 2, \cdots)$ *is everywhere dense on* Γ, *then the Dirichlet problem for* Γ *cannot have more than one continuous solution.*

Proof. The solution $u(x, y) = \phi(x) + \psi(y)$ defines functions $\phi(x)$, $\psi(y)$ up to a constant. We shall show that under the conditions of the theorems, knowing $\phi(x)$ at one point $a \in \Gamma$ makes it possible to determine $\phi(T^n a)$, $\psi(T^n a)$ at all the points $T^n a$ $(n = 0, 1, \cdots)$ (we write $\phi(a)$ and $\psi(a)$ for $\phi(x)$, $\psi(y)$, where x, y are the coordinates of the point $a \in \Gamma$).

Knowing $\phi(a)$, it is easy to find

$$\psi(Pa) = f(Pa) - \varphi(a),$$

since the abscissae of a and Pa are the same. Then we may determine

$$\varphi(Ta) = f(Ta) - \psi(Pa),$$

using the fact that the ordinates of the points Pa and Ta coincide. Further, in the same way we obtain ϕ, ψ at all the points $T^n Pa$, $T^n a$. They form a set everywhere dense on Γ, so that continuous functions which coincide at these points of Γ coincide everywhere. The theorem is proved.

In the case when D is the rectangle $0 \leq x + y \leq l$, $0 \leq y - x \leq t$, the transformation T is, in particular, a rotation. Indeed, if we introduce on the contour Γ a parameter

$$\vartheta = \frac{2\alpha\pi}{\sqrt{2}\,(l + t)},$$

where α is the length measured along the contour from the point 0 to a (Figure 11), then our transformation

$$T : T\vartheta = \vartheta + \frac{2\pi t}{t + l}$$

is a rotation through the angle $2\pi t/(t + l)$. If D is an ellipse, then it is not difficult to introduce on Γ a parameter such that in it the transformation may be written as a rotation. Indeed, we map the ellipse affinely onto a disk. The straight lines in the coordinate directions go into two families of parallel lines, while two lines of different families form an angle of $\pi\mu$, in general not a right angle. Evidently, when the ellipse is subjected to the transformation T, the circumference rotates by an angle $2\pi\mu$ (Figure 10).

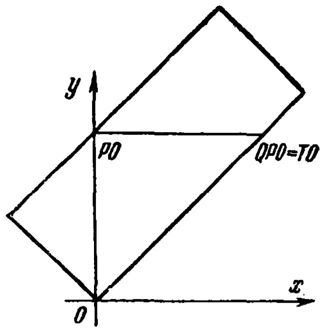

Figure 11

If Γ is a curve of bounded curvature, then T is a twice differentiable transformation, from which, by Denjoy's theorem, we have the result that for an irrational rotation number μ the mapping T of the set $T^n a$ is everywhere dense on Γ. Hence we have the following theorem.

Theorem 11 (see [5], [24]). *If Γ has bounded curvature and μ is irrational, then the Dirichlet problem can have only one continuous solution.*

Remark. Using the theorem on points of density, it is easy to prove that under the conditions of our theorem there can be only one measurable solution. On the other hand, the method of proof of Theorem 10 makes it possible, for irrational μ, to construct as many solutions as desired, but, generally speaking, nonmeasurable ones.

14.3. Detailed investigation of the rectangle.

Theorem 12 (see [23], [17]). *Suppose that on the boundary Γ of the rectangle $0 \leq x + y \leq l$, $0 \leq y - x \leq t$, there is given a function $f(\vartheta)$ which is $(p + \epsilon)$ times differentiable along the boundary. Then for all $\mu = t/(t + l) \in M_k$ satisfying the inequality $|\mu - m/n| > K/|n|^3$ for any m and n and some $K > 0$, the Dirichlet problem with the indicated boundary functions has a $p - 1$ times differentiable* *solution, and the problem relative to $f(\vartheta)$ is correctly posed. In the case of analyticity of f the solution for the same μ is analogous.*

For certain irrational μ, even in spite of the analyticity of the function $f(\vartheta)$, the solution may turn out to be

1) *only infinitely differentiable,*
2) *differentiable k, but not $k+1$, times,*
3) *only continuous,*
4) *discontinuous,*
5) *nonmeasurable.*

Proof. If

$$f(\vartheta) = \sum_{n \neq 0} a_n e^{in\vartheta}, \qquad \varphi(\vartheta) = \sum_{n \neq 0} b_n e^{in\vartheta}, \qquad \psi(\vartheta) = \sum_{n \neq 0} c_n e^{in\vartheta},$$

then, since $\phi(\vartheta)$ depends only on x, and $\psi(\vartheta)$ only on y, we have

$$\varphi(\vartheta) = \varphi(-2\pi\mu - \vartheta), \quad b_n = b_{-n} e^{in2\pi\mu},$$
$$\psi(\vartheta) = \psi(-\vartheta), \qquad c_n = c_{-n}.$$

Since $f(\vartheta)$ is real and therefore $a_n = \bar{a}_{-n}$, from the equation $f(\vartheta) = \phi(\vartheta) + \psi(\vartheta)$ we find that

$$b_n + c_n = a_n, \quad b_n e^{-in2\pi\mu} + c_n = \bar{a}_n,$$

or

$$b_n = \frac{\bar{a}_n - a_n}{e^{-2\pi i\mu n} - 1}, \quad c_n = a_n - b_n. \tag{1}$$

Now, when the formal solution is found, the rest of the proof may be carried out by an exact repetition* of the argument of §2.

Remark. It is clear from formula (1) that for all μ it is possible, by truncating the series, to construct an "approximate solution," the degree of approximation of which is greater in proportion as the commensurability of l and t is less. For rational μ the approximation is not higher than the limit imposed by μ, but for strongly noncommensurable l and t we have Theorem 11. This meaning of correctness with respect to a region was introduced by N. N. Vahanija in the paper [28].

We may assert that the dependence of the solution on μ is *monogenic* (see §7).

14.4. **General case.** If the boundary of D is such that the transformation T may be represented as a rotation in a parameter which is a smooth function of the

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*Added in proof. In an article [42] by P. P. Mosolov, published while the present article was at the press, the statement analogous to that of Theorem 12 was proved for an arbitrary linear differential equation with constant coefficients in which all the derivatives are of even order.

point on the boundary, then evidently for each such contour all the arguments of subsection 14.3 are applicable, and in the case of a "sufficiently irrational μ" the Dirichlet problem has a smooth solution.

As an example there is the ellipse, for which the parameter was constructed in subsection 14.2. Now in the general case of irrational μ, in spite of the arbitrary degree of smoothness of Γ, one cannot guarantee the smoothness of the parameter in which the transformation T becomes a rotation, although by Denjoy's theorem such a parameter exists. F. John [5] showed that with a *continuous* change of variables x, y of the form $x \to u(x)$, $y \to v(y)$ ("preserving the equation $\partial^2 w/\partial x \partial y = 0$") it is possible to map a region for which T has an irrational μ onto a rectangle or onto an ellipse with the same μ. However this substitution, generally speaking, is only continuous, and it may convert smooth boundary conditions on the curve into nonsmooth boundary conditions on the ellipse.

We note that if Γ is an analytic curve, then P and Q, and thus T and T^n, are analytic mappings. But if Γ is also analytically close to an ellipse, then in an appropriate parameter the transformation will be analytically close to a rotation. Therefore it follows from Theorem 2 that *among the curves for which $\mu \in M_k$, all curves sufficiently close to the ellipse are analogous to the ellipse in respect to the solvability of the Dirichlet problem.*

In exactly the same way one may formulate other problems on mappings of the circumference in these terms. In particular, if the transformation T has a cycle, then the Dirichlet problem with zero boundary conditions has a nonzero solution (at least piecewise constant; for more details see [24]).

The Dirichlet problem for the string equation is a problem on eigenvalues for the two-dimensional Sobolev equation

$$\frac{\partial^2 \Delta u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(see [24], [27], [29], [30]). The values of λ which belong to the spectrum are those for which the mapping T_λ, constructed for the curves Γ_λ, has a cycle (here by Γ_λ we mean the curve Γ subjected to a dilation depending on λ).

From the results of §10 it follows that if the cycle is stable, then all the curves close to Γ_λ yield an analogous cycle, and accordingly the point λ belongs to the spectrum, together with a neighborhood. An example of a curve Γ generating a transformation with a stable cycle was constructed by R. A. Aleksandrjan [24]. On the basis of §10 we may show that such curves may lie in any neighborhood of any curve Γ.

The Dirichlet problem for the wave equation with given values on the ellipsoid was recently investigated by R. Dencev [32].

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THE STABILITY OF THE EQUILIBRIUM POSITION OF A HAMILTONIAN SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS IN THE GENERAL ELLIPTIC CASE.*

V. I. ARNOL'D

1. Let $p = q = 0$ be a fixed point of the system

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{1}$$

where $H(p, q, t)$ is an analytic function of p, q, t and periodic in t with period 2π. A case is called *elliptic* if the equilibrium position is stable in the first (linear) approximation. Then, as was shown by Birkhoff [1], by a proper choice of the variables p, q, t the Hamiltonian assumes the form

$$H = \lambda r + c_2 r^2 + \ldots + c_n r^n + \widetilde{H}(p, q, t), \tag{2}$$

where $2r = p^2 + q^2$, $\widetilde{H} = 0(r^{n+1})$ is an analytic function of p, q, t, $n \geq 2$ and arbitrary. We call a case a *general elliptic* case if among the constants c_l $(2 \leq l < \infty)$ is different from zero.

2. Examples are known where the equilibrium position is unstable, and λ is rational [2]. We investigate the case for λ irrational. Let us denote by Λ_K the set of those λ for which the inequality

$$|\lambda n - m| > \frac{K}{(|m| + |n|)^2} \tag{3}$$

is satisfied for all integral $m, n > 0$. Denote by Λ the union of points of compactness of all the sets Λ_K. As is known, on a straight line the complement of Λ is of measure zero [3].

Theorem 1. *If $\lambda \in \Lambda$ then the equilibrium position* $0, 0$ *of the system of equations* (1) *with the Hamiltonian $H(p, q, t)$ of the general elliptic type* (2) *is stable.*

Theorem 2. *Under the conditions of Theorem* 1 *in an arbitrary neighborhood of the circumference of $p = q = 0$ of the p, q, t-space, there exists an analytic invariant torus T_μ whose equation is $r = r(\phi, t)$ $(\phi = \operatorname{arctg} p/q)$. On the torus T_μ it is possible to introduce the analytic coordinate $\psi(\phi, t)$ such that on the torus T_μ the equation* (1) *will take the form $\dot{\psi} = \mu$. The set formed by the tori T_μ has a positive measure in the p, q, t-space.*

Theorem 3. *Let the Hamiltonian be of the form*

$$H(r, \varphi, t) = H_0(r) + \widetilde{H}(r, \varphi, t), \tag{4}$$

where $dH_0/dr = \mu + \Omega(r)$, $\mu \in \Lambda_K$, $\Omega(0) = 0$ and the function

$$\widetilde{H} = \sum_{m^2+n^2 \neq 0} H_{mn}(r)\, e^{i(m\varphi + nt)}$$

is analytic for $|\operatorname{Im} \phi, t| \leq \rho$, $|r| \leq \rho_r = \delta^k$ and satisfies the inequality

$$|\widetilde{H}| \leq M = \delta^N, \tag{5}$$

and the function $\Omega(r)$ for $|r| \leq \rho_r$ is analytic and

$$\delta^a = \theta \leqslant \left|\frac{d\Omega}{dr}\right| \leqslant \Theta = \delta^{-b}. \tag{6}$$

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Here $\delta > 0$ is some constant; N, k, a, b are natural numbers. If the inequalities

$$2k + 28 + 2a + 4b < N < 3k - 14 - 2b; \quad \delta < 10^{-6} K^2; \ \delta < 0.1 \rho, \tag{7}$$

are satisfied, then there exist functions $R(\phi, t)$, $\Psi(\phi, t)$ of period 2π in ϕ and t, which are analytic for $|\mathrm{Im}\,\phi, t| \leq 0.1\,\rho$ and such that on the torus $r = R(\phi, t)$ the equations

$$\dot{\varphi} = \frac{\partial H}{\partial r}, \quad \dot{r} = -\frac{\partial H}{\partial \varphi}$$

imply $\dot{\psi} = \mu$ (here $\psi = \phi + \Psi$).

Theorem 1 follows from Theorem 2, since the tori T_μ separate the circumference $r = 0$ from the rest of the p, q, t-space. Theorem 2 follows from Theorem 3. It is not difficult to see than in the conditions of Theorem 2 there exist arbitrary small toroidal rings $|r - r_0| \leq \rho_r$ around the circumference $r = 0$, to which Theorem 3 is applicable, if r is replaced by $r - r_0$.

3. The last two theorems can ge generalized to systems with n degrees of freedom. However the resulting invariant $(n + 1)$-dimensional tori do not separate the $(2n + 1)$-dimensional p, q, t phase space and the question of stability remains open. Analogous theorems can be proven for the circumferences of equilibrium positions of autonomous Hamiltonian systems. In this case in the $(2n - 1)$-dimensional manifolds $H(p, q) = h$ lie the tori of dimension n. From this follows:

Theorem 4. *The equilibrium position of an autonomous system of Hamilton's equations of two degrees of freedom in the general elliptic case is stable if $\lambda_2/\lambda_1 \in \Lambda$.*

Here the general elliptic case is that case when in proper coordinates the analytic function H has the form [1]

$$H(p_1, p_2, q_1, q_2) = \lambda_1 r_1 + \lambda_2 r_2 + H_0(r_1, r_2) + \widetilde{H}(p_1, p_2, q_1, q_2),$$

where $H_0(r_1, r_2) = \sum_{i+j=2}^{n} c_{ij} r_1^i r_2^j$, $\widetilde{H} = O(r_1 + r_2)^{n+1}$, $2r_i = p_i^2 + q_i^2$ and $h(\epsilon) = H_0(\epsilon\lambda_2, -\epsilon\lambda_1)$ does not vanish identically.

It can also be shown that an arbitrary analytic canonic transofrmation of the plane into itself in a neighborhood of a fixed point of the general elliptic type is stable if its rotation number $\lambda \in \Lambda$. Theorems 2 and 3 admit a corresponding generalization even in the many-dimensional case.

4. Let us outline the proof of Theorem 3. This theorem is a stronger version of A. N. Kolmogorov's theorem on the preservation of a conditionally periodic motion for small variations of the Hamiltonian [4]. The invariant torus is found, as in A. N. Kolmogorov's work, by Newton's method of approximation. This method gives fast convergence which can not be destroyed by the small denominator which appears in formula (9).

Fundamental lemma. *Under the assumptions (4)–(7) of Theorem 3 there exist an analytic function $\widetilde{F}(\bar{r}', \phi, t) = \sum_{m^2+n^2 \neq 0} F_{mn}(\bar{r}')\, e^{i(m\phi + nt)}$ and a number $\bar{r}^*$ such that the canonical transformation*

$$\bar{\phi} = \phi + \partial\widetilde{F}/\partial\bar{r}')\ \bar{r} = \bar{r}' - \bar{r}^*, \ r = \bar{r}' + \partial\widetilde{F}/\partial\phi$$

transforms the Hamiltonian (4) to the form $\bar{H}(\bar{r}, \bar{\phi}, t) = \bar{H}_0 + \widetilde{\bar{H}}$, where $d\bar{H}_0/d\bar{r} = \mu + \bar{\Omega}(\bar{r})$, $\bar{\Omega}(0) = 0$ and the function

$$\widetilde{\bar{H}}(\bar{r}, \bar{\varphi}, t) = \sum_{m^2+n^2 \neq 0} \bar{H}_{mn}(\bar{r})\, e^{i(m\bar{\varphi}+nt)}$$

is analytic for $|\mathrm{Im}\,\bar{\phi}, t| \leq \bar{\rho} = \rho - 3\delta$, $|\bar{r}| \leq \bar{\rho}_r = \bar{\delta}^k$ and satisfies the inequality

$$|\widetilde{\overline{H}}| \leqslant \overline{M} = \delta^N,$$

and $\overline{\Omega}(\overline{r})$ is analytic for $|\overline{r}| \leq \overline{\rho}_r$ and

$$\bar{\delta}^a = \bar{\theta} \leqslant \left|\frac{d\overline{\Omega}}{d\overline{r}}\right| \leqslant \overline{\Theta} = \bar{\delta}^{-b}.$$

In these formulas $\overline{\delta} = \delta^{1\frac{1}{2}}$.

Theorem 3 is derived from the fundamental lemma without difficulty, since the error of the sth approximation M_s is not greater than $M^{(1\frac{1}{2})^s}$.

Not having the possibility to give the proof of the fundamental lemma, we shall only show the method of constructing $\widetilde{F}$ and $\overline{r}^*$. As is known $\overline{H}(\overline{r}, \overline{\phi}, t) = \overline{H}'(\overline{r}', \overline{\phi}, t) = \hat{H}(\overline{r}', \phi, t)$ where we denoted

$$\hat{H}(\bar{r}', \varphi, t) = H(r(\bar{r}', \varphi, t), \varphi, t) + \frac{\partial \widetilde{F}(\bar{r}', \varphi, t)}{\partial t}.$$

It is obvious that

$$\hat{H}(\bar{r}', \varphi, t) = H_0(\bar{r}') + \hat{S}_1 + \hat{S}_2 + \hat{S}_3,$$

where

$$\hat{S}_1(\bar{r}', \varphi, t) = \mu\frac{\partial \widetilde{F}}{\partial \varphi} + \frac{\partial \widetilde{F}}{\partial t} + \widetilde{H};$$

$$\hat{S}_2(\bar{r}', \varphi, t) = H_0(r) - H_0(\bar{r}') - \mu(r - \bar{r}'), \quad |\hat{S}_2| = |\Omega|\left|\frac{\partial \widetilde{F}}{\partial \varphi}\right|; \tag{8}$$

$$\hat{S}_3(\bar{r}', \varphi, t) = \widetilde{H}(r) - \widetilde{H}(\bar{r}'), \quad |\hat{S}_3| = \left|\frac{\partial \widetilde{H}}{\partial r}\right|\left|\frac{\partial \widetilde{F}}{\partial \varphi}\right|.$$

The function $\widetilde{F}$ is defined by the condition $\hat{S}_1 \equiv 0$:

$$F_{mn} = \frac{iH_{mn}}{\mu m + n}. \tag{9}$$

Passing to the variables $\overline{r}'$, $\overline{\phi}$, t we obtain

$$\overline{H}'(\bar{r}', \bar{\varphi}, t) = H_0'(\bar{r}') + S_{2_0}(\bar{r}') + S_{3_0}(\bar{r}') + \widetilde{\overline{H}}'(\bar{r}', \bar{\varphi}, t) = \overline{H}_0' + \widetilde{\overline{H}}',$$

where $\widetilde{\overline{H}}'(\overline{r}', \overline{\phi}, t) = \widetilde{S}_2 + \widetilde{S}_3$ unites the variable terms of the Fourier series functions in $\overline{\phi}$, t,

$$S_i(\bar{r}', \bar{\varphi}, t) = \widetilde{S}_i + S_{i_0}(\bar{r}') = \hat{S}_i(\bar{r}', \varphi(\bar{r}', \bar{\varphi}, t)\, t), \quad (i = 2, 3).$$

Now $\overline{r}^*$ is determined from the equation

$$\left.\frac{d\overline{H}_0'}{d\bar{r}'}\right|_{\bar{r}^*} = \mu.$$

Here the inequality (6) is used in estimating $\overline{r}^*$.

When the inequalities (7) are satisfied, the quantity $\widetilde{\overline{H}}$, estimated by the use of formulas (8) does not exceed $M^{1\frac{1}{2}} = \overline{M}$.

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GENERATION OF ALMOST PERIODIC MOTION FROM A FAMILY OF PERIODIC MOTIONS*

V. I. ARNOL'D

1. The motion of a point x, y on a torus T is called almost periodic if

$$\frac{dy}{dx} = \lambda, \tag{1}$$

where λ is an irrational constant, and x and y are coordinates on the torus, such that $(x + k, y + l)$ and (x, y) are the same point T. Let us consider the family of neighboring differential equations

$$\frac{dy}{dx} = \lambda + a + \epsilon f(x, y) \tag{2}$$

where a and ϵ are parameters and $f(x, y)$ is an analytic function.

Recently [1] I showed that if the "perturbation" $\epsilon f(x, y)$ is small, then one may find an $a(\epsilon)$, such that equation (2) for $a = a(\epsilon)$, may be brought into the form (1) by an analytic change of variable.

In this article we consider the degenerate case $\lambda = 0$; the unperturbed motion takes place along parallels of the torus y = constant, i.e., periodically. It turns out that for a large number of small perturbations such a motion becomes almost periodic. It is interesting to study this changeover, because systems which are almost degenerate frequently occur in mechanics.

The difficulties which appear in bringing the equation to form (1) because of the presence of small denominators, are overcome by means of successive approximations of newtonian type. A. N. Kolmogorov [2] first adopted this method in constructing almost periodic motions of a system with hamiltonian

$$H(p, q) = H_0(p_1, \cdots, p_n) + \epsilon \widetilde{H}(p_1, \cdots, p_n, q_1, \cdots, q_n) + \cdots.$$

Essential in the present article is the introduction of certain new (but simple) arithmetical facts; their appearance is connected with the degeneracy of the unperturbed system. Generalizing Theorem 2 of our article, we are able to extend the results [2] to systems with hamiltonian

$$H = H_0(p_1, \ldots, p_k) + \varepsilon H_1(p_1, \ldots, p_n, q_1, \ldots, q_k) + \varepsilon^2 \widetilde{H}(p_1, \ldots, p_n, q_1, \ldots, q_n) + \ldots \quad (k < n).$$

2. Let us denote by Λ_θ the set of points λ, such that

$$|\lambda n + m| > \theta |\lambda| n^{-2}$$

for all integers m and n, $n \neq 0$. Let Λ denote the union of the sets Λ_θ for all $\theta > 0$. It is easy to

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prove the following

Lemma. *Zero is the limit point of the set* $\Lambda_\theta (0 < \theta < 0.25)$ *and a point of density of the set* Λ.

Theorem 1. *On the torus T, let there be given the following differential equation,*

$$\frac{dy}{dx} = \epsilon f(x, y), \tag{3}$$

depending on the parameter ϵ, *with function* $f(x, y)$ *analytic. Let* $\int_0^1 f(x, y)\,dx > 0$ *for all* y. *Then for every sufficiently small* $\lambda \in \Lambda_\theta$ *there can be found an* $\epsilon(\lambda)$ *and a change of variable* $z = z_\lambda(x, y)$, *which is analytic in* x *and* y, *such that equation* (3) *assumes the form* $dz/dx = \lambda$. *The set* $\epsilon(\lambda)$ $(\lambda \in \Lambda)$ *has positive measure; zero is its point of density.*

The proof of Theorem 1 is based on the following idea: first, one performs the change of variable suggested by the usual asymptotic methods, and then applies

Theorem 2. *The conclusion of Theorem 1 is true for the differential equation on the torus*

$$\frac{dy}{dx} = \epsilon c + \epsilon^2 F(x, y, \epsilon), \tag{4}$$

where $c > 0$ *is a constant, and the function* $F(x, y, \epsilon)$ *is analytic.*

3. We shall show how to express equation (3) in form (4). Let us express $f(x, y)$ as the sum of an *averaged* part $\bar f(y) = \int_0^1 f(\xi, y)\,d\xi$ and a *variable* part $\tilde f(x, y) = f(x, y) - \bar f(y)$. First we introduce a coordinate on the torus

$$y_1 = y_1(x, y, \epsilon) = y + \epsilon h(x, y) \tag{5}$$

to ensure that the variable part dy_1/dx will be of order ϵ^2. Clearly,

$$\frac{dy_1}{dt} = \epsilon \bar f + \epsilon\left(\tilde f + \frac{\partial h}{\partial x}\right) + \epsilon^2 f \frac{\partial h}{\partial y},$$

so that in (5) it is necessary to set $h(x, y) = -\int_0^x \tilde f(\xi, y)\,d\xi$. Then

$$\frac{dy_1}{dx} = \epsilon \bar f(y_1) + F_1(x, y_1, \epsilon), \tag{6}$$

where $F_1 = \epsilon[\bar f(y_1) - \bar f(y)] + \epsilon^2 f\,\partial h/\partial y$. Since $\bar f(y_1) - \bar f(y) = -\epsilon \bar f' \int_0^x \tilde f'(\xi, y)\,d\xi$, F_1 in (6) contains the factor ϵ^2.

Let us now transform the coordinate y_1 in such a way that the term of order ϵ on the right side becomes constant. Since, by the condition of Theorem 1, $\bar f > 0$, one may set

$$y_2(y_1) = \int_0^{y_1} \frac{c\,d\xi}{\bar f(\xi)}, \quad \frac{1}{c} = \int_0^1 \frac{dy}{\bar f(y)}.$$

The constant c is defined by the condition $y_2(1) - y_2(0) = 1$. Now the equation (3) becomes

$$\frac{dy_2}{dt} = \epsilon c + F_2(x, y_2, \epsilon),$$

where $F_2 = y_2' F_1(x, y_1(y_2), \epsilon)$ contains the factor ϵ^2, and

$$c = \left[\int_0^1 \frac{dy}{\int_0^1 f(x, y)\,dx}\right]^{-1}.$$

4. Coming to the proof of Theorem 2, we shall outline the first step in constructing the variable $z(x, y) = y + k(x, y)$. It is evident that

$$\frac{dz}{dx} = \lambda + \left(\varepsilon^2 F + \frac{\partial k}{\partial x} + \lambda \frac{\partial k}{\partial y}\right) + \varepsilon^2 F\,\frac{\partial k}{\partial y} \qquad (\lambda = \varepsilon c), \tag{7}$$

so in the first approximation we determine k from the condition

$$\epsilon^2 F + \frac{\partial k}{\partial x} + \lambda \frac{\partial k}{\partial y} = \epsilon^2 F_{00}.$$

The Fourier coefficients of the function $k(x, y) = \sum_{m^2+n^2 \neq 0} k_{mn} e^{2\pi i (mx+ny)}$ are expressed in terms of the Fourier coefficients of the function $F(x, y)$:

$$2\pi k_{mn} = \frac{i\epsilon^2 F_{mn}}{m + \lambda n}.$$

Let us fix ϵ, and assume that the *winding number* [3] of equation (4) is $\lambda \in \Lambda_\theta$. Then $k(x, y)$ and its derivatives are of order $\epsilon^2 |\frac{F}{\lambda}|$, i.e., of order ϵ. The equation (7) gives

$$\frac{dz}{dx} = \lambda + \epsilon^2 F \frac{\partial k}{\partial y} + \epsilon^2 F_{00}.$$

Since $\epsilon^2 F \, \partial k/\partial y$ is of order ϵ^3, and the winding number of equation (7) is λ, $\epsilon^2 F_{00}$ has order ϵ^3. Thus, in the new coordinates, the "perturbation" $\epsilon^2 (F_{00} + F \, \partial k/\partial y)$ has the order $\epsilon^3 = (\epsilon^2)^{3/2}$, which ensures rapid convergence of the successive approximations.

5. The constructions of §4 enable us to prove the following:

Fundamental lemma. *Assume that the differential equation*

$$\frac{dy}{dx} = \lambda + F(x, y) \tag{8}$$

on the torus T has the winding number $\lambda \in \Lambda_\theta$, and that the function $F(x, y)$ for $|\mathrm{Im}\, x, y| < \rho$ is analytic and $|F| < M$. We shall assume that for some $\delta > 0$ the following inequalities are satisfied:

$$\delta < 0.1\rho; \quad \delta < 2^{-7}\theta; \quad M < \delta^4 |\lambda|. \tag{9}$$

Then there exists a change of variable $y = y(x, z)$ analytic for $|\mathrm{Im}\, x, z| < \rho_1 = \rho - 3\delta$, such that

$$\frac{dz}{dx} = \lambda + F_{\mathrm{new}}(x, z).$$

where the function F_{new} for $|\mathrm{Im}\, x, z| < \rho_1$ is analytic, and $F_{\mathrm{new}} < M_1 = \frac{M^2}{\delta^4 |\lambda|}$.

Under the conditions of Theorem 2 we have that $c_0 > 0$, $\epsilon_0 > 0$, $\rho > 0$, $N > 0$, and for $|\epsilon| < \epsilon_0$, $|\mathrm{Im}\, x, y| < \rho$, the function $F(x, y, \epsilon)$ is analytic, $|F| < N$, and the winding number of equation (4) is $\lambda(\epsilon) > c_0 \epsilon$ $(\epsilon > 0)$.

Let us fix θ, $0 < \theta < 0.25$; in an arbitrary neighborhood of zero there exists a point $\lambda \in \Lambda_\theta$. Let $\lambda = \delta^{20}$ be such a point, and

$$\delta < c_0, \quad \delta^{19} < \epsilon_0, \quad \delta < 0.1\rho, \quad \delta < 2^{-7}\theta, \quad \delta < N^{-1/2}. \tag{10}$$

Then, $\epsilon = \epsilon(\lambda) < \lambda c_0^{-1} < \delta^{19}$, and we have the conditions of the fundamental lemma, with $M = \delta^{36} > N\epsilon^2$. From conditions (10) it follows that the inequalities (9) are satisfied, and in agreement with the fundamental lemma we have

$$M_1 = \delta^{48} = M^{4/3}.$$

Using the fundamental lemma s times, we obtain $M_s = M^{4s/3}$, so that the proof of convergence of the approximations for $s \to \infty$ does not present any difficulties.

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SOME REMARKS ON FLOWS OF LINE ELEMENTS AND FRAMES*

V. I. ARNOL'D

It is well known that many problems in mechanics can be reduced to geodesic flows (see [1–4]). In this note we define more general dynamic systems in the spaces of line elements and frames of a Riemannian manifold—the isotropic flows. These include flows connected with curves of constant geodesic curvature and with the motion of a charged particle on a smooth surface in the presence of a magnetic field.

1. Let M be an n-dimensional Riemannian manifold. By a k-frame or k-hedron on M we mean a pair $\omega = (x, \xi^k)$, where $x \in M$ is the carrier, and ξ^k is an ordered set $(\xi_1, \cdots, \xi_k)$ of pairwise orthogonal unit tangent vectors to M at the point x. The frames with a common carrier form a homogeneous space Ξ_k, while the set of all the frames on M form a space Ω_k. The volume element in the space Ω_k is given by the formula $d\Omega = dMd\Xi$, where dM is the volume element in M, and $d\Xi$ denotes the invariant measure in the space Ξ_k.

By a flow of k-frames we mean a one-parameter group S^t of transformations of the space Ω_k: $\omega \to S^t\omega$. By the trajectories of the flow we mean the lines Γ on M formed by the carriers $x(t)$ of the elements $S^t\omega$. By a tangent flow we mean a flow of n-frames in which the frame $S^t\omega$ is the frame accompanying the trajectory $x(t)$. Let v denote the velocity of $x(t)$ on Γ, and let $k_1, \cdots, k_{n-1}$ be the curvature of Γ. Clearly, a tangent flow is determined by the functions $v(\omega)$, $k_1(\omega), \cdots, k_{n-1}(\omega)$ on Ω_n.

Definition. By an isotropic flow we mean a tangent flow in which the velocity v is constant, and in which the curvature $k_1(x), \cdots, k_{n-1}(x)$ depends only on the carrier x but does not depend on the direction of the vectors $\xi_1, \cdots, \xi_n$ in the frame ω.

In particular, a geodesic flow is isotropic: $v \equiv 1$, $k_i \equiv 0$.

2. An isotropic flow is a dynamical system with an invariant integral. As is known, the measure $d\Omega$ is an invariant measure for a geodesic flow.

Theorem 1. *The transformations S^t of any isotropic flow preserve the measure $d\Omega$.*

The proof is based on the fact that the infinitesimal transformation S^{dt} consists of the infinitesimal transformation of the geodesic flow and an infinitesimal rotation.

For $n = 2$ an isotropic flow is determined by the geodesic curvature $k(x)$ of the trajectory through the point x of the surface M. Such a flow is isomorphic to a flow of fixed energy in a dynamical system with Lagrangian $L_2 + L_1$, containing terms that are linear and quadratic in the velocity. In this case Theorem 1 follows from Liouville's theorem. Excluding cyclical coordinates it is easy to investigate motion along a curve of constant curvature on a surface of rotation. These curves have been studied by Minding [5] and Darboux [6].

3. The case of cyclical flows, in which $k_1, \cdots, k_{n-1}$ are constant, on manifolds of constant curvature K, is also easy. One considers the Lobačevskiĭ plane ($n = 2$, $K = -1$). The usual methods permit one to pass to the case of an arbitrary surface of constant negative curvature [2].

Cycles of curvature k in the Lobačevskiĭ plane are of three kinds: proper cycles ($k^2 > 1$), oricy-

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cles $(k^2 = 1)$ and hypercycles $(k^2 < 1)$. In the Poincaré model these are, respectively, disjoint circles, tangent circles, and circles meeting at an angle α; $\cos\alpha = k$.

Theorem 2. *Every hypercyclical* $(k^2 + K < 0)$ *flow on a surface of constant negative curvature* K *is isomorphic to a geodesic flow.*

The proof is based on the fact that a hypercycle of curvature k is an equidistant curve Γ_r, that is, has the constant distance r from some geodesic Γ (where $k = -\sqrt{-K}\,\mathrm{th}\sqrt{-K}\,r$). With the equidistant curves Γ_r on an arbitrary surface one can connect those tangent flows that are isomorphic to geodesic flows. On surfaces of constant curvature the equidistant flow Γ_r is cyclical.

4. Similar considerations in Lobačevskiĭ space lead to the following assertion:

Theorem 3. *Every cyclic flow on an n-dimensional manifold of constant* negative curvature -1 belongs to one of the following three types:*

1. *The flow is isomorphic to a generalized geodesic flow* $(k_1 = 0)$.
2. *The flow is isomorphic to the generalized oricyclic flow* $(k_1 = 1,\ k_2 = 0)$.
3. *The flow is isomorphic to one in which the carrier* $S^t\omega$ *does not move.*

The ordinary geodesic flow (in which $v = 1,\ k_1 = k_2 = \cdots = k_{n-1} = 0$) is of the first type; all these flows are quite similar (see Theorem 5). Among the flows of the second type a similar role is played by the ordinary oricyclic flow $(v = k_1 = 1;\ k_2 = \cdots = k_{n-1} = 0)$. Flows of the third type are not ergodic; the ergodic components are tori of dimension r; in general $r = [n/2]$; on each torus the flow has a discrete spectrum with r generators.

The problem of deciding the type of a given flow is settled in the following manner:

Theorem 4. *In a* $2r$*-dimensional space of curvature* -1 *a cyclical flow is of the first, second or third type according to whether* k_1^2 *is smaller than, equal to or larger than*

$$\varkappa^2 = 1 + \frac{k_2^2}{k_3^2} + \frac{k_2^2 k_4^2}{k_3^2 k_5^2} + \ldots + \frac{k_2^2 k_4^2 \ldots k_{2r-2}^2}{k_3^2 k_5^2 \ldots k_{2r-1}^2}\,.$$

In a $(2r+1)$*-dimensional space all flows with* $k_{2r} \neq 0$ *are of type* 1, *while if* $k_{2r} = 0$ *the flow is of type* 1, 2 *or* 3 *depending on whether* k_1^2 *is smaller than, equal to or larger than* κ^2.

5. The methods used in a recent work of Ya. G. Sinaĭ [4, 7] devoted to ordinary geodesic flows can also be applied to generalized geodesic flows.

Theorem 5. *A flow* S^t *of type* 1 *on a compact manifold of constant negative curvature is a K-system* [4, 7]. *The orispherical flow* [4, 7] *is conjugate to* S^t.

The entropy [8] of the flow S^t can be calculated. Let $h(k_1, \cdots, k_{n-1})$ be the entropy for a unit of time of the cyclic flow with velocity 1 and curvature $k_1, \cdots, k_{n-1}$ on a manifold of constant curvature -1. Then

$$h(k_1, \ldots, k_{n-1}) = h(0)\,v,$$

where $h(0) = h(0, \cdots, 0)$ is the entropy of a geodesic flow, calculated in [4], and $v(k_1, k_2, \cdots, k_{n-1})$ is the velocity of movement in the generalized geodesic flow isomorphic to S^t.

In particular, for $n = 2$ we have

$$h(k) = h(0)\sin\alpha = h(0)\sqrt{1-k^2},$$

while for $n = 3$ the number $v^2 = x$ is the positive root of the equation

$$x^2 + (k_1^2 + k_2^2 - 1)\,x + k_2^2 = 0.$$

* In each two-dimensional direction.

In the case of a surface $(n = 2)$, $h \to 0$ as $k^2 \to 1$. This leads one to think that the entropy of the oricyclic flow $h(1)$ is equal to zero. This has been proven by B. M. Gurevič [9]. Probably the entropy of any flow of type 2 is equal to zero.

6. The spectra of the geodesic and oricyclic flows were first found in [3, 10] by using an algebraic construction due to I. M. Gel'fand and S. V. Fomin; this construction depends on a group G and on certain of its subgroups: compact K, discrete D and one-parameter g^t.

Theorem 6. *If G is the group of motions of the n-dimensional Lobačevskiĭ space and K is the rotation group of the $(n - k)$-dimensional Euclidean space, then the corresponding dynamical system is isomorphic to one of the cyclic flows of k-frames on a manifold of constant negative curvature; conversely, all such flows are obtained in this manner.*

The proof is based on the connection between curves of constant curvature and one-parameter subgroups of the group of motions; conjugate groups correspond to isomorphic systems.

7. The group of rotation numbers [11, 12] provides an interesting topological characterization of dynamical systems. I. M. Gel'fand and I. I. Pyateckiĭ-Šapiro calculated the rotation numbers for the geodesic and oricyclic flows on surfaces of constant negative curvature [13].

Theorem 7. *If an isotropic flow on a Riemannian manifold, that is not the two-dimensional torus or the Klein bottle, is ergodic, then all rotation numbers are zero and the flow has no nonconstant continuous eigenfunctions.*

The proof is based on the fact that the trajectories of an ergodic isotropic flow pass equally often through each point in each direction.

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УСПЕХИ МАТЕМАТИЧЕСКИХ НАУК

НАУЧНЫЕ СООБЩЕНИЯ И ЗАДАЧИ*

ПРИЗНАК НОМОГРАФИРУЕМОСТИ С ПОМОЩЬЮ ПРЯМОЛИНЕЙНОГО АБАКА ДЕКАРТА

В. И. Арнольд

Прямолинейным абаком Декарта называют [1] три однопараметрических семейства прямых на плоскости (u^1, u^2)

$$u^1 = f_1(x^1), \tag{1}$$
$$u^2 = f_2(x^2), \tag{2}$$
$$u^2 = h_1(z)\,u^1 + h_2(z) \equiv T_z u^1, \tag{3}$$

где x^1, x^2, z — параметры семейств (1) — (3).

Рассмотрим какие-нибудь x^1, x^2, и пусть z таково, что прямая (3) проходит через точку $u^1 = f_1(x^1)$, $u^2 = f_2(x^2)$. Такое z есть функция x^1 и x^2. Функции $z(x^1, x^2)$, которые можно получить таким способом, называются *номографируемыми на прямолинейном декартовом абаке* [1]).

Рассматривая формулы (1), (2) как формулы преобразования плоскости (x^1, x^2) в плоскость (u^1, u^2), можем сказать, что система линий уровня $z = c$ номографируемой функции $z(x^1, x^2)$ может быть превращена в семейство прямых (3) изменением (1), (2) масштаба по осям x^1, x^2. Мы получим необходимое для такой возможности геометрическое свойство системы линий уровня функции $z(x^1, x^2)$.

Определение. Множество точек $X_0, X_1, \ldots, X_j, \ldots, X_n$ плоскости (x^1, x^2) называется *молнией*, если отрезок X_{j-1}, X_j $(j = 1, \ldots, n)$ параллелен оси x^i ($i = 2$, если j четно, $i = 1$, если j нечетно). Молния *замкнута*, если $X_n = X_0$.

Теорема. *Пусть функция $z(x^1, x^2)$ номографируема на прямолинейном абаке Декарта и* I, II, III, IV — *ее четыре линии уровня* ($X \in$ I, *если* $z(X) = z_1$ *и т. д.*). *Если точки молнии*

$$X_0, X_1, \ldots, X_{20} \tag{4}$$

[1]) Легко видеть, что это те самые функции, которые допускают номограмму из выравненных точек формы Коши (т. е. с двумя прямыми шкалами). Они называются также номографируемыми порядка 4 жанра 1 (см. [1]).

* Editor's note: V.I. Arnol'd: A test for nomographic representability using Decartes' rectilinear abacus. Published in Uspekhi Mat. Nauk 16:4 (1961)

принадлежат этим линиям уровня, а именно

$$\left.\begin{array}{c} X_1, X_6, X_9, X_{12}, X_{15}, X_{18} \in \mathrm{I}; \quad X_2, X_5, X_8, X_{11}, X_{16}, X_{19} \in \mathrm{II}; \\ X_4, X_7, X_{14}, X_{17} \in \mathrm{III}; \quad X_0, X_3, X_{10}, X_{13}, X_{20} \in \mathrm{IV}, \end{array}\right\} \qquad (5)$$

то молния (4) *замкнута.*

Доказательство основано на известном алгебраическом факте.

Лемма. *Пусть A, B, C, D — невырожденные линейные преобразования прямой. Тогда*

$$(ABA^{-1}B^{-1})(CDC^{-1}D^{-1})(ABA^{-1}B^{-1})^{-1}(CDC^{-1}D^{-1})^{-1} = E. \qquad (6)$$

Доказательство. Каждое преобразование имеет вид $x \to ax + b$, где a — коэффициент растяжения. Очевидно, коэффициент растяжения произведения преобразований равен произведению их коэффициентов растяжения. Поэтому коэффициент растяжения каждой скобки в (6) равен 1, т. е. каждая скобка выражает сдвиг. Любые два сдвига прямой перестановочны. Это и записано в (6). Полагая $D = A$, получаем

Следствие.

$$ABA^{-1}B^{-1}CAC^{-1}A^{-1}BAB^{-1}CA^{-1}C^{-1} = E. \qquad (7)$$

Доказательство теоремы. Пусть линии уровня I, II, III, IV при преобразовании (1), (2) перешли в прямые I′ $(u^2 = T_1 u^1 \equiv h_1(z_1) u^1 + h_2(z_1))$, II′, III′, IV′. Тогда молния (4) перешла в молнию на плоскости (u^1, u^2)

$$U_0, U_1, \ldots, U_{20}, \qquad (4')$$

а условия (5) запишутся в виде

$$u_1^2 = T_1 u_1^1; \ \ldots; \ u_0^2 = T_4 u_0^1; \ \ldots; \ u_{20}^2 = T_4 u_{20}^1, \qquad (5')$$

где u_j^1, u_j^2 — координаты точки U_j.

Так как по определению молнии $u_{2k}^2 = u_{2k+1}^2$; $u_{2k}^1 = u_{2k-1}^1$, находим

$$T_4 u_0^1 = T_1 u_1^1, \quad u_1^1 = T_1^{-1} T_4 u_0^1,$$

то, продолжая, мы выразим через u_0^1 последовательно

$$u_2^1 = u_1^1; \ u_3^1 = T_4^{-1} T_2 T_1^{-1} T_4 u_0^1; \ \ldots; \ u_{20}^1 = \prod_k T_{jk}^{\alpha_k} u_0^1 \equiv$$
$$\equiv T_2^{-1} T_1 T_3^{-1} T_1 T_1^{-1} T_2 T_1^{-1} T_3 T_4^{-1} T_1 T_2^{-1} T_1 T_1^{-1} T_4 \times$$
$$\times T_1^{-1} T_2 T_3^{-1} T_1 T_2^{-1} T_1 T_1^{-1} T_3 T_4^{-1} T_1 T_1^{-1} T_2 T_1^{-1} T_4 u_0^1. \qquad (8)$$

Обозначим

$$T_2^{-1} T_1 = A, \quad T_3^{-1} T_1 = B, \quad T_4^{-1} T_1 = C.$$

Тогда тождество (7) выражает, что $\prod_k T_{jk}^{\alpha_k}$ в (8) есть E, т. е. $u_{20}^1 = u_0^1$, следовательно, молния (4′) замкнута. Очевидно, тогда и молния (4) должна быть замкнутой.

Замечание. Если функция $z(x^1, x^2)$ — номографируемая порядка 3, т. е. $h_2(z)$ в (3) есть нуль, то прямые (3) образуют пучок и потому все преобразования T_z перестановочны. Пусть опять I, II, III — три линии уровня $z(x^1, x^2)$, тогда *молния*

$$X_0, X_1, \ldots, X_6$$

замкнута, если

$$X_0,\ X_3,\ X_6 \in \mathrm{II};\quad X_1,\ X_4 \in \mathrm{III},\quad X_2,\ X_5 \in \mathrm{I}$$

(известное «условие шестиугольника», см., например, [2], рис. 12).

Для доказательства составим, аналогично (8), произведение

$$u_6^1 = T_1^{-1} T_3 T_2^{-1} T_1 T_3^{-1} T_2 u_0^1.$$

Так как все T перестановочны, то $u_6^1 = u_0^1$, что и требовалось.

Признаки номографируемости шестого порядка недавно получены Н. Д. Айзенштат, И. А. Вайнштейном и М. А. Крейнесом [3], а для номографируемости пятого порядка — З. М. Кишкиной и М. А. Крейнесом, [4]. Они носят характер *неравенств*; неизвестно, существуют ли равенства, подобные установленным выше, необходимые для номографируемости пятого и шестого порядков.

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СИБИРСКИЙ МАТЕМАТИЧЕСКИЙ ЖУРНАЛ

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ЗАМЕЧАНИЯ О ЧИСЛАХ ВРАЩЕНИЯ*

Дискретный спектр эргодических динамических систем, определенных дифференциальными уравнениями, связан в известных примерах, с наличием нетривиальной группы чисел вращения. Впервые числа вращения ввел Пуанкаре ([1]), изучавший интегральные кривые на поверхности тора. Общее определение дал недавно Шварцман ([2]).

Мы покажем, что числа вращения эргодических геодезических потоков всегда* равны нулю. Для геодезических потоков на поверхности постоянной отрицательтельной кривизны равенство нулю чисел вращения доказано И. М. Гельфандом и И. И. Пятецким-Шапиро, которые использовали аппарат теории представлений групп ([3]).

В действительности равенство нулю всех чисел вращения связано просто с тем, что почти каждая геодезическая эргодического потока идет по многообразию в одну сторону так же часто, как и в любую другую, в том числе прямо противоположную. Это соображение позволяет вычислить также числа вращения некоторых более общих потоков линейных элементов и реперов (см. ([6])).

§ 1. Определения и обозначения

Пусть на компактном римановом многообразии Ω дано гладкое** векторное поле A. Дифференциальные уравнения

$$\frac{d\omega}{dt} = A(\omega) \tag{1}$$

определяют однопараметрическую группу преобразований

$$\omega_0 \longrightarrow S^t\omega_0 = \omega(\omega_0, t).$$

Если эти преобразования имеют инвариантную меру μ, $\mu(\Omega) = 1$, с положительной плотностью, то говорят, что созокупность (Ω, S^t, μ) есть определенная дифференциальными уразнениями (1) динамическая система. Эргодической называется такая динамическая система, у которой каждое инвариантное множество в Ω имеет меру 0 или 1.

Пусть (Ω, S^t, μ) — определенная уравнениями (1) эргодическая динамическая система. Число λ принадлежит аддитивной группе Λ чисел вращения, если на Ω существует гладкая, однозначная с точностью до целого

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* Исключение может составить эргодический геодезический поток на двумерном торе, если он существует.

** Здесь и далее слово «гладкий» означает непрерывно дифференцируемый.

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* Editor's note: V.I. Arnol'd: Remarks on winding numbers. Published in Sibirsk. Mat. Ž. 2:6 (1961), 807–813

слагаемого функция $F = F(\omega)$, такая что

$$\int_{\Omega} (A,\ \operatorname{grad} F)\, d\mu = \lambda. \tag{2}$$

Можно показать (см. [2], [3]), что при почти всех $\omega \in \Omega$ имеем

$$\lim_{|t| \to \infty} \frac{F(S^t\omega) - F(\omega)}{t} = \lambda, \tag{3}$$

так что числа вращения можно определить этим равенством.

Если Ω есть пространство линейных элементов ω компактного риманова пространства M, то определяемая уравнениями (1) система называется потоком линейных элементов. В этом случае $\omega \in \Omega$ есть пара (x, ξ), где носитель $x \in M$, а ξ — единичный вектор, касательный к M в x. Носители $x(t)$ элементов $S^t\omega$ образуют линию — траекторию потока на M. Мы будем рассматривать только касательные потоки, у которых касательная к траектории $x(t)$ в каждой точке имеет направление $\xi(t)$ элемента $S^t\omega$. К касательным потокам относятся геодезический поток, у которого траектории суть геодезические линии M, а также изотропные потоки и потоки эквидистант, определенные в [6].

Инвариантная мера касательного потока, как мера на пространстве линейных элементов, задается формулой

$$d\mu = d\sigma\, ds(x),$$

где $d\sigma$ — элемент объема на M, а элемент $ds(x)$ определяет некоторую «условную» меру на сфере $S(x)$ линейных элементов с общим носителем x. В случае геодезического потока, а также изотропных и эквидистантных потоков, мера $ds(x)$ индуцирована метрикой сферы $S(x)$.

Определение. Мера ds на сфере S называется симметричной, если симметричные относительно центра сферы множества имеют одинаковую меру. Мера $d\mu$ на Ω симметрична, если все меры $ds(x)$ симметричны. Касательный поток симметричен, если его инвариантная мера симметрична и величины скорости движения по симметричным направлениям одинаковы.

Очевидно, все геодезические и изотропные потоки симметричны.

§ 2. Числа вращения симметричных потоков

Теорема 1. *Пусть M — компактное риманово многообразие, отличное от двумерного тора, и (Ω, S^t, μ) — эргодический симметричный поток в пространстве Ω линейных элементов M. Тогда все числа вращения потока равны нулю.*

Доказательство основано на следующем известном топологическом факте.

Лемма 1. *Пусть M — отличное от двумерного тора компактное риманово многообразие, Ω — пространство его линейных элементов. Тогда слабые одномерные гомологии переносятся на Ω с M. Это значит, что если $F(\omega)$ — гладкая функция, однозначная на Ω с точностью до целого слагаемого, то на Ω существует такая гладкая функция $G(\omega) = g(x)$, зависящая только от носителя x элемента ω, что $F(\omega) - G(\omega)$ однозначна на всем Ω.*

Для доказательства леммы заметим, что Ω есть косое произведение n-мерного многообразия M на $(n-1)$-мерную сферу, и если $n>2$, то умножение на односвязную сферу S^{n-1} не влияет на одномерные гомологии. Если же $n=2$ и M не тор, то, хотя умножение на окружность и дает новый одномерный цикл, он слабо гомологичен нулю.

При доказательстве теоремы 1 используется еще связь между некоторыми скалярными произведениями векторов касательных пространств к M и к Ω. Пусть $x(t)$ — траектория на M касательного потока, заданного уравнениями (1) в Ω, проходящая при $t=0$ через носитель x элемента $\omega=(x, \xi)$ по направлению ξ. Обозначим через $\dot{x}=\dot{x}(\omega)$ вектор $\left.\frac{dx(t)}{dt}\right|_{t=0}$ касательного к M в x пространства.

Лемма 2. *Пусть $f(x)$—гладкая функция на M, $F(\omega)=F(x, \xi)$—функция на Ω, определенная равенством $F(x, \xi)=f(x)$. Тогда*

$$(\dot{x}, \operatorname{grad} f)=(A, \operatorname{grad} F).$$

Для доказательства этой леммы достаточно выбрать в окрестности $x \in M$, $\xi \in S(x)$ и $\omega \in \Omega$ координаты. Выкладка упрощается при применении нормальных координат; мы ее опустим.

Доказательство (теоремы 1). 1. Если M не двумерный тор, то на основании леммы 1 любое число вращения заданного уравнениями (1) эргодического потока в пространстве Ω можно представить в виде

$$\lambda=\int_{\Omega}(A, \operatorname{grad} F)\, d\mu, \tag{4}$$

где $F(\omega)=f(x)$ зависит лишь от $x \in M$. Действительно, если функция $F(\omega)$ однозначна на Ω, то, на основании (3), ей отвечает $\lambda=0$. Поэтому-заменяя F в (2) функцией $f(x)$ с помощью леммы 1, получим нужное представление (4).

2. Предполагая, что $F(\omega)=f(x)$ и применяя лемму 2, находим:

$$\lambda=\int_{\Omega}(\dot{x}(\omega), \operatorname{grad} f(x))\, d\mu.$$

Интегрируя сначала по сфере $S(x)$, получим:

$$\lambda=\int_{M}\int_{S(x)}(\operatorname{grad} f_1, \dot{x})\, ds(x)\, d\sigma=\int_{M}\left(\operatorname{grad} f_1, \int_{S(x)} \dot{x}\, ds(x)\right) d\sigma.$$

Так как для симметричного потока, очевидно, $\int_{S(x)} \dot{x}\, ds(x)=0$, то теорема 1 доказана.

Следствие. *Если собственная функция эргодического симметричного потока на отличном от двумерного тора многообразии непрерывна, то она постоянна.*

Доказательство. Пусть $\varphi_\lambda(\omega)$ — такая функция, т. е. $\varphi_\lambda(S^t\omega)= =e^{i\lambda t}\varphi_\lambda(\omega)$. По теореме Шварцмана [2] о собственных функциях, λ есть число вращения. Значит $\lambda=0$, а тогда $\varphi_\lambda(\omega)$ постоянна, ввиду эргодичности потока.

Замечания. Так как изотропные и геодезические потоки симметричны, теорема 1 применима, в частности, к ним. Впрочем, в случае геодезического потока доказательство можно упростить, применяя равенство (3) к двум половинам одной траектории.

Эргодические геодезические потоки существуют. На компактных многообразиях постоянной отрицательной кривизны и на компактных поверхностях строго отрицательной кривизны геодезические потоки эргодичны. Эти потоки обладают непрерывным спектром (см. ([4,5,7])). Как мы видели, для геодезического потока из эргодичности вытекает отсутствие непрерывных собственных функций. Вероятно, измеримых собственных функций тоже не может быть.

Заметим еще, что исключенный в теореме 1 случай двумерного тора, возможно, и не реализуется: геодезический поток на двумерном торе вряд ли может быть эргодическим.

§ 3. Числа вращения потоков реперов

Пусть M — n-мерное риманово компактное пространство. Репером $\xi^k = (\xi_1, \ldots, \xi_k)$ на M называется совокупность k упорядоченных ортогональных единичных векторов касательного к M в точке x (носителе репера) пространства. Реперы ξ^k на M образуют риманово пространство реперов Ω_k. Мы будем рассматривать потоки реперов ξ^k, т. е. определенные уравнениями (1) динамические системы в этом пространстве. Особенно интересны случаи $k = 1$ (поток линейных элементов) и $k = n$ (поток n-эдров).

Первый вектор репера ξ_1 назовем направлением репера. Поток реперов назовем касательным, если направление репера является направлением определяемой им траектории $x(t)$ на M; здесь $x(t)$ — носитель репера $S^t\xi^k$. Очевидно, введенные в ([6]) изотропные потоки реперов, в том числе геодезический поток реперов, а также эквидистантные потоки, являются касательными.

Инвариантная мера потока реперов, как мера на Ω_k, имеет вид:

$$d\mu = d\sigma\, d\xi(x),$$

где $d\sigma$ — мера на M, а $d\xi(x)$ — некоторая «условная» мера на пространстве $\Xi_k(x)$ реперов ξ^k с общим носителем x. В случае геодезического потока, а также изотропных и эквидистантных потоков, мера $d\xi(x)$ индуцирована инвариантной относительно движений $\Xi_1(x)$ метрикой и потому симметрична. Вообще, симметричные меры в случае потоков реперов определим следующим образом.

Пусть $A \subseteq \Xi_1$. Обозначим через $\varphi(A)$ множество всех реперов $\xi^k = (\xi_1, \ldots, \xi_k)$, у которых $\xi_1 \in A$. Меру $d\xi$ на пространстве Ξ_k реперов ξ^k назовем симметричной, если при любом A меры множеств $\varphi(A)$ и $\varphi(-A)$ одинаковы. Мера $d\mu$ на Ω_k симметрична, если все меры $d\xi(x)$ симметричны. Касательный поток симметричен, если его инвариантная мера симметрична и величины скорости движения по противоположным направлениям одинаковы.

Очевидно, геодезические и изотропные потоки симметричны. Так же, как теорема 1, доказывается

Теорема 2. *Все числа вращения эргодического симметричного потока реперов на отличном от двумерного тора и от бутылки Клейна компактном римановом пространстве равны* 0.

Соответствующие обобщения лемм 1 и 2 очевидны. Из них, как в § 2, получаем:

$$\lambda = \int_M (\operatorname{grad} f, \int_{\Xi_k(x)} \dot{x}\, d\xi(x))\, d\sigma.$$

Далее, внутренний интеграл представим в виде:

$$\int_{\Xi_k(x)} \dot{x}\, d\xi(x) = \int_{\Xi_1(x)} \int_{\varphi(\xi)} \dot{x}\, d\eta(\xi)\, ds(x), \tag{5}$$

где $\eta(\xi)$ — условная мера на множестве $\varphi(\xi)$ реперов с первым вектором ξ. Для симметричного потока, очевидно, интеграл

$$\int_{\varphi(\xi)} \dot{x}\, d\eta(\xi)$$

есть нечетная функция ξ, поэтому весь интеграл (5) равен 0, чем и доказывается теорема 2.

Замечание 1. Требование симметричности можно варьировать; важно только, чтобы интеграл (5) обращался в нуль.

Замечание 2. Если n-мерное пространство M ориентируемо, то поток n-эдров ξ^n и Ω_n не может быть эргодичным. В этом случае естественно разбить Ω_n на пространства Ω_n^+ и Ω_n^- n-эдров разной ориентации; очевидно, теорема 2 приложима к потокам в каждом из них в отдельности.

Замечание 3. Пространство биэдров бутылки Клейна совпадает с пространством линейных элементов тора, поэтому при обобщении леммы 1 возникает новое исключение. При $n > 2$ первое число Бетти Ξ_k всегда равно 0.

§ 4. Числа вращения подобных потоков

Динамические системы (Ω_1, S_1^t, μ_1) и (Ω_2, S_2^t, μ_2), заданные дифференциальными уравнениями, подобны, если существует гомеоморфное отображение T, переводящее Ω_1 в Ω_2, S_1^t в S_2^t, μ_1 в μ_2 так, что

$$T\Omega_1 = \Omega_2; \quad S_2^t = TS_1^t; \quad \mu_1(E) = \mu_2(TE).$$

Теорема 3. *Если системы* (Ω_1, S_1^t, μ_1) *и* (Ω_2, S_2^t, μ_2) *эргодичны и подобны, то группы их чисел вращения совпадают.*

Для доказательства достаточно воспользоваться равенством (3), которое может служить определением чисел вращения. Каждой непрерывной, однозначной с точностью до целого слагаемого, функции $F_1(\omega_1)$ на Ω_1 сопоставим функцию $F_2(\omega_2) = F_1(T^{-1}\omega_1)$ на Ω_2, обладающую, очевидно, теми же свойствами. Так как таким путем можно получить любую функцию этого рода на Ω_2, то из равенства

$$F_1(S_1^t\omega) - F_1(\omega) = F_2(S_2^t T\omega) - F_2(T\omega),$$

в силу (3), вытекает, что каждое число вращения одной системы является числом вращения другой.

Потоки эквидистант (см. ([6])) подобны геодезическому. Поэтому теорема 3 позволяет перенести результаты §§ 2 и 3 на потоки эквидистант отличного от тора и бутылки Клейна многообразия.

§ 5. Числа вращения с точки зрения метрической теории динамических систем

Пусть динамическая система (Ω, S^t, μ) задана уравнениями (1) на компактном римановом многообразии Ω. В пространстве $L_2(\Omega)$ интегрируемых с квадратом по мере μ функций на Ω определим операторы сдвига формулами $U^t f(\omega) = f(S^t\omega)$. Пусть D—инфинитезимальный оператор системы, т. е.

$$D = \lim_{t\to 0} \frac{U^t - E}{2\pi i t}.$$

Теорема 4. *Чтобы λ было числом вращения эргодической системы* (Ω, S^t, μ), *необходимо и достаточно существование такой гладкой функции $\varphi(\omega)$, всюду равной 1 по модулю, что $\lambda = (D\varphi, \varphi)$.*

Доказательство. Если $\varphi(\omega)$ — гладкая функция, то

$$D_\varphi = \frac{1}{2\pi i} \frac{d}{dt} \varphi(S^t\omega)\Big|_{t=0}.$$

Если еще $|\varphi| = 1$, то $\overline{\varphi} = \varphi^{-1}$. Тогда

$$(D\varphi, \varphi) = \frac{1}{2\pi i} \int_\Omega \frac{d\varphi}{dt} \overline{\varphi}\, d\mu = \frac{1}{2\pi i} \int_\Omega \frac{d \ln \varphi}{dt} d\mu. \tag{6}$$

Ввиду однозначности $\varphi(\omega)$ функция $\Phi(\omega) = \dfrac{\ln\varphi}{2\pi i}$ гладкая, определенная с точностью до целого слагаемого, поэтому, $\dfrac{d}{dt}\Phi(S^t\omega)\Big|_{t=0}$ — однозначная гладкая функция на Ω. Очевидно, она равна $(A, \operatorname{grad}\Phi)$, где A — правая часть уравнения движения (1). Таким образом, из (6) следует, что

$$(D\varphi, \varphi) = \int_\Omega \frac{d\Phi}{dt} d\mu = \int_\Omega (A, \operatorname{grad}\Phi)\, d\mu = \lambda.$$

Наоборот, если $\lambda = \int_\Omega (A, \operatorname{grad} F)\, d\mu$, то, положив $\varphi = e^{2\pi i F}$, получим, очевидно, $(D\varphi, \varphi) = \lambda$.

Замечание 1. Возможно, что множество значений $(D\varphi, \varphi)$ на равных по модулю 1 гладких φ совпадает со множеством всех значений $(D\varphi, \varphi)$ на функциях, равных по модулю 1, для которых интеграл $\int_\Omega D\varphi\cdot\overline{\varphi}\, d\mu$ имеет смысл. Если верно хотя бы, что для любой равной 1 по модулю функции $\varphi \in L_2(\Omega)$, для которой $D\varphi \in L_2(\Omega)$, существует непрерывная функция φ_ε с модулем 1, такая что

$$|(D\varphi, \varphi) - (D\varphi_\varepsilon, \varphi_\varepsilon)| < \varepsilon,$$

то из теоремы 1 вытекает, например, что любой эргодический геодезический поток на отличном от двумерного тора компактном римановом многообразии имеет чисто непрерывный спектр.

Замечание 2. Неизвестно, является ли группа Λ чисел вращения, т. е. значений $(D\varphi, \varphi)$ на гладких функциях, равных по модулю 1, метрическим инвариантом системы. Заменив здесь требование гладкости метрическим требованием (например, $D\varphi \in L_2$ или $D\varphi \in L_1$), при условии непрерывности $\frac{d}{dt}\varphi(S^t\omega)$, мы получим метрический аналог чисел вращения. Неизвестно, является ли какой-нибудь из этих инвариантов спектральным. На двумерном торе существуют эргодические аналитические динамические системы, для которых уже множество чисел вращения шире множества собственных значений ([8]).

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ON THE BEHAVIOR OF AN ADIABATIC INVARIANT UNDER SLOW PERIODIC VARIATION OF THE HAMILTONIAN*

V. I. ARNOL'D

1. Let a dynamic system depend on the slowly varying parameter $\lambda = \epsilon t$; then the Hamiltonian $H(p, q; \lambda)$ contains the time t and is not conserved. A function $J(p, q; \lambda)$ is called an adiabatic invariant of the system if for small ϵ the quantity $J(t) = J[p(t), q(t); \lambda(t)]$ changes slightly during the time $t \sim 1/\epsilon$ (changes in λ, H are finite here).

Let us consider the phase plane p, q for a fixed value of the parameter λ. The energy level $H(p, q; \lambda) = H(p_0, q_0; \lambda)$ passing through the point p_0, q_0 bounds a certain domain. Let us denote the magnitude of the area of this domain by $2\pi I(p_0, q_0; \lambda)$. It can be shown that I is an adiabatic invariant [1, 2].

It does not generally follow from the adiabatic invariance of a quantity that it varies slightly in unbounded time for a fixed small ϵ. This is associated with the possibility of accumulating small changes in the adiabatic invariant. For example, let us consider the linear vibrational system

$$\ddot{x} = -\omega^2 x\,(1 + \alpha \cos \epsilon t).$$

As is known, for certain ϵ (namely, $\epsilon \approx 2\omega/k$; $k = 1, 2 \cdots$) parametric resonance is possible and $I(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Here, the rate of change of the system parameters ϵ can evidently be as small as desired.

It appears, however, that in a nonlinear system with a slowly varying, periodic analytic Hamiltonian $H(p, q; \lambda)$, the adiabatic invariant is conserved perpetually: for any $\eta > 0$ there is found an $\epsilon_0(\eta) > 0$ such that there results from $|\epsilon| < \epsilon_0$

$$|I(t) - I(0)| < \eta$$

for all $-\infty < t < \infty$.

The linear system occupies an exceptional position because the frequency of its vibration is independent of the amplitude. In a nonlinear system the frequency changes as the amplitude increases and the vibrations do not succeed in increasing as the resonance condition $\epsilon \approx 2\omega/k$ is violated.

The proof of the perpetual adiabatic invariance of an operation is projected in the following paragraphs. It can be shown, by an analogous method, that the adiabatic invariant I_y of an autonomous, vibrational system with two degrees of freedom and the analytic Hamiltonian

$$H = \frac{\dot{x}^2 + \dot{y}^2 + U(\epsilon x, y)}{2}$$

is perpetually conserved. It is only necessary that the frequency ratio ω_x/ω_y should depend in a first approximation on the vibration amplitude y for fixed total energy $H = h$.

In particular, a field with the potential

$$U = \omega^2 y^2 \quad (\omega = 1 + \lambda^2,\ \lambda = \epsilon x)$$

is a trap, for $\epsilon \ll 1$, which is capable of perpetually detaining a particle with the initial conditions

* Editor's note: translation into English published in Soviet. Math. Dokl. 3 (1962)
Translation of V.I. Arnol'd: On the behavior of an adiabatic invariant under slow periodic variation of the Hamiltonian. Dokl. Akad. Nauk SSSR 142:4 (1962), 758–761

$x_0, y_0, \dot{x}_0, \dot{y}_0$ on the order of one. This results from the perpetual adiabatic invariance of the quantity $I_y = \frac{\dot{y}^2 + U}{2\omega}$.*

2. Let the analytic Hamiltonian H of a vibrational system with one degree of freedom depend periodically with period 2π on the slow time $\lambda = \epsilon t$. For a fixed value $(\dot{\lambda} = 0)$ of the parameter λ in the effect I – angle ω variables, the function H has the form $H_0(I; \lambda) = H_0(I; \lambda + 2\pi)$, $\dot{I} = 0$ and the torus $I = \text{const}$ in the p, q, λ space (where p, q, λ and $p, q, \lambda + 2\pi$ are cemented) is invariant. A phase point moves along a meridian of the torus ($\lambda = \text{const}$) with the frequency

$$\dot{w} = \frac{\partial H_0}{\partial I} = \omega(I, \lambda).$$

Let $\bar{\omega}(I)$ denote the *mean frequency* $\frac{1}{2\pi} \oint \omega(I, \lambda)\, d\lambda$ and let $T_{\bar{\omega}}$ denote the torus itself.

If $\epsilon \neq 0$ is small, then it can be considered, in a first approximation, that the motion proceeds over the torus $T_{\bar{\omega}}$, where the longitude varies slowly $(\dot{\lambda} = \epsilon)$ and the latitude w varies rapidly with the slowly-varying frequency $\dot{w} = \omega(I, \lambda)$. It appears that for a sufficiently small ϵ and a frequency $\bar{\omega}$ "sufficiently incommensurable" with ϵ, *there actually exists an invariant torus* $T_{\bar{\omega}}(\epsilon)$ *close to* $T_{\bar{\omega}}$. This torus is filled, conditionally, by periodic trajectories with frequencies ω and ϵ.

For fixed invariant two-dimensional tori $T_{\bar{\omega}}(\epsilon)$ divide the three-dimensional space into thin toroidal layers. Each trajectory, starting from such a layer, is entirely included within it. The thickness of the layers tends to zero along with ϵ; hence, the proof of the perpetual adiabatic invariance of I is simple after the tori $T_{\bar{\omega}}(\epsilon)$ have been found.

These tori are sought by the Newton method [3]. "Small denominators" $m\bar{\omega} + n\epsilon$ appear in the appropriate Fourier series. Certain of them are small because of the approximate commensurability of the frequencies $\bar{\omega}$ and ϵ, others because the frequency ϵ is small (degenerate). The difficulty associated with degeneration is overcome on the basis of the same considerations as in the note [4], which is a non-Hamiltonian analog of the present note.

3. **Preliminary canonic transformation.** As is known [1, 2], the Hamiltonian in the action I – angle w variables has the form

$$H(I, w, \lambda) = H_0(I, \lambda) + \epsilon H_1(I, w, \lambda) \quad (\lambda = \epsilon t). \tag{1}$$

We assume that the functions H_0 and H_1, having the periods 2π in w and λ, are analytic in a complex neighborhood of the toroidal layer $I_1 \leq I \leq I_2$.

Theorem 1. *Let the frequency* $\omega(I, \lambda) = \partial H_0/\partial I$ *not go to zero in the layer under consideration. Then there exist positive numbers* ϵ_0, r_0, ρ_0 *and analytic functions* P, Q, T *of the variables* I, w, λ *independent of* ϵ, *such that:*

1. *The functions* P, $Q - \lambda$ *and* $T - w$ *have the periods* 2π *in* w *and* λ.

2. *The canonic equations with the Hamiltonians* (1) *are equivalent to the canonic equations with the Hamiltonians*

$$k(P, Q, T) = \epsilon k_0(P) + \epsilon^2 k_1(P, Q, T) + \cdots, \tag{2}$$

which are analytic for $|\epsilon| \leq \epsilon_0$, $|\text{Im } Q, T| \leq \rho_0$, $|P - P_0| \leq r_0$ *and which have the periods* 2π *in* Q *and* T.

3. *The principal part of the Hamiltonian* (2) *is* $\epsilon k_0(P)$, *where the function* $k_0(P)$ *is inverse to*

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* *Note added in proof.* By the same method, the perpetual adiabatic invariance of the magnetic moment in an axisymmetric magnetic trap [6] can be proved.

$\bar{H}_0(I) = \frac{1}{2\pi} \oint H_0(I, \lambda)\, d\lambda,$ *so that* $\bar{H}_0(k_0(P)) \equiv P.$

Let us first introduce the new time $T = w$. As is known [5], the integral curves of a Hamiltonian system in the I, w, λ space are invariantly associated with the differential form

$$I\, dw - H(I, w, \epsilon t)\, dt = -\frac{1}{\epsilon}(H\, d\lambda - \epsilon I\, dw). \tag{3}$$

Multiplication of the form by a constant does not change this relation. Let us consider the independent variables in (3) to be H, λ, rather than I, w, λ. Solving (1) with respect to I, we obtain

$$I(H, \lambda, w) = I_0(H, \lambda) + \epsilon I_1(H, \lambda, w) + \cdots. \tag{4}$$

Let us introduce the notation* $p = H$, $q = \lambda$, $T = w$, $K = \epsilon I$, so that

$$K(p, q, T) = \epsilon I_0(p, q) + \epsilon^2 I_1(p, q, T) + \cdots. \tag{5}$$

Then

$$H\, d\lambda - \epsilon I\, dw = p\, dq - K(p, q, T)\, dT,$$

hence [5] the systems with Hamiltonians (1) and (5) are equivalent.

Let us note that in §2 the frequency $\omega(I, \lambda)$ varied with time. Using the canonic transformation $p, q \longrightarrow P, Q$, let us change the coordinate $q = \lambda$ (which has the meaning of time) so that the frequency with respect to the unchanged time Q would become the constant $\bar{\omega}(I)$. To do this, let us introduce the action P – angle Q variables into the system with the Hamiltonian $I_0(p, q)$.

If $S(q, P)$ is a generating function, then the transformation is defined by the equations

$$p = \frac{\partial S}{\partial q}; \quad Q = \frac{\partial S}{\partial P}.$$

Let us select S so as to satisfy the Jacobi-Hamilton equation

$$I_0\left(\frac{\partial S}{\partial q}, q\right) = k_0(P)$$

with the as yet unknown function $k_0(P)$. According to (4), we find $\partial S/\partial q = H_0(k_0(P), q)$ or

$$S = \int^{q} H_0(I, \lambda)\, d\lambda, \text{ where } I = k_0(P). \tag{6}$$

The periodicity condition $Q(p, q + 2\pi) = Q(p, q) + 2\pi$ yields now

$$\oint \frac{\partial H_0}{\partial I} \frac{dk_0}{dP}\, d\lambda = 2\pi; \quad \frac{dk_0}{dP} \frac{d\bar{H}_0}{dI} = 1. \tag{7}$$

Equality (7) will be satisfied if the function inverse to $\bar{H}_0(I)$ is taken as $k_0(P)$. Then the generating function (6) introduces the variables P, Q which satisfy Theorem 1. Hence $k(P, Q, T) = K(p, q, T)$.

4. Construction of invariant tori of the system with the Hamiltonian (2) is carried out by successive approximations of Newtonian type [3, 4]. Let $\epsilon \neq 0$. We will say that the number $\bar{\omega}$ is sufficiently incommensurable with ϵ and let us write $\bar{\omega} \in \Omega(\epsilon)$, if

$$|m\bar{\omega} + n\epsilon| > |\epsilon|\,(|m| + |n|)^{-2}$$

for all integers m, n, $|m| + |n| > 0$. Let $\bar{\Omega}(\epsilon)$ denote the complement to $\Omega(\epsilon)$ on the ω axis. It is easy to prove:

Lemma. *The measure of* $\bar{\Omega}(\epsilon)$ *does not exceed* 10ϵ.

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* Do not confuse with p, q, from §§1, 2!

An $I_1 = k_0(P_1)$ and a definite frequency $\overline{\omega}_1 = \omega(I_1)$ correspond to each value $P = P_1$. Let us assume that $d\overline{\omega}/dI \neq 0$ (which is equivalent to the conditions $d^2\overline{H}_0/dI^2 \neq 0$; $d^2k_0/dP^2 \neq 0$). From the lemma presented, it is not difficult to deduce that the measure of the set of those P_1 for which $\overline{\omega}_1 \in \overline{\Omega}(\epsilon)$, approaches zero along with ϵ.

Theorem 2. *Let the Hamiltonian* (2) *be analytic for* $|\epsilon| < \epsilon_0$ *in the following neighborhood of the torus* $P = P_1$:

$$|P - P_1| \leq r_1; \quad |\mathrm{Im}\, Q,\ T| \leq \rho_0.$$

Let us assume that there is compliance in this neighborhood with the inequalities:

$$|k_0| \leq M, \quad |k_1 + \epsilon k_2 + \cdots| \leq M, \quad \left|\frac{d^2 k_0}{dP^2}\right| \geq \theta > 0.$$

Then there exists an $\epsilon_1(r_1, \rho_0, M, \theta) > 0$ *such that if the frequency* $\overline{\omega}_1$ *is sufficiently incommensurable with* ϵ *for a certain* $\epsilon < \epsilon_0, \epsilon_1$, *then there exist analytic functions* $F_\epsilon(Q, T)$, $G_\epsilon(Q, T)$ *such that the torus* $P = F_\epsilon(Q, T)$ *is invariant and* $dG_\epsilon/dT = \epsilon/\overline{\omega}_1$ *on it. The functions* $F_\epsilon(Q, T)$ *and* $G_\epsilon(Q, T) - Q$ *have period* 2π *in* Q *and* T *and as* $\epsilon \to 0$ *tend to* P_1 *and zero, respectively.*

The proof will not fit within the span of the present note (see [3, 4]). As has been explained in §2, there results from Theorem 2

Theorem 3. *Let a vibrational system have the analytic Hamiltonian* (1) *in the action-angle variables and let* $\partial H_0/\partial I \neq 0$, $d^2\overline{H}_0/dI^2 \neq 0$ *everywhere in the toroidal layer* $I_1 \leq I \leq I_2$. *Then for any* $\eta > 0$ *there is found an* $\epsilon_2 > 0$ *such that if* $|\epsilon| < \epsilon_2$ *and* $I_1 + \eta \leq I(0) \leq I_2 - \eta$, *then for all* $-\infty < t < +\infty$ *there will be* $|I(t) - I(0)| < \eta$.

5. Theorem 3 is also valid when the *Hamiltonian varies conditionally-periodically*, namely, when $H(p, q, \lambda_1, \cdots, \lambda_n)$ in the variables of §1 depends on several angular parameters λ which vary each with its own frequency $\dot{\lambda}_i = \epsilon\mu_i$. Let us assume that the μ_i are strongly incommensurable:

$$|\Sigma\, \mu_i n_i| > C(\Sigma\, |n_i|)^{-\nu}, \text{ if } \Sigma\, |n_i| > 0, \tag{8}$$

for some $C, \nu > 0$.

Transforming to the "time" $T = w$, we obtain the Hamiltonian (5) in the form $K(h, q, T)$ where $h = \Sigma\, \mu_i p_i$. Because of condition (8), the transformation $p, q \to P, Q$ is possible. The P_i enter into the Hamiltonian (2) only in the form of the combination $H = \Sigma\, \mu_i P_i$. Similarly to Theorem 2, it is possible to find invariant sets $H = F_\epsilon(Q, T)$ to which correspond invariant $(n + 1)$-dimensional tori in the original $(n + 2)$-space $p, q, \lambda_1, \cdots, \lambda_n$.

6. The case of several degrees of freedom is of considerable difficulty. Theorem 2 admits of the necessary generalization but it is not generally successful in reducing the Hamiltonian to the form (2). The fact is that the ratio of the frequencies of the high-speed motions depends on the phase of the slow motion. The system of equations on the three-dimensional torus

$$\dot{x} = \mu_1(z) + \epsilon f(x, y, z); \ \dot{y} = \mu_2(z) + \epsilon g(x, y, z); \ \dot{z} = \epsilon \tag{9}$$

(x, y, z are angular coordinates of a point on the torus) yields a simple example of this phenomenon; it is impossible to rectify the trajectory (9) by a small change of variables along with ϵ.

Consequently, in the case of a general system with n separate variables and slowly-varying periodic coefficients, it is doubtful if there are tori filled by conditionally-periodic trajectories. Even if such $(n + 1)$-dimensional tori were found, they would not divide the $(2n + 1)$-dimensional p_i, q_i, λ space and would not permit the proof of the perpetual adiabatic invariance of the action variables.

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ON THE GROWTH OF FUNCTIONS WHICH ARE HARMONIC WITHIN A CYLINDER AND BOUNDED, TOGETHER WITH THEIR NORMAL DERIVATIVE, ON ITS SURFACE

I. S. ARŠON AND M. A. EVGRAFOV

In the articles of M. A. Evgrafov and I. A. Čegis [1, 2] theorems for harmonic functions were proved which were analogous to the Phragmen-Lindelöf theorem for analytic functions. In these works the harmonic functions $u(x, x_1, x_2)$ were considered in the half-cylinder

$$0 < x < \infty, \ (x_1, x_2) \subset D, \tag{1}$$

where the region D in the article [1] is a disc, and in the article [2] a rectangle. The results obtained in [1, 2] consisted in the following:

If on the lateral surface of the half-cylinder (1) $u(x, x_1, x_2) = 0$ and $\frac{\partial}{\partial n} u(x, x_1, x_2)$ is bounded, and in the interior of it

$$|u(x, x_1, x_2)| < M \exp\exp \frac{\pi}{h+\eta} x, \quad \eta > 0, \ x > 0,$$

where h is the diameter in the case of a disc, and in the case of a rectangle its smaller side, then $u(x, x_1, x_2) \equiv 0$.

It was also shown there that the constants obtained are exact.

The proofs were based on one uniqueness theorem for the Dirichlet series.

The object of the present note is a proof of a similar result, but for the case of a half-cylinder (1) with an arbitrary region D. (As to the region D we shall only assume that the lateral surface of the half-cylinder (1) is Ljapunov surface.)

The proof of the theorem given below is completely independent from the Dirichlet series.

Theorem. *If a function $u(x, x_1, x_2)$ harmonic in the half-cylinder* (1) *satisfies on the boundary Γ of this cylinder the conditions*

$$|u(x, x_1, x_2)| + \left|\frac{\partial}{\partial n} u(x, x_1, x_2)\right| = O(x^{-\alpha}), \quad 0 < \alpha < 1, \ x \to \infty, \tag{2}$$

SMALL PERTURBATIONS OF THE AUTOMORPHISMS OF THE TORUS*

V. I. ARNOL'D AND Ja. G. SINAĬ

1. Let the torus T^2 be realized in the form of the unit square in the plane (x_1, x_2) with pairwise identified sides. A transformation $x \to Ax = \bar{x}$, given by the integer valued matrix $A = \|a_{ij}\|$ with determinant ± 1 acting on the torus according to the formula $\bar{x}_i = \sum_j a_{ij}x_j \pmod 1$, $i = 1, 2$, is called an automorphism of T^2.

We assume that $\|a_{ij}\|$ has two real eigenvalues which differ mod 1. Then, if $(\alpha_1, 1)$ is the eigenvector of A^* corresponding to the eigenvalue λ_1, $|\lambda_1| < 1$, then the system of lines on the locus

$$dx_2 + \alpha_1 dx_1 = 0 \tag{1}$$

satisfies the following conditions with respect to A;

I. Each straight line Γ of the family (1) is transformed by A into another straight line $A\Gamma$ of (1), i.e. the family (1) is invariant under A.

II. There exists a $\mu_1 > 1$, such that the lengths $s(l)$ and $s(Al)$ of the segments l and Al on Γ and $A\Gamma$ satisfy

$$s(Al) \geq \mu_1 s(l). \tag{2}$$

Similarly, if one takes the system of lines

$$dx_2 + \alpha_2 dx_1 = 0, \tag{1'}$$

where $(\alpha_2, 1)$ is the eigenvector of A^* corresponding to the eigenvalue λ_2, $|\lambda_2| > 1$, then this system satisfies properties I and II′ where II′ is formulated exactly as II except that instead of (2) one has

$$s(Al) \leq \mu_2 s(l), \quad 0 < \mu_2 < 1. \tag{2'}$$

2. It turns out that the property of A possessing a family of curves satisfying I and II (I′ and II′) is coarse, i.e. it is preserved under small perturbations of A by nonlinear terms. Indeed, let $A_\epsilon = A + \epsilon B(x)$, i.e. $x \to A_\epsilon x = Ax + \epsilon B(x)$, where $B(x) = (b_1(x_1, x_2), b_2(x_1, x_2))$, where $b_i(x)$ is periodic in each argument with period 1 and is a thrice continuously differentiable function.

Theorem 1. *For sufficiently small $\epsilon > 0$ there exists a system of curves*

$$dx_2 + \tilde{\alpha}_1(x, \epsilon)\, dx_1 = 0, \tag{3}$$

satisfying I *and* II *with respect to A_ϵ. $\tilde{\alpha}(x, \epsilon)$ has continuous derivatives of bounded variation in x, and is continuous in ϵ. There are no closed curves among the solutions of* (3).

Proof. We employ the method of successive approximation. Let us assume that the curves $dx_2 + \alpha_1^n(x, \epsilon)\, dx_1 = 0$ have already been constructed. We apply A_ϵ to them. If the matrix $\|a_{ij} + \epsilon\, \partial b_i/\partial x_j\|^{-1}$ has the form $\|\bar{a}_{ij}\| + \epsilon \|g_{ij}(x, \epsilon)\|$, where $\bar{a}_{ij}$ are elements of A^{-1} and $\|g_{ij}(x, \epsilon)\|$, is a bounded matrix which depends continuously on x and ϵ, then the system of curves we obtain can be written in the form $dx_2 + \alpha_1^{n+1}(x, \epsilon)\, dx_1 = 0$, where

$$\alpha_1^{n+1}(x, \varepsilon) = \frac{(\bar{a}_{11} + \varepsilon g_{11}(x, \varepsilon))\, \alpha_1^n(A_\varepsilon^{-1}x) + (\bar{a}_{21} + \varepsilon g_{21}(x, \varepsilon))}{(\bar{a}_{12} + \varepsilon g_{12}(x, \varepsilon))\, \alpha_1^n(A_\varepsilon^{-1}x) + (\bar{a}_{22} + \varepsilon g_{22}(x, \varepsilon))}. \tag{4}$$

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* Editor's note: translation into English published in Soviet. Math. Dokl. 3 (1962). Translation of V.I. Arnol'd and J.G. Sinai, On small perturbations of the automorphisms of a torus. Dokl. Akad. Nauk SSSR 144:2 (1962), 695–698, Corrections in Dokl. Akad. Nauk SSSR, 1963, 150:5, 958

Lemma 1. *If* $\max_x |\alpha_1^n(x,\epsilon) - \alpha_1| < \delta$, *then there exist a* μ, $0 < \mu < 1$ *and a* $C < \infty$, *depending only on* A *and* B, *such that* $\max_x |\alpha_1^{n+1}(x,\epsilon) - \alpha_1| \le \mu\delta + C\epsilon$.

Proof. Let us rewrite (4) as follows:

$$\alpha_1^{n+1}(x,\varepsilon) = \frac{\bar a_{11}\alpha_1 + \bar a_{21} + (\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\,\bar a_{11} + \varepsilon\,(g_{11}\alpha_1^n + g_{21})}{\bar a_{12}\alpha_1 + \bar a_{22} + (\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\,\bar a_{12} + \varepsilon\,(g_{12}\alpha_1^n + g_{22})}.$$

It is not difficult to deduce, from the fact that $(\alpha_1, 1)$ is an eigenvector of A^*, that

$$\bar a_{11}\alpha_1 + \bar a_{21} = \alpha_1/\lambda_1, \quad \bar a_{12}\alpha_1 + \bar a_{22} = 1/\lambda_1.$$

Hence

$$\alpha_1^{n+1}(x,\varepsilon) = \frac{\alpha_1 + \lambda_1\,[(\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\,\bar a_{11} + \varepsilon\,(g_{11}\alpha_1^n + g_{21})]}{1 + \lambda_1\,[(\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\,\bar a_{12} + \varepsilon\,(g_{12}\alpha_1^n + g_{22})]},$$

$$|\alpha_1^{n+1}(x,\varepsilon) - \alpha_1| = \left|\lambda_1 \frac{(\alpha_1 - \alpha_1^n)(-\bar a_{12}\alpha_1 + \bar a_{11}) + \varepsilon\,[g_{11}\alpha_1^n + g_{21} - \alpha_1(g_{12}\alpha_1^n + g_{22})]}{1 + \lambda_1\,[(\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\,\bar a_{12} + \varepsilon\,(g_{12}\alpha_1^n + g_{22})]}\right|.$$

To finish the proof, it is sufficient to take into consideration that, analogously to the foregoing, $-\alpha_1\bar a_{12} + \bar a_{11} = \lambda_1$.

It follows from Lemma 1 that for every $\delta > 0$, one can find an $\epsilon_1(\delta)$ such that $\max_x |\alpha_1^n(x,\epsilon) - \alpha_1| < \delta$ for all n, if $\epsilon < \epsilon_1(\delta)$.

Let us evaluate now $|\alpha_1^{n+1}(x,\epsilon) - \alpha_1^n(x,\epsilon)|$. It is easy to find by using (4) that

$$\alpha_1^{n+1}(x,\varepsilon) - \alpha_1^n(x,\varepsilon) = D_n(x,\varepsilon)\,[\alpha_1^n(A_\varepsilon^{-1}x,\varepsilon) - \alpha_1^{n-1}(A_\varepsilon^{-1}x,\varepsilon)],$$

where

$$D_n(x,\varepsilon) = \frac{\bar a_{11}\bar a_{22} - \bar a_{12}\bar a_{21} + \varepsilon^2(g_{11}g_{22} - g_{12}g_{21}) + \varepsilon\,(g_{11}\bar a_{22} + g_{22}\bar a_{11} - g_{21}\bar a_{12} - g_{12}\bar a_{21})}{[(\bar a_{12}\alpha_1^n + \bar a_{22}) + \varepsilon\,(g_{12}\alpha_1^n + g_{22})]\,[(\bar a_{12}\alpha_1^{n-1} + \bar a_{22}) + \varepsilon\,(g_{12}\alpha_1^{n-1} + g_{22})]}.$$

But, in view of Lemma 1, for every $\delta > 0$ and $\epsilon < \epsilon_2(\delta)$ we have $|a_{12}\alpha_1^n + \bar a_{22} - 1/\lambda_1| \le \delta$ for all n. It follows, since $\bar a_{11}\bar a_{22} - \bar a_{12}\bar a_{21} = \pm 1$ and $|\lambda_1| < 1$, that there exists a ρ, $0 < \rho < 1$ such that

$$\max_x |\alpha_1^{n+1}(x,\varepsilon) - \alpha_1^n(x,\varepsilon)| \leqslant \rho \max_x |\alpha_1^n(x,\varepsilon) - \alpha_1^{n-1}(x,\varepsilon)|.$$

Consequently $\alpha_1^n(x,\epsilon)$ converges uniformly on T^2. Let us put $\tilde\alpha_1(x,\epsilon) = \lim_{n\to\infty} \alpha_1^n(x,\epsilon)$. It is easy to see that the system of curves Γ_1

$$dx_2 + \bar\alpha_1(x,\varepsilon)\,dx_1 = 0 \tag{5}$$

satisfies properties I and II with respect to A_ϵ. The continuous dependence of $\tilde\alpha_1(x,\epsilon)$ on ϵ is obvious. V. I. Oseledec has established the existence of derivatives of bounded variations of $\tilde\alpha_1(x,\epsilon)$ with respect to x by methods which are similar to the above.

Let us prove now that there are no closed curves among the curves of (5). Suppose that such a curve exists. Then, in view of property II, the length of the $A_\epsilon^{-n}\Gamma$ satisfies $s(A_\epsilon^{-n}\Gamma) \le \mu_1^{-n} s(\Gamma)$. But, then $A_\epsilon^{-n}\Gamma$ cannot be a curve of (5), since it cannot then satisfy $|\tilde\alpha_1 - \alpha_1| < \delta$ which is satisfied by all curves of (5). Thus the theorem is proved.

Remark. Similarly, one can prove the existence of curves Γ_2, $dx_2 + \tilde\alpha_2(x,\epsilon)\,dx_1 = 0$, satisfying

I and II′.

3. Let us consider two (nonsimultaneous) equations on the torus

$$dx_2 = f_i(x_1, x_2)\, dx_1 \quad (i = 1, 2) \tag{6}$$

where the first derivatives of the functions $f_i(x_1, x_2)$ of period 1 in x_1 and x_2 are of bounded variation. For equations of the form (6) Poincaré [1] has defined the rotation numbers ω_i.

Theorem 2. *Let ω_1 and ω_2 be irrational and*

$$-\infty < c_1 < f_1 < C_1 < c_2 < f_2 < C_2 < \infty, \tag{7}$$

where $c_1, \cdots, C_2$ are constants. Then there exists a homeomorphism of the torus $x \longleftrightarrow y$, straightening the integral curves of both equations (6), i.e. transforming them into straight lines Γ'_i: $dy_2 = \omega_i dy_i$ $(i = 1, 2)$.

Proof. 1^0. Let us denote by $\Gamma_i(x_0)$ the integral curves of (6) passing through the point $x_0 = (x_1^0, x_2^0)$ and by $\Gamma'_i(y_0)$ the line $y_2 - y_2^0 = \omega_i(y_1 - y_1^0)$. Let p_1, p_2 be two integer points. Let us consider the point $q(p_1, p_2) = \Gamma_1(p_1) \cap \Gamma_2(p_2)$ in the x-plane and $q'(p_1, p_2) = \Gamma'_1(p_1) \cap \Gamma'_2(p_2)$ in the y-plane. We define the mapping $y \to x$ by making $q(p_1, p_2)$ correspond to $q'(p_1, p_2)$.

2^0. **Lemma 2.** *The mapping $q' \to q$ is uniformly continuous.*

Proof. According to Denjoy's theorem [2], there exists a continuous transformation of the torus transforming curves Γ_1 into lines Γ'_1. Hence, for every $\epsilon > 0$ there exists a $\delta_1(\epsilon) > 0$ such that if the distance between the two lines $\Gamma'_1(p_1)$ and $\Gamma'_1(p_3)$ is less than δ_1, then the distance between $\Gamma_1(p_1)$ and $\Gamma_1(p_3)$ is less than ϵ everywhere. Similar reasoning applied to Γ'_2 and Γ_2 gives $\delta_2(\epsilon)$. In view of (7), if the distance between $q'(p_1, p_2)$ and $q'(p_3, p_4)$ is less than $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$, then the distance between $q(p_1, p_2)$ and $q(p_3, p_4)$ is less than $K\epsilon$ (where K depends only on C_1 and c_2). This completes the proof of the lemma.

3^0. Since, in view of [2] and (7) the set of points $q(p_1, p_2)$ and $q'(p_1, p_2)$ is everywhere dense, one can extend the mapping $q' \to q$, by continuity, to the whole y-plane. The homeomorphism of the plane thus obtained defines the desired homeomorphism of the torus since the lines Γ'_i are transformed into the curves Γ_i. This completes the proof of Theorem 2.

4. **Theorem 3.** *An ergodic automorphism of the two-dimensional torus is structurally stable.* This means that, under the conditions of Theorem 1, there exists, for sufficiently small ϵ, an automorphism of the torus $x \longleftrightarrow y$ which transforms a perturbed automorphism into a nonperturbed one:*

$$y(A_\epsilon x) = Ay(x). \tag{8}$$

Proof. A_ϵ has, for small ϵ, one fixed point O_ϵ in the x-plane. Let us take it as the origin in the x-plane. Let us construct the homeomorphism of Theorem 2 along the curves Γ_i obtained in §2 (5). Since these are not closed on the torus, the rotation number ω_i is irrational. On the other hand, $\omega_i(\epsilon)$ depends continuously on ϵ. Hence they are constants. Hence the lines Γ'_i have the directions of the eigenvectors A.

The curves $\Gamma_i(O_\epsilon)$ go into themselves under A_ϵ while the lines $\Gamma'_i(O)$ go into themselves under A. Here A maps $q'(p_1, p_2)$ into $q'(Ap_1, Ap_2)$ and A_ϵ maps $q(O_\epsilon + p_1, O_\epsilon + p_2)$ into $q(O_\epsilon + Ap_1, O_\epsilon + Ap_2)$. Hence (8) is satisfied for $x = q(p_1, p_2)$ and therefore by continuity for all x.

Remark. If A_ϵ is analytic then according to [3] the curves Γ_1 or Γ_2 can be straightened sepa-

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*Or 'coarse' in the terminology of Andronov-Pontrjagin.

rately by an analytic transformation. However, the homeomorphism constructed in §3 may be nondifferentiable. Indeed, the eigenvalues of A_ϵ at O_ϵ may be different from the eigenvalues of A. If the homeomorphism $x \longleftrightarrow y$ is absolutely continuous, then A_ϵ has, for sufficiently small ϵ, an invariant measure which is absolutely continuous with respect to the Lebesgue measure and is metrically isomorphic to A. We do not know, however, whether the condition of absolute continuity is fulfilled even for measure preserving analytic perturbations.

5. We have succeeded in proving the following in the n-dimensional case:

Theorem 4. *If the matrix* $A = \|a_{ij}\|$ *has* n *real eigenvalues,* k *of which are greater than* 1 *while the remaining ones are less than* 1, *and* $A_\epsilon = A + \epsilon B$, *then there exists, on the torus* T^n, *a system of* $(n-k)$*-dimensional nonclosed, smooth surfaces, invariant under* A_ϵ *and such that for every piece* l *of the surface the volume* $V(l) \leq \mu' \, V(A_\epsilon l)$, *for* μ', $0 < \mu' < 1$.

Theorem 5. *Let the n-dimensional torus* T^n *be split into* $(n-1)$*-dimensional, smooth surfaces* $\Gamma: x_n = g(x_1, \cdots x_{n-1})$ *and let the functions* g *be twice continuously differentiable. If none of the surfaces is closed then there exists a homeomorphism of the torus* $x \longleftrightarrow y$ *straightening the surfaces* Γ, *i.e. transforming them into the planes* $\Gamma' : y_n = \omega_1 y_1 + \cdots + \omega_{n-1} y_{n-1} + C$.

Proof. 1^0. Let $\Gamma(x_n)$ be the surface Γ going through the point $(0, \cdots, 0, x_n)$. We denote by $Q_p(x_n)$ (where $p = (p_1, \cdots, p_{n-1})$ the point $(p, x_n) \in \Gamma(x)$. Let $p_1, \cdots, p_{n-1}$ be integers. Then one can consider Q_p as a transformation of the circle $p = 0$ onto itself. It is easy to see that $Q_{p+q} = Q_p Q_q$ since all of Q_p are commutative.*

2^0. **Lemma 3.** *Let there be given a finite number of commuting, twice differentiable, homeomorphisms of the circle* $Q_1, Q_2, \cdots, Q_r$. *Then either there exists a homeomorphism of the circle converting them into rotations or there exists an* N *such that* $Q_1^N, \cdots, Q_r^N$ *have a common fixed point.*

Indeed, if even one of the transformations, say Q_1, has an irrational rotation number [1], then it is a rotation by an irrational angle for some choice of the parameter on the circle. Since all of the transformations Q commute with $Q_1^k (k = 1, 2, \cdots)$, they commute with all rotations and hence are themselves rotations. If all the rotation numbers are rational, then for some N each of the transformations $R_1 = Q_1^N, \cdots R_r = Q_r^N$ have fixed points. Hence $\lim_{n_1 \to \infty} R_1^{n_1} \lim_{n_2 \to \infty} R_2^{n_2} \cdots \lim_{n_r \to \infty} R_r^{n_r} x_0 = z$. It is easy to see that z is a fixed point of all of $R_1, R_2, \cdots, R_r$.

3^0. We apply Lemma 3 to the transformations $Q_i = Q_{1,0,\cdots,0}, \cdots, Q_{0,\cdots,0,1}$. If all Q^N have a common fixed point z then the surface $\Gamma(z)$ is closed, contrary to the conditions of the theorem. Therefore, there exists a parameter $\phi(x_n)$ such that $\phi(Q_i x_n) = \phi(x_n) + \omega_i$. Let us now define $x_0(x)$ by $x \in \Gamma(x_0)$ and put $y_i = x_i (1 \leq i < n)$, $y_n = \phi(x_0(x)) + \omega_1 x_1 + \cdots + \omega_{n-1} x_{n-1}$. It is easy to see that $x \longleftrightarrow y$ is the required homeomorphism of the torus.

Theorem 3 was presented to the authors in the form of a conjecture by S. Smeĭl. The authors express their thanks to him as well as to D. V. Anosov and È. G. Belaga for useful discussions.

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*The same method allows the study of decompositions of fibre bundles with the circle as a fibre. The question reduces to determining the representations of the fundamental group of the base in the group of mappings of the circle on itself.

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OPTIMAL PROCESSES WITH CYCLIC CONSTRAINTS

R. GABASOV

Time optimal problems [1] for systems of difference equations are studied below. The constraints on the control action are of a specific nature which we shall call cyclic.

1. Let the behavior of a control system be described by the equation

$$x(n+1) = Ax(n) + bu(n), \tag{1}$$

where $x = \{x_1, x_2, \cdots, x_l\}$ is an l-dimensional vector, $A = \{a_{ij}\}$ is a constant $l \times l$ matrix, $b = \{b_1, b_2, \cdots, b_l\}$ is a constant vector, and $u(n)$ is the control function.

The optimal problem, which will be discussed below, consists of the following. Let there be given the initial condition $x(0) = \{x_1(0), x_2(0), \cdots, x_l(0)\}$. We are required to find a function $u(n)$ such that the point $x(n)$, moving along a trajectory of equation (1), will hit the origin in the least number of steps K^0. In addition, the control function must satisfy one of the following constraints.

Given the number $\omega > 0$, let us define the number N from the condition $N\omega < K^0 \leq (N+1)\omega$, and let us assume that $u(n) = 0$ when $K^0 \leq n \leq (N+1)\omega$.

Problem I

$$\max_{0 \leqslant k \leqslant N} \sum_{i=k\omega}^{(k+1)\omega-1} |u_i| \leqslant 1. \tag{2}$$

Problem II

$$\sum_{k=0}^{N} \max_{k\omega \leqslant i \leqslant (k+1)\omega-1} |u_i| \leqslant 1. \tag{3}$$

Problem III

$$\max_{0 \leqslant k \leqslant N} \left(\sum_{i=k\omega}^{(k+1)\omega-1} |u_i|^p \right)^{1/p} \leqslant 1, \quad p > 1. \tag{4}$$

Problem IV

$$\sum_{i=1}^{\omega-1} \max |u_{k\omega+i}| \leqslant 1. \tag{5}$$

Note. We shall take $u_n = u(n)$.

Problems I–IV differ from the usual optimal control problem in that the constraints on the control function are of a cyclic nature (the duration of the cycle is ω steps). Each of the four problems can be given a physical interpretation. For example, it is possible to have the following physical problem.

THE CLASSICAL THEORY OF PERTURBATIONS AND THE PROBLEM OF STABILITY OF PLANETARY SYSTEMS*

V. I. ARNOL'D

1. The theory of perturbations enables us to predict planetary motion for many years ahead with all necessary accuracy. However, qualitative questions on the behavior of a system during an infinite time interval, for example the problem of stability, could not be solved by the theory of perturbations. Planetary motion is described in this theory by series of the form

$$\sum_{m,n} a_{mn} \cos [(m\omega_1 + n\omega_2)\, t + \phi_{mn}].$$

In Laplace's time the appearance in higher approximations of the "secular terms" of the form $at^{\alpha}\cos \omega t$ and bt^{β} was considered inevitable. This led to attempts to prove the instability of the solar systems (see [1]). However, by the time of Poincaré ([2,3], see also [4,5]) it became clear that it is possible to construct a perturbation theory in such a way that the series of an arbitrary approximation should contain only trigonometric terms. On the other hand, it turned out that the above-mentioned series diverge, and so the question of stability remained open.

The divergence of these series is connected with a kind of a resonance—the approximate commensurability of frequencies. For example, the frequencies of Jupiter and Saturn $\omega_1 \approx 299''.1$, $\omega_2 \approx 120''.5$ almost satisfy the relation $2\omega_1 = 5\omega_2$. The perturbation is expressed by means of the series

$$\sum_{m,n} \frac{a_{mn}}{m\omega_1 + n\omega_2} \cos [(m\omega_1 + n\omega_2)\, t + \phi_{mn}].$$

Since the denominator $2\omega_1 - 5\omega_2$ is very small, one observes a large perturbation of a long period (see [4]). For the majority of pairs ω_1, ω_2 the quantities $|m\omega_1 + n\omega_2|$ do not vanish, and even exceed $K(|m| + |n|)^{-2}$ for some $K > 0$ and all integral $m, n > 0$ (see [6]). This leads to the hypothesis that for the majority of initial conditions the planetary system is stable. This hypothesis, however, was not proved, on account of difficulties of several types. Poincaré [2,3] suggested a number of model problems which contained some of these difficulties separately.

Nontrivial problems with small denominators were solved for the first time only in 1941 by Siegel [8]. Siegel also gave (in certain cases; see [9]) a rigorous proof of the fact, known to Poincaré, that approximations of the theory of perturbations may converge only in isolated special cases.

An important step forward was made in 1954 by Kolmogorov, who applied a method of Newton's type and constructed a convergent version of the theory of perturbations in the so called nondegenerate case (see [10,11]). The results of [10] have numerous applications; however, the majority of problems of celestial mechanics belong to the degenerate case.

After overcoming separately the difficulties connected with degeneracies of various types in the model problems [12,13,14] by combining Newton's method with classical asymptotic methods, it became possible to apply the developed technique to the problem of planetary motion, where all

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difficulties are encountered together. In the present article we give results obtained along this line.*

2. Let us consider, for simplicity, the plane three-body problem of masses M, m_1, m_2, where $m_1, m_2 \ll M$. The perturbation theory gives the following picture of motion [3,4]. In the zeroth approximation the planets m_1, m_2 do not perturb each other and move along Keplerian ellipses with semimajor axes a_1, a_2 and eccentricities e_1, e_2. The directions of semimajor axes are determined by the angles ω_1, ω_2 (the lengths of the perihelia). In the zeroth approximation a_k, e_k, ω_k are preserved during the entire motion. Let us consider an important case when m_k, e_k are small, $a_2 - a_1 > c$ and the planets move around M in one direction.

In the first approximation the mutual influence of planets leads only to a small "trembling" of a_k, e_k, ω_k around constant values. In the second approximation one observes a slow, but unbounded (secular) motion of perihelia. This slow variation of e_k, ω_k may be described in the following manner. Let us characterize the Keplerian ellipse by a vector which is directed along the semimajor axis and is proportional to the eccentricity. It turns out that for each of the planets m_k this vector is the sum of two uniformly rotating vectors ξ_{k1}, ξ_{k2} whose angular velocities ν_1, ν_2 are small and equal for both planets. The planetary motion along ellipses which vary in the above described manner will be called Lagrangian.

Our basic result consists of the fact that, if the masses and the eccentricities of the planets are sufficiently small, then for the majority of initial conditions there will exist a Lagrangian motion which differs little from the true motion during the entire infinite time interval.

Let us consider the centers of gravity of the bodies to be stationary. Then the system has four degrees of freedom. Let $0 < c_1 < C_1 < c_2 < C_2 < \infty$ be constants. The conditions $c_1 < a_1 < C_1$; $c_2 < a_2 < C_2$; $e_1, e_2 < \delta$ define a domain G_δ in an 8-dimensional phase space. A point of G_δ uniquely defines the initial coordinates and velocities, and consequently the entire motion. Let α_1, α_2 be constants and $m_1 = \mu\alpha_1 M$, $m_2 = \mu\alpha_2 M$.

Theorem 1. *For an arbitrary $\eta > 0$ there will exist an $\epsilon > 0$ such that if $\delta, \mu < \epsilon$, then the majority of points of the domain G_δ (the exception consists of points which form a set of measure smaller than η mes G_δ) move in such a way that:* 1) *the point always remains in the domain G_δ;* 2) *it moves conditionally periodically, everywhere sweeping out an analytic four-dimensional torus in G_δ;* 3) *it always remains closer than η from a point of the phase space, which represents a certain Lagrangian motion.*

Remark 1. Analogous theorems on "metric stability" are valid for the plane problem of n bodies and for the space problem of three bodies. Their generalization to the space problem for $n > 3$ bodies requires certain additional calculations (connected with the exclusion of knots).

Remark 2. The exceptional set in Theorem 1 extends to infinity, is connected and everywhere dense. Taking into account, on the one hand, these considerations, and on the other, the known fact of the existence of "scuttles" in the distribution of small planets, it may be assumed that the motion of planets is topologically unstable.

3. Let us give a sketch of the proof of Theorem 1. As it is known [3], the Hamilton's function for the plane problem of three bodies has the form

$$F = F_0(\Lambda) + (\mu)\, F(\Lambda, \xi, \eta) + (\mu)\, \widetilde{F}(\Lambda, \lambda, \xi, \eta), \tag{1}$$

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*Some of these were announced in lectures on July 11, 1961 during the IV All-Union Mathematical Conference and on November 27, 1961 during the conference on theoretical astronomy.

where $\Lambda = (\Lambda_1, \Lambda_2)$, $\lambda = (\lambda_1, \lambda_2)$; $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$ are canonically conjugate variables, Λ_k corresponding to the semimajor axes, the angles λ_k to the phases, and (ξ_k, η_k) to the vectors $(e_k \cos \omega_k, e_k \sin \omega_k)$. In formula (1) the bar indicates the average with respect to λ_1, λ_2, $(\mu)F = \mu F_1 + \mu^2 F_2 + \cdots$. The functions F_0, $\bar{F}$, $\tilde{F}$ are analytic; $\tilde{F}$ has the period 2π with respect to λ and is in the average equal to zero. For small ξ, η the functions $\bar{F}$ and $\tilde{F}$ may be expanded in a convergent Taylor's series in ξ, η, and moreover, F contains terms in even powers only.

By a preliminary canonical transformation $\Lambda, \lambda, \xi, \eta \rightarrow \Lambda', \lambda', \xi', \eta'$ ("the averaging out with respect to the fast variables," see §4) it is possible to reduce F in the major part of the domain G_δ to the form

$$F' = F_0(\Lambda') + (\mu)\bar{F}(\Lambda', \xi', \eta') + F_2(\Lambda', \lambda', \xi', \eta'), \tag{2}$$

where F_2 is a perturbation of the order μ^2.

Secular motion is determined by the Hamilton's function $(\mu)\bar{F}(\xi', \eta')$, where Λ_k are regarded as parameters. The point $\xi' = \eta' = 0$ is a stable position of equilibrium with respect to the linear approximation. Introducing after Birkhoff [7] in the neighborhood of zero new canonical variables r, ϕ, we reduce $\bar{F}$ to the form* $\bar{F} = \bar{F}_2(r) + R_3(r, \phi)$, where $r = (r_1, r_2)$ are quantities of the order e^2; $\phi = (\phi_1, \phi_2)$ are angular variables; $(\mu)\bar{F}_2 = \nu_1 r_1 + \nu_2 r_2 + c_{11} r_1^2 + 2c_{12} r_1 r_2 + c_{22} r_2^2$ and R_3 begins with r^3, i.e., e^6. The canonical transformation $\Lambda', \lambda', \xi', \eta' \rightarrow \Lambda'', \lambda'', r, \phi$ reduces F' to the form

$$F'' = F_0(\Lambda'') + (\mu)\bar{F}_2(\Lambda'', r) + F_2''(\Lambda'', \lambda'', r, \phi), \tag{3}$$

where $F_2'' = F_2 + R_3$ is a perturbation of the order $\mu^2 + \mu r^3$.

The conditionally periodic solutions of equations with Hamilton's function (3) are found by a convergent iterative method of Newton's type. In connection with it two difficulties arise. The first is connected with the limiting degeneration at $r = 0$, and is overcome in the same way as in [12]. The second difficulty, a characteristic degeneration at $\mu = 0$, is connected with the presence of fast and slow motions. At $\mu = 0$ the Keplerian motion is described by two "fast" frequencies λ_1, λ_2, and for $\mu \neq 0$ in the Lagrangian motion there also appear two "slow" frequencies ν_1, ν_2 (of the order μ) (cf. [13,14]).

The verification of the satisfaction of conditions (6) of dependence of frequencies on the momenta for the Hamilton's function (3) was performed by means of direct calculations, which made use of the expansion of F in powers of e and a_1/a_2 [15].

4. Let us formulate the generalization of results [10] for the case of the characteristic degeneration when the Hamilton's function has the form

$$H = H_0(p_1, \cdots, p_k) + \epsilon H_1(p_1, \cdots, p_n, q_1, \cdots, q_n) \quad (k < n), \tag{4}$$

has the period 2π with respect to each of the variables $q_1, \cdots, q_n$ and is analytic** when p varies in a certain domain G and $|\mathrm{Im}\, q| < \rho$. For $\epsilon = 0$ the canonical equations with the Hamilton's function (4) describe a conditionally periodic motion $\dot{q}_i = \omega_i$ $(i \leq k)$, $\dot{q}_{k+1} = \cdots = p_n = 0$ with frequencies $\omega_i = \partial H_0 / \partial p_i$. For small ϵ it is possible to assume, by neglecting the "trembling," that the slow variation of $q_{k+1}, \cdots, p_n$ is in time influenced only by the average value of H_1 with respect to the

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*In the case of two planets $\bar{F}(\xi', \eta')$ may be exactly reduced to the form $\bar{F}(r)$. Our arguments are applicable also in the case of $n > 2$ planets.

***Added in proof.* In connection with the recent articles of J. Moser, the existence of several hundred derivatives is sufficient.

fast variables*

$$\overline{H}_1(p;\ q_{k+1},\cdots,q_n) = (2\pi)^{-k}\oint H_1(p,q)\,dq_1\cdots dq_k.$$

We will now assume** that $\overline{H}_1$ is independent of the phases of slow motions. Then the Hamilton's function (4) may be represented in the form

$$H = H_0(p_1,\cdots,p_k) + \epsilon\overline{H}_1(p_1,\cdots,p_n) + \epsilon\widetilde{H}_1(p_1,\cdots,p_n,q_1,\cdots,q_n). \tag{5}$$

Theorem 2. *Let for* $p \in G$ *the conditions*

$$\det\left|\frac{\partial^2 H_0}{\partial p_i \partial p_j}\right| \neq 0\ (i,j=1,\ldots,k);\quad \det\left|\frac{\partial^2 \overline{H}_1}{\partial p_i \partial p_j}\right| \neq 0\ (i,j=k+1,\ldots,n) \tag{6}$$

be satisfied. Denote by T *the torus layer* $\operatorname{Im} p = \operatorname{Im} q = 0,\ p \in G,\ q_i \in [0, 2\pi)$.

Then for an arbitrary $\eta > 0$ *there exists an* $\epsilon_0 > 0$ *such that if* $|\epsilon| < \epsilon_0$, *then in* T *there exist analytically invariant n-dimensional tori which carry the trajectories of conditionally periodic motions. These tori fill* T *with accuracy up to the remainder of a measure smaller than* η mes T.

Theorem 2 shows that *for small* ϵ *for the majority of initial conditions the motion of the system with the Hamilton's function* (5) *on an infinite time interval differs but little from the conditionally periodic motion of* n *frequencies* $\dot{q}_i = \partial\overline{H}/\partial p_i\,(\overline{H} = H_0 + \epsilon\overline{H}_1)$, $\dot{p} = 0$ *with suitable initial conditions.*

The proof of Theorem 2 is analogous to the arguments of [13], where the case $k = 1$ is considered. For $k > 1$ the small denominators appear already in the first stage of the proof, during the averaging out with respect to the fast variables. In order not to deal with an infinitely large number of resonances it is convenient (cf., for example, [16]) to take into account in the perturbation $\widetilde{H}_1$ only the harmonics up to the Nth order; if $N \sim |\ln \epsilon|$, then the higher harmonics give the sum of the order ϵ^2.

Let Ω be the domain in the space $\omega = (\omega_1,\cdots,\omega_k)$, into which G goes under the map $p \to \partial H_0/\partial p$. Denote by Ω_{KN} the set of those ω for which $|(\omega, n)| > K|n|^{-s}$ ($s = k+1$, $|n| = |n_1| + \cdots + |n_k|$) for an arbitrary integral nonzero vector n, $|n| < N$. Denote by G_{KN} the inverse image of Ω_{KN} and by $G_{KN} - d$ the set of points which belong to G_{KN} with the d-neighborhood.

During the proof of Theorem 2 a certain number $\delta > 0$ is chosen sufficiently small, and then an $\epsilon = \delta^T$ is chosen, where T is a sufficiently large constant which depends only on the number of degrees of freedom n. The first step of the proof consists of establishing the following lemma.

Lemma. *In the conditions of Theorem 2 let* $|\epsilon H_1| < M$, $|\partial^2 H_0/\partial p_i \partial p_j| < \theta$. *Let the numbers* γ, δ *satisfy the inequalities*

$$2\gamma < \rho,\ \delta < \min\left(K, \tfrac{1}{3}, \gamma/4, 1/\theta, e^{2k} n^{-2}(8k)^{-2k}\right),\ M < \delta^{2k+7}.$$

Then in the domain $P \in G_{KN} - 2\delta$, $|\operatorname{Im} Q| < \rho - 2\gamma$ *(here* $N = \frac{1}{\gamma}\ln\frac{1}{M}$*) there exists an analytic canonical one-to-one transformation* $p, q \leftrightarrow P, Q$, *which reduces* H *to the form*

$$H = H_0(P) + \epsilon\overline{H}_1(P) + H_2(P, Q),$$

where $|P - p|,\ |Q - q| < M\delta^{-2k-5}$; $|H_2| < M^2\delta^{-4k-10}$.

Since H_2, therefore, has in the domain $G_{KN} - 2\delta$ the order ϵ^2, and the magnitude of the

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*This remark, which goes back to Gauss, constitutes the essence of the known "method of averaging" [16]. The following may be considered as one of the variations of the basis of this method for an infinite time interval.

**This is the case in the problem of planetary motion; see formula (3).

components of the domain $G_{KN} - 2\delta$ is of the order $|\ln \epsilon^{-1}|$, then the proof of Theorem 2 from the lemma is conducted analogously to the proof of Theorem 2 in [13].

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Письмо в редакцию*

В моей работе «О представлении непрерывных функций трех переменных суперпозициями непрерывных функций двух переменных», опубликованной в «Математическом сборнике» (т. 48 (90) (1959), 3—74), имеется ошибка, на которую мое внимание любезно обратил Боте (H. G. Bothe). Утверждение индуктивной леммы (стр. 30), что можно пристроить к дереву отрезок так, чтобы выполнялось требование 1.a) (незамыкание молний), уже при $n = 1$ противоречит теореме Паппа. Этого затруднения можно избежать, если строить дерево не из отрезков прямых, а из надлежащим образом искривленных простых дуг. Техническое проведение этой идеи довольно громоздко. Текст со всеми необходимыми исправлениями будет опубликован издательством Deutsche Verlag der Wissenschaften в серии Mathematische Forschungsberichte в переводе Боте, которому я очень благодарен за внимание и бдительность.

В. И. Арнольд

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УСПЕХИ МАТЕМАТИЧЕСКИХ НАУК

ДИНАМИЧЕСКИЕ СИСТЕМЫ И ПРЕДСТАВЛЕНИЯ ГРУПП НА СТОКГОЛЬМСКОМ МАТЕМАТИЧЕСКОМ КОНГРЕССЕ*

Динамические системы — одно из наиболее бурно развивающихся в настоящее время направлений математики. На Стокгольмском конгрессе собственно динамическим системам было посвящено три получасовых обзорных доклада и несколько 15-минутных сообщений. В этой заметке будет рассказано о содержании некоторых из этих докладов и о других вопросах, так или иначе касающихся теории динамических систем и затронутых на Стокгольмском математическом конгрессе.

1. Проблемы устойчивости. Вопрос об устойчивости движений в консервативных нелинейных системах, имеющих большое значение для механики и астрономии, остался нерешенным в посвященных ему классических работах А. Пуанкаре, А. М. Ляпунова и Дж. Д. Биркгофа. В последние годы, начиная с работ К. Л. Зигеля [1] и особенно после доклада А. Н. Колмогорова на Амстердамском конгрессе 1954 г. [2], в этой области имеется существенное продвижение.

Основную трудность в проблемах устойчивости представляют «малые знаменатели» в рядах теории возмущений:

$$\sum_{m,\, n\neq 0} \frac{a_{mn}}{m\omega_1+n\omega_2} \cos\left[(m\omega_1+n\omega_2)\, t+\varphi_{mn}\right]. \qquad (1)$$

Здесь некоторые знаменатели $m\omega_1+n\omega_2$ обращаются в 0 при соизмеримых ω_1 и ω_2. Даже если ω_1 и ω_2 несоизмеримы, среди знаменателей встречаются сколь угодно малые. Однако для большинства (в смысле меры Лебега) пар ω_1, ω_2 величина $|m\omega_1+n\omega_2|$ превосходит $C(|m|+|n|)^{-3}$ при некотором $C>0$. Поэтому знаменатели $m\omega_1+n\omega_2$ лишь немного ухудшают сходимость ряда $\sum a_{mn}$. В теории возмущений ряды вида (1) встречаются в каждом приближении; А. Н. Колмогоров предложил вариант теории возмущений, основанный на методе Ньютона, в котором ошибка каждого следующего приближения порядка квадрата ошибки предыдущего. Получается столь быстрая сходимость, что умеренно малые (допускающие указанную выше оценку снизу) знаменатели не могут ее разрушить.

Это направление исследований было представлено на Стокгольмском конгрессе получасовым докладом Ю. Мозера и 15-минутным докладом В. И. Арнольда. Недавно Мозеру удалось, комбинируя предложенны

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Колмогоровым вариант метода Ньютона с подходящим процессом сглаживания, восходящим к Нэшу, перенести теоремы, доказанные ранее для дифференциальных уравнений с аналитическими коэффициентами, на случай коэффициентов, дифференцируемых конечное число раз.

Приведем, в частности, результат Мозера [3] об устойчивости неподвижной точки сохраняющего площадь отображения плоской области на себя. Эта задача была поставлена еще Пуанкаре в связи с его исследованиями ограниченной задачи трех тел. Как известно, к этой задаче можно свести вопрос об устойчивости периодических движений в консервативной системе с двумя степенями свободы.

Для устойчивости необходимо, во всяком случае, чтобы собственные значения линейной части преобразования в неподвижной точке имели модуль 1: $\lambda = e^{2\pi i\alpha}$, $\overline{\lambda}=e^{-2\pi i\alpha}$. Мозер доказал устойчивость в следующих предположениях:

1) преобразование 333 раза дифференцируемо,

2) $\alpha \neq \frac{m}{3}, \frac{m}{4}$, где m целое,

3) $c \neq 0$, где c — некоторый инвариант, вычисляемый по значениям нескольких производных преобразования в нуле.

До работ Мозера устойчивость была доказана лишь в предположении [4], что

1) преобразование аналитично,

2) α иррационально,

3) $c \neq 0$.

В своем докладе Мозер указал также, что эти исследования были применены в США к задачам об адиабатическом инварианте в связи с интересом к движению заряженных частиц в магнитном поле. Аналогичные советские работы [4]—[6], опубликованные годом раньше, в США еще неизвестны.

Доклад В. И. Арнольда был посвящен задачам теории возмущений, связанным с вырождением, в том числе задаче многих тел (см. [4]—[6]).

2. Грубые системы. При сколь угодно малом изменении коэффициентов дифференциального уравнения особая точка типа «центр» может превратиться в фокус; но фокус при достаточно малых изменениях остается фокусом. В этом смысле фокус «структурно устойчив» или «груб» (по терминологии А. А. Андронова и Л. С. Понтрягина [7]), а центр — нет. Точнее, система грубая, если при малом (с производными) изменении поля направлений она остается гомеоморфной себе, т. е. можно взаимно однозначно и непрерывно отобразить фазовое пространство возмущенной и невозмущенной систем друг на друга так, чтобы траектории перешли в траектории.

Интерес к грубым системам станет понятен, если учесть, что в приложениях коэффициенты уравнений никогда не известны абсолютно точно. Андронов и Понтрягин исследовали грубые системы на плоскости; в таких системах все неподвижные точки — узлы, фокусы или седла, а все периодические решения — предельные циклы. В последнее время интерес к этой области вновь пробудился, появились работы Пейксото, Маркуса и др.

Существенно свежей была идея американского тополога С. Смэйла, высказавшего в 1961 г. во время пребывания в Москве гипотезу, что грубой может быть и консервативная система (а именно, например, автоморфизм тора). Так как в консервативной системе асимптотически устойчивые движения невозможны, это принципиально новый тип грубой системы (рассматриваемая система имеет всюду плотные траектории, а также обладает всюду плотным множеством периодических траекторий; она эргодична и имеет весьма сильное перемешивание, а также положительную энтропию; вся эта сложная картина структурно устойчива).

Указанная гипотеза Смэйла ко времени Стокгольмского конгресса была доказана; Д. В. Аносов [8] нашел широкий класс грубых консервативных систем, включающий, в частности, геодезические потоки на поверхностях отрицательной кривизны. 15-минутный доклад Аносова был зачитан на конгрессе Я. Г. Синаем и вызвал большой интерес.

Получасовой доклад Смэйла был также посвящен проблеме структурной устойчивости. Докладчик высказал ряд интересных гипотез.

Основная гипотеза состоит в том, что структурно устойчивые системы составляют в пространстве всех динамических систем на данном многообразии открытое всюду плотное множество.

Эта гипотеза подтверждается всеми известными сейчас примерами. Смэйл предложил некоторый подход к доказательству основной гипотезы.

Рассмотрим динамическую систему с дискретным временем, т. е. дифференцируемое отображение T. Во всех известных примерах структурно устойчивые T обладают следующими свойствами:

1) Все неподвижные точки x_i степеней T элементарны, т. е. модули собственных значений их линейных частей не равны 1.

Тогда через каждую точку x_i проходят два инвариантных многообразия x_i^+ и x_i^-, стягивающихся к x_i при $t \to +\infty$ и $-\infty$.

2) Любые два многообразия x_i^+ и x_j^+ пересекаются трансверсально, а x_i^+ и x_j^+ (x_i^- и x_j^-) не пересекаются.

3) Любая точка при $t \to \infty$ стремится к замыканию множества точек x_i.

Указанный выше подход Смэйла к основной гипотезе о структурной устойчивости состоит в расчленении ее на две гипотезы:

A) Из свойств типа 1)—3) вытекает структурная устойчивость.

B) В любой окрестности любой системы есть система со свойствами 1)—3).

Нам представляется, что основная трудность в доказательстве предложения B). Зейферт давно уже высказал гипотезу, что всякое векторное поле на трехмерной сфере имеет замкнутую интегральную кривую. Однако это не доказано: неизвестно даже, можно ли получить замкнутую кривую малым изменением поля.

3. Теория слоений. Интересными были беседы с Г. Рибом (G. Reeb) (Франция), одним из основателей теории «foliations» (в качестве русского термина можно предложить «слоение», так как наиболее подходящий термин «расслоение» уже занят).

Пусть M есть n-мерное дифференцируемое многообразие; рассмотрим разбиение M на k-мерные (вообще говоря, незамкнутые) дифференцируемые многообразия («слои»). Это разбиение называется слоением (слоением без особенностей), если окрестность каждой точки M может быть дифференцируемо и взаимно однозначно отображена на куб n-мерного евклидова пространства $x_1, \ldots, x_n$ так, что слои отображаются в k-мерные плоскости $x_{k+1}=c_{k+1}, \ldots$

Понятие слоения шире, чем расслоенное пространство (= расслоение = = косое произведение), так как слои вообще не образуют топологического пространства (базы). Пример слоения: обмотка тора с иррациональным числом вращения. Теория слоений — это теория поверхностей, определенных дифференциальными уравнениями; обычная качественная теория обыкновенных уравнений—теория слоений с 1-мерными слоями. Многие факты качественной теории, установленные Пуанкаре для $n=2$, $k=1$, наиболее естественно переносятся на слоения с n — 1-мерными (а не 1-мерными) слоями.

При изучении поведения решений обыкновенных дифференциальных уравнений в комплексной области, естественно, возникают слоения с двумерными слоями. Другие интересные слоения образованы орбитами групп Ли, не транзитивно действующих на многообразии M.

В качестве одного из приложений укажем на существенную роль слоений, определяемых многообразиями x^+ и x^-, для исследования структурной устойчивости (см. п. 2). Теория слоений очень молода; одним из первых ее результатов было установление следующего факта: на трехмерной сфере существует бесконечно дифференцируемое слоение с двумерными слоями (Г. Риб [9]) и не существует аналитического (А. Хэфлигер [10]).

4. Эргодическая теория на Стокгольмском конгрессе. На конгрессе присутствовало довольно много математиков, занимающихся эргодической теорией. Так, там были Чакон, Орнштейн, Л. Грин, Маркус, Хан, Ауслендер, Хедлунд, Биллингсли, Якобс и др. Но большинство из них докладов не делало. Дело в том, что сравнительно незадолго до конгресса в Новом Орлеане (США) проходил симпозиум по эргодической теории. Как удалось выяснить из частных разговоров, в Америке весьма интенсивно занимаются обобщением эргодических теорем для пространств с бесконечной мерой. Это, по-видимому, и было самым новым из того, что докладывалось на этом симпозиуме. Кроме того, там же был прочитан реферат советских работ об энтропии динамических систем (Биллингсли) и Якобс дал свое доказательство одного результата Рохлина об энтропии неэргодического автоморфизма.

На Стокгольмском конгрессе собственно к эргодической теории относилось три доклада: 1) Л. Грина, 2) А. Маркуса, Л. Ауслендера и Ф. Хана (все США) и 3) Я. Г. Синая. В первых двух доложенных работах исследуются некоторые динамические системы, порожденные однопараметрическими подгруппами движений на однородных пространствах (общая теоретико-групповая конструкция таких динамических систем принадлежит Гельфанду и Фомину [11]). Так, доклад Маркуса, Ауслендера и Хана содержал исследование динамических систем на компактных нильмногообразиях, т. е. на многообразиях, получающихся факторизацией нильпотентных групп Ли по дискретным подгруппам. Ими была показана эргодичность и вычислен спектр таких динамических систем. При участии Грина им удалось доказать еще и минимальность этих динамических систем, т. е. существование единственной инвариантной меры.

Возможно, что динамические системы на нильмногообразиях следует считать наиболее естественным аналогом для случая непрерывного времени автоморфизмов с квазидискретным спектром, для которых довольно полная классификационная теория построена Л. М. Абрамовым [12]. Доклад Грина был посвящен различным типам динамических систем на разрешимых многообразиях.

Доклад Я. Г. Синая был посвящен обзору последних советских работ по энтропии и K-системам. Был еще и ряд интересных результатов, пограничных между эргодической теорией и другими разделами математики, в частности, теории вероятностей. Так, М. Розенблатт (США) рассмотрел следующий вопрос. Пусть $f(t)$ — четная положительно определенная функция. Рассмотрим собственные значения интегрального оператора

$$\int_{-T}^{T} f(t-\tau)\varphi(\tau)\,d\tau=\lambda\varphi(t).$$

М. Розенблатт при некоторых предположениях находит асимптотику j-го по величине собственного значения этого оператора при $T\to\infty$. Некоторые предварительные результаты на эту тему имеются в книге Гренандера и Сеге «Формы Теплица и их применения», а также были получены в несколько ином виде М. С. Пинскером.

В. Рудин (США) сделал доклад о продолжении положительно определенных обобщенных функций в случае n переменных. М. Г. Крейн еще довольно давно показал, что заданная на отрезке [0, 1] положительно определенная функция может быть продолжена на всю прямую с сохранением положительной определенности. Оказывается, что при $n\geqslant 2$ существуют положительно определенные функции, заданные внутри единичного n-мерного куба и не продолжаемые на все пространство. Это связано с существованием при $n\geqslant 2$ положительных полиномов, непредставимых в виде суммы квадратов (теорема Гильберта).

П. Мазани сделал доклад о разложении Вольда для полугрупп изометрических операторов гильбертова пространства. По-видимому, ему остались неизвестными старые результаты А. И. Плеснера [13], из которых его результаты вытекают.

5. Представления групп. В работе конгресса принимали участие многие специалисты по теории представлений: Г. Макки, Дж. Фелл, Р. Кейдисон, Ф. Маутнер, Л. Эренпрайс, Дж. Эрнест, А. Стейн и др. (США), Ж. Диксмье и Ф. Брюа (Франция), И. Сатаке и Р. Такахаши (Япония). К сожалению, не смогли приехать Хариш-Чандра (США) и Р. Годман (Франция). Из советских математиков доклады представили И. М. Гельфанд, М. А. Наймарк и А. А. Кириллов. Почти все зарубежные математики не делали докладов. Тем не менее в течение всего конгресса интенсивно работала «кулуарная секция» теории представлений, где обсуждались последние достижения и перспективы дальнейшего развития теории.

Вот основные направления, в которых работают сейчас зарубежные математики.

1) Вопросы, связанные с понятием виртуальной подгруппы и ее представлений. Введением в эту область может служить обзор Г. Макки по теории

представлений, перевод которого помещен в сборнике «Математика» 6 : 6 (1962). Отметим, что здесь могут оказаться полезными язык и методы гомологической алгебры. Так, описание представлений виртуальных подгрупп эквивалентно вычислению одномерной группы когомологий с коэффициентами в некоторых специальных G-пучках. Это направление связано также с задачей классификации динамических систем. Каждой эргодической динамической системе с непрерывным (соответственно с дискретным) временем соответствует виртуальная подгруппа аддитивной группы вещественных (соответственно целых) чисел. И. М. Гельфанд высказал гипотезу о том, что динамическая система полностью определяется набором всех неприводимых представлений соответствующей виртуальной подгруппы.

2) Изучение двойственного объекта для групп и C^*-алгебр. Наиболее крупные достижения в этой области принадлежат Феллу, Диксмье и Глимму. В частности, из совокупности их результатов вытекает совпадение трех важных классов C^*-алгебр, выделенных разными авторами по разным причинам: класса GCR-алгебр в смысле Капланского, алгебр с гладкой двойственной в смысле Макки и алгебр типа I в смысле фон Неймана — Диксмье. Для этого класса алгебр наиболее естественно строится теория характеров, доказана однозначность разложения произвольного представления в прямой интеграл неприводимых представлений, получено много результатов о топологической и борелевской структуре множества неприводимых представлений (см. [14] и указанную там литературу). Из результатов в этом направлении отметим работы Фелла [15], [16], в которых дано полное описание топологии в двойственном пространстве для группы A_n комплексных унимодулярных матриц $(n+1)$-го порядка и изучена структура групповой C^*-алгебры для группы A_1.

3) Дальнейшее изучение представлений вещественных полупростых групп Ли. Здесь имеется много частных результатов, посвященных представлениям отдельных групп. Отметим, например, работы Диксмье и Такахаши о представлениях группы де Сите (группы движений четырехмерного пространства Лобачевского). Общей теорией представлений вещественных полупростых групп Ли много занимается сейчас Хариш-Чандра.

4) Представления простых групп над полем p-адических чисел. Первые результаты здесь были получены Ф. Маутнером, который нашел сферические функции, связанные с группой унимодулярных матриц второго порядка. Дальнейшее продвижение было сделано в работах Брюа [17], который перенес результаты Маутнера на некоторые представления классических групп над полем p-адических чисел.

Отметим, что все эти результаты носят отрывочный характер. Ни для одной группы под полем p-адических чисел не получено описание всех представлений. Неизвестно даже, принадлежат ли эти группы типу I.

Очень большой интерес вызвал обзорный доклад И. М. Гельфанда, посвященный теории представлений[1]). Американские математики организовали размножение текста доклада, не дожидаясь его опубликования в трудах

[1]) И. М. Гельфанд не присутствовал на конгрессе, и его доклад был прочитан профессором Макки (США).

конгресса. Так как на русском языке доклад не опубликован, мы изложим здесь кратко его содержание.

Первая часть доклада посвящена следующей общей задаче. Рассматривается группа G и однородное пространство X с группой движений G и стационарной подгруппой Г. Пусть $L_2(X)$ означает пространство функций на X с суммируемым квадратом по мере, инвариантной относительно движений. Задача состоит в разложении пространства $L_2(X)$ на минимальные инвариантные относительно G компоненты и в изучении возникающих «сферических» функций. В случае, когда G — полупростая группа Ли, а Г — некоторая специальная подгруппа (единичная, максимальная компактная или картановская), эта задача была решена в работе И. М. Гельфанда и М. И. Граева [18] с помощью метода орисфер. В докладе рассматривается случай, когда группа G полупростая, а подгруппа Г дискретная и такая, что $X = G/\Gamma$ компактно или имеет конечный объем. Оказывается, что метод орисфер полезен и в этой ситуации, хотя его применение существенно усложняется. Этот метод приводит к очень интересным функциям, которые автор называет дзета-функциями данного однородного пространства. Эти функции возникают следующим образом. Пусть Ω — транзитивное семейство компактных орисфер в X. (Общее определение орисфер см., например, в работе [19].) Сопоставим каждой финитной функции на X ее интегралы по орисферам из Ω. Мы получим линейное отображение пространства функций на X в некоторое пространство H функций на Ω: $f(x) \to \varphi(\omega)$. В этом пространстве H, естественно, определены два скалярных произведения: одно индуцировано из $L_2(X)$, другое — из $L_2(\Omega)$. Оказывается, что между этими скалярными произведениями существует зависимость $(\varphi, \psi)_2 = (M\varphi, \psi)_1$, где M — некоторый ограниченный положительный оператор в H. Разложим H в сумму подпространств H_k, в каждом из которых представление G кратно неприводимому представлению T_k. Так как оператор M, очевидно, перестановочен с движениями, то в каждом H_k он задается матрицей, порядок которой равен кратности T_k в H_k. Мы получаем функцию, сопоставляющую каждому индексу k некоторую матрицу. Это и есть дзета-функция, связанная с X. Таких функций существует столько, сколько имеется транзитивных семейств компактных орисфер в X. Оказывается, что введенная таким образом дзета-функция тесно связана с обычной дзета-функцией Римана и ее обобщениями. В докладе приводится явное выражение для дзета-функции, связанной с группой вещественных унимодулярных матриц n-го порядка и подгруппой целочисленных матриц.

Вторая часть доклада посвящена представлениям простых групп над конечным полем (групп Шевалле — Диксона). Известная конструкция Гельфанда — Наймарка, дающая все представления комплексных групп, в случае конечного поля недостаточна для описания всех представлений. В докладе описывается новый подход к описанию представлений, основанный на изучении категории представлений, индуцированных одномерными представлениями некоторых подгрупп (подробнее см. в работе [20], [21]). Этот подход позволяет, например, в единой форме описать представления группы унимодулярных матриц n-го порядка над любым конечным полем (а также над полем вещественных или комплексных чисел). Автор надеется, что на самом деле

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его конструкция применима и к другим полям. В докладе приводится также явная формула для дзета-функции, связанной с группами Шевалле — Диксона.

В сообщении А. А. Кириллова говорилось о последних результатах, полученных в теории представлений нильпотентных групп Ли. Подробное изложение этих результатов опубликовано в [22].

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PROOF OF A THEOREM OF A. N. KOLMOGOROV ON THE INVARIANCE OF QUASI-PERIODIC MOTIONS UNDER SMALL PERTURBATIONS OF THE HAMILTONIAN*

V.I. ARNOL'D

Contents

One of the most remarkable of A.N. Kolmogorov's mathematical achievements is his work on classical mechanics of 1954. A simple and novel idea, the combination of very classical and essentially modern methods, the solution of a 200 year-old problem, a clear geometrical picture and great breadth of outlook – these are the merits of the work. Its deficiency has been that complete proofs have never been published.

In the present paper, written for Kolmogorov's 60th birthday, an attempt is made to remedy this deficiency. All the basic ideas are set out in §1; it is my hope that the expert reader will be able to construct the proofs from them. The remaining sections are written more formally: [1] in §2 the various theorems and lemmas are formulated, and §3 contains the proofs, based on the techniques of §4.

It is assumed that the reader is familiar with the foundations of classical mechanics. It is worth noting, however, that the methods we expound are applicable not only to conservative dynamical systems, but also to more general systems of differential equations (cp. [17], [14]).

§1. Introduction

1.1 Integrable and non-integrable problems of dynamics. We shall examine conservative dynamical systems with n degrees of freedom, defined

[1] The list of notation given at the end of § 4 should be of help to the reader.

* Editor's note: translation into English published in Russian Math. Surveys 18 (1963) Translation of V.I. Arnol'd: Proof of a theorem of A.N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. Uspekhi Mat. Nauk 18:5 (1963), 13–40

by the canonical equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \qquad (p = (p_1, \ldots, p_n);\ q = (q_1, \ldots, q_n)) \tag{1}$$

with an analytic Hamiltonian $H(p, q)$. The classical methods of dynamics [1] only enable us to investigate the so-called integrable cases.

EXAMPLE 1. Suppose that the phase-space is the direct product of an n-dimensional torus by a domain of n-dimensional euclidean space. Let $q (\mathrm{mod}\ 2\pi)$ be the angular coordinates on the torus, let p vary in the euclidean space, and let the Hamiltonian depend only on p: $H = H(p)$. Hamilton's equations (1) take the form

$$\dot{p} = 0, \quad \dot{q} = \omega(p) \qquad \left(\omega = (\omega_1, \ldots, \omega_n) = \frac{\partial H}{\partial p}\right)$$

and are immediately integrable. Every torus p = const. is invariant; if the frequency ratios ω_i are incommensurable ($k_1\omega_1 + \ldots + k_n\omega_n = 0$ with integral k_i implies that $k_i = 0$), then the motion is called *quasi-periodic* with n frequency ratios $\omega_1, \ldots, \omega_n$; it is easy to prove that the trajectory $p(t)$, $q(t)$ is everywhere dense in the torus. The variables p, q of example 1 are called *operator – angle variables.*

A great many integrable problems are known at present. The solution of all these problems with n degrees of freedom is based on the fact that there exist (and can be found) n single-valued first integrals in involution.[1]

It can be shown [2] that the existence of such integrals has as a consequence the following picture of the behaviour of the trajectories in the $2n$-dimensional phase-space p, q. A certain singular $(2n-1)$-dimensional set divides the phase-space into invariant domains. Each of these is stratified into invariant n-dimensional manifolds. If the domain is bounded, then these manifolds are tori carrying the quasi-periodic motions. In such a domain we can introduce the operator-angle coordinates of example 1. If n first integrals in involution have already been found, then the canonical transformation introducing the angle variables is given by a quadrature.

EXAMPLE 2. Integrable problems: The two-body problem. The problem of motion under attraction to two fixed centres. The motion of a free point in a geodesic on a triaxial ellipsoid. A heavy, symmetric, rigid body fixed at a point on its axis. A free rigid body not subject to a gravitational field. Linear oscillations.

Non-integrable[2] problems: the n-body problem, including the so-called planar, bounded, circular problem of three bodies. The motion of a free point in a geodesic on a convex surface. A heavy, asymmetric, rigid body. Non-linear oscillations with $n > 1$ degrees of freedom.

The discovery of integrable cases was the major interest of the XIXth century (Jacobi, Liouville, Kovalevska and others). But with Poincaré's work it became clear that the general dynamical system is non-integrable,

1 Functions $f(p, q)$ and $g(p, q)$ are said to be in involution if their Poisson bracket $\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right)$ vanishes identically.

2 More precisely, "not integrated", as the proofs of non-integrability are complicated, and have been carried out rigorously only in particular cases (cf. [1], [3]).

the integrals are not only unknown, but do not exist at all, because the trajectories, in the large, do not lie on invariant n-dimensional manifolds.

1.2 Perturbation Theory. Let us suppose that the system is disturbed from an integrable motion by a small "perturbation"; in the notation of example 1

$$H(p,\ q)=H_0(p)+\varepsilon H_1(p,\ q)+\ldots, \tag{2}$$

where ε is small and H_1 has period 2π in q. According to Poincaré [4] the investigation of this case is the fundamental problem of dynamics. How does the perturbation εH_1 influence the behaviour of the trajectories, as $t\to\infty$? Will there still be invariant tori? Will the trajectory at least remain close to the torus p = const.?

A comparison of the integrable and non-integrable problems of example 2 shows the importance of these questions for mechanics. A complete answer to them would contain, in particular, a solution of the problem of the stability of the planetary system.

A special application of perturbation theory was developed long ago in astronomy for the approximate investigation of trajectories. If the canonical transformation $p,\ q\to p',\ q'$ takes H into the form

$$H(p,\ q)=H_0'(p')+\varepsilon^2 H_1'(p',\ q')+\ldots, \tag{3}$$

then during a time-interval $t\sim\varepsilon^{-1}$ the motion $p'(t),\ q'(t)$ will differ from the quasi-periodic motion described by $H_0'(p')$ by a quantity $\sim\varepsilon$. Returning to $p,\ q$ we obtain for $p(t),\ q(t)$ approximate expressions with errors of order ε, for a time-interval $t\sim\varepsilon^{-1}$. If greater precision is required, we may look for a substitution $p',\ q'\to p'',\ q''$ taking H to the form

$$H(p,\ q)=H_0''(p'')+\varepsilon^3 H_1''(p'',\ q'')+\ldots$$

The error will now be $\sim\varepsilon^3 t$. If the successive approximations converge, then in the limit we obtain $H(p,\ q)=H_0^{\infty}(p^{\infty})$, i.e. the system is integrable: the tori $p^{\infty}(p,\ q)$ = const. are invariant and are filled by the trajectories of quasi-periodic motions.

In carrying out the programme we have just described we encounter two difficulties:

1°. Small denominators. We look for a canonical transformation $p,q\to p',q'$ of the form $p=p'+\dfrac{\partial S}{\partial q}$, $q'=q+\dfrac{\partial S}{\partial p'}$, $S(p',\ q)=\sum_{k\neq 0} S_k(p')\,e^{i(kq)}$. The function $H(p,\ q)$, in the new coordinates $p',\ q'$, will be written in the form

$$H_0(p)+\varepsilon\bar{H}_1(p)+\varepsilon\tilde{H}_1(p,\ q)+\ldots=$$
$$=H_0(p')+\varepsilon\bar{H}_1(p')+\varepsilon\left[\frac{\partial H_0}{\partial p}\frac{\partial S}{\partial q}+\tilde{H}_1\right]+\varepsilon^2\ldots$$

In order to obtain (3) we must eliminate the dependence on q of terms of order ε, i.e. we must have $\left(\omega,\ \dfrac{\partial S}{\partial q}\right)+\tilde{H}_1=0$ or

$$S_k(p') = \frac{ih_k(p')}{(\omega, k)}, \quad \text{where} \quad \widetilde{H}_1(p, q) = \sum_{k \neq 0} h_k(p) e^{i(k, q)}. \qquad (4)$$

For certain "resonance" values ω the denominator (ω, k) is arbitrarily small for suitable k. These small denominators cast suspicion on the validity of our formal transformations.

2^0. Divergence of Approximations. There are cases where the series of approximations terminates and therefore converges. Such cases were investigated in detail by Birkhoff [5]. However, Siegel [3] showed that in general, including the present case, the approximations diverge. The structure of the trajectories described in example 1 would follow from the convergence. In fact, the trajectories of the perturbed system cannot lie on the invariant tori.

Let us suppose that $\det \left|\frac{\partial \omega_i}{\partial p_j}\right| \neq 0$. Then in any neighbourhood of an invariant torus of the perturbed system there is an n-dimensional torus on which all the trajectories are closed. Under a small perturbation this n-dimensional manifold of closed trajectories collapses, in general. Consequently the series arising from the perturbation method fails, in general, to converge in any domain of the phase space.

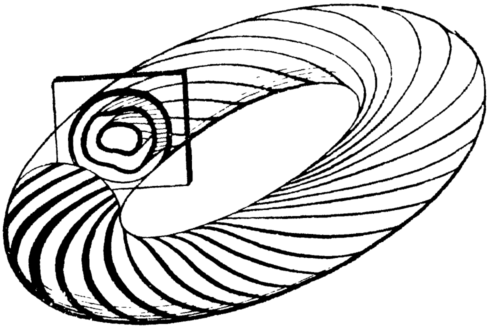

Fig. 1.

In spite of their great efforts over a long period Poincaré, Birkhoff, Siegel and others have not succeeded in making use of perturbation theory to obtain precise qualitative conclusions about the behaviour of the majority of the trajectories, as $t \to \infty$. The non-integrable problems of dynamics appeared inaccessible to the tools of modern mathematics.

1.3 The theorem of A.N. Kolmogorov. Essential progress was made in 1954, when A.N. Kolmogorov proved [6], [7] that *if* $\det \left|\frac{\partial^2 H_0}{\partial p_i \partial p_j}\right| \neq 0$, *then under a small analytic perturbation the majority of the invariant tori do not collapse but are only slightly deformed.* These tori form a nowhere dense, closed set whose complement has measure small with ε.

Apart from this little is known about the behaviour of the trajectories. In the case of a system with two degrees of freedom the phase space is four-dimensional. The three-dimensional invariant manifold H = const. is divided by two-dimensional invariant tori of the perturbed system. The complementary domain has the form of gaps through which the trajectories cannot leave, since they cannot intersect the invariant tori (see Fig. 1). For $n > 2$ the invariant n-dimensional tori do not separate the $(2n-1)$-dimensional energy levels H = const., and trajectories from the gaps can a priori go to infinity.

We now indicate some applications of A.N. Kolmogorov's theorem and its generalizations.

EXAMPLE 3.[1] The theorem is applicable to *the problem of the motion of a point on an analytic surface* close to a surface of rotation or to a triaxial ellipsiod and also enables us to establish *the stability of the motion of a planetoid* in the planar, bounded, circular three body problem.

The theorem is not applicable, however, in those cases where the motion of the unperturbed system is described by a smaller number of frequencies than in the perturbed system (so-called degeneracy), since in these cases $\det\left|\frac{\partial^2 H_0}{\partial p_i\, \partial p_j}\right| \equiv 0$. The so-called limiting degeneracy occurring in the theory of oscillations when investigating the stability of equilibrium configurations and periodic motions is not covered. The paper [6] stimulated a series of studies in this direction [8], [9].

EXAMPLE 4. The *stability of equilibrium configurations* and periodic motions is established for systems with two degrees of freedom in the so-called general elliptic case [8], in particular the *stability of the Lagrange periodic solution* of the planar, bounded, circular three body problem [9].

The n-dimensional invariant tori for the perturbation of a degenerate system carrying out a motion with $k < n$ frequencies arise from sets of k-dimensional tori completed by quasi-periodic trajectories with k "fast" and $n - k$ "slow" frequencies. This phenomenon was studied in [10], [11], [12].

EXAMPLE 5. The *eternal adiabatic invariance* of the variables of the system is established for a non-linear oscillating system with one degree of freedom subject to a slow periodic variation of the Hamiltonian. The *eternal retention of a charged particle* is established for an axially symmetric magnetic field [11]. In the n-body problem it is proved that if the masses, eccentricities, and inclinations of the planets are sufficiently small, then for the majority of initial conditions the motion is quasi-periodic and *the major arcs of the orbits always remain near their initial positions*, (in the case of three bodies the eccentricities need only be bounded from above, but not necessarily by a very small constant) [12].

The recent important papers by J. Mozer [13], [14] abandon the requirement of analyticity of the Hamiltonian and substitute instead the requirement that several hundred derivatives exist. This progress is very significant and rather unexpected. Moser makes use of a method of Newton's type, proposed by A.N. Kolmogorov, in combination with a smoothing process analogous to that introduced by Nash [15].

1.4. Newton's method. The following variant of perturbation theory is at the basis of Kolmogorov's proof. Let us suppose that the perturbation $\varepsilon H_1 + \ldots$ in (2) admits the estimate $|\varepsilon H_1 + \ldots| \ll M \ll 1$ for $|\mathrm{Im}\, q| \leqslant \rho$. Then the coefficients h_k decrease in geometric progression

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[1] We restrict ourselves here to examples explicitly considered by A.N. Kolmogorov in his lectures in Moscow in 1957 and in Paris in 1956.

like $Me^{-|k|\rho}$. As is known from the theory of diophantine approximation, for the majority of ω the small denominators (ω, k) admit a lower bound of the form $|(\omega, k)| \geqslant K|k|^{-(n+1)}$. With the aid of this bound we obtain for $\varepsilon^2 H_1' + \ldots$ a bound of the form $M^2 \delta^{-\nu}$ for $|\mathrm{Im}\ q'| \leqslant \rho - \delta$. The next approximation gives an error of order M^4, and so on, the error squaring each time, as is typical of Newton's method of tangents [16]. A rapid convergence of this nature overcomes the influence of the small denominators, and the series converges for the majority of the ω. To prove convergence we choose a sufficiently small $\delta_1 > 0$, sufficiently large T, and with $M \leqslant \delta_1^T$ we establish for the s-th approximation the inequalities $|H^{(s)}| \leqslant \delta_s^T = M_s$ for $|\mathrm{Im}\ q^{(s)}| \leqslant \rho_s$, $\rho_s > \rho_\infty > 0$, $\delta_{s+1} = \delta_s^{3/2}$ $(s = 1, 2, \ldots)$.

Kolmogorov published only a sketch of a proof of his theorem[1]. A detailed proof is set out below, but it should be noted that the author of the present paper is alone responsible for the unwieldy details of the proof. These details are probably very different from those of the original proof. Our account is constructed so that it can easily be generalized to the more complicated cases [8]-[12]. We restrict ourselves to the analytic case and do not make use of Mozer's results.

In the construction proposed by Kolmogorov each invariant torus of the perturbed system is found by the aid of a sequence of approximations constructed in decreasing neighbourhoods of the unknown torus. Also, in the formula (4) the collection of frequencies ω is fixed in advance and does not depend on p.

In our proof we return to the original idea of the perturbation method, but do not fix ω and consider it in (4) as a function $\omega(p')$. In order to avoid dealing with an infinite number of small denominators at once we restrict ourselves in each approximation to a finite number N_s of harmonics, each time relating the leading harmonics to the terms of highest order. Thanks to this we are able to manage without Borel's monogenic functions in obtaining a bound for the measure of the complementary domain.

1.5 Unsolved problems. The methods developed here can certainly be applied to various concrete problems of dynamics, for example the investigation of moon orbits, asymmetrical tops, and the discovery of the so-called magnetic surfaces. However, I want to dwell on some problem of a more fundamental nature (see also [20]).

1°. *Zones of Instability.* How do the trajectories that begin in the "gaps" of 1.3 behave? Can they, for $n > 2$, depart very far from the torus p = const.? In particular, are the equilibrium configurations and periodic solutions of general elliptic type stable when the number of degrees of freedom n exceeds 2? The simplest problem is the canonical mapping of the four-dimensional space.

For $n = 2$ the motion in a zone of instability is of a different character (see [5], [21]). As a rough model we can consider the permutation of the sub-intervals $\Delta_1 = [0, a)$, $\Delta_2 = [a, b)$, $\Delta_3 = [b, 1)$ of the interval $[0, 1)$, into the order $\Delta_3, \Delta_2, \Delta_1$.

2°. *Large Perturbations.* Quasi-periodic motions are observed only for very small values of the perturbation parameter ε. Do they occur also for large perturbations? For the n-body problem, with any values for the

1 For the first detailed account of the method see [17].

masses, does there exist a set of initial conditions of positive measure giving rise to bounded motions?

The following typical problems are also interesting in their own right: the reduction to rotations of analytic mappings of a circle onto itself (cp. [17]), and the Floquet theory for linear differential equations with quasi-periodic coefficients (cp. [22], [23]).

3°. *Dynamical systems of classical mechanics.* By a *dynamical system* we understand a one-parameter group of measure-preserving transformations of a smooth manifold, defined by differential equations ([5], [7]). Suppose that the canonical equations with Hamiltonian $H(p, q)$ have the first integrals $F_1 = H, F_2, \ldots, F_k$ ($F_i(p, q)$ – univalent functions). Then on each invariant manifold $M : F =$ const. we have a dynamical system. For example geodesic streams (see, for example, [24]) and quasi-periodic motions (see 1.1) can be considered in this way. A series of other systems has recently been studied ([25], [26]). Do they occur in mechanics, in particular for $H = T + U$ (where T is the kinetic and U the potential energy)? What restrictions must be imposed on the topology of the manifold M? These questions are related to the study of canonical and contact structures on manifolds, and demand an examination in the large of the theorems of classical dynamics (see [2]).

§2. Formulation of the Theorems

2.1. THEOREM 1. *Suppose that the Hamiltonian function $H(p, q)$ is analytic in the domain $F : p \in G$, $|\operatorname{Im} q| \leqslant \rho$ and has period 2π in $q = q_1, \ldots, q_n$. Let $H = H_0(p) + H_1(p, q)$, where*[1] *in the domain F*

$$\det \left| \frac{\partial^2 H_0}{\partial p_i \, \partial p_j} \right| \not\equiv 0. \tag{1}$$

Then for any $\varkappa > 0$ there exists $M = M(\varkappa, \rho, G, H_0) > 0$ such that if in F we have

$$|H_1| \leqslant M, \tag{2}$$

then the motion defined by the canonical equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}, \tag{3}$$

has the following properties:

1°. *There exists a decomposition* $\operatorname{Re} F = F_1 + F_2$, *where F_1 is invariant (i.e. together with the point p, q contains the trajectory $p(t)$, $q(t)$ of the motion (3) passing through it), and F_2 is small:* mes $F_2 \leqslant \varkappa$ mes F.

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[1] Instead of (1) it is sufficient that the determinant

$$\det \begin{vmatrix} \dfrac{\partial^2 H_0}{\partial p_i \, \partial p_j} & \dfrac{\partial H_0}{\partial p_i} \\ \dfrac{\partial H_0}{\partial p_j} & 0 \end{vmatrix}$$

of order $n + 1$ should not vanish.

2°. *F_1 is composed of invariant n-dimensional analytic tori I_ω, defined parametrically by the equations*

$$p = p_\omega + f_\omega(Q), \quad q = Q + g_\omega(Q), \tag{4}$$

where f_ω, g_ω are analytic functions of period 2π in $Q = Q_1, \ldots, Q_n$, and ω is a parameter determining the torus I_ω.

3°. *The invariant tori I_ω differ little from the tori $p = p_\omega$:*

$$|f_\omega(Q)| < \varkappa, \quad |g_\omega(Q)| < \varkappa. \tag{5}$$

4°. *The motion (3) on the torus I_ω is quasi-periodic with n frequencies $\omega_1, \ldots, \omega_n$:*

$$\dot{Q} = \omega, \text{ where } \omega = \left.\frac{\partial H_0}{\partial p}\right|_{p_\omega} \tag{6}$$

Theorem 1 is proved in §3 with the aid of the construction provided by the following inductive theorem.

2.2 THEOREM 2. *Consider the function $H(p, q)$, the domains F, G, Ω, and the positive numbers D, M, Θ, θ, ρ, β, γ, δ, κ. We suppose that:*

1°. *In the domain $F(p \in G, |\mathrm{Im}\, q| \leqslant \rho \leqslant 1)$ the function*

$$H = H_0(p) + H_1(p, q)$$

is analytic and

$$|H_1| \leqslant M, \quad \theta|dp| \leqslant |dA| \leqslant \Theta|dp|, \quad 0 < \theta < 1 < \Theta < \infty,$$

where A is a diffeomorphism $p \to \omega = \dfrac{\partial H_0}{\partial p}$ of the domain G onto the domain Ω of the type D (see §4, 4.1).

2°. *The inequality*

$$\delta < \delta^{(5)}(n, \theta, \Theta, \varrho, \varkappa, D) =$$
$$= \min\{\delta^{(1)}(n; 0{,}5\theta; 2\Theta);\ \delta^{(2)}(n, \varkappa, \varrho);\ \delta^{(3)}(n, \theta, \Theta, \varkappa, D);\ (\delta^{(4)}(\varkappa, \theta)\}$$

is satisfied, where $\delta^{(1)}$ is defined in 2.3,

$$\delta^{(2)} = \min\{10^{-4n}\varrho^{4n};\ 4^{-4n};\ \varkappa\}, \quad \delta^{(4)} = \varkappa(2+\theta^{-1})^{-1},$$
$$\delta^{(3)} = \min\{e^{2n}(32n^2+100n)^{-2n};\ (6+14\Theta)^{-1};\ 4^{-n-2}\varkappa\theta^n\Theta^{-n}D^{-1}n^{-1}\}.$$

3°. *Let $\beta = \delta^3$, $\gamma = \delta^{\frac{1}{4}n}$, $M = \delta^T$, $T = 8n + 24$. Suppose also that $\delta_1 = \delta$, and for $s \geqslant 1$ put*

$$\delta_{s+1} = \delta_{s-1}^{3/2}, \qquad \beta_s = \delta_s^3, \qquad \gamma_s = \delta_s^{\frac{1}{4}n}, \qquad M_s = \delta_s^T.$$

Then under the hypotheses 1°, 2°, 3° there exists a sequence of domains $F_0 = F, F_1, F_2, \ldots$ of the form F_s: $P_s \in G_s$, $|\mathrm{Im}\, Q_s| \leqslant \rho_s$ and a sequence of canonical diffeomorphisms $B_s : P_s, Q_s \to P_{s-1}, Q_{s-1}$ of the domains F_s into F_{s-1} such that for $s \geqslant 1$

1. 1. $|B_s - E| < \beta_s$, $|dB_s| < 2|dx_s|$, $F_s \subset F_{s-1} - \beta_s$,

$$\varrho_s > \frac{\varrho}{3}.$$

2. *For $p, q = B_1 B_2 \ldots B_s(P_s, Q_s)$ where $P_s, Q_s \in F_s$ we have*

$$H(p,\ q) = H^{(s)}(P_s,\ Q_s) = H_0^{(s)}(P_s) + H_1^{(s)}(P_s,\ Q_s),$$

$$|H_1^{(s)}| \leqslant M_{s+1}, \quad \left|\frac{\partial H_1^{(s)}}{\partial x_s}\right| < \delta_s\beta_{s+1}, \quad \left|\frac{\partial^2 H_1^{(s)}}{\partial x_s^2}\right| < \delta_s \quad (x_s = P_s,\ Q_s)$$

3. *The mapping* $A_s : P_s \to \frac{\partial H_0^{(s)}}{\partial P_s}$ *is a diffeomorphism of the domain* G_s, *such that*

$$\underline{\theta}\,|dP_s| \leqslant |dA_s| \leqslant \overline{\Theta}\,|dP_s|, \quad |A_s - A_{s-1}| < \beta_s\delta_s, \ \textit{where}\ \underline{\theta} = 0{,}5\theta,\ \overline{\Theta} = 2\Theta.$$

4. $\text{mes}\,(G - G_s) \leqslant \varkappa\ \text{mes}\,G$.

The proof of Theorem 2 is given in §3 and makes use of the inductive structure B_s, F_s; each step being based on the following lemma.

2.3 The inductive lemma. *We consider the function* $H(p,\ q)$, *domains* F, G, H, *and positive numbers* Θ, θ, ρ, β, γ, δ, M, K. *We suppose that*

1°. *In the domain* $F(p \in G,\ |\text{Im}\ q| \leqslant \rho)$ *the function*

$$H(p,\ q) = H_0(p) + H_1(p,\ q)$$

is analytic and

$$|H_1| \leqslant M, \quad \theta\,|dp| \leqslant |dA| \leqslant \Theta\,|dp|, \quad 0 < \theta < 1 < \Theta < \infty,$$

where A *is a diffeomorphism* $p \to \omega = \frac{\partial H_0}{\partial p}$ *of the domain* G *onto the domain* Ω.

2°. *The numbers* β, γ, δ, K *satisfy the inequalities*

$$\delta < \delta^{(1)}(n,\ \theta,\ \Theta) = \min\{\delta^0(n,\ 2\Theta);\ 2^{-1}n^{-1}\theta\},$$
$$10\delta < 2\gamma < \varrho \leqslant 1, \quad 3\beta < 2\delta, \quad 2\beta < K,$$

where $\delta^{(0)}$ *is defined in* 2.4.

3°. $M < \delta^\nu K\beta^2$, *where* $\nu = 2n + 3$.

Then there exists a domain $F' : P \in G_1 \subset G$, $|\text{Im}\ Q| \leqslant \rho' = \rho - 3\gamma$ *and a canonical diffeomorphism* $B : P,\ Q \to p,\ q$ *of the domain* F' *into* F, *such that*

1. $|B - E| < \beta$, $|dB| < 2\,|dx|$, $F' \subset F - \beta\ (x = P,\ Q)$.

2. $H(p,\ q) = \overline{H}(p) + H_2(P,\ Q)$ *where in the domain* $P,\ Q \in F'$

$$|H_2| < M' = M^2\delta^{-2\nu}\beta^{-2},$$

$$\left|\frac{\partial H_2}{\partial x}\right| < \frac{M'}{\beta}, \quad \left|\frac{\partial^2 H_2}{\partial x^2}\right| < \frac{2M'}{\beta^2}.$$

3. *The mapping* $A' : P \to \frac{\partial \overline{H}}{\partial p}$ *is a diffeomorphism of* G_1 *onto* Ω_1, *and* $\theta'\,|dP| < |dA'| < \Theta'\,|dP|$, *where* $\theta' = \theta(1 - \delta)$, $\Theta' = \Theta(1 + \delta)$ *and* $|A' - A| < \beta\delta$, *and in the notation of* 2.4

$$\Omega_1 = \Omega_{KN} - d,$$

where

$$d = (5 + 7\Theta)\,\beta, \quad N = \frac{1}{\gamma}\log\frac{1}{2M}.$$

4. $\text{mes}\,(G - G_1) < \theta^{-n}\,\text{mes}\,(\Omega - \overline{\Omega}_1)$, *where*

$$\overline{\Omega}_1 = \Omega_{KN} - \overline{d}, \quad \overline{d} = (6 + 7\Theta)\,\beta.$$

The proof is given in §3. The following lemma is crucial to the argument.

2.4 Fundamental lemma. *Consider the function* $H(p, q)$, *the domains* F, G, Ω, *and the positive numbers* Θ, θ, ρ, β, γ, δ, M, K. *We suppose that*

1°. *In the domain* $F(p \in G, |\operatorname{Im} q| \leqslant \rho)$ *the function*

$$H(p, q) = \overline{H}(p) + \widetilde{H}(p, q)$$

is analytic and

$$|\widetilde{H}(p, q)| \leqslant M,$$

$$\oint \widetilde{H}_1(p, q)\, dq = 0, \quad \theta|dp| \leqslant |dA| \leqslant \Theta|dp| \qquad (0<\theta<1<\Theta<\infty),$$

where A *is a diffeomorphism* $p \to \omega$ *of* G *onto the domain* Ω *(whose points are* $\omega = \frac{\partial\overline{H}}{\partial p}$*).*

2°. *The numbers* β, γ, δ, K *satisfy the inequalities*

$$\delta < \delta^{(0)}(n, \Theta) = \min\{L_1^{-1}, L_2^{-1}, L_3^{-1}\Theta^{-1}; L_4^{-1}\},$$
$$10\delta < 2\gamma < \varrho \leqslant 1, \quad 3\beta < 2\delta, \quad 2\beta < K,$$

where $L_i(n)$ *is defined in* 3.1.

3°. $M < \delta^{\nu} K \beta^2$, *where* $\nu = 2n + 3$.

We set $N = \frac{1}{\gamma}\log\frac{1}{M}$ *and* $G_{KN} = A^{-1}\Omega_{KN}$, *where* Ω_{KN} *consists of those* ω *for which* $|(\omega, k)| \geqslant K|k|^{-(n+1)}$ *for all integral* k, $0 < |k| < N$.

Then in the domain $P \in G_{KN} - 2\beta$, $|\operatorname{Im} Q| \leqslant \rho - 2\gamma$ *there exists a diffeomorphism* $B: P, Q \to p, q$ *such that*

1. $|B - E| < \beta$, $|dB| < 2|dx| \quad (x = P, Q)$.

2. $H(p, q) = \overline{H}(P) + H_2(P, Q)$, *where* $(p, q) = B(P, Q)$ *and for* $P \in G_{KN} - 2\beta$, $|\operatorname{Im} Q| \leqslant \varrho - 2\gamma$, $|H_2(P, Q)| < M^2\delta^{-2\nu}\beta^{-2}$.

§3. Proofs

3.1. Proof of the fundamental lemma. 1°. The canonical transformation with generating function $Pq + S(P, q)$

$$p = P + S_q, \quad Q = q + S_p \tag{1}$$

takes $H(p, q)$ into the form

$$H(p, q) = \overline{H}(P) + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4(P, Q), \tag{2}$$

where

$$\Sigma_1 = (\omega(P), S_q) + [\widetilde{H}(P, q)]_N,$$
$$\Sigma_2 = \overline{H}(p) - \overline{H}(P) - \left(p - P, \frac{\partial\overline{H}}{\partial P}\right),$$
$$\Sigma_3 = \widetilde{H}(P, q) - [\widetilde{H}(P, q)]_N,$$

$$\Sigma_4 = \widetilde{H}(p, q) - \widetilde{H}(P, q),$$
$$\widetilde{H}(P, q) = \sum{}' h_k(P) e^{i(k, q)},$$
$$[\widetilde{H}(P, q)]_N = \sum_{0<|k|<N} h_k(P) e^{i(k, q)}, \quad \omega(P) = \frac{\partial \overline{H}(P)}{\partial P},$$

and the variables p, q are given in terms of P, Q by (1).

2°. In order that $\Sigma_1 \equiv 0$ we set

$$S(P, q) = \sum_{0<|k|<N} S_k(P) e^{i(k, q)},$$

where

$$S_k(P) = \frac{ih_k(P)}{(\omega(P), k)}. \tag{3}$$

For $P \in G_{KN}$ we have $|(\omega, k)| \geqslant K|k|^{-(n+1)}$ $(0 < |k| < N)$. By 2°A of 4.2 and condition 1°. of Lemma 2.4 we have $|h_k| \leqslant Me^{-|k|\rho}$. Hence by (8) 1°. of 4.2 we have

$$|S_k(P)| \leqslant \frac{Me^{-|k|\rho}}{k} \frac{L_0}{\delta^{\nu_1}} e^{|k|\delta} = M \frac{L_0}{K\delta^{\nu_1}} e^{-|k|(\rho-\delta)},$$

where $\nu_1 = n + 1$, $L_0 = \nu_1^{\nu_1} e^{-\nu_1}$ and so by 2°B of 4.2 we have, for $P \in G_{KN}$ and $|\mathrm{Im}\, q| \leqslant \rho - 2\delta$,

$$|S(P, q)| < \frac{ML_5}{K\delta^{\nu_2}} \qquad (L_5 = 4^n L_0, \ \nu_2 = 2n + 1).$$

3°. Since $M < \delta^{\nu} K \beta^2$, $\delta < L_2^{-1}$ $(L_2 = 16nL_5,\ \nu = 2n + 3)$ we have $\frac{ML_5}{K\delta^{\nu_2}} < \frac{\beta^2}{16n}$. Thus, by 3°. of 4.3, the equations (1) define a canonical diffeomorphism B of the domain

$$P \in G_{KN} - 2\beta, \ |\mathrm{Im}\, Q| \leqslant \varrho - 5\delta < \varrho - 2\delta - 3\beta$$

(since $3\beta \leqslant 2\delta$), moreover

$$|B - E| < \frac{ML_5}{K\beta\delta^{\nu_2}} < \beta, \quad |dB| < 2|dx|, \quad |P - p| < \frac{ML_5}{K\delta^{\nu_2+1}}$$
$$(x = P, Q).$$

4°. We estimate the quantity Σ_2 by Taylor's formula (4°. of 4.2). If

$$P \in G_{KN} - 2\beta, \ |\mathrm{Im}\, Q| \leqslant \varrho - 5\delta,$$

then from $\frac{\partial^2 \overline{H}}{\partial P_i \partial P_j} \leqslant \Theta$ it follows (in view of $\delta < L_3^{-1}\Theta^{-1}$, $L_3 = 0,5L_5^2 n^2$) that

$$|\Sigma_2| \leqslant \frac{M^2}{K^2} \frac{\Theta n^2 L_5^2}{2\delta^{2\nu_2+2}} < \frac{M^2}{K^2\delta^{2\nu_2+3}}.$$

5°. The estimate for Σ_3 is established in C) 2°. 4.2. Since $|h_k| \leqslant Me^{-|k|\rho}$, for $P \in G$, $|\mathrm{Im}\, Q| \leqslant \rho - \gamma - \delta$, $N = \frac{1}{\gamma} \log \frac{1}{M}$ we have (in view of $\delta < L_4^{-1}$, $L_4 = 2\left(\frac{2n}{e}\right)^n$)

$$|\Sigma_3| < \frac{M^2L_4}{\delta^{\nu_1}} < \frac{M^2}{\delta^{\nu_1+1}}.$$

6°. The estimate for Σ_4 is obtained from the Lagrange formula (4°. 4.2). From

$$P \in G_{KN} - 2\beta, \qquad |\operatorname{Im} Q| \leqslant \varrho - 5\delta$$

there follows

$$|\operatorname{Im} q| \leqslant \varrho - 4\delta \text{ and} |P - p| < \beta;$$

and so

$$P + \lambda(p - P) \in G_{KN} - \beta(|\lambda| \leqslant 1)$$

so that by Cauchy (3°. 4.2),

$$\left|\frac{\partial \widetilde{H}}{\partial P}\right| \leqslant \frac{M}{\beta}.$$

For $\delta < L_2^{-1}$, $L_2 = 16 L_5 n$ we have

$$|\Sigma_4| \leqslant \frac{M^2L_5n}{K\beta\delta^{\nu_2+1}} < \frac{M^2}{K\beta\delta^{\nu_2+2}}.$$

7°. Combining the estimates for Σ_2, Σ_3, Σ_4 and using the conditions $2\beta < K$, $\delta < L_1^{-1} = 12^{-1}$, $\gamma > 3\delta$, $\nu = 2n + 3$, $\nu_1 = n + 1$, $\nu_2 = 2n + 1$ we obtain, for $P \in G_{KN} - 2\beta$, $|\operatorname{Im} Q| < \rho - 2\gamma \leqslant \rho - 5\delta - \gamma$ the inequalities

$$|\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4| < M^2 \left[\frac{\delta^{-(2\nu_2+3)}}{K^2} + \delta^{-(\nu_1+1)} + \frac{\delta^{-(\nu_2+2)}}{K\beta}\right] < M^2\delta^{-2\nu}\beta^{-2},$$

as required. The constants L_i are given by

$$L_1 = 12, \quad L_2 = 4^{n+2}ne^{-(n+1)}(n+1)^{n+1}, \quad L_3 = 2^{4n-1}n^2e^{-(2n+2)}(n+1)^{2n+2},$$
$$L_4 = 2^{n+1}e^{-n}n^n.$$

3.2. Proof of the inductive lemma. 1°. We set

$$H_1(p, q) = \overline{H}_1(p) + \widetilde{H}_1(p, q),$$

where

$$\oint \widetilde{H}_1(p, q)\, dq = 0.$$

Then

$$H(p, q) = \overline{H}(p) + \widetilde{H}_1(p, q),$$

where

$$\overline{H}(p) = H_0(p) + \overline{H}_1(p).$$

Consider the mapping $A' : p \to \frac{\partial \overline{H}}{\partial p} = A(p) + \Delta(p)$. By the conditions 2°., 3°. of Lemma 2.3 we have $M < \frac{\delta\theta}{2n}\beta^2$, $\delta < 1$, $\theta < 1$. Consequently, by Cauchy (3°. 4.2), for $p \in G - \beta$ we have $|\Delta| < \frac{M}{\beta} < \beta\delta$, $|d\Delta| < \delta\theta|dp|$. We now apply the lemma of 5°. 4.3 concerning the variation of the frequency, putting $b = 3\beta$. By 5°. 4.3 A' maps $G = A'^{-1}\Omega_1$ diffeomorphically onto

$\Omega_1 = \Omega_{KN} - d$ where $d = (5 + 7\theta)\beta$ and $G_1 + 3\beta$ is mapped into Ω_{KN}. Furthermore, the conclusions 3 and 4 of the inductive lemma are also valid.

2°. We now apply the basic lemma to the function $H(p, q) = \overline{H}(p) + \widetilde{H}_1(p, q)$ in the domain F. Since, by 1°., $|\widetilde{H}_1| \leqslant 2M$ and $\theta'|dp| < |dA'| < \Theta'|dp|$, the conditions of the inductive lemma imply that we may apply the basic lemma. This latter gives us a diffeomorphism $B : P, Q \to p, q$ of the domain $P \in G_{KN} - 2\beta$, $|\mathrm{Im}\, Q| < \rho - 2\gamma$ and the inequality $|H_2(P, Q)| < M'$. Hence, by Cauchy (3°. 4.2), for $P \in G_{KN} - 2\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 3\gamma < \rho - 2\gamma - \beta$ we obtain the estimates of conclusion 2 of the inductive lemma.

3°. According to 1°., for $P, Q \in F' : P \in G_1$, $|\mathrm{Im}\, Q| \leqslant \rho - 3\gamma$ there follows $P \in G_{KN} - 3\beta$. But then, by 2°., the conclusions 1 and 2 of the inductive lemma follow, and this completes the proof.

3.3. Proof of Theorem 2. 1°. Put $K = \varkappa\,\delta_1$. We shall show that under the conditions of Theorem 2 the inductive lemma is applicable. In fact, the condition 1°. of the inductive lemma follows from condition 1°. of Theorem 2. $\delta < \delta^{(1)}$ of the inductive lemma follows from the inequality $\delta < \delta^{(5)}$ of Theorem 2. Since $\delta < \delta^{(2)}$, condition 2°. of the inductive lemma is satisfied because

$$\gamma < 0{,}1\varrho;\ \gamma < 4^{-1};$$

$$10\delta < 10\delta^{1/2}\delta^{1/4n} < 10\cdot 4^{-2n}\gamma < 2\gamma, \quad 3\delta^3 < 3\cdot 4^{-1}\delta < 2\delta,\ 2\delta^3 < \delta^2 < \delta\varkappa = K.$$

Finally, condition 3°. follows from the inequality $8n + 24 = T > \nu + 2 + 6 = 2n + 11$.

2°. Since $\delta < \delta^{(2)}$, we easily obtain for $s \geqslant 1$ the inequalities

$$\delta_s + \delta_{s+1} + \ldots < 2\delta_s, \quad 3(\gamma_s + \gamma_{s+1} + \ldots) < 6\gamma_s \leqslant 6\gamma_1 < 2\ \frac{\varrho}{3}. \tag{4}$$

By (4), if we put $\theta_0 = \theta$, $\Theta_0 = \Theta$, $\rho_0 = \rho$, $\theta_s = \theta_{s-1}(1 - \delta_s)$, $\Theta_s = \Theta_{s+1}(1 + \delta_s)$, $\rho_s = \rho_{s-1} - 3\gamma_s$ $(s = 1, 2, \ldots)$, we obtain the inequalities

$$\theta_s > \underline{\theta} = \frac{\theta}{2}, \quad \Theta_s < \overline{\Theta} = 2\Theta, \quad \varrho_s > \varrho_\infty = \frac{\varrho}{3}. \tag{5}$$

It is easy to verify that for $s \geqslant 1$ the numbers β_s, γ_s, δ_s, M_s, K satisfy the inequalities of condition 2°. of the inductive lemma with the constants θ_{s-1}, Θ_{s-1}, ρ_{s-1}. This was established in 1°. for $s = 1$.

3°. From $\delta < \delta^{(3)}$ it follows in view of the inequality 1°. 4.2, for $N_s = \dfrac{1}{\gamma_s}\log\dfrac{1}{2M_s}$ that

$$\delta_s N_s^n \leqslant \delta_s\,(\delta_s^{\frac{1}{4}}{}^n \log \delta_s^{-(T+1)})^n \leqslant \delta_s\Big(4n\,(T+1)\,e^{-1}\delta_s^{\frac{1}{2}n}\Big)^n \leqslant \delta_s^{\frac{1}{2}}(4n\,(T+1)\,e^{-1})^n < 1. \tag{6}$$

Put

$$\sigma_s = \sum_{N_{s-1} \leqslant m < N_s} m^{-2}.$$

Since $\sum \sigma_s < 2$, $\delta < \delta^{(3)}$ and (6) imply the inequalities

$$\sum_{s=1}^{\infty} [K\sigma_s + (6 + 7\Theta_s)\,\beta_s N_s^n] < \sum_{s=1}^{\infty} [K\sigma_s + \delta_s] < 4\delta_1 < \varkappa\overline{D}^{-1}, \tag{7}$$

where

$$\bar{D} = \left(2\frac{\Theta}{\theta}\right)^n LD, \qquad L = n2^{n+2}..$$

4°. Suppose that the quantities

$$A_{s-1},\ F_{s-1},\ G_{s-1},\ H^{(s-1)}(P_{s-1}, Q_{s-1}),\ \Omega_{s-1},\ \theta_{s-1},\ \Theta_{s-1},\ \varrho_{s-1} \qquad (8_{s-1})$$

and β_s, γ_s, δ_s, M_k satisfy the conditions of the inductive lemma. Then that lemma defines the quantities

$$A',\ B,\ F',\ G_1,\ \bar{H}(P) + H_2(P, Q),\ \Omega_1,\ \bar{\Omega}_1,\ \theta',\ \Theta',\ \varrho',$$

which we denote (in the notation of 2°.), respectively, by

$$A_s,\ B_s,\ F_s,\ G_s,\ H^{(s)}(P_s, Q_s) = H_0^{(s)}(P_s) + H_1^{(s)}(P_s, Q_s),\ \Omega_s,\ \bar{\Omega}_s,\ \theta_s,\ \Theta_s,\ \varrho_s. \quad (8_s)$$

From conclusion 2 of the inductive lemma in the form $T = 8n + 24$ we obtain, in F_s,

$$|H_1^{(s)}| < M_s^2\delta_s^{-2\nu}\beta_s^{-2} = \delta_s^{2T-4n-12} = \delta_s^{\frac{3}{2}T} = \delta_{s+1}^T = M_{s+1}. \qquad (9)$$

The conclusions 1, 2, 3 of the inductive lemma with 2°. and (9) imply that if the quantities (8_{s-1}) and β_s, γ_s, δ_s, M_s, K satisfy the conditions of the inductive lemma, then so do the quantities (8_s) and β_{s+1}, γ_{s+1}, δ_{s+1}, M_{s+1}, K.

5°. But by 1°. the quantities (8_0) (where $A_0 = A$, $F_0 = F$, $G_0 = G$, $H^{(0)} = H$, $P_0 = p$, $Q_0 = q$ and β_1, γ_1, δ_1, M_1, K satisfy the conditions of the inductive lemma. Consequently it can be applied for all s, and so we obtain the quantities (8_s) for $s = 1, 2, \ldots$. The conclusions 1, 2 and 3 of Theorem 2 now follow from conclusions 1, 2 and 3 of the inductive lemma in the form (5). We have, however, not yet proved that F_s is non-empty. This follows from conclusion 4 of Theorem 2, which we shall now prove.

6°. By conclusions 3 and 4 of the inductive lemma we have

$$\text{mes}\,(G_{s-1} - G_s) \leqslant \underline{\theta}^{-n}\,\text{mes}\,(\Omega_{s-1} - \bar{\Omega}_s), \qquad (10)$$

where $\bar{\Omega}_s = (\Omega_{s-1})_{KN_s} - \bar{d}_s$, $\bar{d}_s = (6 + 7\Theta_s)\beta_s$ and Ω_{s-1} are obtained from Ω by means of the formulae

$$\Omega_0 = \Omega, \qquad \Omega_m = (\Omega_{m-1})_{KN_m} - d_m,$$

$$d_m = (5 + 7\Theta_s)\,\beta_s, \qquad N_m = \frac{1}{\gamma_m}\log\frac{1}{2M_m}\ (m = 1, 2, \ldots, s-1).$$

Since $d_m > 0$, $1 < N_1 < N_2 < \ldots$ and the domain $\Omega = \Omega_0$ is of type D by condition 1°. of Theorem 2, we have by the arithmetical lemma (3°. of 4.1)

$$\text{mes}\,(\Omega_{s-1} - \Omega_s) \leqslant DL\,[K\sigma_s + (6 + 7\Theta_s)\,\beta_s N_s^n]\,\text{mes}\,\Omega. \qquad (11)$$

But since $\text{mes}\,\Omega \leqslant \Theta^n\,\text{mes}\,G$, (10), (11), (7) lead to

$$\text{mes}(G - G_s) =$$

$$= \sum_{m=1}^{s}\ \text{mes}\ (G_{m-1} - G_m) \leqslant \sum_{m=1}^{s}[K\sigma_m + (6 + 7\Theta_m)\,\beta_m N_m^n]\,\bar{D}\,\text{mes}\,G \leqslant \varkappa\,\text{mes}G. \quad (12)$$

Consequently conclusion 4 of Theorem 2 is established.

3.4. Proof of Theorem 1. In this proof all the variables are to be taken as real, unless otherwise stated.

1°. Because of inequality (1) of Theorem 1, for any $\varkappa > 0$ we may find positive numbers θ, Θ, D, m depending only on $\varkappa$, G, H_0 such that the domain G can be represented in the form $G^{(1)} \cup \ldots \cup G^{(m)} \cup \overline{G}$, where $\operatorname{mes} \overline{G} < \varkappa/2$ and each domain $G^{(i)}$ is transformed diffeomorphically by the mapping $A : p \to \dfrac{\partial H_0}{\partial p}$ into a domain $\Omega^{(i)}$ of type D (see 1°. of 4.1); moreover the inequalities

$$\theta |dp| \leqslant |dA| \leqslant \Theta |dp| \qquad (0 < \theta < 1 < \Theta < \infty).$$

are satisfied in each of the domains $G^{(i)}$.

2°. If we can find $M(\varkappa, \rho, G^{(i)}, H_0)$ in each of the domains $G^{(i)}$, then

$$M(\varkappa, \varrho, G, H_0) = \min_i M\left(\frac{\varkappa}{2m}, \varrho, G^{(i)}, H_0\right)$$

gives the proof of Theorem 1. We shall therefore assume henceforth that condition 1°. of Theorem 2 is satisfied in the domain G. We shall prove Theorem 1 assuming $M = \delta_1{}^T$, $T = 8n + 24$, $\delta_1 < \delta^{(5)}$ $(n, \theta, \Theta, \rho, \varkappa, D)$, where the constant $\delta^{(5)}$ is defined in Theorem 2. In view of (2) §1 the conditions of Theorem 2 are satisfied and so its conclusion holds.

3°. *Convergence.* Theorem 2 yields the sequences F_s and B_s. From $\beta_s = \delta_s^3 < 4^{-s}$ and the conclusion 1 of Theorem 2 all the conditions of the lemma on convergence of 1°. of 4.4 follow. According to this lemma the sequence of diffeomorphisms $S_s \equiv B_1 B_2 \ldots B_s$ $(s = 1, 2, \ldots)$ on the compactum

$$F_\infty = \bigcap_{s \geqslant 0} F_s \quad \left(P_\infty \in G_\infty, \ |\operatorname{Im} G_\infty| \leqslant \varrho_\infty, \text{ where } G_\infty = \bigcap_{s \geqslant 0} G_s, \ \varrho_\infty \geqslant \frac{\varrho}{3}\right)$$

converge to a certain mapping S_∞. From conclusion 4 of Theorem 2 $\operatorname{mes} G_\infty \geqslant (1 - \varkappa) \operatorname{mes} G$. But S_s are canonical transformations and so preserve measure, so that $\operatorname{mes} S_s F_\infty = \operatorname{mes} F_\infty$. By 4°. 4.4 it follows that

$$\operatorname{mes} S_\infty F_\infty \geqslant \lim_{s \to \infty} \operatorname{mes} S_s F_\infty = (2\pi)^n \operatorname{mes} G_\infty \geqslant (2\pi)^n (1 - \varkappa) \operatorname{mes} G = (1 - \varkappa) \operatorname{mes} F. \tag{13}$$

We set $F' = S_\infty F_\infty$ and prove the assertions 1-4 of Theorem 1.

4°. *Invariance.* It follows from conclusion 3 of Theorem 2 that the sequence of diffeomorphisms A_s converges on G_∞ to a mapping A_∞, where

$$|A_s - A_\infty| \leqslant \sum_{m=s+1}^{\infty} \beta_m \delta_m < \frac{1}{2} \beta_{s+1}. \tag{14}$$

Let us write the canonical equations with Hamiltonian $H^{(s)}(P_s, Q_s)$ in the form

$$\dot{x}_s = X_s(x_s), \quad \text{where } x_s = P_s, Q_s. \tag{15_s}$$

The transformations S_s are canonical and so if $x_s(t)$ is the solution of the equations (15_s), then $x(t) = S_s x(t)$ satisfies (15_0). We shall show that if $x_\infty = P_\infty$, $Q_\infty \in F_\infty$ putting $X_\infty = 0$, $A_\infty(P_\infty)$ and $x_\infty(t)$ is the solution of the equation (15_∞) with initial conditions in F_∞, then $x_\infty(t) \in F_\infty$ and $S_\infty x_\infty(t)$ belongs to F and satisfies (15_0).

We use lemma 3°. of 4.4. Suppose that for $x_s = P_s, Q_s$, $\overline{X}_s(x_s) = 0$, $A_s(P_s)$. By conclusion 2 of Theorem 2 we have $|X_s - \overline{X}_s| < \frac{1}{2}\beta_{s+1}$ in F_s. From (14) we obtain $|X_\infty - X_s| < \beta_{s+1}$ in F_∞. Also, from the conclusions 2 and 3 of Theorem 2 we obtain

$$\left|\frac{\partial X_s}{\partial x_s}\right| < 2n\delta_s + \overline{\Theta} < n + \overline{\Theta}.$$

Lemma 3°. of 4.4 now shows that $S_\infty x_\infty(t)$ satisfies (15_0) ≡ 3 §2.

5°. Let us introduce the notation $p_\omega = A^{-1}A_\infty p_\infty$, where $P_\infty \in G_\infty$. Since (see 4°.) $|AP_\infty - A_\infty P_\infty| < \beta$, we have $|AP_\infty - Ap_\omega| < \beta_1$ and by lemma 4°. of 4.3 $|P_\infty - p_\omega| < \beta_1\theta^{-1}$. Also, by the lemma of 3°. on convergence we have $|S_\infty - E| < 2\beta$. Thus, for P_∞, $Q_\infty \in F_\infty$ we have (from the condition $\delta < \delta^{(4)}$ of Theorem 2)

$$|S_\infty(P_\infty, Q_\infty) - (p_\omega, Q_\infty)| < \beta_1(2 + \theta^{-1}) < \varkappa. \tag{16}$$

6°. The equations $p, q = S_\infty(A_\infty^{-1}\omega, Q)$ for each fixed $P_\infty = A_\infty^{-1}\omega \in G_\infty$ can be written in the form (4) of the conclusion 2 of Theorem 1. They define the torus T_ω and the coordinates $Q = Q_\infty$ on it. The invariance of I_ω is proved in 4°. and also the equation (6) of conclusion 4 of Theorem 1. The analyticity of S_∞ with respect to Q_∞ follows from the uniform convergence of S_s for each fixed $P_\infty \in G_\infty$ in the complex domain $|\mathrm{Im}\, Q_\infty| \leqslant \rho_\infty$. Conclusion 3 of Theorem 1 follows from (16). This completes the proof of Theorem 1.

§4. Technical Lemmas

4.1. The arithmetical lemma. This lemma expresses the incommensurability of two random numbers.

1°. *Domains of type D.* Let Ω be a bounded domain in a space ω and let the boundary of Ω consist of a finite number of pieces of smooth manifolds. It is easy to see there exists a constant $D > 0$ such that for any $d_2 > d_1 > 0$

$$\mathrm{mes}\,[(\Omega - d_1)\setminus(\Omega - d_2)] \leqslant D(d_2 - d_1)\,\mathrm{mes}\,\Omega. \tag{1}$$

We shall say that Ω is a domain of type D.

An $h/2$-neighbourhood of a hyperplane will be called a *strip* Γ of width h. For example, the inequality $|(k, \omega)| \leqslant a$ defines a strip Γ, and if $|k_i| \geqslant 1$ then the width of Γ is not greater than $2a$. It is easy to see that in an n-dimensional space ω

$$\mathrm{mes}\,(\Omega \cap \Gamma) \leqslant Dnh\,\mathrm{mes}\,\Omega. \tag{2}$$

Let $\Omega' \subseteq \Omega$. We shall say that Ω' is of type N in Ω if $\Omega' = (\Omega - d)\setminus\bigcup(\Gamma_i)$ where $d > 0$ and $\bigcup\Gamma_i$ is a union of not more than N strips of any width. Clearly, for $d_2 > d_1 > 0$

$$(\Omega' - d_1)\setminus(\Omega' - d_2) \subseteq [(\Omega - d - d_1)\setminus(\Omega - d - d_2)] \cup \{[(\bigcup\Gamma_i + d_2)\setminus(\bigcup\Gamma_i + d_1)] \cap \Omega\},$$

and so by (1) and (2) we have

$$\text{mes}\,[(\Omega'-d_1)\diagdown(\Omega'-d_2)] \leqslant D\,(1+hN)\,(d_2-d_1)\,\text{mes}\,\Omega. \tag{3}$$

2°. *Integral Points.* The number of different vectors k with integral coordinates $k_1, \ldots, k_n$, in an n-dimensional space, for which $|k| = m \geqslant 1$ does not exceed $2^n m^{n-1}$. The number of vectors with $|k| \leqslant m$ does not exceed $2^n m^n$.

The proofs are obvious.

3°. *The arithmetical lemma.* Let Ω be a domain of type D. Denote by Ω_{KN} (where $K > 0$, $N > 1$) the set of $\omega \in \Omega$ for which

$$|(k, \omega)| \geqslant K\,|k|^{-\nu} \qquad (\nu = n+1) \tag{4}$$

for arbitrary integral vectors k, $0 < |k| < N$. Let $d_1, d_2, \ldots > 0$ and $1 < N_1 < N_2 < \ldots$ We introduce the domains Ω_s by the relations

$$\Omega_0 = \Omega, \quad \Omega_s = \Omega'_{s-1} - d_s, \quad \Omega'_{s-1} = (\Omega_{s-1})_{KN_s} \qquad (s = 1, 2, \ldots).$$

LEMMA. *For any $s \geqslant 1$ and any $d > 0$ we have*

$$\text{mes}\,[\Omega_{s-1}\diagdown(\Omega'_{s-1}-d)] \leqslant LD\,[K\sigma_s + dN_s^n]\,\text{mes}\,\Omega, \tag{5}$$

where

$$\sigma_s = \sum_{N_{s-1}\leqslant m<N_s} m^{-2}, \quad N_0 = 1, \qquad L = 2^{n+2}n.$$

PROOF. First let us convince ourselves that

$$\text{mes}\,[\Omega_{s-1}\diagdown\Omega'_{s-1}] \leqslant LDK\sigma_s\,\text{mes}\,\Omega. \tag{6}$$

In fact, (4) defines a strip Γ_k of width not greater than $2K|k|^{-\nu}$ by 1°. The strip Γ_k with $|k| = m$ is, by 2°., not greater than $2^n m^{n-1}$. (2) leads to

$$\sum_{|k|=m} \text{mes}\,(\Omega \cap \Gamma_k) \leqslant LDKm^{-2}\,\text{mes}\,\Omega.$$

Summing over m for $N_{s-1} \leqslant m < N_s$ we get (6). Also there are not more than $2^n m^n$ distinct k with $k \leqslant m$. Thus Ω'_{s-1} is of type $2^n N_s^n$ in Ω and according to (3)

$$\text{mes}\,[\Omega'_{s-1}\diagdown(\Omega'_{s-1}-d)] \leqslant d\,(1+2n2^nN_s^n)\,D\,\text{mes}\,\Omega. \tag{7}$$

(5) now follows from (6) and (7).

4.2. Analytical lemmas. These lemmas enable us to study the Fourier coefficients and derivatives of analytic functions in terms of the functions themselves, and conversely.

1°. *Inequalities. For any* $m > 0$, $\nu > 0$, $\delta > 0$

$$m^\nu \leqslant \left(\frac{\nu}{e}\right)^\nu \frac{e^{m\delta}}{\delta^\nu}, \quad \log\frac{1}{\delta} \leqslant \frac{\nu}{e}\left(\frac{1}{\delta}\right)^{\frac{1}{\nu}}. \tag{8}$$

In fact, the function $f(x) = x - \nu\log x$ has a minimum for $x = \nu$, so that $\frac{e^x}{x^\nu} \geqslant \frac{e^\nu}{\nu^\nu}$. Putting $x = m\delta$, $x = |\log\delta|$ we obtain (8).

2°. *Fourier coefficients. Let*

$$f(q) = \sum_k f_k e^{i(k,q)}.$$

A) *If for* $|\text{Im}\, q| \leqslant \rho$ *we have* $|f(q)| \leqslant M$, *then* $|f_k| \leqslant Me^{-|k|\rho}$.

B) *If* $|f(q)| \leqslant Me^{-|k|\rho}$, *then for* $|\text{Im}\, q| \leqslant \rho - \delta$ *(where* $0 < \delta < \rho < 1$)

$$|f(q)| < 4^n\delta^{-n}M.$$

To prove A) we have to shift the contour of integration in the formula

$$f_k = (2\pi)^{-n} \int f(q)e^{-i(k,q)}\,dq$$

to $i\rho$. To prove B):

$$|f| \leqslant \sum_k Me^{-|k|\delta} = M\Big(1 + 2\sum_{m>0} e^{-m\delta}\Big)^n = M(1+e^{-\delta})^n(1-e^{-\delta})^{-n} < 4^n\delta^{-n}M,$$

since $(1 + e^{-\delta})(1 - e^{-\delta})^{-1} < 4\delta^{-1}$ for $0 < \delta < 1$.

We introduce the notation

$$R_N f = \sum_{|k|\geqslant N} f_k e^{i(k,q)}.$$

C) *If*

$$|f_k| \leqslant Me^{-|k|\rho} \text{ and } 2\delta < \gamma,\ \delta+\gamma < \rho < 1,$$

then for

$$|\text{Im}\, q| \leqslant \rho - \delta - \gamma$$

we have

$$|R_N f| < \left(\frac{2n}{e}\right)^n \frac{M}{\delta^{n+1}} e^{-N\gamma}.$$

In fact, taking 2°. of 4.1 and (8) into account we have

$$|R_N f| < M \sum_{m\geqslant N} (2m)^n e^{-m(\delta+\gamma)} <$$

$$< M\left(\frac{2n}{e\delta}\right)^n \sum_{m\geqslant N} e^{-m\gamma} < \left(\frac{2n}{e}\right)^n \frac{M}{\delta^n}\frac{e^{-N\gamma}}{1-e^{-\gamma}} < \left(\frac{2n}{e}\right)^n \frac{M}{\delta^{n+1}} e^{-N\gamma},$$

since $1 - e^{-\gamma} > \delta$ for $\gamma > 2\delta$, $\delta < 0,5$.

3°. *Cauchy estimates.* If for $x \in U$ the function $f(x)$ is analytic and $|f(x)| \leqslant M$, then for $x \in U - \delta$

$$\left|\frac{\partial f}{\partial x}\right| \leqslant \frac{M}{\delta}, \quad \left|\frac{\partial^2 f}{\partial x^2}\right| \leqslant \frac{2M}{\delta^2}.$$

The proof is by means of the Cauchy integral

$$f(x) = \frac{1}{2\pi i}\oint \frac{f(\xi)\,d\xi}{\xi - x}.$$

4°. *The Lagrange and Taylor formulae. If in the segment ab of the space* $x = x_1, \ldots, x_n$ *the function* $f(x)$ *(vector-valued, in general:* $f = f_1, \ldots, f_m$) *satisfies the inequality* $|df| \leqslant C|dx|$, *then* $|f(b) - f(a)| \leqslant C|b - a|$. *In particular,*

$$|f(b) - f(a)| \leqslant cn|b - a|,$$

if $\left|\dfrac{\partial f_i}{\partial x_j}\right| \leqslant c$. *If in the domain* $|x_i - a_i| \leqslant |b_i - a_i|$ *the inequality*

$\frac{\partial^2 f}{\partial x_i \partial x_j} \leqslant \Theta$ *is everywhere valid, then*

$$\left| f(b) - f(a) - \left(\frac{\partial f}{\partial x}\Big|_a, \ b-a \right) \right| \leqslant \frac{\Theta n^2}{2} |b-a|^2.$$

For the proofs it suffices to write down the increments as integrals of derivatives.

4.3. Geometrical Lemmas. These lemmas guarantee the unique invertibility of the changes of variables.

1° *ε-displacement. Let U be a closed domain in the euclidean space R and let A be a continuous mapping of U into R such that $|Ax - x| < \varepsilon$. Then its image AU contains $U - \varepsilon$.*

In fact, let $x_0 \in (U - \varepsilon)\backslash AU$. Then the mapping

$$A^*x = x_0 + \varepsilon |Ax - x_0|^{-1}(Ax - x_0)$$

is continuous for $|x - x_0| \leqslant \varepsilon$. Consequently the degree $d(t)$ [27] of the mapping A^* of the sphere $S_{t\varepsilon} : |x - x_0| = t\varepsilon$ $(0 < t \leqslant 1)$ onto the sphere S_ε does not depend on t. But $d(1) = 1$, hence $x_0 \in AU$.

2°. *The inversion of the ε-displacement. Under the conditions of 1°. suppose that $|dx| \neq 0$ whenever $|dA| \neq 0$. Then A is a diffeomorphism of the domain $U - 4\varepsilon$.*

For let $x, y \in U - 4\varepsilon$ and $Ax = Ay = z$. The sphere D of radius 2ε with centre at the mid-point of the segment xy lies in $U - \varepsilon$. The image $Axy \subset D$ of the segment xy is a closed arc containing z which can be shrunk in D to z. Since $|dA| \neq 0$, it follows, by 1°. that the segment xy shrinks to a point, leaving its ends fixed. Hence $x = y$.

3°. *The canonical transformation. Let G and U be domains of n-dimensional euclidean spaces P and q. If the function $S(P, q)$ is analytic in $G \times U$ and $|S| \leqslant M \leqslant \beta^2 16^{-1} n^{-1}$, then the substitutions $p = P + S_q$, $Q = q + S_P$ define a canonical diffeomorphism $B : P, Q \to p, q$ of the domain $P \in G - 2\beta$, $Q \in U - 3\beta$ such that $|B - E| \leqslant M\beta^{-1}$, $|dB| < 2|dx|$ $(x = P, Q)$, $|P - p| \leqslant M\delta^{-1}$ for $Q \in U - 3\beta - \delta$.*

PROOF. For each $P \in G - \beta$, by Cauchy (3°. of 4.2), we have, for the mapping $B_P : q \to q + S_P$

$$|S_P, S_q| \leqslant M\beta^{-1} < 0{,}2\beta, \quad |S_{PP}, S_{Pq}, S_{qq}| \leqslant 2M\beta^{-2} < 4^{-1}n^{-1}.$$

and so

$$|B_P - E| \leqslant M\beta^{-1} < 0{,}2\beta, \quad |dB_P| \neq 0.$$

According to 2°. B_P is a diffeomorphism of the domain $q \in U - 1{,}8\beta$ and by 1°. its image contains the domain $Q \in U - 2\beta$. Hence for $P \in G - \beta$, $Q \in U - 2\beta$ we have a mapping $B : P, Q \to p, q = P + S_q, B_P^{-1}(Q)$ (where we substitute $q = B_P^{-1}(Q)$ in S_q after differentiating). For $P \in G - 2\beta$, $Q \in U - 3\beta$ we obtain

$$|B - E| \leqslant M\beta^{-1} < 0{,}2\beta, \quad |dB - dx| < 0{,}5|dx|.$$

3°. now follows from 2°. and the Cauchy estimates (3°. of 4.2).

4°. *Let A be a mapping of the sphere $U_\varepsilon(x_0) : |x - x_0| \leqslant \varepsilon$ and $\theta|dx| \leqslant |dA| \leqslant \Theta|dx|$. Then*

$$U_{\theta\varepsilon}(Ax_0) \subseteq AU_\varepsilon(x_0) \subseteq U_{\Theta\varepsilon}(Ax_0).$$

In fact, the right-hand inclusion follows from the Lagrange formula (4°. of 4.2). Let $y(t) = y_0 + ty$, where $y_0 = Ax_0$, $0 < t < \infty$. For small t there is a continuous branch $A^{-1}(y(t)) \subseteq U_\varepsilon(x_0)$, where $A^{-1}(y_0) = x_0$. Let $\bar{t}$ be the greatest t for which $A^{-1}(y(t)) \subseteq U_\varepsilon(x_0)$; then $|A^{-1}(\bar{y}) - x_0| = \varepsilon$ at the point $\bar{y} = y(t)$. But by the Lagrange formula (4°. of 4.2) we have

$$|A^{-1}\bar{y} - A^{-1}y_0| \leqslant \theta^{-1}|\bar{y} - y_0|,$$

and so $|\bar{y} - y_0| \geqslant \theta\varepsilon$, as required.

5°. *Lemma concerning the variation of the frequency ratios. Let A be be a diffeomorphism of the domain G of the space p onto the domain Ω of the space ω such that $\theta|dp| \leqslant |dA| \leqslant \Theta|dp|$. Further, let A' be a mapping of the domain $G - \beta : p \to A'(p) = A(p) + \Delta(p)$, where $|\Delta(p)| < \beta$, $|d\Delta| < \delta\theta|dp|$. Let Ω_0 be a subdomain of Ω and $\beta > 0$, $b > 0$, $0 < \delta < 1$. Then there exist domains G', G_1 such that*

1) $G \supseteq G - \beta \supseteq G' \supseteq G_1 + b \supseteq G_1$,

2) *A' is a diffeomorphism of G', and $\theta|dp| < |dA'| < \Theta|dp|$, where $\theta' = \theta(1 - \delta)$, $\Theta' = \Theta(1 + \delta)$.*

3) *$A'G_1 = \Omega_1 = \Omega_0 - d$ where $d = 2\Theta b + (S + \Theta)\beta$, $A'(G_1 + b) \subseteq \Omega_0$.*

4) mes $(G - G_1) \leqslant \theta^{-n}$ mes $(\Omega - \bar{\Omega}_1)$, *where* $\bar{\Omega}_1 = \Omega_0 - \bar{d}$, $\bar{d} = 2\Theta b + (6 + \Theta)\beta$.

PROOF. Evidently, for $\omega \in A(G - \beta)$ we have $|A'A^{-1} - E| < \beta$ and $|dA'A^{-1}| > (1 - \delta)|d\omega|$. According to 2°. for $\omega \in A(G - \beta) - 4\beta$ the mapping $A'A^{-1}$ is a diffeomorphism, and by 1°. we have $A'A^{-1}[A(G - \beta) - 4\beta] \supseteq A(G-\beta) - 5\beta$. Consequently A' is a diffeomorphism in the domain $G'' = A^{-1}[A(G - \beta) - 4\beta]$, and $A'G'' \supseteq A(G - \beta) - 5\beta$. But by 4°. $A(G - \beta) \supseteq \Omega - \Theta\beta$, hence $A'G'' \supseteq \Omega' = \Omega - (5 + \Theta)\beta$. Writing $G' = A'^{-1}\Omega'$, $G_1 = A'^{-1}\Omega_1$, where $\Omega_1 = \Omega_0 - d$, $d = 2\Theta b + (5 + \Theta)\beta$ we have $\Omega_1 + 2\Theta b \subseteq \Omega'$ and by 4°. $G_1 + b \subseteq G'$. The conclusions 2) and 3) are now evident. According to 1°. $AG_1 = AA'^{-1}\Omega_1 \supseteq \Omega_1 - \beta = \bar{\Omega}_1$ hence

$$\text{mes}\,(G - G_1) = \int\limits_{\Omega - AG_1} \left(\det\left|\frac{\partial A}{\partial p}\right|\right)^{-1} d\omega \leqslant \theta^{-n}\,\text{mes}\,(\Omega - AG_1) \leqslant \theta^{-n}\,\text{mes}\,(\Omega - \bar{\Omega}_1)$$

4.4. Convergence lemmas. 1°. *Suppose that a sequence of domains F_s is given and diffeomorphisms $B_s : F_s \to F_{s-1}$ $(s = 1, 2, \ldots)$ such that*

1) $|B_s - E| \leqslant d_s$,

2) $F_s \subseteq F_{s-1} - d_s$,

3) $|dB_s| \leqslant 2|dx|$,

4) $d_s \leqslant c4^{-s}$.

Then the sequence $S_s = B_1B_2 \ldots B_s$ $(s = 1, 2, 3, \ldots)$ converges uniformly in $F_\infty = \bigcap F_s$ to a continuous mapping S_∞ such that $|S_\infty - E| \leqslant c$.

PROOF. Let $x \in F_s$. Then by 1) $|B_sx - x| \leqslant d_s$. It follows from 2) that the Lagrange formula (4°. of 4.2) may be applied to the segment x, B_sx and the mapping S_{s-1}. 3) implies that $|dS_{s-1}| \leqslant 2^s|dx|$ and so by 4)

$$|S_sx - S_{s-1}x| = |S_{s-1}B_sx - S_{s-1}x| \leqslant 2^sd_s \leqslant 2^{-s}c,$$

as required.

2°. *Let F denote a d-neighbourhood of the segment $x = x_0 + vt$, $0 \leqslant t \leqslant \frac{d}{\varepsilon}$. Let $X(x)$ be a smooth vector field on F, and $|X - v| \leqslant \varepsilon$. Denote by $x(t)$ the solution of the equation $\frac{dx}{dt} = X(x)$ satisfying the initial condition $x(0) = x_0$. Then $|x(t) - (x_0 + vt)| \leqslant d$ for $0 \leqslant t \leqslant \frac{d}{\varepsilon}$.*

PROOF. Consider $y(t) = x(t) - (x_0 + vt)$. $|y(t_0)| = d$ and for $0 \leqslant t \leqslant t_0$, $|y(t)| < d$. Since $\frac{dy}{dt} \leqslant \varepsilon$ for $t < t_0$ and $y(0) = 0$, by the Lagrange formula $|y(t_0)| \leqslant \varepsilon t_0$ and so $t_0 \geqslant \frac{d}{\varepsilon}$, as required.

3°. *Suppose that, under the conditions of 1°., we are given a smooth vector field $X_0(x)$ in F_0 defining a motion $S_0^t(x) : \frac{d}{dt} S_0^t(x) = X_0(S_0^t x)$, $S_0^0(x) = x$. Naturally there arise the motions $S_s^t = S_s^{-1} S_0^t S_s$ and the corresponding fields X_s on F_s: $X_s(x) = \frac{d}{dt}(S_s^t x)\Big|_{t=0}$. Let us suppose that*

5) *the sequence $X_s(x)$ converges for $s \to \infty$, $x \in F_\infty$ to $X_\infty(x)$, and in F_∞ we have $|X_s - X_\infty| \leqslant d_{s+1}$.*

6) *the segment $x = x_0 + vt$, $0 \leqslant t \leqslant 1$ belongs to F_∞, and in this segment $X_\infty = v$.*

7) $\left|\frac{\partial X_s}{\partial x}\right| \leqslant \Theta$ *in F_s, where the constant Θ is independent of s.*

Under the hypotheses 1)-7) we have for $0 \leqslant t \leqslant \frac{1}{1+\Theta}$

$$S_0^t(S_\infty, x_0) = S_\infty(x_0 + vt) \subseteq F_0.$$

PROOF. We shall show that for $0 \leqslant t \leqslant \frac{1}{1+\Theta}$

$$|S_s^t x_0 - (x_0 + vt)| \leqslant d_{s+1}. \tag{9}$$

According to 5) and 6) $|X_s - v| \leqslant d_{s+1}$ on the segment $x_0 + vt$ $\left(0 \leqslant t \leqslant \frac{1}{1+\Theta}\right)$. By 2) the d_{s+1}-neighbourhood of this segment belongs to F_s. In this neighbourhood, using the Lagrange formula (4°. of 4.2) and 7), we have $|X_s - v| \leqslant (1 + \Theta)d_{s+1}$. Putting, in 2°.; $d = d_{s+1}$ and $\varepsilon = (1 + \Theta)d_{s+1}$ we obtain (9). (9) and (2) show that the segment $(x_0 + vt, S_s^t x_0)$ belongs to the domain F_s. Using the Lagrange formulae, 3) and 4) we see that $|S_s S_s^t x - S_s(x_0 + vt)| \to 0$, as $s \to \infty$, for $0 \leqslant t \leqslant \frac{1}{1+\Theta}$, as required (because $S_s S_s^t = S_s^t S_s$).

4°. *The measure of the limit. Let F be compact and S_s $(s = 1, 2, \ldots)$ be a sequence of continuous mappings of F onto F_s in a euclidean space R, converging uniformly to the mapping S_∞ onto $S_\infty F$. Then*

$$\operatorname{mes} S_\infty F \geqslant \overline{\lim} \operatorname{mes} F_s.$$

In fact, mes $(S_\infty F + \delta) <$ mes $F + \varepsilon$ for any $\varepsilon > 0$ and $\delta < \delta(\varepsilon)$. By virtue of the uniform convergence we have $F_0 \subseteq S_\infty F + \delta$ for $s > s(\delta)$ and so, as required: mes $F_s \leqslant$ mes $(S_\infty F + \delta) <$ mes $S_\infty F + \varepsilon$.

4.5. Notation. 1°. *Functions.* All the functions we consider are taken to be complex-analytic, and real for real values of the argument. We consider complex n-dimensional spaces of canonically conjugate variables $p = p_1, \ldots, p_n$; $q = q_1, \ldots, q_n$ (also denoted by $x = x_1, \ldots, x_{2n} = p, q = p_1, \ldots, q_n$) and the space of frequencies $\omega = \omega_1, \ldots, \omega_n$. We take the maximum of the moduli of the coordinates as a norm in these spaces. The functions we consider have period 2π in q_i and may be expanded in Fourier series

$$f(q) = \sum f_k e^{i(k,q)} = f_0 + \tilde{f}(q) = f_0 + \sum' f_k e^{i(k,q)} \qquad \Big(\sum \equiv \sum_{-\infty<k<+\infty}, \ \sum' \equiv \sum_{k\neq 0}\Big),$$

where $(k, q) = k_1q_1 + \ldots + k_nq_n$, and k is a vector with integral coordinates K_i. In the space conjugate to q, consisting of the orders of the harmonics k, we take $|k| = |k_1| + \ldots + |k_n|$ as norm.

We use abbreviated notation of the form

$$f_p \equiv \frac{\partial f}{\partial p} \equiv \frac{\partial f}{\partial p_1}, \ \ldots, \ \frac{\partial f}{\partial p_n} \equiv f_{p_1}, \ \ldots, \ f_{p_n},$$

where $f(p)$ is a numerical or vector-valued function $f(p_1, \ldots, p_n)$.

2°. *Domains.* Let U be a compact complex domain, i.e. a bounded domain in a complex numerical space, together with its boundaries. If $d > 0$, then $U + d$, $U - d$ denote the d-neighbourhood of U and the set of points contained in U together with a d-neighbourhood. $U_1 \bigcup U_2$ denotes the union, $U_1 \bigcap U_2$ the intersection, $U_1 \backslash U_2$ is the part of U_1 not in U_2. $U_1 \subseteq U_2$ means that every point n of U_1 ($x \in U_1$) also belongs to U_2 ($x \in U_2$). Re U denotes the intersection of the domain U with the real space, Im denotes the imaginary part. mes U denotes the Lebesgue measure of Re U [28] even if the domain U is complex.

A compact domain in the space p is denoted by the letter G, and in the space ω by Ω (both complex). F denotes the domain in the space $x = p, q$ defined by the conditions $p \in G$, $|\text{Im } q| \leqslant \rho$. The points q and $q + 2\pi k$ are identified as in §1, so that mes $F = (2\pi)^n$ mes G.

3°. *Mappings.* The mappings we consider are given by analytic functions. A diffeomorphic mapping, or a diffeomorphism, of the compact domain U_1 onto U_2 is a one-to-one mapping that together with its inverse is continuously differentiable at each point of U_1 (or of U_2, respectively). The differential of the mapping A at the point x is the linear operator $dA = \dfrac{\partial A}{\partial x} dx$.

A denotes a diffeomorphism of the domains G and Ω: B and S are diffeormorphisms of the domains F, and are canonical transformations (see for example, [1]). E denotes the identity transformation $x \to x$.

4°. *Constants.* The numbers θ, Θ, ρ, κ, D are positive constants. The numbers β, γ, δ, M, K are very small in comparison with the previous positive constants, also $\gamma \gg \delta \gg \beta \gg M$.

The numbers N are large and positive.

L, ν, T denote absolute positive constants (i.e. depending only on the number of degrees of freedom).

The index s numbers the approximations.

§5. Appendix. The rotatory motion of a heavy asymmetric rigid body

We shall show that Theorem 1 of §2 ensures the stability of a rapid rotation of a heavy asymmetric rigid body fixed at an arbitrary point O. We shall see that *the magnitude and inclination to the horizontal of the angular momentum vector* **M** *always remain near their initial values* (Fig. 2). In particular, *if a body is undergoing a rapid rotation about the major or the minor inertial axis,*[1] *then the angular velocity vector* Ω *will in the body always remain near that axis,* and in space will slowly precess about the direction of the gravitational force. Furthermore, the magnitude of the angular velocity Ω and the angle of inclination of the axis to the horizontal will always remain near their initial values (Fig. 3).

When discussing a rapid rotation we assume that the potential energy of the body in the gravitational field Π is small in comparison with the kinetic energy of rotation T. We shall find it more convenient to take not $T \gg 1$, but $\Pi = \varepsilon U \ll 1$, i.e. motion in a weak gravitational field (which mathematically, of course, is equivalent to a rapid rotation). The total energy will be

$$H = T + \varepsilon U.$$

For the unperturbed motion ($\varepsilon = 0$) we take the motion in the absence of a gravitational force, i.e. the Euler-Poinsot motion.

5.1. The Euler-Poinsot case. A rigid body with a fixed point is a system with 3 degrees of freedom and a 6-dimensional phase space. In the absence of a gravitational force there exist 4 independent single-valued first integrals

$$T,\ M_x,\ M_y,\ M_z \tag{1}$$

(the energy and 3 components of the vector **M**). These 4 functions of position in the phase-space change their values for a given motion. The points of the 6-dimensional phase-space for which the 4 functions have given values (1) form, in general, a two-dimensional manifold V. We shall show that these manifolds $V(T, \mathbf{M})$ are tori.

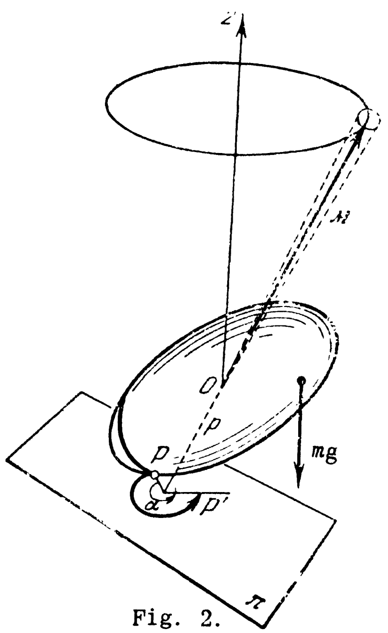

Fig. 2.

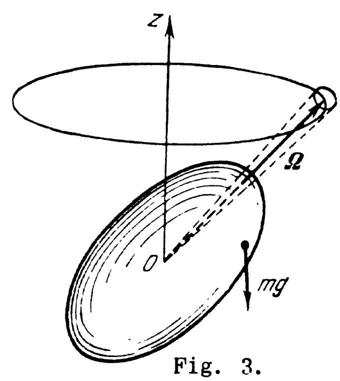

Fig. 3.

[1] At the fixed point O. In Figs. 2 and 3 the body is replaced by its inertial ellipsoid at O.

In fact, the manifold V is invariant and so the phase-velocity vector at each point of V touches V; consequently V admits of a vector field without singular points. It is evident that V is orientable and compact. The only compact two-dimensional orientable manifold admitting a tangential vector field without singular points is, as is well known, the torus.

It is also well known and easy to prove (see 5.2) that the phase point moves on the torus V in a quasi-periodic motion with two frequencies ω_1 and ω_2. In order to explain the significance of the frequencies ω_1 and ω_2 let us turn to the representation of the motion found by Poinsot (see Fig. 2).

The ellipsoid of inertia, with centre at O, in the motion of the body rolls, without sliding, on the fixed plane π. The plane π is perpendicular to the vector **M** and is distant p from O, where

$$p^2 = \frac{2T}{M^2}, \qquad M^2 = M_x^2 + M_y^2 + M_z^2. \tag{2}$$

Suppose that initially the ellipsoid touches the plane π at the point P. In an oscillation the point of contact varies, describing a closed curve on the ellipsoid (polhode). After a time τ the point of contact with the plane again arrives at the initial point P. On the plane π, however, the point of contact will not be P, but P'; the ellipsoid has rotated about the axis **M** through an angle α.

The frequencies ω_1 and ω_2 are given by the formulae

$$\omega_1 = \frac{2\pi}{\tau}, \quad \omega_2 = \frac{\alpha}{\tau}. \tag{3}$$

The quantities ω_1 and ω_2 are determined by the values of the integrals T and M^2, and their ratio $\alpha/2\pi$ depends only on the geometrical parameters.[1]

$$\alpha = \alpha(p, a, b, c),$$

where $a \geqslant b \geqslant c$ are the principal semi-axes of the initial ellipsoid.

5.2. Reduction to a problem with two degrees of freedom. As in the Euler-Poinsot case, so when there is a gravitational force, the number of degrees of freedom is easily reduced from 3 to 2. We choose as generalized coordinates the Euler angles φ, ϑ, ψ taking as fixed axis Oz the vertical, and as fixed axes the principal axes of the inertial ellipsoid at the fixed point O. The Lagrange function

$$L = T(\dot\varphi, \dot\vartheta, \dot\psi; \vartheta, \psi) - \varepsilon U(\vartheta, \psi)$$

does not contain the cyclic coordinate φ. The corresponding impulse

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = \frac{\partial T}{\partial \dot\varphi} = M_z$$

remains constant. We may consider the function

$$H = T(p_\vartheta, p_\psi; \vartheta, \psi; M_z) + \varepsilon U(\vartheta, \psi) \tag{4}$$

as the Hamiltonian function of the system with two degrees of freedom ϑ, ψ depending on the parameter M_z. We fix the value of M_z and shall frequently not indicate the dependence of the function on this parameter.

---

[1] The quantities p, a, b, c have the dimensions $(\text{mass})^{-1/2}$ $(\text{length})^{-1}$.

We first examine the case $\varepsilon = 0$ (Euler-Poinsot case). The system is integrable owing to the existence of two first integrals

$$H = T = \text{const}, \quad M^2(p_\vartheta, p_\psi; \vartheta, \psi; M_z) = M_x^2 + M_y^2 + M_z^2 = \text{const}. \quad (5)$$

Points with the same values of T and M^2 form a two-dimensional[1] invariant torus $V(T, M^2)$ in the 4-dimensional phase-space p , p_ψ; ϑ, ψ. Each such torus corresponds to a certain Euler-Poinsot motion; the phase point moves in the corresponding quasi-periodic motion with frequencies (3).

For the integration of the system (4) when $\varepsilon = 0$ it is convenient to introduce operator-angle variables[2] by the canonical transformation

$$p_\vartheta, \; p_\psi; \; \vartheta, \; \psi \longrightarrow I_1, \; I_2; \; w_1, \; w_2 \quad (6)$$

The quantities I_1 and I_2 depend only on T and M^2, and so T can be represented as a function

$$T = H_0(I_1, I_2),$$

whose derivatives with respect to I_1 and I_2 are the frequencies (3):

$$\omega_1 = \dot{w}_1 = \frac{\partial H_0}{\partial I_1}, \quad \omega_2 = \dot{w}_2 = \frac{\partial H_0}{\partial I_2}.$$

In the presence of gravity ($\varepsilon \neq 0$) the function (4) takes the form

$$H = H_0(I_1, I_2) + \varepsilon U(I_1, I_2; w_1, w_2), \quad (7)$$

where the "perturbation" εU has period 2π in ω_1 and ω_2.

5.3. The investigation of the system with Hamiltonian function (7). The function (7) has the form (1) §2. We shall show in 5.4 that the condition (2) §2 of non-degeneracy is satisfied. Theorem 1 is therefore applicable, and the system with Hamiltonian (7) has invariant tori for sufficiently small ε, so that the majority of initial conditions the system is quasi-periodic.

But since the system (7) has two degrees of freedom, we can say more. Consider the ratios of the frequencies of the unperturbed system, ω_1/ω_2. As we shall show in 5.4, for fixed energy T this ratio $\alpha/2\pi$ varies from torus to torus. It follows then that for sufficiently small $\varepsilon(\Delta)$ the perturbed system has invariant tori for each energy-level and in any neighbourhood of any point of the phase space. These two-dimensional invariant tori divide the three-dimensional invariant energy-levels (see Fig. 1). Thus, even for those initial conditions that do not hit on an invariant torus of the perturbed system the phase-point for the motion always remains enclosed between two such neighbouring tori. It follows from 3°. of

[1] We exclude only those values of T and M^2 for which the body can rotate about an axis of inertia, i.e. in (2), $p = a$, b, or c.

[2] The canonical transformation (6) is defined by the formulae

$$p_\vartheta = \frac{\partial S}{\partial \vartheta}, \quad p_\psi = \frac{\partial S}{\partial \psi}; \quad w_1 = \frac{\partial S}{\partial I_1}, \quad w_2 = \frac{\partial S}{\partial I_2},$$

where

$$S(I_1, I_2; \vartheta, \psi) = \int p_\vartheta \, d\vartheta + p_\psi \, d\psi. \quad (8)$$

Here the integral is taken along a curve lying on the torus ($V(T, M^2)$ and does not depend on the path of integration (in the small). $2\pi I_1$ and $2\pi I_2$ are the values of the integral (8) with respect to the base cycle of the torus.

Theorem 1 that for sufficiently small ε the change in $I_1(t)$, $I_2(t)$ for an infinite time will be arbitrarily small.

But M^2 is a function of I_1, I_2. We are therefore led to the conclusion that *for a rapid rotation of an asymmetric rigid body the magnitude of the angular momentum vector* **M** *varies little during an infinite time interval.*

More precisely we have the following

THEOREM. *For every* $\lambda > 0$ *there exists* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$, *then for any* t, $-\infty < t < \infty$,, *we have* $|M^2(t) - M^2(0)| < \kappa$.

In view of the conservation of the component M_z the inclination of M to the horizontal varies little. However, the azimuth of the vector **M** will not remain constant as in the Euler-Poinsot case. The vector **M** carries out a slow precession about the vertical, and the body rotates about **M** approximately according to Poinsot (Fig. 2).

In the 6-dimensional phase-space $p_\varphi p_\vartheta p_\psi$; φ, ϑ, ψ the two-dimensional invariant tori we have found correspond to three-dimensional invariant tori and quasi-periodic motions with three frequencies ω_1, ω_2, ω_3 (rotation, nutation, precession). The precession frequency ω_3 is small with ε, and for $\varepsilon = 0$ the three-dimensional tori collapse into the two-dimensional tori V considered in 5.1.

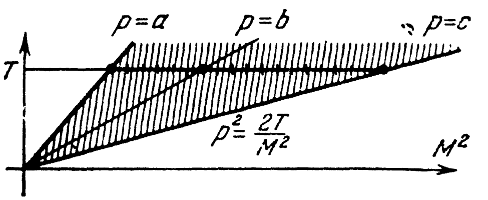

Fig. 4.

5.4. Verification of the non-degeneracy conditions. It remains to check that the condition (2) of §2, which is of the form (Fig. 4):

$$\left|\frac{\partial^2 H_0}{\partial I_i \partial I_j}\right| = \frac{\partial(\omega_1, \omega_2)}{\partial(I_1, I_2)} = \frac{\partial(\omega_1, \omega_2)}{\partial(\omega_1, \alpha)} \frac{\partial(\omega_1, \alpha)}{\partial(T, p)} \frac{\partial(T, p)}{\partial(I_1, I_2)} \neq 0 \tag{9}$$

is satisfied. If $T \neq 0$ and the ellipsoid of inertia is not a sphere $(a > c)$, then evidently

$$\frac{\partial(\omega_1, \omega_2)}{\partial(\omega_1, \omega_2/\omega_1)} = \frac{\partial(\omega_1, \omega_2)}{\partial(\omega_1, \alpha)} \neq 0, \quad \frac{\partial(T, p)}{\partial(I_1, I_2)} \neq 0. \tag{10}$$

By the similarity arguments it is clear that $\omega_1 = K(p)\sqrt{T}$, where $K = K(p;\ a,\ b,\ c) \neq 0$. Consequently

$$\frac{\partial(\omega_1, \alpha)}{\partial(T, p)} = \frac{K}{2\sqrt{T}} \frac{d\alpha}{dp} . \tag{11}$$

It remains to prove that the ratio of the frequencies $\alpha(p)/2\pi$ does

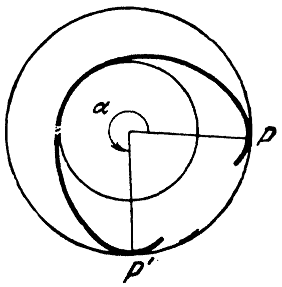

Fig. 5.

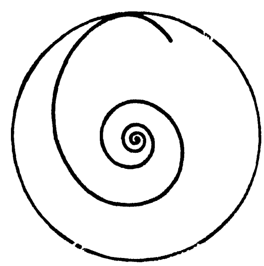

Fig. 6.

not reduce to a constant for any values of the principal semi-axes $a \geqslant b \geqslant c > 0$, $a > c$. For the proof we take advantage of the fact that in a rotation through the angle α (see Fig. 2) the curve described by the point of contact with the plane π (herpolhode, see Fig. 5) joins up with itself. It is then easy to see that if $a > c$

$$\lim_{p \to b} \alpha(p) = \infty. \tag{12}$$

In fact, for $p = b$ the herpolhode is a spiral with an infinite number of turns (Fig. 6) and so (12) follows.

Since $\alpha(p) \not\equiv \infty$, we have by (12) $\frac{d\alpha}{dp} \not\equiv 0$, and so by (11) and (10) we obtain (9).

In this way, *for* $a > c$, *the non-degeneracy condition* (9) *is satisfied and the ratio of the frequencies* $\alpha/2\pi$ *varies with* p *for fixed* T (see Fig. 4).

We have excluded from the discussion the case $a = b = c$, but if $a = b = c$, then wherever the centre of gravity is situated, the body is a symmetric top (Lagrange case).

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КАЧЕСТВЕННЫЕ ВОПРОСЫ НЕБЕСНОЙ МЕХАНИКИ

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МАЛЫЕ ЗНАМЕНАТЕЛИ И ПРОБЛЕМА УСТОЙЧИВОСТИ В КЛАССИЧЕСКОЙ И НЕБЕСНОЙ МЕХАНИКЕ*

§ 1. Вопросы устойчивости, связанные с расходимостью рядов теории возмущений, давно занимают математиков [1—6]. В последние годы в этой области достигнут некоторый прогресс. Цель настоящего сообщения — изложить результаты и методы недавних исследований поведения решений дифференциальных уравнений динамики на бесконечном промежутке времени [7—10].

Неприятной особенностью рассматриваемых задач является то, что сколь угодно малое изменение начальных условий способно за бесконечно большое время совершенно изменить картину движения. В некоторых случаях тем не менее удается точно исследовать движение (которое оказывается условно-периодическим) для большинства начальных условий.

Следует подчеркнуть, что хотя кое-где и будут употребляться термины «тела», «планеты», речь идет о чисто математической задаче о движении материальных точек, строго подчиняющихся законам Ньютона. Мы не останавливаемся сейчас на интересном вопросе о соотношении между движением идеализированных систем на бесконечном промежутке времени и поведением реальных систем в течение большого конечного промежутка времени.

§ 2. Малые знаменатели. Основной причиной расходимости рядов теории возмущений являются малые знаменатели, связанные с приближенной соизмеримостью частот движений. Например, частоты Юпитера $\omega_1 = 299.''1$ и Сатурна $\omega_2 = 120.''5$ почти удовлетворяют соотношению $2\omega_1 = 5\omega_2$. В ряду, выражающем возмущение, член

$$\frac{a_{2,-5}}{2\omega_1 - 5\omega_2} \cos[(2\omega_1 - 5\omega_2)t + \alpha]$$

велик из-за наличия малого знаменателя $2\omega_1 - 5\omega_2$. Этим объясняется известное большое долгопериодическое возмущение планет друг другом.

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Пусть ω_1, ω_2 — два наугад взятых действительных числа. Кажется невероятным, чтобы они оказались соизмеримыми, так что $p\omega_1 = q\omega_2$. Мало вероятно также, что выражения $|p\omega_1 + q\omega_2|$ (p, $q = \pm 1, \pm 2 \ldots$) будут малы при небольших p, q. Малые знаменатели в рядах теории возмущений с большой вероятностью не слишком малы — такого рода соображения, неоднократно обсуждавшиеся астрономами XIX в., с развитием теории меры получили строгое обоснование [11]. Мы придадим сейчас точный смысл словам: «наугад взятое из отрезка $0 \leqslant \omega \leqslant 1$ число ω иррационально».

Мы покажем, что вероятность получить число ω, приближаемое хотя бы одной дробью p/q с ошибкой меньше c/q^3, не превосходит $4c$. Действительно, столь близкие к p/q числа ω образуют «резонансную зону» ширины $2c/q^3$. Зафиксируем q и рассмотрим зоны, соответствующие $p=0, 1, \ldots, q$; все эти зоны вместе имеют длину $2c/q^2$. Теперь будем менять $q = 1, 2, \ldots$; тогда найдем, что общая длина всех резонансных зон меньше

$$\sum_{q=1}^{\infty} \frac{2c}{q^2} = \frac{\pi^2}{3} c < 4c.$$

Таким образом, если выкинуть из отрезка $0 \leqslant \omega \leqslant 1$ резонансные зоны общей мерой меньше 0,04, то для оставшихся точек ω при любых целых p, $q \neq 0$ будет

$$\left| \omega - \frac{p}{q} \right| > \frac{0{,}01}{q^3}.$$

Точно так же в случае малых знаменателей вида $p\omega_1 + q\omega_2$ находим

$$|p\omega_1 + q\omega_2| > \frac{c}{(|p| + |q|)^2}$$

для большинства пар ω_1, ω_2.

Так как, с другой стороны, числители a_{pq} в коэффициентах рядов теории возмущений убывают обычно в геометрической прогрессии, то сами эти ряды

$$\sum_{p,q} \frac{a_{pq}}{p\omega_1 + q\omega_2} \cos [(p\omega_1 + q\omega_2) t + \alpha_{pq}]$$

сходятся для большинства ω_1, ω_2 с такой же почти скоростью, как ряд $\sum a_{pq}$.

§ 3. Условно-периодические движения. Рассмотрим поверхность тора (баранки) с угловыми координатами φ, ψ, меняющимися от 0 до 2π. Дифференциальные уравнения

$$\frac{d\varphi}{dt} = \omega_1; \quad \frac{d\psi}{dt} = \omega_2$$

описывают «равномерное» движение точки φ, ψ по тору. Это движение называется условно-периодическим, если частоты ω_1 и ω_2 несоизмеримы. В этом случае траектория движения заполняет тор всюду плотно.

Представим себе, что мы наблюдаем изменение со временем $A[\varphi(t), \psi(t)]$ какой-нибудь функции $A(\varphi, \psi)$, заданной на торе, т. е. периодической по φ, ψ. Легко видеть, что A как функция времени выражается рядом вида

$$A(t) = \sum a_{pq} \cos[(p\omega_1 + q\omega_2)t + \alpha_{pq}], \tag{1}$$

где a_{pq} — коэффициенты Фурье $A(\varphi, \psi)$.

Мы будем далее рассматривать движение точки, изображающей систему с n степенями свободы, в $2n$-мерном фазовом пространстве координат и импульсов. При некоторых условиях мы найдем, что большинство точек в определенной области фазового пространства движется условно-периодически по n-мерным торам. Изменение со временем координат точек, составляющих систему, будет тогда выражаться рядами вида (1) (с n частотами $\omega_1, \ldots, \omega_n$).

§ 4. Интегрируемые системы. Можно показать, что в любой системе, имеющей n первых интегралов в инволюции и совершающей ограниченное движение, почти все фазовые точки движутся условно-периодически. Фазовое пространство можно разбить на области, имеющие вид произведения n-мерного тора на область n-мерного евклидова пространства. Положение фазовой точки в такой области можно характеризовать канонически сопряженными переменными $p_1, \ldots, p_n$, из которых $q_1, \ldots, q_n$ имеют смысл углов, так что изменение их на 2π не меняет точки фазового пространства. Переменные p, q можно в интегрируемой задаче выбрать так, чтобы функция Гамильтона H $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ зависела только от $p_1, \ldots, p_n$ (в задаче двух тел p, q — элементы Делоне);

$$H(p, q) \equiv H_0(p_1, \ldots, p_n). \tag{2}$$

Тогда уравнения Гамильтона принимают вид

$$\dot{p}_i = 0; \qquad \dot{q}_i = \omega(p) \equiv \frac{\partial H_0}{\partial p_i}. \tag{3}$$

Уравнения $p = \text{const}$ выделяют в $2n$-мерном пространстве p, q n-мерный тор. В силу уравнений (3) точка $q(t)$ движется по нему условно-периодически с частотами $\omega_i(p)$. В общем случае эти частоты не удовлетворяют никакому соотношению

$$k_1\omega_1 + \ldots + k_n\omega_n \equiv (k, \omega) = 0$$

с ненулевыми целыми коэффициентами. В противном случае говорят о вырождении [12].

Если якобиан частот по «импульсам» отличен от нуля

$$\det\left|\frac{\partial\omega_i}{\partial p_j}\right| \equiv \det\left|\frac{\partial^2 H_0}{\partial p_i \partial p_j}\right| \neq 0, \tag{4}$$

то вырождение называют случайным; малым изменением постоянных p можно сойти с тора, на котором частоты соизмеримы, на невырожденный.

Если семейство торов $p = \text{const}$ содержит отдельные торы размерности $k < n$ (например, рассмотрим в трехмерном пространстве окружность и семейство двумерных торов, вложенных друг в друга и стягивающихся к ней; здесь $k = 1$, $n = 2$), то вырождение называют предельным.

Если же якобиан (4) тождественно обращается в нуль, то говорят о собственном вырождении. В этом случае H можно привести к виду

$$H(p, q) \equiv H_0(p_1, \ldots, p_k), \qquad k<n.$$

Такая функция Гамильтона описывает условно-периодическое движение с $k < n$ частотами

$$\omega_i = \frac{\partial H_0}{\partial p_i};$$

траектории заполняют k-мерные торы $p_i = \text{const}$ $(i = 1, \ldots, n)$; $q_i = \text{const}$ $(i = k + 1, \ldots, n)$.

К указанному типу относится задача двух тел при ньютоновском законе притяжения ($k = 1$). Вообще же в небесной механике встречаются вместе все три случая вырождения. Мы рассмотрим их последовательно в § 7—9.

§ 5. Теория возмущений. В теории возмущений рассматривается случай, когда функция Гамильтона отличается от (2) малым добавком

$$H = H_0(p_1, \ldots, p_n) + \varepsilon H_1(p_1, \ldots, p_n; q_1, \ldots, q_n) + \ldots, \tag{5}$$

где возмущающая функция $\varepsilon H_1 + \ldots$ имеет период 2π по $q_1, \ldots, q_n$. Теория основана на гипотезе, что подходящей канонической заменой переменных можно привести функцию Гамильтона к интегрируемому виду (2). Таким образом, предполагается, что и у системы (5) фазовое пространство разбивается на инвариантные n-мерные торы (близкие к торам $p = \text{const}$).

Если указанная замена существует и аналитична по ε, то ее можно искать в виде ряда по степеням ε, коэффициенты которого — ряды Фурье по q, зависящие от p. Как известно, коэффициенты таких рядов можно последовательно определить, приравнивая члены с ε^0, ε^1 и т. д. [3, 12]. При этом появляются малые знаменатели, так что ряды каждого приближения сходятся не при всех p.

Более того, можно показать [3,13], что в с е п р и б л и ж е н и я в м е с т е р а с х о д я т с я, а с а м а г и п о т е з а о п о в е д е н и и р е ш е н и й н е в е р н а.

Действительно, для интегрируемой системы (2) в любой малой области пространства p найдется точка p_0, для которой все частоты ω_i соизмеримы, а движение периодично. Следовательно, существует n-мерное многообразие начальных условий (тор $p = p_0$), начиная с которых система точно возвращается в прежнее положение через время T. Исследуя уравнения в вариациях, можно показать, что, вообще говоря, у возмущенной системы (5) такого инвариантного многообразия не будет. Следовательно, система не может быть интегрируемой и приближения теории возмущений не могут сходиться.

Трудности, связанные с малыми знаменателями, долгое время не были преодолены. Лишь в последнее время К. Л. Зигелю [14], А. Финчи [15, 16], А. Н. Колмогорову [7] и А. Е. Гельману [17] независимо друг от друга удалось решить ряд задач, содержащих эти трудности.

§ 6. Метод Ньютона. Из развитых названными учеными методов наиболее сильным оказался метод Ньютона, примененный А. Н. Колмогоровым. Мы изложим здесь один из вариантов этого метода, позволяющий сходящимися последовательными приближениями отыскивать условно-периодические движения в общей задаче теории возмущений.

Мы откажемся от разложений по степеням ε и рассмотрим функцию Гамильтона

$$H(p, q) = H_0(p) + \widetilde{H}_1(p, q); \quad \widetilde{H}_1(p, q) = \sum_{k \neq 0} H_k(p) e^{i(k, q)}. \tag{6}$$

Зададимся заранее определенными значениями частот $\omega_1, \ldots, \omega_n$ и будем искать инвариантный n-мерный тор $p = p(q)$ и координаты на нем $Q = Q(q)$ так, чтобы $\dot{Q}_i = \omega_i$. В качестве нулевого приближения возьмем $p(q) = p_0$, $Q_0(q) = q_0$, где $\left.\frac{\partial H_0}{\partial p_i}\right|_{p_0} = \omega_i$. Мы предположим, что

$$|(k, \omega)| > c|k|^{-n}; \ (|k| = |k_1| + \ldots + |k_n|; (k, \omega) = k_1\omega_1 + \ldots + k_n\omega_n) \quad (7)$$

при любых целых k_i, $|k| > 0$ (ср. § 2).

Первый шаг метода Ньютона весьма близок к обычной теории возмущений. Сделаем в окрестности тора $p = p_0$ каноническую замену переменных $p, q \to P, Q$ с производящей функцией

$$Pq + S(P, q) = Pq + \sum_{k \neq 0} S_k(P) e^{i(k, q)} \quad (8)$$

(где коэффициенты S_k будут определены ниже). Тогда

$$p = P + \frac{\partial S}{\partial q}; \quad Q = q + \frac{\partial S}{\partial P} \quad (9)$$

и функция Гамильтона (6) примет вид

$$H(p, q) = H_0(P) + \left[H_1(P, q) + \left(\omega, \frac{\partial S}{\partial q}\right)\right] + \\ + \left[H_0(p) - H_0(P) - \left(\left.\frac{\partial H_0}{\partial p}\right|_{p_0}, p - P\right)\right] + \\ + [H_1(p, q) - H_1(P, q)], \quad (10)$$

где p, q всюду должны быть выражены через P, Q с помощью (8), (9).

Определим теперь $S(P, q)$ условием

$$H_1(P, q) + \left(\omega, \frac{\partial S}{\partial q}\right) \equiv 0, \quad (11)$$

или, что эквивалентно,

$$S_k(P) = \frac{iH_k(P)}{(\omega, k)}. \quad (12)$$

Тогда (10) примет вид

$$H(p, q) \equiv H'(P, Q) = H_0(P) + H_2(P, Q), \quad (13)$$

где функция

$$H_2 = \left(\left(\frac{\partial H_0}{\partial p}\bigg|_{p_1} - \frac{\partial H_0}{\partial p}\bigg|_{p_0} \right), (p-P) \right) + \left(\frac{\partial H_1}{\partial p}\bigg|_{p_2}, (p-P) \right) \tag{14}$$

(p_1 и p_2 — точки между p и P) гораздо меньше, чем H_1.

Действительно, ввиду (7) и (12) можно ожидать, что при $H_1 \sim \varepsilon$ будет $S \sim \varepsilon$; следовательно, $|p - P|$ также порядка ε, и в ε-окрестности p_0 величина H_2 будет порядка ε^2 (см. (14)). Точно так же среднее значение $\overline{H}_2(P)$ функции H_2 по Q будет порядка ε^2.

Выберем в указанной окрестности ту точку $P = P_0$, для которой

$$\frac{\partial}{\partial P}(H_0 + \overline{H}_2)|_{P_0} = \omega.$$

Такая точка существует при условии (4) на расстоянии порядка ε^2 от p_0. Теперь в ε-окрестности P_0 функция (13)

$$H'(P, Q) = H_0'(P) + \widetilde{H}_2(P, Q); \quad (H_0' = H_0 + \overline{H}_2; \ \widetilde{H}_2 = H_2 - \overline{H}_2)$$

имеет снова вид (6), но возмущение $\widetilde{H}_2$ — уже порядка ε^2.

Таким образом, вместо возмущения порядка ε осталось возмущение порядка ε^2. Приближения, в которых невязка следующего приближения порядка квадрата невязки предыдущего, типичны для быстро сходящихся методов типа ньютоновского метода касательных.

Повторяя наши рассуждения применительно к $H'(P, Q)$, получим последовательность замен переменных $p, q \to P, Q \to \ldots$, определенных во все более узких вложенных друг в друга торовых кольцах $p - p_0 \sim 1$; $P - P_0 \sim \varepsilon; \ldots$

При этом возмущение $\widetilde{H}_s$, остающееся после s-го приближения, будет порядка ε^{2^s}. Столь быстрая сходимость парализует влияние малых знаменателей, появляющихся в каждом приближении при вычислении коэффициентов S_k по формуле (12).

§ 7. Сходимость метода Ньютона. Здесь будет намечено доказательство того факта, что при достаточно малом возмущении $\widetilde{H}_1(p, q)$ последовательные приближения, построенные в предыдущем параграфе, сходятся. Соответствующие оценки при аккуратном проведении довольно утомительны [18], и мы укажем только основную идею.

Предположим, что при $|\operatorname{Im} q| \leqslant \rho_1$ функция $\widetilde{H}_1(p, q)$ аналитична и $|\widetilde{H}_1| \leqslant M_1$.

Мы построим последовательность $\rho_1 > \rho_2 > \ldots > 0{,}5\rho_1$ и будем оценивать величину

$$M_s = \sup |\widetilde{H}_s|, \qquad |\operatorname{Im} Q_s| < \rho_s,$$

через величину M_{s-1}.

Используя (7), (12), (14), нетрудно показать, что при достаточно малом $\delta > 0$ и некотором $r > 0$, зависящем только от числа степеней свободы n, в области $|\operatorname{Im} Q_s| < \rho_{s-1} - \delta$ будет

$$|\widetilde{H}_s| < \frac{M_{s-1}^2}{\delta^r}.$$

Начиная с числа δ_1, определим $\delta_2 = \delta_1^{3/2}, \ldots, \delta_s = \delta_{s-1}^{3/2}, \ldots$. Если δ_1 достаточно мало, то

$$\sum_{s=1}^{\infty} \delta_s < 0{,}5\rho_1.$$

Далее, пусть $M_1 < \delta_1^k$ и $\rho_s = \rho_{s-1} - \delta_{s-1}$. Тогда $\rho_s > 0{,}5\rho_1$ и $M_1^2/\delta_1^r < \delta_1^{2k-r}$.

Чтобы последняя величина была меньше $M_2 = \delta_2^k = (\delta_1^{3/2})^k$, достаточно взять $2k - r > 1{,}5k$; $k > 2r$.

Итак, выберем, как указано выше, $\delta_1 > 0$ и положим $k = 2r + 1$ и $M_1 < \delta_1^k$. Тогда при $\rho_s = \rho_{s-1} - \delta_{s-1}$ будет $\rho_s > \frac{1}{2}\rho_1$ и $M_2 < \delta_2^k, \ldots, M_s < \delta_s^k \ldots$.

Отсюда легко выводится сходимость последовательности замен переменных при $|\operatorname{Im} q| < 0{,}5\rho_1$.

Аналогичными рассуждениями можно доказать сходимость других вариантов метода Ньютона. Например, удобно в (12) в качестве ω брать не постоянные $\left.\frac{\partial H_0}{\partial p}\right|_{p_0}$, а функции $\omega(p) = \frac{\partial H_0}{\partial p}$.

При этом следует ограничиться в s-м приближении конечным числом гармоник N_s, растущим с s, при определении производящей функции (8).

Тот или другой вариант позволяет доказать следующее [7].

Теорема 1. Пусть функция Гамильтона (6) аналитична в некоторой области $R: p \in G$, $|\operatorname{Im} q| < \rho$ и выполнено условие невырожденности (4). Тогда при достаточно малых M_1 точки области R, исключая множество малой вместе с M_1 меры, движутся условно-периодически по n-мерным торам, близким к торам $p = \text{const}$.

Заметим, что при $n=2$ найденные двумерные инвариантные торы разделяют трехмерное инвариантное многообразие уровня энергии $H(p, q)=h$ на узкие торовые слои; траектория, начавшаяся в таком слое, никогда не может из него выйти и, следовательно, устойчива по Лагранжу. К рассматриваемому случаю принадлежит, в частности, плоская ограниченная круговая задача трех тел. В большинстве же задач небесной механики теорема 1 не применима из-за предельного или собственного вырождения при малых эксцентриситетах или массах. Мы рассмотрим эти два вида вырождения ниже.

§ 8. Устойчивость положений равновесия и периодических движений. Простейшие виды предельного вырождения — положения равновесия (число частот $k=0$) и периодические движения ($k=1$). В случае чисто мнимых характеристических показателей известные методы Пуанкаре и Ляпунова не позволяют решить вопрос об устойчивости. Между тем только этот случай и представляет интерес для консервативных систем, в которых асимптотически устойчивые движения невозможны в силу наличия положительного интегрального инварианта (теорема Лиувилля). Метод Ньютона позволил исследовать вопрос об устойчивости в случае двух степеней свободы.

Изучение окрестности периодического решения может быть, как известно [5], сведено к исследованию окрестности положения равновесия системы с периодическими коэффициентами.

Пусть $H(p, q; t) \equiv H(p, q; t+2\pi)$ — аналитическая* функция Гамильтона системы, имеющей решение $p=q=0$, которое является положением равновесия, устойчивым в линейном приближении. Следуя Биркгофу [5], каноническим преобразованием $p, q \to r, \varphi$ приведем H в окрестности нуля к виду

$$H = \lambda_0 + \lambda_1 r + \lambda_2 r^2 + H_3(r, \varphi; t), \qquad (15)$$

где $r = O(p^2+q^2)$, φ — угловое переменное, $H_3 = O(r^3)$ и λ_0, λ_1, λ_2 — постоянные. Приведение к виду (15) возможно при условии $\lambda_1 \neq k/l$.

Теорема 2. Пусть $\lambda_2 \neq 0$. Тогда для почти всех (в смысле меры Лебега) значений λ_1 положение равновесия $p=q=r=0$ системы с функцией Гамильтона (15) устойчиво [9].

Аналогичные теоремы доказаны для положений равновесия автономных систем с двумя степенями свободы. В случае $n>2$ степеней свободы удается только найти занимающие большую часть окрестности n-мерные инвариантные торы. Но они не

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* Во время печатания настоящего сборника появилась работа Ю. Мозера [20], который показал, что в теореме 2 можно ограничиться 333-кратной дифференцируемостью и условиями $\lambda_1 \neq \frac{k}{3}, \frac{k}{4}$; $\lambda_2 \neq 0$.

делят фазовое пространство, и вопрос об устойчивости остается открытым.

Результаты [9] применимы, в частности, к лагранжевым периодическим решениям. Недавно А. М. Леонтович [19] доказала, что лагранжево периодическое решение плоской круговой ограниченной задачи трех тел устойчиво для почти всех значений отношений масс, для которых оно устойчиво по первому приближению.

§ 9. Собственное вырождение. В задачах небесной механики, как правило, невозмущенное движение описывается меньшим числом частот, чем возмущенное. Например, пусть две планеты масс m_1, m_2 движутся вокруг центрального тела массы M. При $m_1=m_2=0$ планеты не влияют друг на друга и совершают кеплерово движение, связанное с двумя «быстрыми» частотами обращения вокруг M. При m_1, $m_2\neq 0$ вследствие векового движения узлов и перигелиев появляются новые, «медленные» частоты.

Трудности, связанные с появлением малых частот, в простейшей модельной задаче преодолены в заметке [10]. Комбинируя соображения [7], [9], [10], можно найти условно-периодические решения задачи n тел.

Оказывается, если массы планет достаточно малы, то для большинства начальных условий, при которых эксцентриситеты и наклонения достаточно малы, движение будет условно-периодическим, эксцентриситеты и наклонения будут вечно оставаться малыми, а большие полуоси будут вечно колебаться вблизи начальных значений.

Для определенности рассмотрим пространственную задачу трех тел масс $m_1=\mu\alpha_1 M$, $m_2=\mu\alpha_2 M$ и M (где α_1, α_2, M — постоянные, а μ — малый множитель). Центр тяжести всех трех тел будем считать неподвижным; тогда число степеней свободы сведется к 6, а размерность фазового пространства — к 12. Точку фазового пространства будем характеризовать 12 эллиптическими элементами обеих планет m_1, m_2, в том числе величинами больших полуосей a_1, a_2, эксцентриситетов e_1, e_2 и наклонений i_1, i_2 оскулирующих эллипсов, по которым планеты вращаются вокруг M в одну сторону.

Пусть $0<c_1<C_1<c_2<C_2<\infty$ — постоянные. Условия $c_1<a_1<<C_1$; $c_2<a_2<C_2$; $e_1^2+e_2^2<\varepsilon$; $i_1^2+i_2^2<\varepsilon$ выделяют в фазовом пространстве ограниченную область $G(\varepsilon)$. Точка из $G(\varepsilon)$ определяет положения и скорости обеих планет и, следовательно, все движение. Можно доказать методами [7—10] следующее.

Теорема 3. Для любого $\eta>0$ найдется $\varepsilon_0(\eta)>0$ такое, что если $\varepsilon<\varepsilon_0$, $\mu<\varepsilon_0$, то для большинства начальных условий из $G(\varepsilon)$ (исключение составляют точки, образующие множество меры меньше $\eta \operatorname{mes} G(\varepsilon)$ движение условно-периодично, e_k и i_k вечно остаются малыми и a_k вечно остаются вблизи начальных значений.

Можно сказать, что теорема 3 есть метрический аналог «теоремы» Лапласа об устойчивости планетной системы; последняя теорема, как известно, никогда не была доказана. Следует заметить, что исключительное множество начальных условий в теореме 3 хотя и имеет малую меру в $G(\varepsilon)$, все же всюду плотно, связно и простирается в бесконечность. Движение при исключительных начальных условиях нужно еще исследовать; a priori оно может оказаться осциллирующим или даже уходящим в бесконечность. Принимая во внимание известный факт существования «люков» в распределении малых планет, можно предполагать, что для большинства исключительных начальных условий большие полуоси a_k не остаются вечно вблизи начальных значений. Так как сколь угодно малое изменение начальных условий способно перевести типичную точку в исключительную, из справедливости этой гипотезы вытекала бы топологическая неустойчивость планетной системы.

Подход с точки зрения теории меры кажется, однако, более разумным, особенно если учесть, что мы не знаем точно начальных условий, должны учитывать поправки к законам Ньютона и интересуемся движением на больших, а не на бесконечных промежутках времени.

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2 Зак. № 5331

SMALL DENOMINATORS AND PROBLEMS OF STABILITY OF MOTION IN CLASSICAL AND CELESTIAL MECHANICS*

by

V. I. ARNOL'D

Contents

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INTRODUCTION

§1. Results

The difficulty of qualitative problems of classical mechanics is well known. In spite of prolonged efforts by many mathematicians most of these problems still await solution. Only in recent times, beginning with the work of C.L. Siegel (1942) and A.N. Kolmogorov (1954), has some progress been made in solving problems on the stability of motion of dynamical systems. In particular,

1) *The stability of positions of equilibrium and periodic solutions* of conservative systems with two degrees of freedom has been proved in the so-called general elliptic case.

2) *The perpetual adiabatic invariance* of the variable of action has been proved for a slow periodic variation of the parameters of a non-linear oscillatory system with one degree of freedom. It has been established that *a "magnetic trap" with an axial-symmetric magnetic*

field can perpetually retain charged particles.[1]

3) Conditionally periodic motions in the many-body problem have been found. If the masses of n "planets" are sufficiently small in comparison with the mass of the central body, the motion is conditionally periodic for the majority of initial conditions for which the eccentricities and inclinations of the Kepler ellipses are small. *Further, the major semi-axis perpetually remain close to their original values and the eccentricities and inclinations remain small.*

The present paper gives complete proofs of these results which nave previously been published in the form of notes [14]-[17].

The author's interest in small denominators was stimulated by A.N. Kolmogorov's lectures in 1957 and he wishes to take this opportunity to express his deep gratitude to A.N. Kolmogorov for his attention to this paper.

The author was advised on that part of the work concerned with celestial mechanics by V.M. Alekseev. The paper by L.A. Artsimovich and M.A. Leontovich in the autumn of 1958 brought the author's attention to the problem of the perpetual adiabatic invariance of a magnetic moment. Many discussions with B.V. Chirikov helped to make the investigation of this question possible. G.A. Merman carefully read the manuscript and made a number of valuable remarks. L.Yu. Pius verified the calculations of Ch. III. §4. The author is grateful to all those named above.

We shall require results from mechanics not well known among mathematicians, and certain mathematical ideas little known to those engaged in the field of mechanics. They are set out in the following two sections.

§2. Preliminary results from mechanics

1. What are small denominators? Astronomers long ago noted[2] that resonance phenomena connected with the commensurability of frequencies of interacting motions lead to "small denominators" and considerable mathematical difficulties.

EXAMPLE 1. During the course of a day Jupiter moves along its orbit by an amount $\omega_1 = 299''.1$ and Saturn by an amount $\omega_2 = 120''.5$.

The frequencies ω_1, ω_2 are almost commensurable:

$$2\omega_1 - 5\omega_2 \approx 0.$$

The expression $m\omega_1 + n\omega_2$ is found as a denominator in series arising in the theory of perturbations of the form

[1] Although we use the terms "particles" and "planets", we are really dealing with mathematical theorems concerning the behaviour of solutions of definite differential equations. The applicability of these theorems to real systems has to be specially investigated in each individual case.

[2] "Difficulties encountered in celestial mechanics on account of the existence of small divisors and approximate commensurabilities of mean motions are connected with the very nature of things and cannot be avoided" - H. Poincaré [2].

$$\sum_{m,n\neq 0} a_{mn} \frac{e^{i(m\omega_1+n\omega_2)t}}{m\omega_1+n\omega_2}. \tag{0.2.1}$$

Since the time of Laplace the existence has been known of a large long-periodic perturbation of planetary motions around the Sun connected with the small denominator $2\omega_1 - 5\omega_2$.

2. **What are stability problems?** The first and highly stimulating problem of this type (which remains unsolved even today) was the question of the stability of planetary orbits. Do small perturbations of planets on each other give rise after a sufficiently long time to collisions or departure to infinity? [1]

The theory of stability of motion, developed in the well-known works of H. Poincaré and A.M. Lyapunov, made the discovery of asymptotic stability possible. But stability problems of classical mechanics always refer to the "neutral case" of purely imaginary characteristic exponents: asymptomatically stable motions are not then possible owing to the conservation of volume in the phase space (Liouville's theorem). The indicated methods therefore contribute nothing to investigations of the stability of motion of non-linear conservative systems.

The fundamental difficulty encountered in these investigations is connected with the divergence of the series (0.2.1) of the theory of perturbations on account of the small denominators $m\omega_1 + n\omega_2$. H. Poincaré studied the plane restricted three-body problem and showed that this difficulty occurs even in model problems which have a perfectly simple mathematical formulation. G.D. Birkhoff studied one of these problems in detail (see [3]).

EXAMPLE 2 ("Birkhoff's problem"). *An area-preserving analytic mapping T of the neighbourhood of the zero of a p, q-plane onto itself is given. Let zero be a fixed point. Is it stable?*

It is supposed that the linear part of T at zero is a rotation of the p, q-plane.

Recently the answer to this question has been found to be positive; we shall discuss it in §4.

3. In the following chapters we shall use the "frightening formal apparatus of dynamics". The canonical form of the equations of motion is not essential for the application of the methods discussed (see, for example, [18]) but it makes many of the calculations easier. The reader will find it useful to be acquainted with configuration and phase spaces, Lagrangian and Hamiltonian equations, cyclic coordinates and laws of conservation, canonical transformation, Poisson brackets, integral invariants and action-angle variables such as in [4] or [3], [5], and [6]. As a control it is useful to solve the following problem:

PROBLEM. Let a point move by inertia over a surface S (in the absence of the force of gravity). Find invariant two-dimensional manifolds in the phase space and investigate the motion of the phase point over it. Consider the case when S is a) a torus, b) an ellipsoid of revolution, c) a triaxial ellipsoid (cf. [57]).

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1 See the discussion of this question and of "Laplace's theorem" in [31].

§3. Preliminary results from mathematics

1. What is conditionally periodic motion? Let us consider the surface of a torus (anchor ring) (see Fig. 1).

On the surface we introduce "geographical" coordinates: longitude q_1 and latitude q_2. We shall express the angles q_1 and q_2 in radians and consider them to within an integral multiple of 2π. The square $0 \leqslant q_{1,2} \leqslant 2\pi$ in the q_1, q_2-plane can, for example, serve as a map of the torus. It is also convenient to use the whole q_1, q_2-plane divided into squares of side 2π. Each point of the torus has a representation in each square of such a map (Fig. 2).

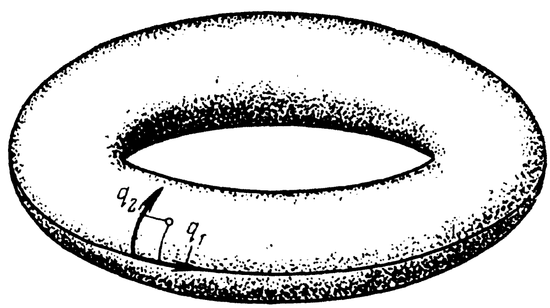

Fig. 1.

Let us consider the point $q_1(t)$, $q_2(t)$ moving along the torus so that its coordinates vary uniformly:

$$\frac{dq_1}{dt} = \omega_1, \qquad \frac{dq_2}{dt} = \omega_2. \tag{0.3.1}$$

On the q_1, q_2-map this motion is represented by a straight line.

If $\frac{\omega_1}{\omega_2} = \frac{m}{n}$, where m and n are whole numbers, then in time $t = 2\pi\frac{m}{\omega_1} = 2\pi\frac{n}{\omega_2}$ the point will return to its original position, having made m revolutions along the parallel and n along the meridian ($m = 2$, $n = 3$ in Fig. 2). In this case (0.3.1) defines a periodic motion.

If, however, $\frac{\omega_1}{\omega_2}$ is irrational, the moving point never returns to its original position. In this case the motion (0.3.1) is called *conditionally periodic* with two frequencies ω_1, ω_2. The trajectory $q_1(t)$, $q_2(t)$ is called a *winding of the torus*.

Closely connected with conditionally periodic motions are conditionally periodic functions. If $F(q_1, q_2)$ is a function on the torus expanded as a Fourier series

$$F(q_1, q_2) = \sum_{m,n=-\infty}^{+\infty} f_{mn} e^{i(mq_1+nq_2)},$$

then its variation in time for the motion (0.3.1) is of the form

$$f(t) = F[q_1(t), q_2(t)] = \sum_{m,\, n=-\infty}^{+\infty} f_{mn} e^{i[(m\omega_1+n\omega_2)t+\varphi_{mn}]}. \tag{0.3.2}$$

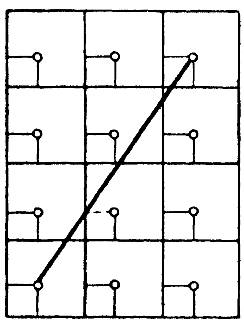

Fig. 2.

The functions (0.3.2) are called *conditionally periodic*. As an example we can take

$f(t) = \cos t + \cos\sqrt{2}\,t$. The appearance of series of the form (0.3.2) in any problem always indicates conditionally periodic motion (0.3.1).

2. Certain properties of conditionally periodic motions.

PROPERTY 1. *The trajectory of a conditionally periodic motion is everywhere dense on the torus.*

This means that, given any domain Δ, the moving point $p(t)$, $q(t)$ will sooner or later find itself in it. Property 1 follows easily from the following fact:

1a) *Let α be an irrational number and Δ be an arc of the circle $|z| = 1$. Then among the points $e^{2\pi i n\alpha}$ $(n = 1, 2, 3, \ldots)$ there are points of Δ.* [1]

We also note that the trajectory of a conditionally periodic motion is uniformly distributed (see [58]): the portion of time from $t = 0$ to $t = T$ which the moving point spends in the domain Δ (see Fig. 3) is proportional to the area of this domain if T is large. [2]

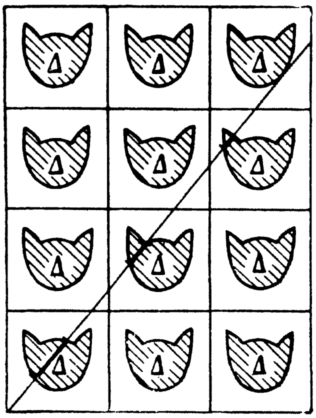

Fig. 3.

PROPERTY 2. *For any Riemann integrable function $F(q_1, q_2)$ the time mean is equal to the space mean:*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\omega_1 t, \omega_2 t)\,dt = \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} F(q_1, q_2)\,dq_1\,dq_2.$$

We give below another example to illustrate Property 2 (the reader may omit this).

3. Lagrange's problem on the mean motion of perihelia. Let the vector $\mathbf{a}(t)$ in the x,y-plane be the sum of three vectors:

$$\mathbf{a}(t) = \mathbf{a}_1(t) + \mathbf{a}_2(t) + \mathbf{a}_3(t),$$

of lengths a_1, a_2, a_3, rotating uniformly with independent[3] angular velocities ω_1, ω_2, ω_3. We denote by $\varphi(t)$ the angle which the vector $\mathbf{a}(t)$ makes with the x-axis (Fig. 4).

PROBLEM. *To find the mean angular velocity of the vector* $\mathbf{a}$:

$$\omega = \lim_{T\to\infty} \frac{\varphi(T)}{T}.$$

SOLUTION. $\omega = \dfrac{\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3}{\alpha_1 + \alpha_2 + \alpha_3}$, *where α_1, α_2, α_3 are the angles of the triangle with sides a_1, a_2, a_3.*

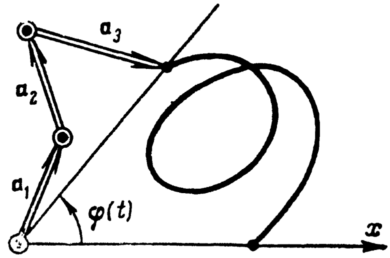

Fig. 4.

The connection with the motion of perihelia can be understood from Ch. III, §1, 2. For the bibliography and history of this problem which was

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1 Can the number 2^n begin with the digit 7? In accordance with 1a) the number 2^n can begin with any combination of digits.

2 With which digits does the number 2^n begin more often: 7 or 8?

3 The numbers ω_1, ω_2, ω_3 are independent if from $k_1\omega_1 + k_2\omega_2 + k_3\omega_3 = 0$, where the k_i are integral, it follows that $k_i = 0$.

solved by Ból, Sierpinski and H. Weyl see [52].

4. In the following chapters it is assumed that the reader is acquainted with the concept of a differentiable manifold (see [55]). In particular, a point of an n-dimensional torus is given by n angular coordinates $q_1, \ldots, q_n$. The reader will find it useful to investigate generalizations of 1. and 2. to the case of an n-dimensional torus. We also use the rudiments of measure theory (see [56]) (the reader should know that the set of rational numbers has measure zero).

Before passing on to the general theory we shall consider a very simple example in which many essential features of the phenomena to be studied are manifest.

§4. The simplest problem of stability

Let us return to Example 2 of §2: *an area-preserving analytic mapping T of the neighbourhood of zero of a p, q-plane onto itself is given. Is the fixed point O stable?*

We shall briefly set out the result of applying general methods (see Chapter I) to this case.

1. Three examples. For linear transformations the problem is solved by calculating the eigenvalues λ_1, λ_2. In view of the preservation of area $\lambda_1\lambda_2 = 1$. If λ_1 and λ_2 are not real then $\bar{\lambda}_1 = \lambda_2$, $|\lambda_{1,2}| = 1$, $\lambda_{1,2} = e^{\pm i\omega}$.

EXAMPLE A. *Let us consider an ordinary rotation A of the p, q-plane through an angle ω about the point O.* Each circle $p^2 + q^2 = \text{const}$ is invariant, i.e. transforms into itself: it rotates as a whole through an angle ω. The trajectory is everywhere dense on the circle if $\omega \neq 2\pi\frac{m}{n}$.

Every linear transformation T with $\lambda_{1,2} = e^{\pm i\omega}$ can be reduced to the form A by a linear change S of the coordinate system:

$$\begin{array}{ccc} p, q & \xrightarrow{T} & p, q \\ S\downarrow & & \downarrow S \\ p', q' & \xrightarrow{A} & p', q' \end{array}$$

Such a mapping T is called an elliptic rotation.

Let us pass on to a non-linear mapping. If the linear part of the given mapping T at zero is an elliptic rotation, T is called a mapping of *elliptic type*.

EXAMPLE B. *Let us consider the mapping* B (Fig. 5) *in which each circle $p^2 + q^2 = 2\tau$ is rotated through an angle*

$$\omega(\tau) = \omega_0 + \omega_1\tau + \ldots \tag{0.4.1}$$

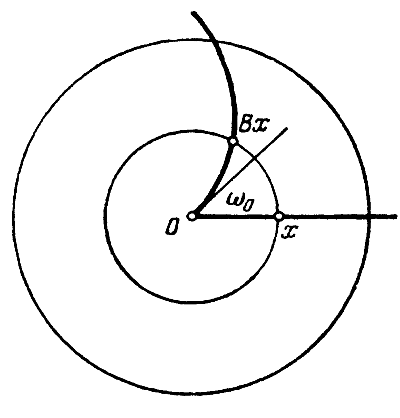

Fig. 5.

This mapping is of elliptic type and is stable.

Let us consider the coordinate system p', q' connected with p, q by the analytic transformation S which preserves area and keeps O in its original position. On the p', q'-plane let us consider the mapping B of Example B.

EXAMPLE C. *Let us write the mapping B in terms of the coordinates p, q. Then we obtain the mapping $C = S^{-1}BS$.*

This mapping is of elliptic type and is stable, since by a change of variables S it is transformed into the stable mapping B.

Is it possible to obtain any mapping of elliptic type with the construction of Example C? This would, in particular, give a positive solution to the problem of stability.

2. Formal solution. It has been known since Birkhoff's time [3] that if questions on the convergence of the series are disregarded, then under the conditions

$$\omega \neq 2\pi \frac{m}{n} \qquad (m = 0, \pm 1, \pm 2, \ldots; n = 1, 2, \ldots) \tag{0.4.2}$$

a mapping T of elliptic type can always be reduced to the form $STS^{-1} = B$ of Example B by a formal change of variables S. S is determined by means of "Birkhoff's series" which are analogous to the series of perturbation theory. In the general case these series are *divergent*. The stability of the mapping T does not follow from the existence of the formal series S.

Nevertheless it is possible to cut short the series S and by a convergent change of variables $S^{(s)}$ reduce T to a form which differs from B by small quantities of arbitrarily high order $O(\tau^s)$. The coefficients $\omega_0, \omega_1, \ldots$ in (0.4.1) which are then obtained do not depend on the method $S^{(s)}$ by which T is reduced to the form B; they are invariants of T with respect to the area-preserving analytic transformations. If $\omega_1 \neq 0$, the mapping T is said to be of *general elliptic type*.

In this case the angle $\omega(\tau)$ through which the circle $\tau = \text{const}$ rotates in the mapping B varies with τ (see (0.4.1)). Therefore some circles are rotated through an angle commensurate with, and others through an angle incommensurate with, 2π.

Given appropriate variables the mapping T close to O can be regarded as a rotation B through a variable angle $\omega(\tau)$ perturbed by very small auxiliary terms. Therefore our problem reduces to a study of those mappings T that differ from B only by perturbations small in comparison with τ^s.

3. Invariant curves. If Birkhoff's series S converge, then the neighbourhood of O wholly consists of invariant curves of T close to the circles $\tau = \text{const}$.

It turns out that, in fact, *the majority of invariant circles of the mapping B on which the angle $\omega(\tau)$ is incommensurate with 2π do not disappear for a small perturbation of B, but are only slightly deformed.* Therefore the fixed point O is surrounded by arbitrarily small analytic closed curves invariant with respect to T, and is therefore *stable*. These curves can be proved to fill a set of positive measure with O as a point of accumulation (see Chapters I and IV).

But these curves do not fill the neighbourhood of O completely and, in general, do not fill any domain: between them there still remain "zones of instability", which arise on perturbation from the circles

τ = const, where $\omega(\tau)$ is commensurate with 2π. On each ray going out from O the invariant curves carve out a track like Cantor's perfect set, but of positive measure.

4. Zones of instability. Let us consider an invariant circle of an "unperturbed" transformation B rotated through an angle $\omega(\tau) = 2\pi \frac{m}{n}$. On an n-fold iteration of B each point of the circle returns to its original position. This property of B is not, generally speaking, retained for a small perturbation and such an invariant circle is "scattered away". G.D. Birkhoff proved that instead of a complete circle of points fixed relative to B^n, T^n has, in general, a finite even number of fixed points close to this circle. Half of these points are of elliptic and half of "hyperbolic" type.[1]

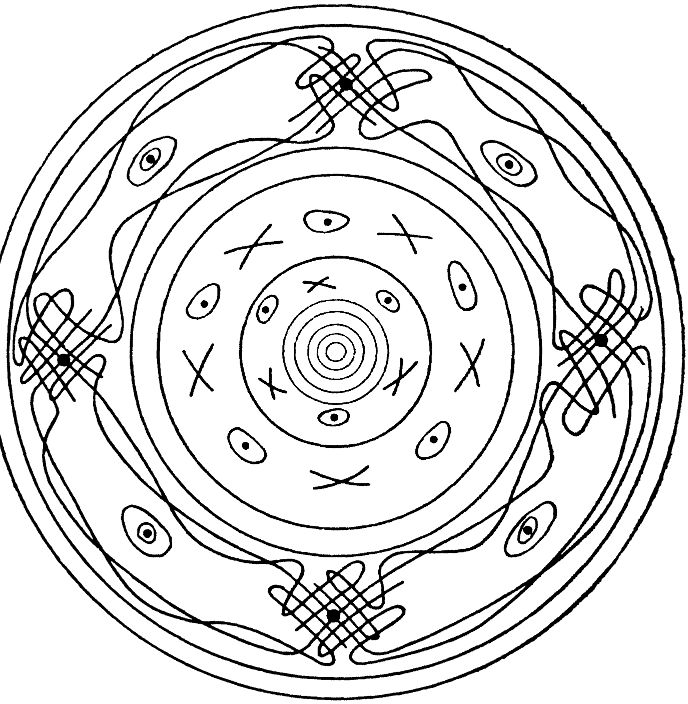

Fig. 6.

As we showed in 3., points of elliptic type are, generally speaking, stable and surrounded by invariant curves not enclosing O (Fig. 6). Consequently in the general case the neighbourhood of O is not stratified into invariant closed curves. The divergence of Birkhoff's series, mentioned above, follows from this (see 2.).

The separatrices of hyperbolic points intersecting each other create an intricate network in the "zones of instability". In the neighbourhood

1 The reader will readily understand their construction by considering the hyperbolic rotation $p \to 2p,\ q \to \frac{1}{2}q$.

of each elliptic point the same picture holds, with invariant curves, zones of instability etc.

5. Conditions of stability. We now state the assumptions under which the existence of invariant curves (see 3.), and the stability of T can be proved.

Conditions used by the author in [14] were that $\frac{\omega_0}{2\pi}$ is irrational, $\omega_1^2 + \omega_2^2 + \ldots \neq 0$, and T is analytic. Then J. Moser [25] showed that, in place of the irrationality of $\frac{\omega_0}{2\pi}$, it is sufficient to require $\omega_0 \neq 2\pi\frac{m}{3}$, $2\pi\frac{m}{4}$, and in place of the analyticity of T, the continuity of the 333rd partial derivatives. For $\omega_0 = 2\pi\frac{m}{3}$ instability is possible, as T. Levi-Cività had already established (see [8] and, on the same subject, the paper by G.A. Merman in [21], pp. 18-41).

§5. Contents of the paper

The paper consists of six chapters. Fundamental ideas and methods are described in Chapter I; the exposition here is of a non-rigorous heuristic character. I hope, however, that the thoughtful reader of Chapter I will be able to establish the proofs of the fundamental results without turning to Chapters IV and V which contain the strict mathematical basis.

Chapters II and III are devoted to the applications of the general theory of Chapter I. The concept of an adiabatic invariant with applications to "magnetic traps" is examined in detail in Chapter II. In Chapter III we are concerned with the problem of many planets. The plane three-body problem in the case of small eccentricities and small planet masses is considered in detail.

Chapters IV and V and also Chapter III with the exception of §§1 and 5 are written formally. The reader will find the list of notation at the end of Chapter V helpful.

The concluding chapter contains a number of separate remarks some of which refer to unsolved problems.

Chapter I

THEORY OF PERTURBATIONS

In this chapter the ideas and constructions are described that will be used in Chapters II and III to solve specific problems of dynamics.

We shall use:

1) methods developed by astronomers of the nineteenth century and completed in the investigations of H. Poincaré [1] (see §§1, 2, 3, 5);

2) the investigations of G.D. Birkhoff on the stability of positions of equilibrium and periodic motions [3] (see §9);

3) a method of successive approximations of Newtonian type suggested by A.N. Kolmogorov [12] (see §4).

The author's investigations [14]-[17] refer to the so-called degeneracy. Two remarks in §§6 and 7 allow the proper degeneracy to be dealt with in §8. Certain cases of limiting degeneracy are examined in §10. [1]

In this chapter we avoid giving accurate formulations and strict proofs of the theorems. We only show the mainsprings of the complicated constructions and strict proofs of Chapter IV.

§1. Integrable and non-integrable problems of dynamics

We shall consider conservative dynamical systems with n degrees of freedom defined by the canonical equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad (p = p_1, \ldots, p_n;\ q = q_1, \ldots, q_n) \qquad (1.1.1)$$

with an analytic Hamiltonian $H(p, q)$. Classical methods of dynamics permit the investigation only of the so-called integrable cases.

EXAMPLE 1. Let us assume that the phase space p, q is the direct product of an n-dimensional torus and a domain of an n-dimensional Euclidian space. Let $q_i \pmod{2\pi}$ be the angular coordinates on the torus and p_i in space and let the Hamiltonian depend only on p: $H = H(p)$. The Hamiltonian equations (1.1.1) take the form

$$\dot{p} = 0, \quad \dot{q} = \omega(p) \quad \left(\omega = \frac{\partial H}{\partial p} = \omega_1, \ldots, \omega_n\right)$$

and are at once integrable. Each torus p = const is invariant; if the frequencies ω are incommensurate (i.e., from $\omega_1 k_1 + \ldots + \omega_n k_n = 0$ with integral k_i it follows that $k_i = 0$), then the motion is called *conditionally periodic* with n frequencies $\omega_1, \ldots, \omega_n$; it is easy to prove that the trajectory $p(t)$, $q(t)$ fills the torus everywhere densely. The variables p, q are called *action-angle variables*.

A fairly large number of integrable problems are known today. The solution of these problems with n degrees of freedom is based on the fact there exist (and can be found) n single-valued first integrals in involution. [2]

It can be shown ([20], Ch. VI, §1) that the existence of these integrals implies the following pattern of behaviour of the trajectories in the $2n$-dimensional phase space p, q. A certain particular $(2n - 1)$-dimensional set divides the phase space into invariant domains each of

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1 At the time when the last of our notes [14]-[17] was printed there appeared the first of two papers by J. Moser [24], [25] showing that in place of the analyticity of perturbations in [12] it is sufficient that several hundred derivatives should exist. We do not use the results and techniques of J. Moser, since the main part of the present paper was written before the publication of [24], [25]. Only in §10 have we taken into account J. Moser's strengthening of the result of [14] (see also Ch. VI, §6).

2 The functions $f(p, q)$ and $g(p, q)$ are in involution if their Poisson bracket $\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}$ vanishes identically.

which is stratified into invariant n-dimensional manifolds. If the domain is bounded, these manifolds are tori supporting conditionally periodic motions. The action-angle coordinates of Example 1 can be introduced into such a domain. If n first integrals in involution have been found, then a canonical transformation introducing action-angle variables is given by quadrature.

EXAMPLE 2. The following are integrable problems: the two-body problem; the attraction of two fixed centres; the motion of a free point along a geodesic on the surface of a triaxial ellipsoid; a heavy symmetric solid body fastened at a point on its axis; an asymmetric solid body fastened at its centre of gravity; linear oscillations.

Non-integrable[1] problems are: the n-body problem including the so-called plane restricted circular three-body problem; the motion of a free point along a geodesic on a convex surface; a heavy asymmetric solid body; non-linear oscillations with $n > 1$ degrees of freedom.

The search for integrable cases was principally dealt with in the nineteenth century (Jacobi, Liouville, Kovalevska and others). But with the work of Poincaré it became clear that a dynamical system in its general form was non-integrable; the integrals were not only not known, but did not exist at all, because the trajectories on the whole did not lie on invariant n-dimensional manifolds.

§2. The classical theory of perturbations

Let us assume that a system differs from an integrable system by small "perturbations"; using the notation of Example 1,

$$H(p, q) = H_0(p) + \mu H_1(p, q) + \dots, \tag{1.2.1}$$

where μ is small and $H_1 + \dots$ is of period 2π with respect to q. According to Poincaré [1] the investigation of this case is a fundamental problem of dynamics. What influence does the perturbation μH_1 have on the behaviour of the trajectories as $t \to \infty$? Are the invariant tori preserved? Does the trajectory remain at least close to the torus p = const?

A comparison of the integrable and non-integrable problems of Example 2 demonstrates the significance these questions have in mechanics. A complete answer to them would contain, in particular, the solution to the problem of the stability of a planetary system.

For an approximate investigation of trajectories for large t a special apparatus was developed long ago in the theory of perturbations of astronomy. If a canonical transformation $p, q \to p', q'$ reduces H to the form

$$H(p, q) = H_0'(p') + \mu^2 H_1'(p', q') + \dots, \tag{1.2.2}$$

1 More carefully, one might say problems in which the integration cannot be performed, since the proofs of non-integrability are complex and are carried out rigorously only in individual cases (see [1], [10]).

then in the course of time $t \sim \frac{1}{\mu}$ the motion $p'(t)$, $q'(t)$ differs from the conditionally periodic motion described by $H_0'(p')$ by a quantity $\sim \mu$. Returning to p, q, we shall obtain for $p(t)$, $q(t)$ approximate expressions with an error of order μ for $t \sim \frac{1}{\mu}$. If greater accuracy is required, the following approximation can be made: $p', p' \to p'', q''$, which reduces H to the form

$$H(p, q) = H_0''(p'') + \mu^3 H_1''(p'', q'') + \ldots$$

The error now $\sim \mu^3 t$. If the successive approximations converge, then in the limit we obtain $H(p, q) = H_0^{(\infty)}(p^{(\infty)})$, i.e. the system is integrable: the tori $p^{(\infty)}(p, q) = \text{const}$ are invariant and are filled with trajectories of conditionally periodic motions.

In carrying out this programme we encounter two difficulties.

1. Small denominators. We shall look for a canonical transformation $p, q \to p', q'$ in the form $p = p' + \mu \frac{\partial S}{\partial q}$, $q' = q + \mu \frac{\partial S}{\partial p'}$, $S(p', q) = \sum_{k \neq 0} S_k(p') e^{i(k, q)}$. The function $H(p, q)$ in terms of the co-ordinates p', q' is written in the form[1]

$$H_0(p) + \mu \overline{H}_1(p) + \mu \widetilde{H}_1(p, q) + \ldots =$$
$$= H_0(p') + \mu \overline{H}_1(p') + \mu \left[\frac{\partial H_0}{\partial p} \frac{\partial S}{\partial q} + \widetilde{H}_1 \right] + \mu^2 \ldots$$

In order to obtain (1.2.2) it is necessary to eliminate the terms of order μ depending on q, i.e. we must have $(\omega, \frac{\partial S}{\partial q}) + \widetilde{H}_1 = 0$ or

$$S_k(p') = \frac{i h_k(p')}{(\omega, k)}, \text{ where } \widetilde{H}_1 = \sum_{k \neq 0} h_k e^{i(k, q)}. \quad (1.2.3)$$

The denominator (ω, k) vanishes for certain "resonance" values of k and for all ω is arbitrarily small for suitable k. These small denominators (ω, k) raise doubts concerning the validity of our formal transformations for $n > 1$.

2. The divergence of approximations. There are cases in which the series for each approximation terminate and therefore converge. Such cases were investigated in detail by Birkhoff [3] (cf. §9). But Siegel [10] has shown that, as a rule, all the approximations taken together, including those in this case, diverge. As a consequence of convergence the structure of the trajectories should be as described in Example 1. In fact, however, the trajectories of the perturbed system cannot lie on invariant tori.

Let us suppose that $\det\left|\frac{\partial \omega}{\partial p}\right| \neq 0$. Then in any neighbourhood of any invariant torus of the unperturbed system there is an n-dimensional torus on which all the trajectories after a certain time become closed. With a small perturbation this n-dimensional manifold of closed trajectories is

---

[1] The line above H_1 denotes averaging with respect to q: $\overline{H}_1(p) = (2\pi)^{-n} \int H_1(p, q) dq$.

in general destroyed. Consequently, the series of the theory of perturbations do not, generally speaking, converge in any domain of the phase space.

The above considerations do not exclude the possibility that invariant tori on which $(\omega, k) \neq 0$ may exist in a perturbed system. These tori cannot, however, fill any domain.

§3. Small denominators

In investigating the influence of the small denominators (ω, k) astronomers have for a long time used certain arithmetic arguments (see [1], [5]). The simplest of these consists in that there are more irrational numbers than rational.

Furthermore, the components of a randomly selected vector ω are incommensurable. Therefore for almost all[1] vectors ω we have $(\omega, k) \neq 0$ for all integers $k \neq 0$.

The following theorem from the theory of Diophantine approximations (cf. [54]) expresses this idea more precisely:

Almost every vector $\omega = \omega_1, \ldots, \omega_n$ *satisfies the inequalities*

$$|(\omega, k)| \geqslant K|k|^{-\nu} \qquad (|k| = |k_1| + \ldots + |k_n|;\ \nu = n+1) \tag{1.3.1}$$

for all integral $k \neq 0$ *and for a certain* $K(\omega) > 0$.

PROOF. Let us consider a bounded domain Ω, a fixed $K > 0$, and integral k. Then the inequality (1.3.1) fails to hold only in the "resonance zone" of width less than $2K|k|^{-\nu}$. The volume of this zone does not exceed $K|k|^{-\nu}D$, where the constant $D > 0$ depends only on Ω.

The number of values of k with $|k| = m$ does not exceed Lm^{n-1} (the constant $L > 0$ depends only on n). Therefore the measure of all resonance zones with $|k| = m$ does not exceed $Km^{-2}DL$ and the measure of all the zones with $|k| > 0$ does not exceed $\sum_{m=1}^{\infty} Km^{-2}DL \leqslant K\bar{D}(\Omega)$, $\bar{D} = 2LD$. As $K \to 0$, the total measure of resonance zones tends to zero and hence the proof of the assertion immediately follows.

Thus, for the majority of vectors ω the small denominators (ω, k) not only do not vanish, but can be estimated from below by a power of $|k|$. From this stems the hope (see [1], [5], [7]) that the series (1.2.3) of the theory of perturbations might converge for the majority of vectors ω: in fact, the Fourier coefficients h_k of the analytic function H_1 decrease in geometric progression.

Let H_1 be analytic for $|\mathrm{Im}\, q| \leqslant \rho$ and let $|H_1(p, q)| \leqslant M$. Then $|h_k| \leqslant Me^{-|k|\rho}$ (see Ch. V, §3, 2.). With the condition (1.3.1) the coefficients $S_k = \dfrac{ih_k}{(\omega, k)}$ decrease in geometric progression almost as rapidly as the h_k: for any $\delta > 0$ we have

$$|S_k| \leqslant \frac{ML}{K\delta^{\nu}} e^{-|k|(\rho-\delta)},$$

[1] All, except for a set of Lebesgue measure zero.

where ν, L are absolute constants ((4.6.5), Chapter IV). Consequently, the series S converges for $|\text{Im } q| < \rho$ and in the somewhat smaller domain $|\text{Im } q| \leqslant \rho - 2\delta$ we have (see (4.6.6), Chapter IV)

$$|S| \leqslant \frac{ML}{K\delta^{\nu}}. \tag{1.3.2}$$

Thus, (1.2.3) converges for almost all ω. However, 1) the functions so obtained depend on p everywhere discontinuously: 2) the convergence of all the approximations $p^{(s)}$, $q^{(s)}$ as $s \to \infty$ remains doubtful.

§4. Newtons Method

In order to overcome the difficulties 1) and 2) above A.N. Kolmogorov [12] made the following two suggestions.

1) *We shall look for only one invariant torus T_{ω^*} of the perturbed system* on which there is conditionally periodic motion with frequencies ω^*. *The set of frequencies ω^* satisfying* (1.3.1) *is fixed in advance.* We look for the torus T_{ω^*} in the neighbourhood of the corresponding invariant torus of the unperturbed system: $p = p^* + \mu \ldots;\ \frac{\partial H_0}{\partial p^*} = \omega^*$.

In the formula (1.2.3) we put ω^* in place of $\omega(p) = \frac{\partial H_0}{\partial p}$. Then $H(p, q)$ expressed in terms of the new variables contains an additional term $\mu\left[(\omega - \omega^*)\frac{\partial S}{\partial q}\right]$. For $|p - p^*| \sim \mu$ this term will be of order μ^2.

2) In the indicated neighbourhood of the torus $p = p^*$ we can introduce new variables p', q' by means of an analytic canonical transformation $p, q \to p', q'$ in which the Hamiltonian $H(p, q) = H_0(p) + H_1(p, q)$ takes the form

$$H(p, q) \equiv H^{(1)}(p', q') = H_0^{(1)}(p') + H_1^{(1)}(p', q'),$$

where $|H_1^{(1)}| \sim |H_1|^2$.

The so arising *quadratic convergence*, typical of Newton's tangent method (see [51]), allows us to find the invariant torus T_{ω^*}. More precisely, the second suggestion above is contained in the following. *Let $|H_1| \leqslant M_1$ for $|\text{Im } q| \leqslant \rho$. By means of* (1.2.3), (1.3.1), (1.3.2) *and* 1) *above we can obtain variables p', q' such that for $|\text{Im } q'| \leqslant \rho - L\delta$, $|p' - p^{*\prime}| \leqslant M_1$ we have*

$$|H_1^{(1)}(p', q')| \leqslant M_2 = \frac{M_1^2}{\delta^{2\nu}}. \tag{1.4.1}$$

Here $L > 0$, $\nu > 0$ are constants depending only on the number of degrees of freedom n; at the point $p^{*\prime}$ we have $\frac{\partial H_0^{(1)}}{\partial p'} = \omega^*$, and $\delta > 0$ can be chosen arbitrarily provided it does not exceed a certain constant depending only on H_0, ω^* and ρ.

We shall now show, with (1.4.1) at our disposal, how to construct converging successive approximations to the invariant torus T_{ω^*}. Since $H^{(1)}(p', q')$ has the same form as $H(p, q)$, we can construct with the help of (1.2.3) the canonical transformations

$$p',\ q' \longrightarrow p'',\ q'' \longrightarrow \dots \longrightarrow p^{(s)},\ q^{(s)} \longrightarrow \dots,$$
$$H(p,q) = H^{(s)}(p^{(s)}, q^{(s)}) = H_0^{(s)}(p^{(s)}) + H_1^{(s)}(p^{(s)}, q^{(s)}).$$

In the contracting domains, defined by the inequalities

$$|p^{(s)} - p^{*(s)}| \leqslant M_s, \quad |\operatorname{Im} q^{(s)}| \leqslant \varrho_s \ \text{(where } M_{s+1} = \frac{M_s^2}{\delta_s^{2\nu}},$$
$$\varrho_{s+1} = \varrho_s - L\delta_{s+1}, \quad \left.\frac{\partial H_0^{(s)}}{\partial p^{(s)}}\right|_{p^{*(s)}} = \omega^* \quad (s = 1, 2, \dots)) \quad (\varrho_0 = \varrho,\ M_1 = M),$$

we have, by (1.4.1),

$$|H_1^{(s)}| \leqslant M_{s+1} = \frac{M_s^2}{\delta_s^{2\nu}}. \tag{1.4.2}$$

We shall now deal with the quantities δ_s. We put $\delta_{s+1} = \delta_s^{3/2}$ $(s = 1, 2, \dots)$. If $M_s < \delta_s^T$ and T is sufficiently large, then by (1.4.2),

$$M_{s+1} \leqslant \delta_s^{2T-2\nu} \leqslant \delta_s^{T\,3/2} = \delta_{s+1}^T. \tag{1.4.3}$$

We take T to be large and fixed, for example $T = 4\nu + 1$ and assume that for $|\operatorname{Im} q| \leqslant \rho$ we have $|H_1(p, q)| \leqslant M_1 = \delta_1^T$, where δ_1 is sufficiently small. Then for all $s = 1, 2, \dots$ we have $|H_1^{(s)}(p^{(s)}, q^{(s)})| \leqslant \delta_s^T$ in the domain $|\operatorname{Im} q^{(s)}| \leqslant \rho_s$, $|p^{(s)} - p^{*(s)}| \leqslant M_s$. In addition, for sufficiently small δ_1 we have $\delta_s > \frac{\rho}{2} > 0$ $(s = 1, 2, \dots)$. It is easy to see that, by (1.4.2) and (1.4.3), the so constructed domains contract to the invariant analytic torus T_{ω^*}.

Thus, we arrive at the following picture of perturbed motion (cf. [13]). We assume that $\det\left|\frac{\partial\omega}{\partial p}\right| = \det\left|\frac{\partial^2 H_0}{\partial p^2}\right| \not\equiv 0$. Then in a small neighbourhood of any point p there are points where the frequencies ω are commensurable and also points for which $\omega(p) = \omega^*$ satisfies (1.3.1). In accordance with this, the canonical equations with Hamiltonian $H_0(p)$ determine everywhere dense conditionally periodic trajectories on some tori $p = \text{const}$, but not on others.

It turns out that, for a small perturbation $H = H_0(p) + \mu H_1(p, q)$ $(\mu \ll 1)$, most of the invariant tori with incommensurable frequencies ω^* satisfying (1.3.1) with fixed K do not disappear, but are merely slightly deformed. The trajectories of the perturbed motion beginning on the deformed torus T_{ω^*} fill it everywhere densely and conditionally periodically. The tori T_{ω^*} form a closed nowhere dense set (between them are gaps filled with the remainders of destroyed tori with commensurable ω). But this invariant nowhere

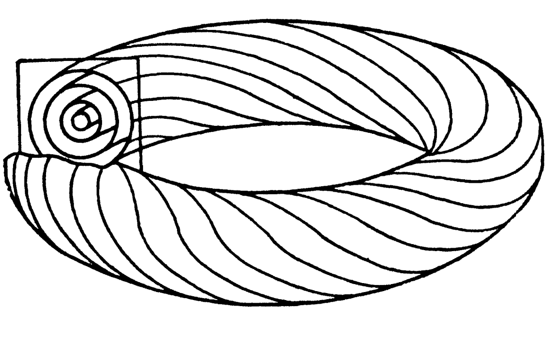

Fig. 7.

dense set has a positive measure which tends to the measure of the whole phase space when $K \to 0$, $\mu \to 0$ (see §7).

In the case $n = 2$ the two-dimensional tori $T_{\omega*}$ divide the three-dimensional invariant "energy level" $H = \text{const}$ (Fig. 7). Therefore a trajectory beginning in a gap between two tori $T_{\omega*}$ cannot pass out of this gap. Thus *for $n = 2$ the existence of invariant tori allows us to reach conclusions regarding the stability of motion.*

In the case $n > 2$ the n-dimensional tori $T_{\omega*}$ do not divide the $(2n-1)$-dimensional manifold $H = \text{const}$, and the "gaps" can extend to infinity. In this case we obtain information about the motion only for a majority of initial conditions.

§5. Proper degeneracy

In the arguments of §4 we assumed that the frequencies $\omega(p) = \frac{\partial H}{\partial p}$ were independent so that $\det\left|\frac{\partial \omega_i}{\partial p_j}\right| \not\equiv 0$. But not infrequently cases of "proper degeneracy" occur when in the unperturbed system the motion is described by a smaller number of frequencies n_0 than the number of degrees of freedom n (see [5]). This is precisely the case in the problem of magnetic traps ($n_0 = 1$, $n = 2$) and in many problems of celestial mechanics, where the two-body problem ($n_0 = 1$, $n = 3$) plays the role of the unperturbed system. In these cases the determinant

$$\det\left|\frac{\partial \omega_i}{\partial p_j}\right| = \det\left|\frac{\partial^2 H_0}{\partial p_i\, \partial p_j}\right|$$

vanishes identically. In the present section we shall consider proper degeneracy from the point of view of the classical theory of perturbations (see §2).

EXAMPLE. Suppose that in the action-angle variables $p_1, \ldots, q_n$ of Example 1 (§1) we have

$$H = H_0(p_1, \ldots, p_{n_0}),$$

where $n_0 < n$. We shall denote the vector of "rapid variables" $p_1, \ldots, p_{n_0}$ by p_0 and the vector of "slow variables" $p_{n_0+1}, \ldots, p_n$ by p_1; q_0 and q_1 have similar meanings.

The canonical equations

$$\dot{q}_0 = \omega_0(p_0), \qquad \dot{p}_0 = 0, \qquad \dot{q}_1 = 0, \qquad \dot{p}_1 = 0$$

(where $\omega_0(p_0) = \frac{\partial H_0}{\partial p_0}$) describe the conditionally periodic motion with n_0 frequencies $\omega_0 = \omega_1, \ldots, \omega_{n_0}$ over the n_0-dimensional invariant torus $p_0 = \text{const}$, $p_1 = \text{const}$, $q_1 = \text{const}$.

We now assume that there exists a perturbation

$$H(p, q) = H_0(p_0) + \mu H_1(p, q) + \ldots \tag{1.5.1}$$

Then the classical theory of perturbations[1] gives the following picture of the motion (with an accuracy $\sim \mu$ for $t \sim \frac{1}{\mu}$). Let us divide H_1 into a "secular part"

$$\overline{H}_1(p, q_1) = \overline{H}_1(p_1, \ldots, p_n; q_{n_0+1}, \ldots, q_n) = (2\pi)^{-n_0} \int H_1 \, dq_0$$

and a periodic part $\widetilde{H}_1(p, q)$:

$$H_1(p, q) = \overline{H}_1 + \widetilde{H}_1.$$

It turns out that the secular and periodic parts of the perturbation play completely different roles. The canonical equations with Hamiltonian $\mu\overline{H}_1$:

$$\dot{p}_1 = -\mu \frac{\partial \overline{H}_1}{\partial q_1} + \ldots, \qquad \dot{q}_1 = \mu \frac{\partial \overline{H}_1}{\partial p_1} + \ldots,$$

determine the slow secular variation of the parameters p_1, q_1, defining an invariant torus. The periodic part $\widetilde{H}_1$ leads only to an additional vibration of the perturbed trajectory about the conditionally periodic motion with slowly changing parameters described by the Hamiltonian $H_0 + \mu\overline{H}_1$.

The indicated picture of the motion is obtained by means of the transformation p_0, $q_0 \to p_0'$, q_0' of §2 if p_1, q_1 are regarded as the parameters.

In order to obtain more precise conclusions regarding the character of the perturbed motion it is necessary to investigate the "averaged" canonical equations with Hamiltonian $\overline{H}_1(p_1, q_1)$ (depending on the parameters p_0). In what follows we shall consider cases when these equations are integrable or nearly integrable, which happens, for example, in the plane three-body problem with small masses or in the general n-body problem with small masses, eccentricities and inclinations.

In the integrable case with appropriate choice of variables p_1, q_1 the secular part $\overline{H}_1 = \overline{H}_1(p_1, \ldots, p_n)$ does not depend on the angular variables q_1 and as a first approximation we arrive at the conditionally periodic motion

$$\dot{q}_0 = \omega_0(p_0), \qquad \dot{p}_0 = 0, \qquad \dot{q}_1 = \omega_1(p), \qquad \dot{p}_1 = 0$$

with n_0 rapid frequencies $\omega_0 = \xi_0$ and $n_1 = n - n_0$ slow frequencies $\omega_1 = \mu\xi_1$. A perturbation arising from $\mu\widetilde{H}_1$ and equal on average to zero is superimposed on this motion.

Our basic result consists in the construction of a rigorous theory of perturbations similar to that described in §4 for the case of a proper degeneracy. We shall show that when μ is sufficiently small, for the majority of initial conditions the perturbed motion is, in fact, conditionally periodic and close to the first approximation described above for all $-\infty < t < +\infty$ (see §8).

We encounter a number of difficulties in attempting to apply the techniques of §4. First and foremost we note that *among the small denominators* (ω, k) *there exist zeros*, since for $\mu = 0$ there are only n_0 frequencies ω_0. Furthermore, in the subsequent approximations n frequencies ω_0, ω_1

[1] In this case it is also called the "method of averaging" (cf. [5], [7]).

determine the small denominators $[(\xi_0, k_0) + \mu(\xi_1, k_1)]$ of which *some are small owing to the approximate commensurability of the frequencies* ω, *but others are small on account of degeneracy.* For $k_0 = 0$ denominators of order μ are obviously obtained.

§6. Remark I

Difficulties connected with degeneracy were overcome in [15], [16], [17], at first for $n_0 = 1$ in the model problem [15] and in the problem of an adiabatic invariant [16], and then also in the general case [17]. Our construction is based on two remarks. In this section we shall consider the first of these, which is sufficient for the investigation of the case $n_0 = 1$.

We noted in §5 that some of the denominators $[(\xi_0, k_0) + \mu(\xi_1, k_1)]$ were small on account of the approximate commensurability of the frequencies ω, but others (for $k_0 = 0$) on account of the smallness of μ. It turns out that typical small denominators $(\xi_0, k_0) + \mu(\xi_1, k_1)$ admit an estimate from below of the form

$$|(\xi_0, k_0) + \mu(\xi_1, k_1)| \geqslant \begin{cases} K|k|^{-\nu}, & \text{if} \quad k_0 \neq 0, \\ \mu K |k|^{-\nu}, & \text{if} \quad k \neq 0 \end{cases} \qquad (\nu = n+1). \qquad (1.6.1)$$

For it is easily verified that, in the ξ_0, ξ_1 space for fixed K, μ, k, the condition (1.6.1) is violated only "in a resonance strip" not wider than $2K|k|^{-\nu}$. The subsequent argument proceeds as in §3.

This simple remark permits the construction of converging approximations to the invariant tori in the case in which the Hamiltonian is of the form

$$H = H_0(p_0) + \mu H_1(p) + H_2(p, q) + \ldots, \qquad H_2 \sim \mu^2 \qquad (1.6.2)$$

For we shall regard $H_0 + \mu H_1$ as the unperturbed function. We introduce new variables p', q' by means of a canonical transformation defined as in §§2 and 4. We assume that $|H_2| \leqslant M$. Then from (1.6.1) and (1.2.3), in a similar way to (1.4.1)-(1.4.3), we find that

$$|S| \sim \frac{M}{\mu}, \qquad |p_0 - p_0'| \sim M^2, \qquad |p - p'| \sim \frac{M}{\mu}, \qquad |q - q'| \sim \frac{M}{\mu}. \qquad (1.6.3)$$

In the new variables

$$H = H_0(p_0') + \mu H_1(p') + H_3(p', q'),$$

where

$$|H_3| \sim \frac{M^2}{\mu} \qquad (1.6.4)$$

(the sign ~ denotes "is of the order of" in a very rough sense: we disregard the divisor $K\delta^{\nu}$, taking M and μ as considerably less than $K\delta^{\nu}$).

Now from (1.6.4) for $|H_2| \leqslant M \sim \mu^2$ we obtain $|H_3| \sim M^{3/2}$. This allows the introduction of Newtonian successive approximations with $M_{s+1} = M_s^{3/2 - \Delta}$, $\Delta > 0$ (for example, $M_{s+1} = M_s^{4/3}$).

It remains to reduce the Hamiltonian (1.5.1) to the form (1.6.2). This operation, which is an averaging over the rapid variables, does not require any new ideas in the case $n_0 = 1$, $n = 2$.

In fact, for $n_0 = 1$ there are no small denominators in (1.2.3) and the generating function of the canonical transformation $p, q \to p', q'$ is determined simply by means of the formula

$$S(p_0', p_1', q_0, q_1) = -\frac{1}{\omega(p_0', p_1', q_1)} \int_0^{q_0} \widetilde{H}_1(p_0', p_1', q_0, q_1)\, dq_0. \qquad (1.6.5)$$

It is easy to verify that in the new variables p', q' (see §2) the Hamiltonian (1.5.1) takes the form

$$H_0(p_0') + \mu \overline{H}_1(p', q_1') + H_2(p', q') + \dots, \quad H_2 \sim \mu^2. \qquad (1.6.6)$$

Furthermore, if $n_0 = 1$, $n = 2$, then $n - n_0 = 1$ and the averaged system with the Hamiltonian $\overline{H}_1$ (where p_0' is a parameter and p_1' and q_1' are canonically conjugate variables) has one degree of freedom and is therefore integrable. By introducing action-angle variables p_1'', q_1'' in place of p_1', q_1' we reduce $\overline{H}_1(p_0', p_1'; q_1')$ to the form $\overline{H}_1''(p_0', p_1'')$ and thereby (1.6.6) to the form (1.6.2).

Problems that can be reduced to the case $n_0 = 1$, $n = 2$ are the behaviour of an adiabatic invariant of an oscillating system with one degree of freedom for a slow periodic variation of the Hamiltonian, and also the adiabatic invariance of the magnetic moment in an axially-symmetric magnetic trap (see Chapter II). Since $n = 2$, the invariant tori divide the phase space and our method allows us to prove that the adiabatic invariant is perpetually conserved and a particle remains perpetually enclosed in the trap.

If $n > 2$, then reduction to the form (1.6.2) is possible on the assumption that the averaged system is integrable or nearly integrable.

In the case $n_0 > 1$ a substantial additional difficulty occurs in reducing the Hamiltonian (1.5.1) to the form (1.6.2) or even (1.6.6). The difficulty is as follows. In averaging over the rapid variables the estimate $|H_2| \sim \mu^2$ is obtained in accordance with §4 only in a domain $\sim \mu$ about the chosen value of p_0^*.

But it is then not possible to proceed with the subsequent approximations. In fact, for $|H_2| < M$ in the formula (1.6.3), estimating the derivatives by Cauchy's formula we obtain $|q_0 - q_0'| \sim \frac{M}{\mu^2}$. Therefore in (1.6.4) $|H_3| \sim \frac{M^2}{\mu^2}$ and for $M \sim \mu^2$ it turns out that $H_3 \sim H_2$ so that Newtonian approximations are not obtained.

§7. Remark 2

The above difficulty can be overcome in the following way. Let us consider again the non-degenerate case (§§2 and 4). The limitation on the domain $|p - p^*| \sim \mu$ is connected with 1) of §4: this introduces into $H_1^{(1)}$ a term of order $|p - p^*|^2$.

Returning to the classical methods of the theory of perturbations (§2) *we disregard* 1) *of* §4 *and take* ω *in formula* (1.2.3) *of* §2 *to be a function* $\omega(p) = \frac{\partial H_0}{\partial p}$. But then, in order not to have to deal with everywhere discontinuous functions of p, *we restrict ourselves in the series of the theory of perturbations to a finite number of harmonics, putting*

$$S = \sum_{0<|k|<N} S_k e^{i(k,q)}.$$

Then a term appears in $H_1^{(1)}$ of the form $\mu\widetilde{H}_1 - \mu[\widetilde{H}_1]_N = \mu \sum_{|k|\geqslant N} h_k e^{i(k,q)}$. In order that this term should be of order μ^2 it is sufficient to take N to be of order $\ln\frac{1}{\mu}$ (for the Fourier coefficients of the analytic function $\widetilde{H}_1$ decrease in geometric progression).

The function S obtained defines a canonical transformation $p, q \to p', q'$ which is small provided that p' belongs to a certain domain G_{KN}. The domain G_{KN} is obtained from the domain G, where H is defined, by eliminating a finite number ($\sim N^n$) of resonance zones: in the domain G_{KN}

$$|(\omega(p), k)| \geqslant K|k|^{-\nu} \qquad (0<|k|<N;\ \nu = n+1). \tag{1.7.1}$$

From (1.7.1) it is seen that the magnitude of the components of the domain G_{KN} is of order $\left(\ln\frac{1}{\mu}\right)^{-n}$, i.e. of order ~ 1. Consequently, the difficulty mentioned at the end of §6 no longer arises: if on averaging over the rapid variables we make use of Remark 2, then we obtain (1.6.2) in a domain of magnitude ~ 1 and consequently (1.6.3).

Remark 2 is also useful in the non-degenerate case: it enables us to do without the apparatus of Borel's monogenic functions [18] on estimating the measure of the invariant set. The paper [19] gives a formal account of this idea. Here we take note of the basic technical features of this application of Remark 2.

It is not difficult to see that if, for $|\mathrm{Im}\, q| \leqslant \rho$, we have $|f(q)| \leqslant M$, then the remainder of the Fourier series admits an estimate of the form $|f(q) - [f(q)]_N| \leqslant M^2$ for $|\mathrm{Im}\, q| \leqslant \rho - \gamma$ if $N = \frac{1}{\gamma}\ln\frac{1}{M}$ (for more detail see 3) in Ch. V, §3, 2.).

Let us now assume that the Hamiltonian $H_0(p) + H_1(p, q)$ for $|\mathrm{Im}\, q| \leqslant \rho$ from the domain G satisfies the inequality $|H_1| \leqslant M$. Our constructions depend on the parameters β, γ, δ and K which henceforth will be chosen sufficiently small and such that $\gamma \gg \delta \gg \beta \gg M$.

On taking account of our Remark 2 the arguments of §§2 and 4 make it possible to introduce new variables p', q' so that $H = H_0^{(1)}(p') + H_1^{(1)}(p', q')$, where $|H_1^{(1)}| \sim M^2$ in the domain defined by the following conditions: 1) p' with a β-neighbourhood belongs to G_{KN} (see (1.7.1); we shall write $p' \in G_{KN} - \beta$); 2) $|\mathrm{Im}\, q| < \rho - \delta - \gamma$. Here $N = \frac{1}{\gamma}\ln\frac{1}{M}$ and $|H_1^{(1)}| \sim M^2$ denotes

$$|H_1^{(1)}| < \frac{M^2}{\beta^2\delta^{2\nu}} \tag{1.7.2}$$

(for details see the fundamental lemma in [19]).

The quantities β, γ, $\delta > 0$ can be chosen arbitrarily provided they are bounded above by a constant depending on H_0, ρ, K, where K is also arbitrary. For a given $\varkappa > 0$ we take the quantities K, β, γ, δ, M so that for $|H_1| \leqslant M$ the complement to the invariant tori fills only a part of order $\varkappa$ of the whole phase space.

It is easy to see (see the arithmetic lemma in Chapter V) that the domain $G_{KN} - \beta$ of admissible p' differs from G by resonance strips the total measure of which has an upper bound of the type

$$K\sigma + \beta N^n \qquad \left(\sigma = \sum_{1 \leqslant m < N} m^{-2},\ N = \frac{1}{\gamma} \ln \frac{1}{M}\right). \tag{1.7.3}$$

Similar bounds are obtained for the subsequent approximations, but in place of σ the s-th approximation will have $\sigma_s = \sum_{N_{s-1} \leqslant m < N_s} m^{-2}$; consequently $\sum_s \sigma_s < 2$.

The quantities K, β_s, γ_s, δ_s, M_s are chosen so that:

1) $\sum_s (\gamma_s + \delta_s) < \frac{\rho}{2}$ (all the approximations are then possible),

2) $M_{s+1} = M_s^2 \beta_s^{-2} \delta_s^{-2\nu} < M_s^{3/2}$ (the approximations then converge, see (1.7.2)),

3) $\sum_s (K\sigma_s + \beta_s N_s^n) < \varkappa$ (then by (1.7.3) the measure of the complement to the invariant set $\sim \varkappa$).

We put

$$\gamma_s = \delta_s^{\alpha}, \quad \beta_s = \delta_s^3, \quad M_s = \delta_s^T, \quad N_s = \frac{1}{\gamma_s} \ln \frac{1}{M_s}, \quad \delta_{s+1} = \delta_s^{3/2} \ (0 < \alpha \ll 1,\ T \gg 1).$$

We choose T to be sufficiently large so that 2) is satisfied. We choose α sufficiently small so that $\beta_s N_s^n < \delta_s$ (if δ_1 is sufficiently small). Then for sufficiently small δ_1 1) is satisfied. Further, for sufficiently small K and δ_1 3) is satisfied. K is chosen in this way, and then δ_1 is chosen sufficiently small so that all the above requirements are satisfied.

For $M = \delta_1^T$ we now obtain converging approximations to the invariant tori and the measure of the complement to these tori is of order $\varkappa$ in view of 3).

§8. Application to the problem of proper degeneracy

By combining Remarks 1 and 2 all the difficulties of proper degeneracy can be overcome (see §5). We now pass to the following theorem (the conditions of which are satisfied, for example, in the three-body problem when two planets have small masses and the inclinations of their Kepler ellipses are small).

Let the Hamiltonian be analytic and of the form

$$H(p, q) = H_0(p_0) + \mu H_1(p, q) + \dots \qquad (1.8.1)$$

in the domain F of the phase space p, q (where $p = p_0, p_1$; p_0 is a vector of n_0 dimensions, p_1 is a vector of $n_1 = n - n_0$ dimensions; $q = q_0, q_1$ are, respectively, the rapid and slow angular variables (mod 2π)).

We suppose that

1) *the secular part of the perturbation*

$$\overline{H}_1 = (2\pi)^{-n_0} \oint H_1 \, dq_0$$

does not depend on the phases q_1 *of the slow motion*: $\overline{H}_1 = \overline{H}_1(p)$;

2) *the determinants of the frequencies with respect to the impulses of order* n_0 *and* n_1

$$\left| \frac{\partial \omega_0}{\partial p_0} \right| = \left| \frac{\partial^2 H_0}{\partial p_0^2} \right| \not\equiv 0, \qquad \left| \frac{\partial \xi_1}{\partial p_1} \right| = \left| \frac{\partial^2 \overline{H}_1}{\partial p_1^2} \right| \not\equiv 0$$

do not vanish identically.

Then for sufficiently small $|\mu|$ *and the majority*[1] *of initial conditions from F, the motion defined by the canonical equations with Hamiltonian* (1.8.1) *is conditionally periodic. With these initial conditions the motion for all* $-\infty < t < +\infty$ *is close to conditionally periodic motion* (1.8.2) *with* n_0 *rapid frequencies* ω_0 *and* n_1 *slow frequencies* $\omega_1 = \mu \xi_1$:

$$\dot{p}_0 = 0, \quad \dot{q}_0 = \omega_0(p_0) = \frac{\partial H_0}{\partial p_0}, \quad \dot{p}_1 = 0, \quad \dot{q}_1 = \mu \xi_1(p) = \mu \frac{\partial \overline{H}_1}{\partial p_1} \qquad (1.8.2)$$

(*with suitable initial conditions for the system* (1.8.2)).

The proof first of all uses Remark 2 (§7) in order to carry out an averaging over the rapid variables (the transformation $p, q \to p', q'$ of §7 with respect to the variables p_0, q_0 where p_1, q_1 are regarded as parameters). Then, for the second and subsequent approximations, use is made of Remark 1 (§6).

We do not dwell on the details of the proof, because a rather more general result (see Chapter IV) is required in the application to the n-body problem ($n > 3$). This general result refers to the case in which the non-singular degeneracy considered here is combined with a so-called limiting degeneracy.

§9. Limiting degeneracy. Birkhoff's transformation

As "limiting cases" among the n-dimensional invariant tori p = const into which the phase space of an integrable system is stratified, there are frequently individual tori of dimension $k < n$. Let us consider as an example an oscillating system with Hamiltonian $H = \frac{x^2 + \dot{x}^2}{2}$. The trajectories are concentric circles ($n = 1$) in the $x, \dot{x}$-plane; the position of equilibrium is given by $x = \dot{x} = 0$ ($k = 0$).

[1] In the sense of Lebesgue measure.

In such cases we speak of a "limiting degeneracy" (see [5]). Two cases of limiting degeneracy occur particularly frequently (theory of oscillations!): positions of equilibrium ($k = 0$) and periodic motions ($k = 1$). These cases were investigated in detail by Birkhoff [3]. We shall briefly state here the results of Birkhoff, which will be used in the subsequent argument. For the proofs see [3] and [8].

We suppose that the point $p = q = 0$ is a position of equilibrium of the system with Hamiltonian $H(p, q)$ which is expanded in the neighbourhood of the origin as a series

$$H = H_2(p, q) + H_3(p, q) + H_4 + \dots \tag{1.9.1}$$

of powers of $p = p_1, \dots, p_n$, $q = q_1, \dots, q_n$ (the H_m are terms of degree m).

The canonical equations with Hamiltonian H_2 are linear and therefore integrable. We suppose that the position of equilibrium of the linear system with Hamiltonian H_2 is stable (*the so-called elliptic case*). Then there exists a linear canonical transformation $p, q \to p', q'$, reducing H_2 to the form

$$H_2(p, q) = (\lambda, \tau), \quad \text{where} \quad \tau = \tau_1, \dots, \tau_n, \ 2\tau_i = p_i'^2 + q_i'^2;$$

here $\lambda = \lambda_1, \dots, \lambda_n$ is the set of fundamental frequencies of the linear oscillating system with Hamiltonian H_2. In this system the invariant n-dimensional tori are given by the equations $\tau = \text{const} > 0$; if the frequencies λ are arithmetically independent, the motion is conditionally periodic and the trajectories fill these tori everywhere densely. The frequencies λ are the same on all the tori. The action-angle variables τ, φ are polar canonical coordinates connected with p', q' by the formulae

$$p = \sqrt{2\tau}\cos\varphi, \qquad q = \sqrt{2\tau}\sin\varphi. \tag{1.9.2}$$

In a small neighbourhood of the origin the terms $H_3, H_4, \dots$ in (1.9.1) are small in comparison with H_2. Following the ideas of the theory of perturbations we shall try by means of a suitable canonical transformation $p'' = p' + \dots$, $q'' = q' + \dots$ to eliminate H_3 etc. Small denominators (λ, k) will then appear. Furthermore, calculation shows that H_3 can be eliminated, but there still remains part of H_4 and of other even-degree terms. It turns out that, for any integer $2s \geqslant 3$, (1.9.1) can be reduced to the so-called normal form (1.9.4).

We assume that

$$(\lambda, k) \neq 0 \quad \textit{for} \quad |k| = |k_1| + \dots + |k_n| \leqslant 2s - 1. \tag{1.9.3$_s$}$$

Then there exists a canonical transformation $p^{(s)} = p' + \dots$, $q^{(s)} = q' + \dots$, *given by convergent power series*[1] *in the neighbourhood of the origin such that* 1.9.1) *expressed in terms of* $p^{(s)}$, $q^{(s)}$ *has the form*

$$H(p, q) = \overline{H}^{(s)}(\tau^{(s)}) + \widetilde{H}^{(s)}(p^{(s)}, q^{(s)}), \tag{1.9.4}$$

where

$$2\tau_i^{(s)} = (p_i^{(s)})^2 + (q_i^{(s)})^2, \qquad \overline{H}^{(s)} = (\lambda, \tau^{(s)}) + \sum_{i,j} \lambda_{ij}\tau_i^{(s)}\tau_j^{(s)} + \dots$$

---

1 Possibly polynomials.

is a polynomial in τ_i^s *(where* $\lambda_{ij} = \lambda_{ji}$*) and* $\tilde{H}^{(s)} = O(|\tau^{(s)}|^s)$ *is a convergent series in the powers* $p^{(s)}$, $q^{(s)}$, ..., *beginning with terms of degree 2s. The coefficients* λ_i, λ_{ij}, ... *of the polynomial* $\overline{H}^{(s)}$ *do not depend on s nor on the method of reducing* (1.9.1) *to the form* (1.9.4): *they are invariants of the function* (1.9.1) *with respect to the canonical transformations of p, q.*

The system with Hamiltonian $\overline{H}^{(s)}$ is integrable. The action-angle variables are canonical polar coordinates $\tau^{(s)}$, $\varphi^{(s)}$ connected with $p^{(s)}$, $q^{(s)}$ by the formulae (1.9.2). The invariant tori are given by the equations $\tau^{(s)}$ = const. The corresponding frequencies $\lambda_i^{(s)} = \lambda_i + 2\sum_j \lambda_{ij}\tau_j^{(s)}$, generally speaking, change from torus to torus. If one of the determinants (of order n or $n + 1$)

$$|2\lambda_{ij}| = \left|\frac{\partial^2\overline{H}}{\partial\tau^2}\right|_0 \neq 0, \qquad \begin{vmatrix} 2\lambda_{ij} & \lambda_i \\ \lambda_j & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial^2\overline{H}}{\partial\tau^2} & \frac{\partial\overline{H}}{\partial\tau} \\ \frac{\partial\overline{H}}{\partial\tau} & 0 \end{vmatrix} \neq 0 \qquad (1.9.5)_{1,2}$$

is different from zero, we shall say that (1.9.1) is of *general elliptic type*.

The normal form (1.9.4) enables us to investigate the behaviour of trajectories that begin in a small ($\sim\varepsilon$) neighbourhood of the origin. For $t \sim \varepsilon^{-s}$ the trajectory remains close to the trajectory of the integrable system with Hamiltonian $\overline{H}^{(s)}$. The latter trajectory remains in the ε-neighbourhood of the origin for all $-\infty < t < +\infty$. Thus, if $(\lambda, k) \neq 0$ for $k \neq 0$, a trajectory beginning in the ε-neighbourhood of the origin remains close to the origin for at least a time $C_s\varepsilon^{-s}$, where s is arbitrarily large if ε is sufficiently small.

But, as we have already noted in §2, the sequence of series $p^{(s)} = p' + \ldots$, $q^{(s)} = q' + \ldots$ diverges as $s \to \infty$. Therefore the stability of the point 0, 0 does not follow from (1.9.4).

The investigation of the neighbourhood of the periodic motion of a conservative system with n degrees of freedom can [3] be reduced to the investigation of the position of equilibrium of a system with $n - 1$ degrees of freedom for which the Hamiltonian (1.9.1) depends periodically on the time: $H(p, q; t + 2\pi) = H(p, q; t)$. If in place of (1.9.3) it is required that $(\lambda, k) \neq k_0$ for $|k| \leqslant 2s - 1$, $k_0 = 0, \pm 1, \pm 2, \ldots$, there exists a Birkhoff transformation $p, q \to p^{(s)}, q^{(s)}$ that reduces $H(p, q; t)$ to the form (1.9.4). The functions $p^{(s)}(p, q)$, $q^{(s)}(p, q)$ and $\tilde{H}^{(s)}(p^{(s)}, q^{(s)})$ in this case still depend on t (with period 2π).

§10. Stability of positions of equilibrium of Hamiltonian systems

The stability of positions of equilibrium of a conservative system of general elliptic type was first established in the author's paper [14] in the case of two degrees of freedom and on the assumptions $(1.9.3)_\infty$ and $(1.9.5)_2$. A.M. Leontovich [23] has applied this result to the bounded three-body problem and proved the stability of Lagrange's periodic solutions.

Then J. Moser observed that, in place of the irrationality of λ_1/λ_2 (condition $(1.9.3)_\infty$) it is sufficient to take the condition $(1.9.3)_{5/2}$, i.e. $k_1\lambda_1 + k_2\lambda_2 \neq 0$ for $|k_1| + |k_2| \leqslant 4$. If the number of degrees of freedom n is greater than 2, stability remains an open question; it has been proved only for a majority of initial conditions.

Similar results have been obtained (by the author in [14] and then by Moser in [25]) in the non-autonomous case. Stability was proved in two-frequency problems, i.e. for periodic motion of an autonomous system with two degrees of freedom and for the position of equilibrium of a system with one degree of freedom and with a periodic variation of the Hamiltonian.

We mention here the proof of stability of the position of equilibrium of a Hamiltonian system of general elliptic type with two degrees of freedom on the assumption that $k_1\lambda_1 + k_2\lambda_2 \neq 0$ for $|k_1| + |k_2| \leqslant 4$ (see $(1.9.3)_{5/2}$).

In accordance with §9 we are justified in taking the Hamiltonian to be of the form (see (1.9.4))

$$H = H_0(\tau) + H_1(\tau, \varphi), \text{ where } H_0(\tau) = (\lambda, \tau) + \sum \lambda_{ij}\tau_i\tau_j, \qquad |H_1| \leqslant C|\tau|^{5/2}, \tag{1.10.1}$$

and $H_1(\tau, \varphi)$ is analytic with respect to τ, φ in the domain $|\tau_i - \varepsilon| < \varepsilon$, $|\mathrm{Im}\,\varphi| < 1$. We first of all assume that condition $(1.9.5)_1$ is fulfilled. Then on varying τ in the domain $|\tau - \varepsilon| < \varepsilon$ the frequencies $\lambda(\tau) = \frac{\partial H_0}{\partial \tau}$ run through the domain of magnitude of order ε about the point $\lambda = \lambda(0)$.

But it is not difficult to observe that *the majority of points* λ *in the domain of magnitude* ε *admit a lower estimate of the small denominators* (λ, k) *of the form*

$$|(\lambda, k)| \geqslant K\varepsilon|k|^{-\nu} \qquad (|k| > 0,\ \nu = n+1) \tag{1.10.2}$$

for a suitable $K > 0$.

For it is easy to see that the measure of the resonance strip with number k constitutes a part of order $K|k|^{-\nu}$ of the measure of the domain being considered. The subsequent argument follows along the same lines as in §3.

It follows from this remark that for sufficiently small $K > 0$ (not depending on ε) the small denominators $(\lambda(\tau), k)$ have the lower bound (1.10.2) for the majority of τ in the domain G: $|\tau_i - \varepsilon| < \varepsilon$; our conclusion depends in no way on the arithmetic nature of $\lambda = \lambda(0)$.

We now apply the construction of §7 to the function (1.10.1) in the domain G. We put $\beta = \delta^3\varepsilon$. Let $|H_1| \leqslant M$. For sufficiently small δ in the domain $G_{\varepsilon K, N} - \beta$ we obtain from (1.7.2)

$$|H_1^{(1)}(\tau', \varphi')| < \frac{M^2}{\varepsilon^2\delta^{2\nu+6}}.$$

If $M < C\varepsilon^{5/2}$, $M < \delta^T$, then for sufficiently large T and sufficiently small δ it follows from the last formula that

$$|H_1^{(1)}(\tau', \varphi')| < M^{11/10} \quad (\text{in general } < M^{6/5-\Delta},\ \Delta > 0).$$

Therefore we put

$$\gamma_s = \delta_s^{\alpha}, \quad \beta_s = \delta_s^3 \varepsilon, \quad M_s = \delta_s^T, \quad N_s = \frac{1}{\gamma_s} \ln \frac{1}{M_s}, \quad \delta_{s+1} = \delta_s^{11/10}.$$

For sufficiently small α, sufficiently small K, sufficiently large T, sufficiently small δ_1 and $\varepsilon < \delta_1^R = \delta_1^{2/5T}$, the approximations constructed in §7 converge. In an ε-neighbourhood of the origin we find invariant tori with conditionally periodic motions.

Up to this point our arguments have been valid for any number of degrees of freedom n. If, however, $n = 2$, then the two-dimensional tori that are found divide the three-dimensional manifold H = const which contains them. If on each such manifold in any neighbourhood of the origin there is an invariant torus, then the origin is obviously a stable position of equilibrium.

Tori that are given by our construction exist at each level of H = const if the ratio $\frac{\lambda_1(\tau)}{\lambda_2(\tau)}$ varies along the line $H_0(\tau) = 0$. In the case of n degrees of freedom this is expressed by the condition

$$\frac{D\left(\frac{\lambda_1}{\lambda_n}, \ldots, \frac{\lambda_{n-1}}{\lambda_n}\right)}{D(\tau_1, \ldots, \tau_{n-1})} \neq 0 \tag{1.10.3}$$

(where τ_n has been expressed in terms of $\tau_1, \ldots, \tau_{n-1}$ from the equation $H_0(\tau) = 0$).

In its symmetrical form (1.10.3) takes the form $(1.9.5)_2$. Thus stability is obtained for $n = 2$ on the assumptions $(1.9.5)_2$ and $(1.9.3)_{5/2}$. Supposition $(1.9.5)_2$ can be still further weakened (cf. [14])[1].

For $n \geqslant 2$ the invariant tori obtained fill a large part of the ε-neighbourhood of the position of equilibrium. This is proved with the help of estimates of the type (1.7.3) which now express the ratio of the measure of the resonance zones $G \setminus (G_{\varepsilon K, N} - \beta)$ to the measure of the whole domain G. A detailed proof is given in Chapter IV (where we have to put $n_0 = 0$). For the sake of technical convenience this proof is carried out under the assumptions $(1.9.3)_{7/2}$ (i.e. $(\lambda, k) \neq 0$ for $|k| \leqslant 6$) and $(1.9.5)_1$.

The question of stability in the general many-dimensional elliptic case remains open. The simplest unsolved problem is to determine whether a fixed point of a canonical mapping of a four-dimensional space onto itself is stable.

Chapter II

ADIABATIC INVARIANTS

In this chapter we consider the concept of an adiabatic invariant (which has been little studied by mathematicians in spite of its importance

[1] Examples are known which demonstrate that (1.9.3) cannot be disregarded completely (see [4]).

and interest). We investigate the variation of an adiabatic invariant in the course of an infinite interval of time for a small periodic variation of the parameters of an oscillating system with one degree of freedom. It turns out that *if the system is non-linear, this variation is small for all* $-\infty < t < +\infty$ (see [16]).

§1 contains the definition of an adiabatic invariant and of a perpetual adiabatic invariant. In §2 the proof of perpetual adiabatic invariance of action is reduced to the general theorem of the type in Ch. I, §8. §3 contains an outline of a similar procedure for a conservative system with two degrees of freedom. The results obtained are applied in §4 to the investigation of the motion of charged particles in axially-symmetric magnetic fields. We show that *adiabatic magnetic traps are capable of retaining a particle perpetually*. The final section contains some remarks on the many-dimensional case.

§1. The concept of an adiabatic invariant

We give in this section one of the possible mathematical definitions of the concept of an adiabatic invariant. We discuss the question of the accumulation of variations of an adiabatic invariant in linear and non-linear oscillating systems with one degree of freedom.

1. Adiabatic variations. Let us consider a dynamical system in which the parameters may change, for example, a pendulum of variable length. For very slow (in comparison with the motion of the system) variations of the parameters distinctive asymptotic phenomena appear. In the pendulum example the amplitude of the oscillations turns out (in the limit) to be a function of the length: if the length is changed sufficiently slowly according to some arbitrary law, then every time the length returns to its original value the amplitude of the oscillations will be the same as it was initially.

Such slow variations of the parameters are called adiabatic. The concept of adiabatic variation was introduced by physicists in a somewhat vague form. It was supposed that the person changing the parameters of the system did not see it (otherwise he would be able to swing a system by changing the parameters in time with its own motion). The mathematical formulation of this last requirement is a very delicate matter.

Following [26] we shall avoid this difficulty and consider slow variations of the parameter λ of the form $\lambda = f(\mu t)$, where $f(x)$ is a fixed and smooth function and μ is a very small number. Without loss of generality we can regard the so-called "slow time" $\lambda = \mu t$ as a slowly varying parameter and fix the Hamiltonian $H(p, q; \lambda)$. We shall consider very small μ and concern ourselves with the motion of the system in the course of a large finite interval of time $t \sim \frac{1}{\mu}$.

2. The adiabatic invariant. Let us consider a dynamical system with Hamiltonian

$$H(p, q; \lambda) \qquad (p = p_1, \ldots, p_n;\ q = q_1, \ldots, q_n;\ \lambda = \mu t).$$

DEFINITION. The function $J(p, q; \lambda)$ is called an *adiabatic invariant of the system* if for small μ

$$J(t) = J[p(t),\ q(t);\ \lambda(t)]$$

varies little during time $t \sim \frac{1}{\mu}$ (changes in λ and H are here of order 1).

More precisely, J is *adiabatically invariant* if, for any $\kappa > 0$, it is possible to find a $\mu_0 > 0$ such that when $0 < \mu < \mu_0$, then for all t in the interval $0 < t < \frac{1}{\mu}$

$$|J(t) - J(0)| < \varkappa.$$

It is obvious that every exactly self-preserving (invariant) quantity is an adiabatic invariant. Less trivial examples are given below.

I. Let us consider an oscillating system with one degree of freedom and a smooth Hamiltonian $H(p, q; \lambda)$. If we fix the value of the parameter λ, then on the phase plane p, q a family of energy level curves $H(p, q; \lambda) = \text{const}$ can be drawn. Curves passing through the point p_0, q_0 bound a certain domain, the area of which we denote by $2\pi I(p_0, q_0; \lambda)$. It can be shown that I is an adiabatic invariant [5], [6]. The quantity I is called the *variable of action* or the *action*.

In the case of the mathematical pendulum

$$H = \frac{p^2 + \omega^2 q^2}{2},$$

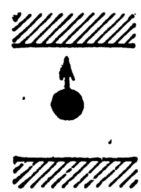

Fig. 8.

$I = \frac{H}{\omega}$, where $\omega = \omega(\lambda)$ is the frequency of the oscillations.

II. As a second example we can consider the motion of a perfectly elastic ball between two slowly moving parallel planes (Fig. 8). The product of the distance between the planes and the velocity of the ball is an adiabatic invariant. This is easily established even by elementary means (cf. [26]). Example II can also be considered as a limiting case of Example I.

III. A further example is obtained by considering the motion of a charged particle in a magnetic field. If the strength of the field B is constant in time and in space, the particle moves along a spiral around a line of force. We resolve the velocity vector of the particle v into its components parallel to and perpendicular to the field:

$$v = v_{\parallel} + v_{\perp}.$$

It can be shown that, for a slow variation of the field $B(\lambda)$, the magnitude of the "magnetic moment"

$$I = \frac{W_{\perp}}{B},$$

where $W_{\perp} = \frac{v_{\perp}^2}{2}$ is the energy of the transverse motion, is an adiabatic invariant (see [28]).

3. Perpetual adiabatic invariant. An adiabatic invariant I varies little during time $t \sim \frac{1}{\mu}$. But during an infinite time it can change very considerably. This is connected with the possibility of accumulating small variations of I. Let us consider, for example, the linear oscillating system (swing)

$$\ddot{x} = -\omega^2(1+\alpha\cos\mu t)\,x \qquad (\alpha \ll 1).$$

It is known [6] that for certain μ (namely, $\mu \approx \frac{2\omega}{k}$, $k = 1, 2, \ldots$) a parametric resonance is possible: $I(t) \to \infty$ as $t \to \infty$. And such a swing can take place for an arbitrarily small rate of change of the parameters μ.

It turns out, however, that *for a slow periodic variation of the Hamiltonian $H(p, q; \lambda)$ of a non-linear oscillating system with one degree of freedom an adiabatic invariant is perpetually conserved: for any $\varkappa > 0$ it is possible to find $\mu_0(\varkappa) > 0$ such that from $|\mu| < \mu_0$ it follows that*

$$|I(t) - I(0)| < \varkappa$$

for all $-\infty < t < +\infty$. This assertion will be proved in §2.

A linear system is an exceptional case, since the frequency of its oscillations does not depend on amplitude. In a non-linear system, on the other hand, if the amplitude is increased, the frequency changes and the oscillations cannot grow, because the resonance condition $\mu \approx \frac{2\omega}{k}$ is violated.

§2. Perpetual adiabatic invariance of action with a slow periodic variation of the Hamiltonian

In this section we consider a non-linear oscillating system with one degree of freedom and an analytic Hamiltonian $H(p, q; \lambda)$ depending periodically on the slow time $\lambda = \mu t$:

$$H(p, q; \lambda + 2\pi) = H(p, q; \lambda).$$

We shall prove the perpetual adiabatic invariance of the action I.

1. Conservative approximation. We first of all consider the roughest zero approximation: λ = const and the system is autonomous. As a conservative system with one degree of freedom it is integrable. It is convenient to describe the periodic motion of the system by means of the action-angle variables I, ω. These variables are introduced by means of the canonical transformation

$$p = \frac{\partial S}{\partial q}, \qquad w = \frac{\partial S}{\partial I} \tag{2.2.1}$$

with the generating function

$$S(I, q) = \int_{H=h}^{q} p\,dq, \tag{2.2.2}$$

where $h = H_0(I)$ is the function inverse to

$$I(h) = \frac{1}{2\pi}\oint_{H=h} p\,dq, \tag{2.2.3}$$

and p in (2.2.2), (2.2.3) denotes the quantity $p(h, q)$ obtained from the equation $H(p, q) = h$.

The formulae (2.2.1) define a canonical transformation $p, q \to I, w$ for every fixed value of λ. The variation of I, w with time is determined by the canonical equations with Hamiltonian $H_0(I)$. Therefore the quantity I is conserved and the angular coordinate w on the circle I = const varies uniformally with frequency

$$\dot{w} = \omega(I) = \frac{\partial H_0}{\partial I}. \tag{2.2.4}$$

All the functions $H(p, q; \lambda)$, $S(I, q; \lambda)$, $p(h, q; \lambda)$, $H_0(I; \lambda)$, $w(I; \lambda)$ depend on the parameter λ which we have omitted in the formulae (2.2.1)-(2.2.4). In the conservative approximation the value of the parameter λ is fixed. This approximation is applicable during time $t \sim 1$.

2. Adiabatic approximation. The action-angle coordinates introduced in 1. are convenient in the case in which the parameter λ varies with time. The transformation $p, q \to I, w$ is canonical but it depends on λ and consequently on the time t. Therefore the variation of I, w with time is determined by the canonical equations with Hamiltonian

$$H(I, w; \lambda) = H_0(I) + \mu H_1(I, w; \lambda), \tag{2.2.5}$$

where $H_1 = \frac{\partial S}{\partial \lambda}$ is a single-valued function of period 2π with respect to w and λ.

The classical theory of perturbations (see Chapter I) gives the following picture of motion in the phase space $p, q; \lambda$. We shall identify points at which the coordinates λ differ from each other by a multiple of 2π. Then the equation I = const determines a two-dimensional torus. We shall call the angular coordinates w and λ on this torus the latitude and longitude, respectively (Fig. 9).

In the conservative approximation ($\mu = 0$) each point of the torus moves along its meridian λ = const with angular velocity $\omega(I; \lambda)$ depending on the longitude. For $\mu \neq 0$ a slow motion ($\dot{\lambda} = \mu$) across the meridian is added and the motion becomes two-frequency. But - in the approximation of perturbation theory - the phase point remains on the invariant torus I = const. This approximation is called adiabatic. It is not difficult to see that the true motion is close to the adiabatic approximation during the time $t \sim \frac{1}{\mu}$. This proves the adiabatic invariance of I.

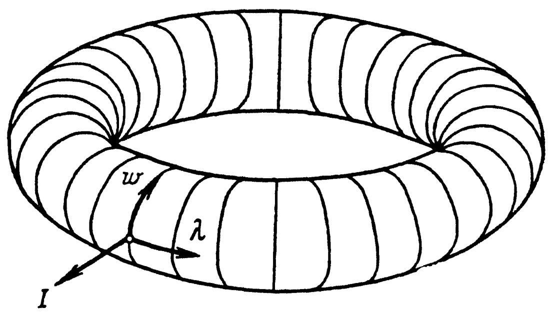

Fig. 9.

We now pass to the proof of *perpetual* adiabatic invariance of I in a non-linear system.

3. Invariant tori. In a linear system $\omega(I)$ = const and $H_0(I) = I\omega$.

We shall assume that the oscillations are non-linear. Let us denote by $\overline{\omega}(I)$ the mean frequency

$$\overline{\omega}(I)=\frac{1}{2\pi}\oint\omega(I;\ \lambda)\,d\lambda=\frac{d\overline{H}_0}{dI},\ \text{where}\ \overline{H}_0(I)=\frac{1}{2\pi}\oint H_0(I;\ \lambda)\,d\lambda.$$

We assume that

$$\frac{d^2\overline{H}_0}{dI^2}=\frac{d\overline{\omega}}{dI}\neq 0. \qquad (2.2.6)$$

With condition (2.2.6) we shall prove the *perpetual* adiabatic invariance of I. With this aim in view we shall find many *truly* (not approximately) invariant tori for the system with Hamiltonian (2.2.5). These tori are close to the tori I = const if μ is small. They are two-dimensional and divide the three-dimensional phase space $p, q;\ \lambda$ into thin layers. If the initial point $p_0, q_0;\ \lambda_0$ lies between two of these tori T_1 and T_2, then the trajectory $p(t),\ q(t);\ \lambda(t)$ lies entirely between T_1 and T_2. We prove the following proposition.

I. *For any $\varkappa > 0$ it is possible to find $\mu_0 > 0$ such that if $|\mu| < \mu_0$, then any point $p_0,\ q_0;\ \lambda_0$ lies between two invariant tori $T_1,\ T_2$, where*

$$|I(p_1,\ q_1;\ \lambda_1)-I(p_2,\ q_2;\ \lambda_2)|<\varkappa,$$

provided $p_1, q_1;\ \lambda_1$ belongs to T_1 and $p_2, q_2;\ \lambda_2$ to T_2.

From Proposition I the perpetual adiabatic invariance of action follows immediately.

THEOREM 1. *Suppose that an oscillating system has the analytic Hamiltonian (2.2.5) in action-angle variables and that $\frac{\partial H_0}{\partial I} \neq 0,\ \frac{\partial^2 H_0}{\partial I^2} \neq 0$ everywhere in the domain $|I - I_0| \leqslant r$. Then for any $\varkappa > 0$ it is possible to find $\mu_0 > 0$ such that if $|\mu| < \mu_0$ and $|I(0) - I_0| \leqslant r - \varkappa$, then for all $-\infty < t < +\infty$ we have $|I(t) - I(0)| < \varkappa$.*

We now proceed to the proof of Proposition I under the assumptions of Theorem 1.

4. Preliminary canonical transformations. We shall attempt to reduce the Hamiltonian (2.2.5) to the form (1.8.1). For this purpose it is convenient to take the phase of the rapid motion w as an independent variable.

LEMMA. *Suppose that the frequency $\omega(I;\ \lambda) = \frac{\partial H_0}{\partial I}$ does not vanish in the domain in question. Then there exist analytic functions $P,\ Q,\ T$ of the variables $I,\ w,\ \lambda$, not depending on μ, such that*

1) *$P,\ Q - \lambda$ and $T - w$ have period 2π with respect to w and λ;*

2) *the canonical equations with Hamiltonian (2.2.5) are equivalent to the canonical equations*

$$\frac{dP}{dT}=-\frac{\partial k}{\partial Q},\qquad \frac{dQ}{dT}=\frac{\partial k}{\partial P}$$

with Hamiltonian

$$k(P,\ Q;\ T)=\mu k_0(P)+\mu^2 k_1(P,\ Q;\ T)+\dots, \qquad (2.2.7)$$

of period 2π with respect to $Q,\ T$ and analytic in the complex neighbourhood of the torus layer $|P - I_0| < r$;

3) *in* (2.2.7) *the function* $k_0(P)$ *is inverse to* $\overline{H}_0(I)$ *so that* $\overline{H}_0(k_0(P)) \equiv P$.

We first of all introduce a new time $T = w$. As is known [4], the integral curves of the Hamiltonian system in the space I, w; λ are invariantly connected with the differential form

$$I\,dw - H(I,\ w;\ \mu t)\,dt = -\frac{1}{\mu}(H\,d\lambda - \varepsilon I\,dw). \tag{2.2.8}$$

Multiplication of this form by a constant does not change the relationship. In (2.2.8) we shall regard H, λ; w as independent variables and not I, w; t. Solving (2.2.5) for I, we obtain

$$I(H,\ \lambda;\ w) = I_0(H,\ \lambda) + \mu I_1(H,\ \lambda;\ w) + \dots$$

We introduce the notation:

$$p' = H, \qquad q' = \lambda, \qquad T = w, \qquad K = \mu I.$$

Then

$$K(p',\ q';\ T) = \mu I_0(p',\ q') + \mu^2 I_1(p',\ q';\ T) + \dots \tag{2.2.9}$$

and

$$H\,d\lambda - \mu I\,dw = p'\,dq' - K(p',\ q';\ T)\,dT,$$

so that the systems with Hamiltonians (2.2.5) and (2.2.9) are equivalent (see [4]).

We note that the frequency $\omega(I;\ \lambda)$ in 2. varies with time. By means of the canonical transformation p', $q' \to P$, Q we change the coordinate $q' = \lambda$ (slow time) so that the frequency with respect to the changed time Q becomes constant $\overline{\omega}(I)$. With this in mind we introduce into the system with Hamiltonian $I_0(p',\ q')$ the action-angle variables P, Q by means of the canonical transformation

$$p' = \frac{\partial S}{\partial q'}, \qquad Q = \frac{\partial S}{\partial P}$$

with the generating function

$$S(P,\ q') = \int^{q'} H_0(I,\ \lambda)\,d\lambda,$$

where $I = k_0(P)$ is a function inverse to $H_0(I)$. Obviously the quantities P, Q; T found satisfy all the requirements of the lemma in 4.

In accordance with this lemma Proposition 1 in 3. follows from the analogous assertion for the system with Hamiltonian (2.2.7).

5. Investigation of the system with Hamiltonian (2.2.7). The function (2.2.7) is not formally within the scope of Ch. I, §8, since it contains the "time" T explicitly. But the conclusions of Ch. I, §8 are valid and can easily be obtained by the method which is outlined in Chapter I and will be worked out in detail in Chapter IV, in a rather more complicated problem. We shall not dwell here on the details of the proof. We shall show that by another method the system (2.2.7) can be reduced to a conservative system with two degrees of freedom. For this purpose we note that the canonical equations

$$\frac{dP}{d\tau}=-\frac{\partial k}{\partial Q},\quad \frac{dR}{d\tau}=-\frac{\partial k}{\partial T},\quad \frac{dQ}{d\tau}=\frac{\partial k}{\partial P},\quad \frac{dT}{d\tau}=1$$

with Hamiltonian

$$R+k(P,\ Q,\ T)=R+\mu k_0(P)+\mu^2\ \dots \tag{2.2.10}$$

and angular coordinates Q, T contain the canonical equations with Hamiltonian (2.2.7). The function (2.2.10) has the form (1.8.1).

An inequality analogous to (1.9.5) and sufficient for the validity of the results of Ch. I, §8 follows from (2.2.6).

Either of these methods can be used to prove the following proposition.

II. *Let* (2.2.7) *be analytic for* $|\mu| < \bar{\mu}$ *in the domain* $|\mathrm{Im}\ Q,\ T| \leqslant \rho$, $|P - P_0| \leqslant r$, *where* $|k_0| \leqslant M$, $|k_1 + (\mu)\ \dots| \leqslant M$, $\left|\frac{d^2k_0}{dP_0^2}\right| \geqslant \theta > 0$. *Then for every* $\varkappa > 0$ *it is possible to find* $\mu_0(\varkappa,\ \bar{\mu},\ r,\ \rho,\ M,\ \theta) > 0$ *such that if* $|\mu| < \mu_0$, *then the real torus layer* F, *where* $|P - P_0| \leqslant r$, *is filled with invariant tori with an accuracy up to a residual of measure less than* $\varkappa$ mes F *and the distance of each of these invariant tori from a certain torus* $P =$ const *is less than* $\varkappa$.

Proposition I, and with it also Theorem 1 of 3., easily follows from Proposition II., by virtue of 4.

§3. Adiabatic invariants of conservative systems

We prove in this section the perpetual adiabatic invariance of variables of action in conservative systems with two degrees of freedom.

1. Adiabatic approximation. Let us consider a conservative dynamical system with two degrees of freedom X, Y. We shall assume that a change in one of these coordinates, for example X, has little influence on the state of the system. It can then be supposed approximately that there exists a system with one degree of freedom Y depending on the slowly varying parameter μX. The variable of action I_Y (see §2, 1.) corresponds to this system. The magnitude of I_Y is shown to be adiabatically invariant in the following sense.

Let us fix a function of four variables $H(\dots)$. Let X, Y, P_X, P_Y be canonically conjugate variables. We consider a dynamical system defined by the Hamiltonian $H(\mu X,\ Y;\ P_X,\ P_Y)$. If μ is small, then the state of the system varies little if X varies by a quantity of order 1. We fix the values of X and P_X. Then H is transformed into the function $H_Y(Y,\ P_Y)$. We denote by $I_Y(Y,\ P_Y)$ the variable of action in the system with Hamiltonian $H_Y(Y,\ P_Y)$. The magnitude of I_Y depends, however, on the parameters μX, P_X. For the original system with Hamiltonian $H(\mu X,\ Y;\ P_X,\ P_Y)$ the action I_Y will be an adiabatic invariant in the sense that the variation of $I_Y[\mu X(t),\ Y(t);\ P_X(t),\ P_Y(t)]$ during time $t \sim \frac{1}{\mu}$ is small together with μ.

$H(\mu X,\ Y;\ P_X,\ P_Y)$ can be written in the form $H = H_X(\mu X,\ P_X;\ I_Y)$. In order to determine approximately the variation of X, P_X with time it is sufficient to form the canonical equations with Hamiltonian $H_X(\mu X,\ P_X)$

(depending on the constant parameter I_Y). If in this system with one degree of freedom the motion is periodic, then a variable of action $I_X \sim \frac{1}{\mu}$ can be introduced. In the adiabatic approximation the motion is composed of rapid oscillations of Y with frequency ω_Y and slow oscillations of X with frequency ω_X of order μ, I_Y and μI_X being conserved.

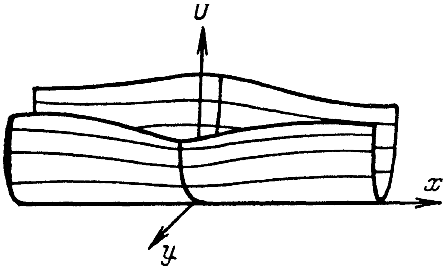

Fig. 10.

This approximation will be proved below. *We shall prove the perpetual adiabatic invariance of I_Y and μI_X on on the assumption that the mean value $\frac{\overline{\omega}_Y}{\omega_X}$ of the ratio $\frac{\omega_Y}{\omega_X}$ depends on I_Y for a fixed total energy h.*

2. Example. Let us consider motion in a "potential ditch" pulled out along the x axis (Fig. 10):

$$H = \frac{P_X^2 + P_Y^2 + U(x, Y)}{2}, \text{ where } U = \omega^2 Y^2, \quad \omega = 1 + x^2, \quad x = \mu X, \quad \mu \ll 1.$$

The quantities introduced in 1. take the form

$$H_Y = \frac{P_Y^2 + U}{2} + \frac{P_X^2}{2}, \qquad I_Y = \frac{P_Y^2 + U}{2\omega_Y}, \quad \omega_Y = \omega = 1 + x^2, \quad \overline{\omega}_Y = \frac{2h}{3I_Y} + \frac{1}{3},$$

$$H_X = \frac{P_X^2 + 2I_Y x^2}{2} + I_Y, \quad \mu I_X = \frac{P_X^2 + 2I_Y x^2}{2\sqrt{2I_Y}}, \quad \omega_X = \mu\sqrt{2I_Y}.$$

Our assumption concerning the dependence of $\frac{\overline{\omega}_Y}{\omega_X}$ on I_Y is fulfilled and therefore I_Y, μI_X are perpetual adiabatic invariants.

At the very bottom of the ditch it is possible to roll away to infinity ($Y = P_Y = I_Y = 0$, $P_X = v$, $X = X_0 + vt$). But if $I_Y \neq 0$, then *the motion takes place in a bounded domain at least for sufficiently small* μ (Fig. 11).

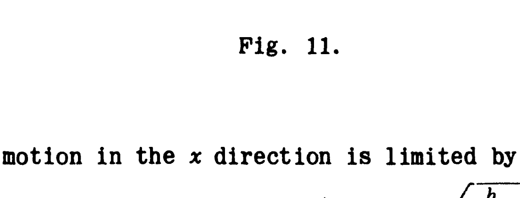

Fig. 11.

For let us fix the values of h and $I_Y \neq 0$ corresponding to the initial conditions we require and then allow μ to tend to zero. For sufficiently small μ we have

$$|I_Y(t) - I_Y(0)| < O(\mu)$$

for all

$$-\infty < t < +\infty.$$

But since

$$h - I_Y(1 + x^2) = \frac{P_X^2}{2} \geqslant 0,$$

motion in the x direction is limited by

$$|x_{\max}| = \sqrt{\frac{h}{I_Y} - 1} + O(\mu).$$

3. Preliminary canonical transformation. We shall reduce the Hamiltonian of the system to the form (2.2.9). The perpetual adiabatic invariance of I_Y and μI_X is easily deduced from the existence of the two-dimensional invariant tori of the latter system since these tori divide the three-dimensional level of energy $H = h$.

LEMMA. *Suppose that the Hamiltonian $H(x, Y; P_X, P_Y)$ $(x = \mu X)$ is analytic and for fixed x, P_X defines an oscillating system with action-angle variables $I_Y(x, P_X; h)$, $w_Y(x, Y, P_X, P_Y)$. Then there exists an analytic substitution expressing x, Y, P_X, P_Y in terms of new variables x', w', P', I' such that:*

1) *The functions x, Y, P_X, P_Y are of period 2π with respect to w. As $\mu \to 0$ the variables x', w', P', I' turn into x, w_Y, P_X, I_Y.*

2) *Along the integral curves of the canonical equations*

$$\frac{dP_X}{dt} = -\frac{\partial H}{\partial X}, \quad \frac{dP_Y}{dt} = -\frac{\partial H}{\partial Y}, \quad \frac{dX}{dt} = \frac{\partial H}{\partial P_X}, \quad \frac{dY}{dt} = \frac{\partial H}{\partial P_Y}$$

the canonical equations

$$\frac{dP'}{dw'} = -\frac{\partial K}{\partial x'}, \quad \frac{dx'}{dw'} = \frac{\partial K}{\partial P'} \tag{2.3.1}$$

with Hamiltonian $K(P', x'; w'; h)$ depending on the parameter h are satisfied.

3) *K is of the form $K = -\mu I'$, where*

$$I' = I_0(P', x'; h) + \mu I_1(P', x'; w'; h) + \dots \tag{2)}$$

is an analytic function of period 2π with respect to w' and $I_0(P', x'; h) = I_Y(x', P'; h)$.

PROOF. From the relationship $H(x, Y; P_X, P_Y) = h$ we can express P_Y in the form $P_Y(x, P_X, Y; h)$ and we put $2\pi I_Y(x, P_X; h) = \oint P_Y\, dY$. This relationship defines the function $h(x, P_X; I_Y)$. The generating function $P'X + S(x, P', Y, I')$, where

$$S = \int^{Y} P_Y[x, P', Y; h(x, P'; I')]\, dY.$$

determines the canonical transformation $X, Y, P_X, P_Y \to X', w', P', I'$ with the help of the formulae

$$X' = X + \frac{\partial S}{\partial P'}, \qquad w' = \frac{\partial S}{\partial I'},$$
$$P_X = P' + \mu\frac{\partial S}{\partial x}, \quad P_Y = \frac{\partial S}{\partial Y}.$$

We also put $x' = \mu X'$. Since

$$H\{x, Y; P', P_Y[x, P', Y; h(x, P'; I')]\} = h(x, P'; I'),$$

we have

$$H\{x, Y; P_X, P_Y[x, P', Y; h(x, P'; I')]\} = h(x, P'; I') + \mu\frac{\partial S}{\partial x}\frac{\partial H}{\partial P_X} + \dots$$

Therefore, expressing X, Y in terms of the new variables, we obtain

$$H(x, Y; P_X, P_Y) = h(x', P'; I') + \mu\frac{\partial S}{\partial x}\frac{\partial H}{\partial P_X} - \mu\frac{\partial S}{\partial P'}\frac{\partial H}{\partial x} + \dots \tag{2.3.3}$$

(2.3.3) can be written in the form ($h_0 \equiv h$)

$$H(x, Y; P_X, P_Y) = H'(x', P', I', w') = h_0(x', P'; I') + \mu h_1(x', P', I', w') + \ldots, \tag{2.3.4}$$

where $\mu h_1 + \ldots$ is an analytic function of period 2π with respect to w'.

We shall measure the time by the phase w'. For this purpose in place of P', X'; I', w'; t as independent variables in the expression $P'dX' + I'dw' - Hdt$ we take, respectively, P', X'; $-H$, t; w' (see [4]). We shall consider h as a parameter and w' as the time. The role of the Hamiltonian is played by $-I'(X', P'; w'; h)$, where I' is determined from the equation

$$H'(\mu X', P', I', w') = h. \tag{2.3.5}$$

The coordinate t is cyclic; discarding the variables $-H$, t we obtain a non-autonomous system with one degree of freedom. We multiply the coordinate X' and the Hamiltonian $-I'$ by the constant μ: $\mu X' = x'$, $-\mu I' = K$. The derivatives of x' and P' with respect to w' are determined by the canonical equations (2.3.1) with Hamiltonian K.

In view of (2.3.5) and (2.3.4) the function I' is of the form (2.3.2) and this proves the lemma.

4. **Proof of the perpetual adiabatic invariance of action.** In accordance with (2.3.2) the function K is of the form (2.2.9). (2.2.6) follows from the condition formulated at the end of 1. Therefore the reasoning of §2 is applicable. It gives invariant tori and the proof of the perpetual adiabatic invariance of I_Y and μI_X.

§4. Magnetic traps

In this section we consider the motion of a charged particle in a magnetic field. It is assumed that the instantaneous radius of the spiral along which the particle moves is small in comparison with the distances at which the field changes significantly. This condition is fulfilled if the field is large or if it is almost constant, or if the velocity of the particle is small. We shall consider the last case (which does not result in any loss of generality).

We show that in axially-symmetric magnetic traps the adiabatic invariant $\frac{W_\perp}{B}$ is perpetually conserved. It therefore follows that such traps are capable of retaining charged particles perpetually.

1. **Equations of motion.** We assume that the magnetic field **B** is determined by a vector potential **A** which, in polar coordinates r, φ, z has only one component $A_\varphi = A(r, z)$. Then the components of the field strength **B** are

$$B_r = -\frac{\partial A}{\partial z}, \quad B_z = \frac{1}{r}\frac{\partial rA}{\partial r}.$$

Therefore the lines of force are determined by the equations

$$rA = \text{const}, \quad \varphi = \text{const}.$$

The Lagrangian of a unit charge of mass 1 with a suitable choice of units is

$$L=\frac{1}{2}(\dot{r}^2+\dot{z}^2+r^2\dot{\varphi}^2)+rA\dot{\varphi}.$$

From this we find the "impulses":

$$p_r=\dot{r}, \qquad p_z=\dot{z}, \qquad p_\varphi=r^2\dot{\varphi}+rA,$$

and the Hamiltonian:

$$H=\frac{1}{2}\left[p_r^2+p_z^2+\frac{(p_\varphi-rA)^2}{r^2}\right].$$

Since φ is cyclic, $p_\varphi = M$ is conserved and it remains to investigate plane motion in the field with potential

$$U(r,\ z)=\frac{(M-rA)^2}{2r^2}, \tag{2.4.1}$$

where M is a fixed constant.

The function (2.4.1) defines a "potential ditch" with the zero bottom along the line of force $rA = M$. In the neighbourhood of this line we have

$$U(r,\ z)=\frac{1}{2}B^2y^2+\ldots, \tag{2.4.2}$$

where y is the distance from the line of force and B is the magnitude of the magnetic field strength on the line of force.

2. Change of variables. In order to apply the results of §3, we introduce curvilinear coordinates x, y into the r, z-plane. We denote by x the arc length along the line of force $rA = M$ from the fixed point 0 to the base of the perpendicular from the point r, z onto this line of force. As in (2.4.2), we shall denote by y the length of this perpendicular (Fig. 12).

Then we have

$$dr^2+dz^2=[1+yk(x)]^2\,dx^2+dy^2,$$

where $k(x)$ is the curvature of the line of force at the point x, 0. Therefore

$$p_x=[1+yk(x)]^2\,\dot{x}, \qquad p_y=\dot{y},$$

and hence

$$H=\frac{1}{2}\left(\frac{p_x^2}{[1+yk(x)]^2}+p_y^2\right)+U, \text{ where } U(x,\ y)=\frac{1}{2}B(x)\,y^2+\ \ldots$$

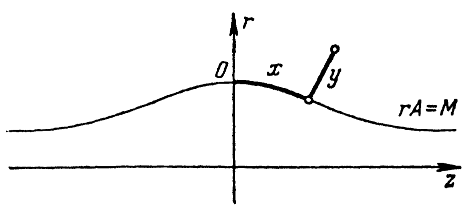

Fig. 12.

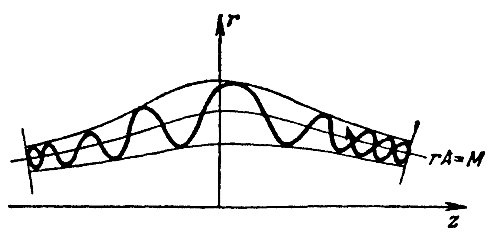

Fig. 13.

The case of interest to us is that in which the radius of the Larmor spiral described by the charge around the line of force is very small in comparison with the characteristic dimensions of the field. In the notation introduced this means that we shall consider values of the constant M and the total energy h such that the inequality

$$U \leqslant h$$

defines a strip of width $\sim\mu$ around the line of force $rA = M$ (Fig. 13). It is therefore convenient to introduce new variables X, Y, P_X, P_Y by means of the relationships

$$x=\mu X, \qquad y=\mu Y, \qquad p_x=\mu P_X, \qquad p_y=\mu P_Y.$$

The way in which these variables change with time is described by the canonical equations with Hamiltonian $H' = \mu^{-2}H$:

$$H'=\frac{1}{2}\left(\frac{P_X^2}{[1+\mu Y k(\mu X)]^2}+P_Y^2\right)+U', \qquad U'(\mu X,\ Y)=\frac{1}{2}B^2(\mu X)\,Y^2+\ldots,$$

which can be written in the form

$$H'=\frac{P_X^2+P_Y^2+B^2(x^2)\,Y^2}{2}+\mu H_1(x,\ Y,\ P_X,\ P_Y)+\ldots,$$

i.e. this function has the form $H'(\mu X,\ Y;\ P_X,\ P_Y;\ \mu)$ similar to that considered in §3. (It is easy to see that the additional dependence on μ is inessential for the applicability of the arguments of §3.)

3. **Perpetual adiabatic conservation of the magnetic moment.** In considering the formulation of the result to be obtained let us define more exactly what asymptotic properties we shall be concerned with. We fix the magnetic field B and also the self-conserving quantity M. Then we fix the initial value[1] of x and finally choose initial values of y, $\dot{x}$, $\dot{y}$ such that $H \sim \mu^2$, $\mu \ll 1$. For this purpose it is necessary to take y, $\dot{x}$, $\dot{y}$ of order μ. We fix $Y, \dot{X}, \dot{Y}$, take $y = \mu Y$, $\dot{x} = \mu\dot{X}$, $\dot{y} = \mu\dot{Y}$ and then let μ tend to 0.

The method of §3 gives us the following result:

I. *If the magnetic field is analytic and* $B(x) > 0$, $B(x) \to \infty$ *as* $|x| \to \infty$, *then for any* $\varkappa > 0$ *it is possible to find* $\mu_0 > 0$ *such that if* $|\mu| < \mu_0$, *then* $|I_Y(t) - I_Y(0)| < \varkappa$ *for all* $-\infty < t < +\infty$, *where*

$$I_Y=\frac{P_Y^2+B^2Y^2}{2B}\ .$$

From this it can be seen at once that a particle for which $I_Y \neq 0$ is *locked in a bounded domain* provided the field B increases infinitely as $x \to \infty$ (a trap with stoppers).

The physical meaning of I_Y will become clear if we consider the moment when $Y = 0$. At this moment $\dot{\varphi} = 0$ and $\frac{\dot{y}^2}{2}=W_\perp$ (see §1, 2., III). Therefore $\mu^2 I_Y=\frac{W_\perp}{B}$. Thus Proposition I can be formulated as follows:

II. *In an axially-symmetric magnetic trap the magnitude of the magnetic moment* $\frac{W_\perp}{B}$ *is a perpetual adiabatic invariant.*

---

1 It is sufficient to fix a bounded domain of initial values of x. The same applies to the fixing of Y, $\dot{X}$, $\dot{Y}$ and M.

§5. The many-dimensional case

This section contains a brief discussion of the possibility of carrying over the results of §§2 and 3 to the case of many degrees of freedom.

1. The multi-parametric problem. Theorem 1 is valid when the Hamiltonian varies conditionally periodically, namely when $H(p, q; \lambda_1, \ldots, \lambda_n)$ depends on several angular parameters λ_i each of which varies with its own frequency $\dot{\lambda}_i = \mu \xi_i$.

We assume that

$$|(k, \xi)| \geqslant K|k|^{-\nu} \quad \text{if} \quad |k| > 0, \tag{2.5.1}$$

for $\nu = n + 1$ and a certain $K > 0$. Then going over to the "time" $T = w$ we obtain the Hamiltonian (2.2.9) in the form $K(\pi', q_i', T)$, where $\pi' = \Sigma \xi_i p_i'$. On account of (2.5.1) we can make the transformation $p', q' \to P, Q$. The quantities P_i enter into the Hamiltonian (2.2.7) only in the form of the combination $\Pi = \Sigma \xi_i P_i$. As in §2, 5. we can find invariant manifolds with equations $\Pi = F_\mu(Q_i, T)$; to them correspond the $(n + 1)$-dimensional tori in the original $(n + 2)$-dimensional space $p, q; \lambda_1, \ldots, \lambda_n$.

2. The case of many degrees of freedom. At this point considerable difficulties appear; thus, the question of the conservation of adiabatic invariants even for $t \sim \frac{1}{\mu}$ has not so far been investigated (see [29]). We can only very briefly dwell on the peculiarities of this case.

The difficulty consists in that the ratio of the frequencies of the rapid motions may depend on the phase of the slow motion. A simple example of this phenomenon is given by the system of equations on a three-dimensional torus

$$\left.\begin{aligned} \dot{x} &= \omega_1(z) + \mu f(x, y, z), \\ \dot{y} &= \omega_2(z) + \mu g(x, y, z), \\ \dot{z} &= \mu \end{aligned}\right\} \tag{2.5.2}$$

(where x, y, z are the angular coordinates of a point on the torus); the trajectories (2.5.2) cannot be rectified by a change of variables.

Therefore invariant tori filled with conditionally periodic motions can scarcely exist in the case of a general integrable system with n degrees of freedom and slowly periodically changing coefficients. If such tori were in fact found, they would be of dimension $n + 1$ and would not divide the $(2n + 1)$-dimensional space $p_i, q_i; \lambda$ so that it would still not be possible to prove the perpetual adiabatic invariance of the variables of action.

Another approach to the many-dimensional case is given in [30].

Chapter III

THE STABILITY OF PLANETARY MOTIONS

With the help of the fundamental theorem of Chapter IV, we investigate in this chapter the class of "planetary" motions in the three-body and many-body problems. We show that, *for the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane, perturbation of the planets on one another produces, in the course of an infinite interval of time, little change on these orbits provided the masses of the planets are sufficiently small* ([17], [21]).

In particular, it follows from our results that *in the n-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities of the bodies belong to this set, the distances of the bodies from each other will remain perpetually bounded.*

Such a hypothesis was put forward long ago by astronomers, but more recently, beginning with Birkhoff, mathematicians engaged on this question have inclined to the opposite view (see [3], [8]).

Precise formulations of the results are given in §1. We shall consider only the plane three-body problem in detail. In §2 suitable coordinates are introduced by means of which the problem is reduced in §3 to the form considered in Chapter IV. §4 contains a verification of the non-degeneracy conditions which are necessary in the application of the fundamental theorem of Chapter IV. In the final section a brief indication is given of the way in which the fundamental theorem of Chapter IV is applied in the investigation of planetary motions in the plane and spatial many-body problems.

Poincaré's book [1] has been extensively used in the writing of this chapter (particularly in §2).

§1. Picture of the motion

In this section we formulate the results of applying the fundamental theorem of Chapter IV to the problem of the motion of n_0 "planets" around a massive central body (3.-5.). 1. and 2. give an account of certain conclusions of the non-rigorous classical theory of perturbations.

1. Kepler motion. Let us consider n_0 material points ("planets") whose masses $m_1, \dots, m_{n_0}$ are small in comparison with the mass M of a material point called the "central body". Suppose that all these points attract one another according to Newton's law, i.e. with a force

$$f = \frac{m_i m_j}{r_{ij}^2} .$$

If $m_1, \dots, m_{n_0}$ are sufficiently small in comparison with M, then "in the zero approximation" the attractions of the planets on one another can be neglected and the central body M can be regarded as fixed. On these assumptions[1] each planet m_k moves independently of all the other planets along a so-called Kepler ellipse with focus at M.

Thus, in the zero approximation the motion of the system is conditionally periodic and is described by n_0 *frequencies of rotation of the planets about M.*

In what follows we shall assume that the planes of all the Kepler ellipses are close to one another and that the "planets" move along them in the same direction.

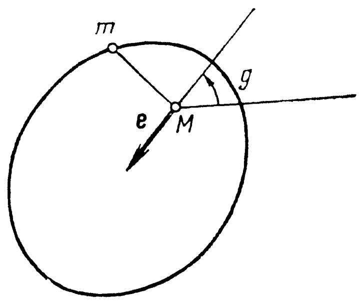

Fig. 14.

The perturbations of the planets on one other result in a difference between the true motion and that described by the zero approximation. The theory of perturbations gives the following picture of the motion.

2. Lagrangian motion. For simplicity let us consider the plane three-body problem ($n_0 = 2$) and suppose that initially the eccentricities e_k ($k = 1, 2$) of the Kepler ellipses are small. The position of the ellipse in the plane is defined by the angle g_k formed by the major axis of the ellipse $2a_k$ and the coordinate axis (g_k is called the *longitude of the perihelion*, Fig. 14).

The perturbed motion can be described as a Kepler motion with variable parameters a_k, e_k, g_k. It turns out that in the first approximation this variation in a_k, e_k, g_k reduces to a small (together with the masses of the planets) "vibration" of $a_k(t)$, $e_k(t)$, $g_k(t)$ about the constant values.

The second approximation contains a small but unbounded (secular) *motion of the perihelia.* This small variation in e_k and g_k can be described in the following way. We characterize the Kepler ellipse by the vector $\mathbf{e}_k$ directed along the major axis and proportional to the eccentricity[2] having the length $\sqrt{m_k}\,\sqrt[4]{a_k}\,e_k$. It turns out that for each planet this vector is the sum of two uniformally rotating vectors: $\mathbf{e}_k = \mathbf{e}_{k1} + \mathbf{e}_{k2}$. The angular velocities ν_1, ν_2 of the vectors $\mathbf{e}_{k1}$, $\mathbf{e}_{k2}$ are small and the same for both planets.

The major semi-axes a_k do not have a secular variation. The motion of planets along ellipses varying in this way shall be called Lagrangian.

The theory of perturbations shows (but does not prove) that *the true motion is close to Lagrangian motion during many rotations of* $\mathbf{e}_{kl}$, *provided the masses of the planets and the initial values of the eccentricities are sufficiently small.*

An analogous picture with n_0 frequencies $\nu_1, \ldots, \nu_{n_0}$ and n_0^2 vectors $\mathbf{e}_{kl}$ is obtained in the plane problem of n_0 planets.

In the space problem the Kepler ellipse is further determined by the inclination i_k (the angle between the plane of the ellipse and the coordinate plane) and the nodal line (the line of intersection of these planes). The secular variation of these quantities is described by means

1 With negative initial energy

2 $\mathbf{e}_k$ is called the Laplace vector.

of the vector i_k of length $\sqrt{m_k}\sqrt[4]{a_k}\, i_k$ directed along the nodal line. It turns out that i_k is also the sum of uniformly rotating vectors but this time there are $n_0 - 1$ of them.

Thus, *the Lagrangian motion is conditionally periodic and to the* n_0 *"rapid" frequencies of the Kepler motion are added* n_0 *(in the plane problem) or* $2n_0 - 1$ *(in the space problem) "slow" frequencies of the secular motions.*

3. The true motion. Our basic result is that *if the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion with suitable initial conditions throughout an infinite interval of time* $-\infty < t < +\infty$.

We shall first of all consider the plane three-body problem. If the centre of gravity is regarded as fixed, the system has four degrees of freedom and an eight-dimensional phase space. As coordinates in this plane we can take, for example, the four quantities three of which (say a_1, e_1, g_1) define the Kepler ellipse of the first planet and the fourth the position of the planet on this ellipse, together with the four analogous quantities for the second planet.

Let us fix the constants α_k, c_k, $C_k > 0$. Let the masses of the bodies be

$$m_k = \mu\alpha_k, \qquad M = 1 \qquad (\mu \ll 1).$$

In the phase space we consider the domain defined by the conditions

$$\alpha_1 \sqrt{a_1}\, e_1^2 + \alpha_2 \sqrt{a_2}\, e_2^2 \leqslant \varepsilon, \tag{3.1.1}$$

$$c_1 \leqslant a_1 \leqslant C_1, \qquad c_2 \leqslant a_2 \leqslant C_2 \quad (\text{where } c_1 < C_1 < c_2 < C_2). \tag{3.1.2}$$

This bounded domain will be denoted by $F(\varepsilon)$.

Basing ourselves on the results of Chapter IV, we shall prove in §§2-4 the following assertion.

THEOREM. *For any* $\kappa > 0$ *it is possible to find* $\varepsilon_0 > 0$ *such that if*

$$\varepsilon < \varepsilon_0, \qquad \mu < \varepsilon^4,$$

then $\mathbf{F}(\varepsilon)$ *can be divided into two parts:*

$$F(\varepsilon) = \mathbf{F}(\varepsilon) + f(\varepsilon),$$

of which one, $\mathbf{F}(\varepsilon)$, *is invariant and the other,* $f(\varepsilon)$, *small:*

$$\operatorname{mes} f(\varepsilon) \leqslant \varkappa \operatorname{mes} F(\varepsilon).$$

Points belonging to $\mathbf{F}(\varepsilon)$ *have a conditionally periodic motion, marking out a four-dimensional analytic invariant torus in* $\mathbf{F}(\varepsilon)$.

If the initial conditions belong to $\mathbf{F}(\varepsilon)$, *then at any moment the true position of the planets differs from the position of points carrying out a certain Lagrangian motion by a quantity less than* κ.

4. The plane problem of n_0 planets. Analogous results are obtained for $n_0 > 2$ planets; this case also can be reduced to the fundamental theorem of Chapter IV. The corresponding calculations are outlined in §5;

they are not given in full in view of their unwieldiness.

In the plane problem the domain $F(\varepsilon)$ in the $4n_0$-dimensional phase space is given by conditions which are generalizations of (3.1.1), (3.1.2):

$$\sum_{k=1}^{n_0} a_k \sqrt{a_k}\, e_k^2 \leqslant \varepsilon, \qquad c_k \leqslant a_k \leqslant C_k \quad (\text{where } C_{k-1} < c_k < C_k). \qquad (3.1.3)$$

The only change in the Theorem of 3. consists in the replacement of "four-dimensional" by "$2n_0$-dimensional".

5. The space problem of n_0 planets. In the space problem the phase space is $6n_0$-dimensional, to the conditions of (3.1.3) we have to add the inequality

$$\sum_{k=1}^{n_0} a_k \sqrt{a_k}\, i_k^2 \leqslant \varepsilon \qquad (3.1.4)$$

and in the theorem of 3. "four-dimensional" should be replaced by "$(3n_0 - 1)$-dimensional". With these changes the theorem of 3. is valid.

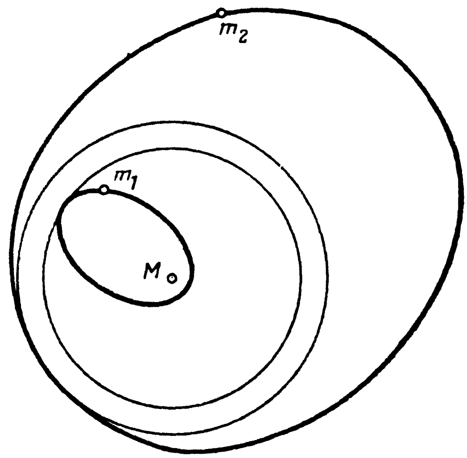

Fig. 15.

6. The case of three bodies. In this case we can to obtain stronger results than those of 3. if conservation of angular momentum (see §5) is used. It turns out that it is not necessary to require the eccentricities to be small; all that is necessary is that they should be small enough to exclude the possibility of collision (Fig. 15). In place of (3.1.1) in the theorem of 3. it is sufficient to require

$$\sum_{k=1}^{2} a_k \sqrt{a_k}\left(1-\sqrt{1-e_k^2}\right) \leqslant \varepsilon_0 (a_k,\ c_k,\ C_k), \qquad (3.1.5)$$

where

$$\varepsilon_0 = \min_{e_1,\, e_2} \sum_{k=1}^{2} a_k \sqrt{a_k}\left(1-\sqrt{1-e_k^2}\right)$$

with the condition $C_1(1 + e_1) = c_2(1 - e_2)$, $0 \leqslant e_1,\ e_2 \leqslant 1$.

If we denote by F the domain defined by (3.1.5), (3.1.2), then the following assertion, which strengthens the theorem of 3., holds.

THEOREM. *For any $\varkappa > 0$ it is possible to find $\mu_0 > 0$ such that if*

$$u < \mu_0,$$

then F can be divided into two parts:

$$F = \mathbf{F} + f,$$

of which one, $\mathbf{F}$, is invariant and the other, f, small:

$$\operatorname{mes} f \leqslant \varkappa \operatorname{mes} F.$$

Points belonging to $\mathbf{F}$ have a conditionally periodic motion, marking out a four-dimensional analytic invariant torus and remaining perpetually in $\mathbf{F}$.

If the initial conditions belong to **F**, *the variation of the major semi-axes during the whole period of the motion does not exceed* $\varkappa$.

7. An analogous theorem is valid for the space three-body problem. In this case $F(\varepsilon)$ is defined by (3.1.5), (3.1.2) and (3.1.4) with sufficiently small ε.

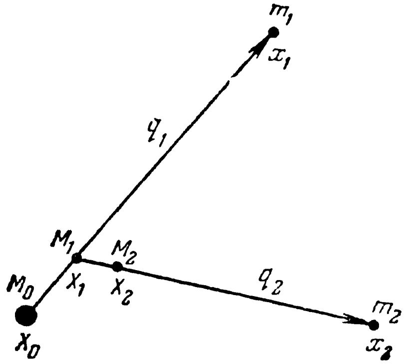

Fig. 16.

§2. Jacobi, Delaunay and Poincaré variables

The systems of canonical variables named after Jacobi $(p,\ q)$, Delaunay $(L,\ G,\ l,\ g)$ and Poincaré $(\Lambda,\ \Gamma,\ \lambda,\ \gamma$ and $\Lambda,\ \xi,\ \lambda,\ \eta)$ are introduced in this section.

1. Jacobi coordinates. Let us consider the plane three-body problem, where the masses M_0, m_1, m_2 have Cartesian radii-vectores x_0, x_1, x_2. We shall use the system of units in which the gravitational constant is equal to 1. Then the Lagrangian takes the form

$$L = \frac{1}{2}\left(M_0\dot{x}_0^2 + m_1\dot{x}_2^2 + m_2\dot{x}_2^2\right) + \frac{m_1M_0}{|x_1 - x_0|} + \frac{m_2M_0}{|x_2 - x_0|} + \frac{m_1m_2}{|x_2 - x_1|}\,. \qquad (3.2.1)$$

We shall denote the centres of gravity of the systems M_0; M_0 and m_1; M_0, m_1 and m_2 by X_0, X_1 and X_2, respectively. On introducing the Jacobi relative coordinates (Fig. 16)

$$q_1 = x_1 - X_0, \qquad q_2 = x_2 - X_1, \qquad (3.2.2)$$

we have

$$L = \frac{1}{2}\left(\mu_1\dot{q}_1^2 + \mu_2\dot{q}_2^2 + M_2\dot{X}_2^2\right) + \frac{m_1M_0}{|q_1|} + \frac{m_2M_0}{|x_2 - x_0|} + \frac{m_1m_2}{|x_2 - x_1|}\,, \qquad (3.2.3)$$

where μ_i are masses given by:

$$\mu_1 = m_1\frac{M_0}{M_1}\,, \quad \mu_2 = m_2\frac{M_1}{M_2} \quad (M_1 = M_0 + m_1, \quad M_2 = M_0 + m_1 + m_2). \qquad (3.2.4)$$

Without loss of generality the centre of gravity of all three bodies can be regarded as fixed. Then the Hamiltonian of the system is

$$H = \frac{1}{2}\left(\frac{p_1^2}{\mu_1} + \frac{p_2^2}{\mu^2}\right) - \frac{m_1M_0}{|q_1|} - \frac{m_2M_0}{|q_2|} - \left[\frac{m_1m_2}{|x_2 - x_1|} + \left(\frac{m_2M_0}{|x_2 - x_0|} - \frac{m_2M_0}{|q_2|}\right)\right], \qquad (3.2.5)$$

where $p_k = \mu_k\dot{q}_k$.

2. Delaunay elements. Let us now turn to the plane problem of the attraction of a material point of mass m by a fixed centre of mass M. In this problem

$$H = \frac{1}{2}\frac{p^2}{m} - \frac{mM}{|q|}\,,$$

where

$$p^2 = (p^{(1)})^2 + (p^{(2)})^2, \qquad |q| = \sqrt{(q^{(1)})^2 + (q^{(2)})^2}\,, \qquad (3.2.6)$$
$$p = p^{(1)},\ p^{(2)}, \qquad q = q^{(1)},\ q^{(2)}.$$

We know that the body m moves along a Kepler ellipse with focus at M. The form of the ellipse is determined by the major semi-axis a and the eccentricity e. The position of the ellipse is determined by the angle g (the longitude of the perihelion) and the position of the body m on the orbit by the angle l (the mean anomaly). The quantities a, e, l, g are called elliptic elements.

As we know (see [1]), there exists a canonical transformation

$$p^{(1)},\ p^{(2)},\ q^{(1)},\ q^{(2)} \longrightarrow L,\ G,\ l,\ g, \tag{3.2.7}$$

introducing the Delaunay elements L, G, l, g, where

$$L = m\sqrt{M}\sqrt{a}\,,\quad G = L\sqrt{1-e^2},\quad H = -\frac{m^3M^2}{2L^2}\,. \tag{3.2.8}$$

We note that G is the angular momentum:

$$G = [mq,\ q] = p^{(1)}q^{(2)} - p^{(2)}q^{(1)}.$$

Let us denote by $S_{m,M}(L,\ G;\ q^{(1)},\ q^{(2)})$ the generating function of the transformation (3.2.7) for which

$$p^{(1)} = \frac{\partial S}{\partial q^{(1)}}\,,\qquad p^{(2)} = \frac{\partial S}{\partial q^{(2)}}\,,\qquad l = \frac{\partial S}{\partial L}\,,\qquad g = \frac{\partial S}{\partial G}\,. \tag{3.2.9}$$

In the plane three-body problem the variables L_k, G_k, l_k, g_k (k = 1, 2) are introduced by the formulae (3.2.9) with generating function

$$S = S_{\mu_1\ M_1'}(L_1,\ G_1;\ q_1^{(1)},\ q_1^{(2)}) + S_{\mu_2,\ M_2'}(L_2,\ G_2;\ q_2^{(1)},\ q_2^{(2)}),$$

where $M_1' = M_1$ and $M_2' = M_2\dfrac{M_0}{M_1}$.

In view of (3.2.4), (3.2.6), (3.2.8), the Hamiltónian (3.2.5) takes the form

$$H = -\frac{\mu_1^3M_1'^2}{2L_1^2} - \frac{\mu_2^3M_2'^2}{2L_2^2} - \left[\frac{m_1m_2}{|x_2-x_1|} + \left(\frac{m_2M_0}{|x_2-x_0|} - \frac{m_2M_0}{|q_2|}\right)\right], \tag{3.2.10}$$

where the coordinates x_0, x_1, x_2, q_2 are expressed in terms of L, G, l, g by means of the relationships (3.2.9), (3.2.2).

3. Expansion as a series of powers of the masses. We now assume that the masses m_1 and m_2 are small in comparison with M_0:

$$M_0 = 1,\qquad m_1 = \mu\alpha_1,\qquad m_2 = \mu\alpha_2, \tag{3.2.11}$$

where α_1, α_2 are finite and μ is a small parameter. We shall denote by $(\mu^k)\ldots$ convergent series of the form $\mu^k f_k + \mu^{k+1} f_{k+1} + \ldots$ It is obvious that

$$\left.\begin{aligned} &M_1 = 1 + (\mu)\ldots,\qquad M_2 = 1 + (\mu)\ldots,\qquad M_2' = 1 + (\mu)\ldots,\\ &\mu_1 = \mu\alpha_1 + (\mu^2)\ldots,\quad \mu_2 = \mu\alpha_2 + (\mu^2)\ldots \end{aligned}\right\} \tag{3.2.12}$$

We shall denote by a_k, e_k the elliptic elements of the "osculating" Kepler ellipses corresponding to L_k, G_k:

$$L_k = \mu_k\sqrt{M_k'}\sqrt{a_k},\quad G_k = L_k\sqrt{1-e_k^2}\,. \tag{3.2.13}$$

It is easy to calculate that the square bracket in (3.2.10) takes the form

$$[\ ] = \mu^2\alpha_1\alpha_2 \left[\frac{1}{|q_1 - q_2|} - \frac{|q_1| s}{|q_2|^2} \right] + (\mu^3)\ldots, \tag{3.2.14}$$

where q_k is the radius-vector of the point moving along a Kepler orbit with elements a_k, e_k; l_k, g_k around the origin, and s denotes the cosine of the angle between the vectors q_1 and q_2. By virtue of (3.2.14) the Hamiltonian (3.2.10) can be written in the form

$$H = -\frac{\mu_1^3 M_1'^2}{2L_1^2} - \frac{\mu_2^3 M_2'^2}{2L_2^2} - \mu^2\alpha_1\alpha_2 \left[\frac{1}{|q_1 - q_2|} - \frac{|q_1| s}{|q_2|^2} \right] + (\mu)^3 \ldots \tag{3.2.15}$$

4. **Poincaré variables,** If H, L and G are all divided by the constant μ, the Hamiltonian form of the equations of motion is preserved. By means of the canonical transformations

$$\mu^{-1}L_k,\ \mu^{-1}G_k;\ l_k,\ g_k \longrightarrow \Lambda_k,\ \Gamma_k;\ \lambda_k,\ \gamma_k \longrightarrow \Lambda_k,\ \xi_k;\ \lambda_k,\ \eta_k. \tag{3.2.16}$$

we introduce the Poincaré variables Λ, Γ, λ, γ and Λ, ξ, λ, η:

$$\left.\begin{aligned} \mu\Lambda_k &= L_k, & \mu\Gamma_k &= L_k - G_k, & \xi_k &= \sqrt{2\Gamma_k}\cos\gamma_k, \\ \lambda_k &= l_k + g_k, & \gamma_k &= -g_k, & \eta_k &= \sqrt{2\Gamma_k}\sin\gamma_k. \end{aligned}\right\} \tag{3.2.17}$$

The variables Λ, Γ are expressed in terms of the elliptic elements by formulae following from (3.2.8) and (3.2.17):

$$\Lambda_k = \beta_k \sqrt{a_k},\quad \Gamma_k = \Lambda_k\left(1 - \sqrt{1 - e_k^2}\right), \text{where } \beta_k = \frac{\mu_k\sqrt{M_k'}}{\mu} = \alpha_k + (\mu)\ldots \tag{3.2.18}$$

The variation of the variables Λ, Γ, λ, γ (or Λ, ξ, λ, η) with time is described by the canonical equations with Hamiltonian $F = \frac{H}{\mu}$ obtained from (3.2.15):

$$F = -\frac{\beta_1'^3}{2\Lambda_1^2} - \frac{\beta_2'^3}{2\Lambda_2^2} - \mu\beta_1\beta_2 \left[\frac{1}{|q_1 - q_2|} - \frac{|q_1| s}{|q_2|^2} \right] + (\mu^2)\ldots, \tag{3.2.19}$$

where $\beta_k' = \frac{\mu_k}{\mu} M_k'^{2/3} = \beta_k + (\mu)\ldots = \alpha_k + (\mu)\ldots$ depends only on the masses of the bodies.

§3. Birkhoff's transformation

In this section new variables $\bar{\bar{\Lambda}}$, $\bar{\bar{\lambda}}$, $\bar{\bar{\xi}}$, $\bar{\bar{\eta}}$ are chosen. The fundamental theorem (Ch. 4, §1) is applied to the Hamiltonian of the three-body problem expressed in terms of these variables.

1. **The Hamiltonian.** We express (3.2.19) in terms of the variables Λ, λ, ξ, η. Obviously,

$$\begin{gathered} F = F_0(\Lambda) + \mu F_1(\Lambda,\ \lambda;\ \xi,\ \eta) + (\mu^2)\ldots, \\ F_0 = -\frac{\beta_1'^3}{2\Lambda_1^2} - \frac{\beta_2'^3}{2\Lambda_2^2},\quad F_1 = -\beta_1\beta_2\ [\ ]. \end{gathered} \tag{3.3.1}$$

Furthermore, F is analytic for Λ_1, Λ_2, $\Lambda_1 - \Lambda_2 \neq 0$, $|\xi|$, $|\eta| < R$ (where R depends on Λ) and F is of period 2π with respect to λ_1, λ_2.

We define the "secular part" of F, i.e. the mean value of F with respect to λ:

$$F_1(\Lambda,\ \lambda;\ \xi,\ \eta) = \bar{F}_1(\Lambda;\ \xi,\ \eta) + \widetilde{F}_1(\Lambda,\ \lambda;\ \xi,\ \eta), \tag{3.3.2}$$

where $\iint \widetilde{F}_1\, d\lambda_1\, d\lambda_2 = 0$. It is easy to see that $\bar{F}$ is even with respect to ξ, η. In $\bar{F}_1$ we shall consider the variables Λ as parameters. The point $\xi = \eta = 0$ is the position of equilibrium of the system with Hamiltonian $\bar{F}_1(\xi,\ \eta)$ (for each fixed Λ). We apply Birkhoff's theory (Ch. I, §9) to $\bar{F}_1(\xi,\ \eta)$.

2. The coordinates ξ', η'. If we disregard the constant $\bar{F}_{10}$, which has no effect on the Hamiltonian equations, the expansion of $\bar{F}_1$ in powers of ξ, η:

$$\bar{F}_1 = \bar{F}_{10} + \bar{F}_{12} + \bar{F}_{14} + \ldots,$$

begins with a negative[1]-definite quadratic form $\bar{F}_{12}$. By a linear canonical transformation $\xi,\ \eta \to \xi',\ \eta'$ we can reduce $\bar{F}_{12}$ to the form

$$\bar{F}_{12} = v_1'\Gamma_1' + v_2'\Gamma_2' \qquad (2\Gamma_k' = \xi_k'^2 + \eta_k'^2 = p_k' q_k';\ p_k = \xi_k + i\eta_k,\ q_k = \xi_k - i\eta_k), \tag{3.3.3}$$

where the coefficients v_k' depend on Λ. We write $\bar{F}_{14}$ in the form

$$\bar{F}_{14} = v_{11}'\Gamma_1'^2 + 2v_{12}'\Gamma_1'\Gamma_2' + v_{22}'\Gamma_2'^2 + \ldots, \tag{3.3.4}$$

where the dots denote terms of the fourth degree in ξ', η' which when expanded in powers of p', q' only give $p_1'^k q_1'^l p_2'^m q_2'^n$ with $(k - l)^2 + (m - n)^2 \neq 0$; the coefficients v_{ij} in (3.3.4) depend on Λ.

3. The coordinates $\xi^{(s)}$, $\eta^{(s)}$. If the Λ are such that

$$k_1 v_1' + k_2 v_2' \neq 0 \quad \text{for} \quad 0 < |k_1| + |k_2| \leqslant 6, \tag{3.3.5}$$

then in accordance with Birkhoff's theory (see Ch. I, §9) there exist further canonical transformations

$$\xi,\ \eta \longrightarrow \xi',\ \eta' \longrightarrow \xi^{(2)},\ \eta^{(2)} \longrightarrow \xi^{5/2},\ \eta^{5/2} \longrightarrow \xi^{(3)},\ \eta^{(3)} \longrightarrow \xi^{7/2}\ \eta^{7/2}, \tag{3.3.6}$$

reducing $\bar{F}_1$ to the form

$$\bar{F}_1 = \bar{\bar{F}}_1^{(s)}(\Gamma^{(s)}) + O(|\xi^{(s)}|^{2s},\ |\eta^{(s)}|^{2s}) \qquad (2\Gamma_k^{(s)} = (\xi_k^{(s)})^2 + (\eta_k^{(s)})^2). \tag{3.3.7}$$

Since $\bar{F}_1$ is even, it follows that $\xi^{(2)}$, $\eta^{(2)} = \xi'$, η'. We introduce the notation:

$$\bar{\xi} = \xi^{5/2} \qquad \bar{\eta} = \eta^{5/2} \qquad 2\bar{\Gamma} = \bar{\xi}^2 + \bar{\eta}^2,$$

$$\bar{\bar{\xi}} = \xi^{7/2} \qquad \bar{\bar{\eta}} = \eta^{7/2} \qquad 2\bar{\bar{\Gamma}} = \bar{\bar{\xi}}^2 + \bar{\bar{\eta}}^2.$$

From the formulae of the transformation ξ', $\eta' \to \bar{\xi}$, $\bar{\eta}$ (see [3], [8]) it is seen that for $2s = 5$ (3.3.7) gives

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1 Our Hamiltonian F differs from the perturbation function of celestial mechanics by a sign.

$$\overline{\overline{F}}_1^{\,5/2} = \overline{v}_0 + \overline{v}_1\overline{\Gamma}_1 + \overline{v}_2\overline{\Gamma}_2 + \overline{v}_{11}\overline{\Gamma}_1^2 + 2\overline{v}_{12}\overline{\Gamma}_1\overline{\Gamma}_2 + \overline{v}_{22}\overline{\Gamma}_2^2, \qquad (3.3.8)$$

where

$$\overline{v}_0 = \overline{F}_{10}, \quad \overline{v}_k = v_k' \text{ and } \overline{v}_{kl} = v_{kl}'$$

(see (3.3.4)).

Consequently, in accordance with Ch. I, §9, in (3.3.7) with $2s = 7$ we have

$$\overline{\overline{F}}_1^{\,7/2} = \overline{v}_0 + \sum \overline{v}_k\overline{\overline{\Gamma}}_k + \sum \overline{v}_{kl}\overline{\overline{\Gamma}}_k\overline{\overline{\Gamma}}_l + \sum \overline{v}_{klm}\overline{\overline{\Gamma}}_k\overline{\overline{\Gamma}}_l\overline{\overline{\Gamma}}_m, \qquad (3.3.9)$$

where the coefficients $\overline{v}_k$ and $\overline{v}_{kl}$ are the same as in (3.3.8) (and $\overline{v}_{kl} = \overline{v}_{lk}$).

4. The canonical transformation $\Lambda, \lambda; \xi, \eta \to \overline{\overline{\Lambda}}, \overline{\overline{\lambda}}; \overline{\overline{\xi}}, \overline{\overline{\eta}}$. We denote by $S(\overline{\overline{\xi}}, \eta)$ the generating function of the transformation $\xi, \eta \to \overline{\overline{\xi}}, \overline{\overline{\eta}}$ (see (3.3.6)). It depends on the parameters Λ: $S = S(\Lambda; \overline{\overline{\xi}}, \eta)$. Let us construct the generating function $\overline{\overline{\Lambda}}\lambda + S(\overline{\overline{\Lambda}}; \overline{\overline{\xi}}, \eta)$. It defines the canonical transformation

$$\Lambda \longrightarrow \overline{\overline{\Lambda}} = \Lambda, \quad \lambda \longrightarrow \overline{\overline{\lambda}} = \lambda + \frac{\partial S}{\partial \overline{\overline{\Lambda}}}, \quad \xi, \eta \longrightarrow \overline{\overline{\xi}}, \overline{\overline{\eta}}.$$

Since λ is changed by a term that does not depend on λ, the division (3.3.2) of the Hamiltonian F_1 into a secular and a periodic part is preserved. We now denote $\overline{\overline{\Lambda}}, \overline{\overline{\lambda}}; \overline{\overline{\xi}}, \overline{\overline{\eta}}; F; \overline{v}, \overline{\overline{\Gamma}}$ by $p_0, q_0; p_1, q_1; H; \lambda, \tau$, respectively. The Hamiltonian (3.3.1) expressed in terms of $\overline{\overline{\Lambda}}, \overline{\overline{\lambda}}, \overline{\overline{\xi}}, \overline{\overline{\eta}}$, in accordance with (3.3.1), (3.3.2), (3.3.7), (3.3.9), then takes the form (4.1.1), (4.1.2), (4.1.3), (4.1.4) of Ch. IV, §1, ($n_0 = 2$, $n_1 = 2$).

5. Conditions of non-degeneracy. We now choose G_0, ρ, R and C in order to satisfy the conditions of the fundamental theorem of Ch. IV, §1.

The transformation (3.3.6) has a singularity at points where condition (3.3.5) is violated. These points form a finite number of analytic submanifolds of the space Λ, since

$$\left|\frac{\partial(v_1', v_2')}{\partial(\Lambda_1, \Lambda_2)}\right| \not\equiv 0. \qquad (3.3.10)$$

The validity of (3.3.10) is proved in §4, 7.

Without loss of generality we can assume that the domain G_0 in the space Λ is situated at a finite distance from the manifolds indicated above. Then the expressions $|k_1 v_1' + k_2 v_2'|$ $(0 < |k_1| + |k_2| \leqslant 6)$ have a lower bound uniform in G_0. In consequence of this, for sufficiently small R and ρ, the function H defined in 4. is analytic in the domain $|p_1 q_1| \leqslant R$, $|\mathrm{Im}\, q_0| \leqslant \rho$, $p_0 \in G_0$. If C is sufficiently large, then condition 3) of the fundamental theorem (Ch. IV, §1) is fulfilled. Condition 4) requires that

$$\left|\frac{\partial^2 F_0}{\partial\Lambda_k\, \partial\Lambda_l}\right| \not\equiv 0, \qquad (3.3.11)$$

$$|\overline{v}_{kl}| \not\equiv 0. \qquad (3.3.12)$$

The validity of (3.3.11) follows from (3.3.1). Inequality (3.3.12) is proved in §4, 7. Thus, with the choice of G_0, ρ, R and C indicated above, all the conditions of the fundamental theorem are fulfilled. Conclusions I-IV of the fundamental theorem give the theorem of §1, 1. of the present chapter.

§4. Calculation of the asymptotic behaviour of the coefficients in the expansion of $\overline{\overline{F}}_1$

It is proved in this section that the determinants (3.3.10) and (3.3.12) do not vanish identically. The proof is based on the calculation of the asymptotic behaviour of these determinants as the ratio of the major semi-axes $\alpha = \frac{a_1}{a_2}$ tends to zero.

1. The expansion of $\overline{|q_1 - q_2|^{-1}}$. From (3.2.19) and (3.3.2) it follows that

$$\overline{F}_1 = -\beta_1\beta_2 \left[\overline{|q_1 - q_2|^{-1}} + \overline{\frac{|q_1| s}{|q_2|^2}} \right].$$

But it is not difficult to see that the mean value of $|q_1| s$ with respect to λ_1 is equal to zero. Therefore

$$\overline{F}_1 = -\beta_1\beta_2 \overline{|q_1 - q_2|^{-1}}. \tag{3.4.1}$$

We shall put $a_1 = \alpha$, $a_2 = 1$ and determine the asymptotic behaviour (3.4.3) as $\alpha \to 0$ of the coefficients ν' in the convergent expansion in powers of the variables ξ', η' (see (3.3.7) and (3.3.8)):

$$\overline{|q_1 - q_2|^{-1}} = \nu'_0 + \nu'_1\Gamma'_1 + \nu'_2\Gamma'_2 + \nu'_{11}\Gamma'^2_1 + 2\nu'_{12}\Gamma'_1\Gamma'_2 + \nu'_{22}\Gamma'^2_2 + R'_4 + R'_5, \tag{3.4.2}$$

where R'_4 consists of fourth degree terms in p', q' of the form $p'^k_1 q'^l_1 p'^m_2 q'^n_2$ with $(k-l)^2 + (m-n)^2 \neq 0$ and R'_5 of fifth and higher degree terms. In 7. we shall obtain without difficulty from (3.4.3) the asymptotic behaviour of the determinants (3.3.10) and (3.3.12).

We shall prove that the coefficients ν' in (3.4.2) are of the form

$$\left.\begin{aligned} &\nu'_0 = 1 + O(\alpha^2), \quad \nu'_1 = \frac{3}{4\beta_1}\alpha^{3/2} \cdot O(\alpha^{7/2}), \quad \nu'_2 = \frac{3}{4\beta_2}\alpha^2 + O(\alpha^{7/2}), \\ &\nu'_{11} = -\frac{3}{8\beta_1^2}\alpha + O(\alpha^3), \quad \nu'_{12} = \frac{9}{8\beta_1\beta_2}\alpha^{3/2} + O(\alpha^{7/2}), \\ &\qquad\qquad \nu'_{22} = \frac{3}{2\beta_2^2}\alpha^2 + O(\alpha^{7/2}). \end{aligned}\right\} \tag{3.4.3}$$

The proof is given in 2. - 5. with the help of the lemmas in 6.

2. Expansion in powers of e_1, e_2. From le Verrier's tables [32] we find that

$$|q_1-q_2|^{-1}=(1^0)+(2^0)\left(\frac{e_1}{2}\right)^2+(3^0)\left(\frac{e_2}{2}\right)^2+(4^0)\left(\frac{e_1}{2}\right)^4+$$
$$+(5^0)\left(\frac{e_1}{2}\right)^2\left(\frac{e_2}{2}\right)^2+(6^0)\left(\frac{e_2}{2}\right)^4+\left[(21^{-1})\left(\frac{e_1}{2}\right)\left(\frac{e_2}{2}\right)+\right.$$
$$\left.+(22^{-1})\left(\frac{e_1}{2}\right)^3\left(\frac{e_2}{2}\right)+(23^{-1})\left(\frac{e_1}{2}\right)\left(\frac{e_2}{2}\right)^3\right]\cos(\gamma_1-\gamma_2)+$$
$$+(31^{-2})\left(\frac{e_1}{2}\right)^2\left(\frac{e_2}{2}\right)^2\cos 2(\gamma_1-\gamma_2)+R_5^0, \quad (3.4.4)$$

where R_5^0 is a series beginning with terms of not lower than the fifth degree in $e\cos\gamma$, $e\sin\gamma$ and where the following notation is used:

$$\left.\begin{aligned}
&(1^0)=\frac{1}{2}A_0^0, \quad (2^0)=A_1^0+A_2^0, \quad (3^0)=A_1^0+A_2^0, \quad (4^0)=3A_3^0+3A_4^0,\\
&(5^0)=2A_1^0+14A_2^0+24A_3^0+12A_4^0, \quad (6^0)=3A_1^0+9A_2^0+9A_3^0+3A_4^0,\\
&(21^{-1})=2A_0^{-1}-2A_1^{-1}-2A_2^{-1}, \quad (22^{-1})=-4A_2^{-1}-18A_3^{-1}-12A_4^{-1},\\
&(23^{-1})=2A_0^{-1}-2A_1^{-1}-22A_2^{-1}-30A_3^{-1}-12A_4^{-1},\\
&(31^{-2})=3A_0^{-2}-3A_1^{-2}+3A_2^{-2}+12A_3^{-2}+6A_4^{-2}.
\end{aligned}\right\} \quad (3.4.5)$$

In formulae (3.4.5) the coefficients A_k^i are given by

$$A_k^i=\frac{\alpha^k}{k!}\frac{d^k b^i}{d\alpha^k}, \text{ where } (1+\alpha^2-2\alpha\cos\varphi)^{-\frac{1}{2}}=\frac{1}{2}\sum_{i=-\infty}^{+\infty} b^i\cos i\varphi, \quad b^i=b^{-i}, \quad (3.4.6)$$

so that the expansions of A_k^i in series of powers of α take the form

$$\left.\begin{aligned}
&A_0^0=2+\frac{\alpha^2}{2}+O(\alpha^4), && A_0^{-1}=\alpha+O(\alpha^3), && A_0^{-2}=\frac{3}{4}\alpha^2+O(\alpha^4),\\
&A_1^0=\alpha^2+O(\alpha^4), && A_1^{-1}=\alpha+O(\alpha^3), && A_1^{-2}=\frac{3}{2}\alpha^2+O(\alpha^4),\\
&A_2^0=\frac{\alpha^2}{2}+O(\alpha^4), && A_2^{-1}=O(\alpha^3), && A_2^{-2}=\frac{3}{4}\alpha^2+O(\alpha^4),\\
&A_3^0=O(\alpha^4), && A_3^{-1}=O(\alpha^3), && A_3^{-2}=O(\alpha^4),\\
&A_4^0=O(\alpha^4), && A_4^{-1}=O(\alpha^5), && A_4^{-2}=O(\alpha^4).
\end{aligned}\right\} \quad (3.4.7)$$

3. Expansion in powers of ξ, η. In accordance with (3.2.18)

$$\frac{e}{2}=\sqrt{\frac{\Gamma}{2\Lambda}}\left(1-\frac{\Gamma}{4\Lambda}+\ldots\right). \quad (3.4.8)$$

On substituting (3.4.8) into (3.4.4) we obtain

$$|q_1-q_2|^{-1}=\nu_0+\nu_1\Gamma_1+\nu_2\Gamma_2+\nu\sqrt{\Gamma_1\Gamma_2}\cos(\gamma_1-\gamma_2)+\nu_{11}\Gamma_1^2+2\nu_{12}\Gamma_1\Gamma_2+$$
$$+\nu_{22}\Gamma_2^2+\left[\varkappa_{13}\sqrt{\Gamma_1\Gamma_2^3}+\varkappa_{31}\sqrt{\Gamma_1^3\Gamma_2}\right]\cos(\gamma_1-\gamma_2)+$$
$$+\varkappa_{22}\Gamma_1\Gamma_2\cos 2(\gamma_1-\gamma_2)+R_5, \quad (3.4.9)$$

where the following notation is used:

$$\left.\begin{aligned} &\nu_0=(1^0)\quad \nu_1=\frac{(2^0)}{2\Lambda_1},\quad \nu_2=\frac{(3^0)}{2\Lambda_2}; && \nu=\frac{(21^{-1})}{2\sqrt{\Lambda_1\Lambda_2}},\\ &\nu_{11}=\frac{(4^0)-(2^0)}{4\Lambda_1^2},\quad 2\nu_{12}=\frac{(5^0)}{4\Lambda_1\Lambda_2}, && \nu_{22}=\frac{(6^0)-(3^0)}{4\Lambda_2^2},\\ &\varkappa_{13}=\frac{2(23^{-1})-(21^{-1})}{8\sqrt{\Lambda_1\Lambda_2^3}},\quad \varkappa_{31}=\frac{2(22^{-1})-(21^{-1})}{8\sqrt{\Lambda_1^3\Lambda_2}}, && \varkappa_{22}=\frac{(31^{-2})}{4\Lambda_1\Lambda_2} \end{aligned}\right\} \quad (3.4.10)$$

and R_5 is a series beginning with terms of not lower than the fifth degree in ξ, η.

In view of the fact that $\Lambda_1 = \beta_1\sqrt{\alpha}$, $\Lambda_2 = \beta_2$, the coefficients in the expansion (3.4.9), on substituting expressions (3.4.5) and (3.4.7) into the formulae (3.4.10), take the form

$$\nu_0=1+O(\alpha^2),\quad \nu_1=\frac{3}{4\beta_1}\alpha^{3/2}+O(\alpha^{7/2}),\quad \nu_2=\frac{3}{4\beta_2}\alpha^2+O(\alpha^4),\quad \nu=O(\alpha^{11/4}), \quad (3.4.11)$$

$$\nu_{11}=-\frac{3}{8\beta_1^2}\alpha+O(\alpha^3),\quad \nu_{12}=\frac{9}{8\beta_1\beta_2}\alpha^{3/2}+O(\alpha^{7/2}),\quad \nu_{22}=\frac{3}{2\beta_2^2}\alpha^2+O(\alpha^4), \quad (3.4.12)$$

$$\varkappa_{13}=O(\alpha^{11/4}),\quad \varkappa_{31}=O(\alpha^{9/4}),\quad \varkappa_{22}=O(\alpha^{7/2}). \quad (3.4.13)$$

4. **Expansion in powers of ξ', η'.** Lemma 2 of 6. is applied to the function

$$H_2=\nu_1\Gamma_1+\nu_2\Gamma_2+\nu\sqrt{\Gamma_1\Gamma_2}\cos(\gamma_1-\gamma_2)$$

From this it follows that

$$H_2=\nu_1'\Gamma_1'+\nu_2'\Gamma_2',\quad 2\Gamma'=\xi'^2+\eta'^2, \quad (3.4.14)$$

where for $x = \xi$, η, p or q with φ determined from the condition

$$(\nu_1-\nu_2)\sin 2\varphi+\nu\cos 2\varphi=0,$$

we have

$$\left.\begin{aligned} x_1&=x_1'\cos\varphi+x_2'\sin\varphi,\\ x_2&=-x_1'\sin\varphi+x_2'\cos\varphi \end{aligned}\right\} \quad (3.4.15)$$

and where, in accordance with (3.4.11) and (3.4.27), (3.4.29) from 6.,

$$\varphi=O(\alpha^{5/4}),\quad \nu_1'=\nu_1+O(\alpha^{7/2}),\quad \nu_2'=\nu_2+O(\alpha^{7/2}). \quad (3.4.16)$$

On substituting (3.4.15) into (3.4.9) we shall obtain the expansion of (3.4.2) in powers of x':

$$|q_1-q_2|^{-1}=\nu_0'+\nu_1'\Gamma_1'+\nu_2'\Gamma_2'+\nu_{11}'\Gamma_1'^2+2\nu_{12}'\Gamma_1'\Gamma_2'+\nu_{22}'\Gamma_2'^2+R_4+R_5, \quad (3.4.17)$$

where, in accordance with (3.4.11), (3.4.14) and (3.4.16),

$$\nu_1'=\frac{3}{4\beta_1}\alpha^{3/2}+O(\alpha^{7/2}),\quad \nu_2'=\frac{3}{4\beta_2}\alpha^2+O(\alpha^{7/2}) \quad (3.4.18)$$

and where R_4 consists of fourth degree terms of the form $p_1'^k q_1'^l p_2'^m q_2'^n$ with $(k-l)^2+(m-n)^2 \neq 0$ and R_6 of terms of degree higher than the fourth. In 5. we shall obtain expressions (3.4.3) for the coefficients ν_{ij}' in (3.4.17).

5. The calculation of ν'_{ij}. On substituting for x expressions (3.4.15) each function of x is expandable as a series in x'. A direct calculation gives for the coefficients of some of these expansions the expressions (3.4.19)-(3.4.21) with relative error $O(\varphi^2)$:

Coefficient of / in the expansion of	x'_1	x'_2
x_1	1	φ
x_2	$-\varphi$	1

(3.4.19)

From (3.4.19), putting $x = p$ or q, we obtain

	$2\Gamma'_1 = p'_1 q'_1$	$p'_1 q'_2$	$p'_2 q'_1$	$p'_2 q'_2 = 2\Gamma'_2$
$2\Gamma_1 = p_1 q_1$	1	φ	φ	φ^2
$p_1 q_2$	$-\varphi$	1	$-\varphi^2$	φ
$p_2 q_1$	$-\varphi$	$-\varphi^2$	1	φ
$2\Gamma_2 = p_2 q_2$	φ^2	$-\varphi$	$-\varphi$	1

(3.4.20)

Using (3.4.20) to calculate the coefficients of the expansion in powers of p', q', we obtain

	$(2\Gamma'_1)^2$	$(2\Gamma'_1)(2\Gamma'_2)$	$(2\Gamma'_2)^2$
$(2\Gamma_1)^2$	1	$4\varphi^2$	φ^4
$(2\Gamma_1)(2\Gamma_2)$	φ^2	1	φ^2
$(2\Gamma_2)^2$	φ^4	$4\varphi^2$	1

(3.4.21)

Consequently the exact matrix of coefficients (3.4.21) differs from the unit matrix by $O(\varphi^2)$.

With the help of (3.4.23), (3.4.24), (3.4.25), Lemma 1 of 6. and formulae (3.4.20), it is easy to calculate that all the coefficients of $(2\Gamma'_1)^2$, $(2\Gamma'_1)(2\Gamma'_2)$ and $(2\Gamma'_2)^2$ in the expansion of the quantities

$$8\sqrt{\Gamma_1^3\Gamma_2}\cos(\gamma_1-\gamma_2),\quad 8\sqrt{\Gamma_1\Gamma_2^3}\cos(\gamma_1-\gamma_2),\quad 8\Gamma_1\Gamma_2\cos 2(\gamma_1-\gamma_2)$$

in powers of p', q' are $O(\varphi)$.

But by (3.4.16) $\varphi = O(\alpha^{5/4})$. Since according to (3.4.12) $\nu_{ij} = O(\alpha)$ and according to (3.4.13) $\kappa_{ij} = O(\alpha^{9/4})$, the coefficients of the expansion (3.4.17) are

$$\nu_{ij} = \nu_{ij} + O(\alpha)\,O(\varphi^2) + O(\alpha^{9/4})\,O(\varphi) = \nu_{ij} + O(\alpha^{7/2})$$

and in consequence of (3.4.12) the ν'_{ij} behave asymptotically like (3.4.3).

6. Two lemmas. We shall use the notation

$$\left.\begin{aligned}\xi_k &= \sqrt{2\Gamma_k}\cos\gamma_k, & p_k &= \xi_k + i\eta_k,\\ \eta_k &= \sqrt{2\Gamma_k}\sin\gamma_k, & q_k &= \xi_k - i\eta_k\end{aligned}\right\}\quad (k=1,\ 2). \tag{3.4.22}$$

LEMMA 1. *The following identities are true:*

$$2\Gamma_k = \xi_k^2 + \eta_k^2 = p_k q_k, \tag{3.4.23}$$

$$4\sqrt{\Gamma_1\Gamma_2}\cos(\gamma_1 - \gamma_2) = 2(\xi_1\xi_2 + \eta_1\eta_2) = p_1 q_2 + p_2 q_1, \tag{3.4.24}$$

$$8\Gamma_1\Gamma_2\cos^2(\gamma_1-\gamma_2) = 2[(\xi_1^2-\eta_1^2)(\xi_2^2-\eta_2^2) + 4\xi_1\xi_2\eta_1\eta_2] = p_1^2q_2^2 + p_2^2q_1^2. \tag{3.4.25}$$

The proof is obvious. By means of Lemma 1 it is easy to prove

LEMMA 2. *Using the notation of* (3.4.22) *the formulae*

$$\left.\begin{aligned}x_1 &= x_1'\cos\varphi + x_2'\sin\varphi,\\ x_2 &= -x_1'\sin\varphi + x_2'\cos\varphi,\end{aligned}\right\} \tag{3.4.26}$$

where x_k denotes ξ_k and η_k or p_k and q_k, define a canonical transformation $\xi,\eta \to \xi',\eta'$. If φ satisfies the relationship

$$\nu' = (\nu_1-\nu_2)\sin 2\varphi + \nu\cos 2\varphi = 0, \tag{3.4.27}$$

the function $H_2 = \nu_1\Gamma_1 + \nu_2\Gamma_2 + \nu\sqrt{\Gamma_1\Gamma_2}\cos(\gamma_1-\gamma_2)$ is expressed in terms of the new variables x' by the formula

$$H_2 = \nu_1'\Gamma_1' + \nu_2'\Gamma_2' \quad (2\Gamma_k' = \xi_k'^2 + \eta_k'^2 = p_k'q_k'), \tag{3.4.28}$$

where

$$\nu_1' = \nu_1 - \frac{\nu}{2}\sin 2\varphi - (\nu_1-\nu_2)\sin^2\varphi,\ \ \nu_2' = \nu_2 + \frac{\nu}{2}\sin 2\varphi + (\nu_1-\nu_2)\sin^2\varphi. \tag{3.4.29}$$

7. The asymptotic behaviour of the determinants (3.3.10), (3.3.12). In consequence of (3.4.1) and (3.3.8) the coefficients $v' = \bar{v}$ are of the form

$$\bar{v}_0 = -\beta_1\beta_2 a_2^{-1}\nu_0',\quad \bar{v}_k = -\beta_1\beta_2 a_2^{-3/2}\nu_k',\quad \bar{v}_{kl} = -\beta_1\beta_2 a_2^{-2}\nu_{kl}' \tag{3.4.30}$$

From (3.4.3) it therefore follows that

$$\begin{vmatrix}\bar{v}_{11} & \bar{v}_{12}\\ \bar{v}_{21} & \bar{v}_{22}\end{vmatrix} = a_2^{-4}\beta_1^2\beta_2^2\begin{vmatrix}\nu_{11}' & \nu_{12}'\\ \nu_{21}' & \nu_{22}'\end{vmatrix} =$$

$$= a_2^{-4}\beta_1^2\beta_2^2\begin{vmatrix}-\frac{3}{8\beta_1^2}\alpha + O(\alpha^{5/2}) & \frac{9}{8\beta_1\beta_2}\alpha^{3/2} + O(\alpha^{5/2})\\ \frac{9}{8\beta_1\beta_2}\alpha^{3/2} + O(\alpha^{5/2}) & \frac{3}{2\beta_2^2}\alpha^2 + O(\alpha^{5/2})\end{vmatrix} =$$

$$= -\frac{117}{64}\frac{\alpha^3}{a_2^4} + O(\alpha^{7/2}) \neq 0. \tag{3.4.31}$$

We shall show further that

$$\frac{\partial(v_1', v_2')}{\partial(\Lambda_1, \Lambda_1)} = \frac{\partial(v_1', v_2')}{\partial(\alpha, a_2)} \Big/ \frac{\partial(\Lambda_1, \Lambda_2)}{\partial(\alpha, a_2)} = \frac{27}{16}\frac{\alpha^3}{a_2^4} + O(\alpha^{9/2}) \not\equiv 0. \quad (3.4.32)$$

In accordance with (3.2.18) for $a_1 = \alpha a_2$

$$\Lambda_1 = \beta_1 \sqrt{a_1} = \beta_1 \sqrt{\alpha a_2}, \qquad \Lambda_2 = \beta_2 \sqrt{a_2}. \quad (3.4.33)$$

Therefore

$$\frac{\partial(\Lambda_1, \Lambda_2)}{\partial(\alpha, a_2)} = \frac{\beta_1\beta_2}{4\sqrt{\alpha}}. \quad (3.4.34)$$

From (3.4.30) and (3.4.3) it follows that

$$\frac{\partial(v_1', v_2')}{\partial(\alpha, a_2)} = \beta_1^2\beta_2^2 \begin{vmatrix} \frac{9}{8\beta_1}\frac{\alpha^{1/2}}{a_2^{3/2}} + O(\alpha^{5/2}) & -\frac{9}{8\beta_1}\frac{\alpha^{3/2}}{a_2^{5/2}} + O(\alpha^{7/2}) \\ \frac{3}{2\beta_2}\frac{\alpha}{a_2^{3/2}} + O(\alpha^{5/2}) & -\frac{9}{8\beta_2}\frac{\alpha^2}{a_2^{5/2}} + O(\alpha^{7/2}) \end{vmatrix} =$$

$$= \frac{27}{64}\frac{\beta_1\beta_2\alpha^{5/2}}{a_2^4} + O(\alpha^4). \quad (3.4.35)$$

On comparing (3.4.35) with (3.4.34) we obtain (3.4.32).

§5. The many-body problem

The particular properties of the plane and space problems of three and many bodies connected with the conservation of angular momentum are discussed below.

1. The plane problem of $n_0 > 2$ planets. The arguments of §§2 and 3 easily carry over to the case of more than two planets. If the number of planets is denoted by n_0, the number of degrees of freedom is $n = 2n_0$ (the centre of gravity of all the bodies is regarded as fixed). The letters Λ, λ, Γ, γ, ξ, η in §§2, 3 denote n_0-dimensional vectors. The inequalities (3.3.10), (3.3.12) can be verified by the same method as in §4 for $n_0 = 2$. We shall not dwell on the details of the calculations which lead to the results of §1, 4.

2. The integral of the moment in the plane three-body problem. The arguments of §§2-4 made no use of the law of conservation of angular momentum which in the notation of §2 is of the form

$$\sum G_k = C.$$

This first integral of the n-body problem corresponds to the first integral of the "averaged system" of canonical equations with Hamiltonian F_1:

$$\sum \Gamma_k = C.$$

On account of the existence of this first integral the averaged system is integrable in the case $n_0 = 2$, and therefore by a suitable choice of variables p_1, q_1 it is possible to reduce $\bar{F}_1$ to the form

$$\bar{F}_1(\Lambda, \lambda; \xi, \eta) = \bar{\bar{F}}_1(p_0; p_1, q_1) \qquad (p_0 = \Lambda) \quad (3.5.1)$$

without any additional terms $\tilde{\bar{F}}_1$ (cf. Ch. IV, §1). The transition from ξ, η to p_1, q_1 is a canonical analytic transformation. For small $|\xi|$, $|\eta|$ the expressions for p_1, q_1 are given by Birkhoff's series (see Ch. I, §9). In this case the series converge, since the averaged system is integrable.

In accordance with (3.5.1) we cannot restrict ourselves to small $|\xi|$, $|\eta|$ and in place of the fundamental theorem of Ch. IV, §1, we can use the result of Ch. I, §8, (this result is simpler than the fundamental theorem and is easily deduced from it). We have to place a restriction on Λ, e in order to exclude the case of collision (cf. (3.1.5)):

$$\Gamma_1+\Gamma_2<\varepsilon_0(\Lambda_{1\max},\ \Lambda_{2\min}). \tag{3.5.2}$$

Condition (3.5.2) distinguishes the domain G in the space Γ, Λ in which the conditions of Ch. I, §8, are fulfilled; on using these conditions we arrive at the result of §1, 6.

3. Delaunay and Poincaré variables in the space problem. The space problem of n_0 planets has $3n_0$ degrees of freedom (as before, we regard the centre of gravity as fixed). Each planet has 6 elliptic elements (see §2, 2.): a, e, i; l, g, h. Here the angle h – the longitude of the node – determines the direction of the nodal line, i.e. the line of intersection of the plane of the Kepler ellipse with the $q^{(1)}$, $q^{(2)}$ plane, and the angle i – the inclination – is the angle between these planes. Finally g – the longitude of the perihelion – is the angle between the nodal lines and the direction of the major semi-axis. The Delaunay elements L, G, H, where $H = G\cos i$ is the projection of the angular momentum on the $q^{(3)}$ axis (as distinct from §2, where H denoted the Hamiltonian), correspond to the angles l, g, h.

In the space problem the angular momentum vector is conserved, giving three first integrals:

$$\sum G_k\sin i_k\cos h_k=\sum\sqrt{G_k^2-H_k^2}\cos h_k=C_1, \tag{3.5.3}$$

$$\sum G_k\sin i_k\sin h_k=\sum\sqrt{G_k^2-H_k^2}\sin h_k=C_2, \tag{3.5.4}$$

$$\sum G_k\cos i_k=\sum H_k=C_3. \tag{3.5.5}$$

The Poincaré elements Λ, Γ, Z; λ, γ, ζ and Λ, ξ, p; λ, η, q are determined by the formulae

$$\mu\Lambda=L,\quad \mu\Gamma=L-G,\quad \mu Z=G-H,\quad \xi=\sqrt{2\Gamma}\cos\gamma,\quad p=\sqrt{2Z}\cos\zeta,$$
$$\lambda=l+g+h,\quad \gamma=-g-h,\quad \zeta=-h,\quad \eta=\sqrt{2\Gamma}\sin\gamma,\quad q=\sqrt{2Z}\sin\zeta.$$

The Hamiltonian averaged with respect to λ

$$\bar{F}_1(\Lambda;\ \xi,\ \eta;\ p,q)=\bar{F}_{10}+\bar{F}_{12}+\bar{F}_{14}+\dots$$

is even with respect to ξ, η, p, q and has a position of equilibrium given by $\xi=\eta=p=q=0$. The quadratic terms are $\bar{F}_{12}=\bar{F}'_{12}(\xi,\eta)+\bar{F}''_{12}(p,q)$ (where $\bar{F}'_{12}(\xi,\eta)$ is $\bar{F}_{12}$ from (3.3.3)), the quadratic form $\bar{F}''_{12}$ reducing to

$$\bar{F}''_{12}=v''_1Z_1+\dots+v''_{n_0}Z_{n_0}\quad (v''_1\equiv 0,\ v''_2,\ \dots,\ v''_{n_0}>0). \tag{3.5.6}$$

The vanishing of the frequencies v_1'' is explained by the effect of the integrals (3.5.3), (3.5.4).

4. The elimination of the node in the three-body problem. As is known (see [1]), the integrals (3.5.3)-(3.5.5) allow the number of degrees of freedom of the system to be reduced by 2. In the case of three bodies it is possible, following Jacobi, to eliminate the variables H, h altogether.

We shall suppose that the coordinate plane $q^{(1)}$, $q^{(2)}$ is perpendicular to the angular momentum vector of the system. Then in (3.5.3)-(3.5.5) we have $C_1 = C_2 = 0$, $C_3 = C$ and

$$\left.\begin{aligned} G_1 \cos i_1 + G_2 \cos i_2 &= C, \\ G_1 \sin i_1 = -G_2 \sin i_2 &= \varepsilon, \\ h_1 - h_2 &= \pi, \end{aligned}\right\} \tag{3.5.7}$$

where C = const and ε is small together with the inclinations. From (3.5.7) it follows that

$$H_1 + H_2 = C, \qquad H_1^2 - H_2^2 = G_1^2 - G_2^2,$$

and therefore

$$H_1 = \frac{C}{2} + \frac{1}{2C}(G_1^2 - G_2^2), \qquad H_2 = \frac{C}{2} - \frac{1}{2C}(G_1^2 - G_2^2). \tag{3.5.8}$$

The coordinates h_1 and h_2 are obviously not contained in the Hamiltonian which can therefore be expressed in terms of the variables L, G; l, g of the plane problem (replacing H_1, H_2 by the expressions (3.5.8)). From the expansions in powers of ε^2:

$$C = G_1 + G_2 - \frac{1}{2}\varepsilon^2\left(\frac{1}{G_1} + \frac{1}{G_2}\right) + \ldots, \quad G_1 - H_1 = \frac{\varepsilon^2}{2G_1} + \ldots, \quad G_2 - H_2 = \frac{\varepsilon^2}{2G_2} + \ldots,$$

it is not difficult to deduce that the function of L, G; l, g obtained (still depending on the parameter C) differs from the Hamiltonian of the plane three-body problem by analytic terms that are small together with ε^2 (i.e. with $G_1 + G_2 - C$).

Thus, the space three-body problem reduces to a certain plane problem which turns into the plane three-body problem when the inclinations tend to 0. By comparison with 2. and using again Ch. I, §8, we arrive at the results of §1, 7.

5. The space problem of many bodies. In the case of more than three bodies there is no such elegant method of reducing the number of degrees of freedom. We shall note here a method that allows the elimination of one degree of freedom, namely the one that corresponds to the zero frequency v_1'' in (3.5.6).

As is known, the Poisson brackets of the components C_1, C_2, C_3 of the angular momentum vector are given by the formulae

$$-(C_1, C_2) = C_3, \qquad -(C_2, C_3) = C_1, \qquad -(C_3, C_1) = C_2.$$

The two functions Φ_1, Φ_2 of C_1, C_2, C_3 can be regarded as canonically conjugate variables if their Poisson bracket $(\Phi_1, \Phi_2) \equiv 1$. We seek functions Φ_1, Φ_2 of the form

$$\Phi_1 = y + \varphi_1(x, y, z), \qquad \Phi_2 = z + \varphi_2(x, y, z),$$

$$\varphi_1(x, y, z) = \varphi(x, y, z), \qquad \varphi_2(x, y, z) = \varphi(x, z, y),$$

where $-C_3 = 1 + x$, $-C_1 = y$, $-C_2 = z$.

For the function $\varphi(x, y, z)$ the following equation is obtained:

$$(\Phi_1, \Phi_2) = \begin{vmatrix} 1+x & y & z \\ \varphi_{1x} & 1+\varphi_{1y} & \varphi_{1z} \\ \varphi_{2x} & \varphi_{2y} & 1+\varphi_{2z} \end{vmatrix} \equiv 1. \qquad (3.5.9)$$

From (3.5.9) it is possible to find, for example, a symmetric solution $\varphi(x, y, z) = \varphi(x, z, y)$ on expanding φ as a series $\varphi = ax + by + cz + \ldots$

Let us now return to the problem of n_0 planets. In view of (3.5.9) it is possible to choose $3n_0$ pairs of canonical variables so that:

1) Λ_k and λ_k comprise n_0 pairs.

2) n_0 pairs correspond to the eccentricities (like G and g, Γ and γ, ξ and η).

3) n_0 pairs correspond to the inclinations (like H and h, Z and ζ, p and q).

4) Φ_1 and Φ_2 comprise one of these latter pairs.

Since Φ_1 and Φ_2 are first integrals, they do not in any way enter into the Hamiltonian, and we obtain a system with $3n_0 - 1$ degrees of freedom. In the expansion corresponding to (3.5.6) there will therefore be $n_0 - 1$ frequencies and $n_0 - 1$ pairs of variables. The method of §3 enables us to reduce the Hamiltonian to the form (4.1.1) (with $n = 3n_0 - 1$, $n_1 = 2n_0 - 1$, where n_0 is the number of planets) and obtain from the fundamental theorem the results of §1, 5.

The rather lengthy calculations involved in the solution of (3.5.9), the construction of variables satisfying conditions 1)-4), and the verification of non-degeneracy conditions analogous to the arguments of §4 will not be discussed here.

Chapter IV

THE FUNDAMENTAL THEOREM

In this chapter we give the precise formulation and complete proof of the theorem on which the arguments of Chapters II and III are based. In the choice of formulation (§1) we were concerned more by the convenience of its application to celestial mechanics than by the generality and finality of the result.

The proof (§§9-14) is rather cumbersome. Apart from the ideas enumerated in Chapter I it is based on a large number of almost trivial inequalities. The most important part of the present chapter is §6, where the fundamental lemma formulated in §4 is proved. The list of notation given at the end of Chapter V will be of help to the reader.

Our inequalities are based on the lemmas of Chapter V. In deriving these inequalities no attempt has been made to achieve elegance or precision in evaluating constants. We have frequently imposed more stringent conditions than necessary (for example, $\mu < \varepsilon^4$ in §1). The reader can thus easily strengthen the results.

§1. Fundamental theorem

Let us consider a function H, a domain G_0 and positive numbers ρ, R, C. We suppose that the following four conditions are fulfilled:

1) *The function $H(p, q)$ (where $p = p_0, p_1$; $q = q_0, q_1$; p_0 is a vector of dimension n_0 and p_1 = a vector of dimension n_1, where $n_0 + n_1 = n$; q_0 are angular variables, $H(p_0, p_1, q_0 + 2\pi, q_1) = H(p_0, p_1, q_0, q_1)$) is analytic in the domain $F: P_0 \in G_0$, $|\mathrm{Im}\ q_0| \leqslant \rho$, $|x_1| \leqslant R$ ($x_1 = p_1, q_1$) and depends on the parameter μ, $0 < \mu \leqslant \mu_0$.*

2) *H is of the form*

$$H = H_0(p_0) + \mu H_1(p, q) + (\mu^2) H_2(p, q), \tag{4.1.1}$$

where

$$H_1(p, q) = \overline{H}_1(p_0, p_1, q_1) + \widetilde{H}_1(p_0, q_0, p_1, q_1), \quad \int \widetilde{H}_1 dq_0 = 0, \tag{4.1.2}$$

with

$$\overline{H}_1(p_0, p_1, q_1) = \overline{\overline{H}}_1(p_0, \tau) + \widetilde{\overline{H}}_1(p_0, p_1, q_1) \tag{4.1.3}$$

and

$$\overline{\overline{H}}_1(p_0, \tau) = \lambda_0 + \sum_{i=1}^{n_1} \lambda_i \tau_i + \sum_{i,j=1}^{n_1} \lambda_{ij} \tau_i \tau_j + \sum_{i,j,k=1}^{n_1} \lambda_{ijk} \tau_i \tau_j \tau_k, \tag{4.1.4}$$

where λ_0, λ_i, $\lambda_{ij} = \lambda_{ji}$ and λ_{ijk} are functions of p_0 and

$$2\tau_i = p_{n_0+i}^2 + q_{n_0+i}^2 \qquad (i = 1, \ldots, n_1). \tag{4.1.5}$$

3) *In F the following inequalities are satisfied (for a certain $C \geqslant 1$):*

$$|(\mu^2) H_2| \leqslant \mu^2 C, \tag{4.1.6}$$

$$|\widetilde{H}_1| \leqslant C, \quad |H_1| \leqslant C, \quad |\overline{H}_1| \leqslant C, \tag{4.1.7}$$

$$|\widetilde{\overline{H}}_1| \leqslant C |x_1|^7. \tag{4.1.8}$$

4) *In G_0*

$$\det \left| \frac{\partial^2 H_0}{\partial p_0^2} \right| \not\equiv 0, \tag{4.1.9}$$

$$\det |\lambda_{ij}(p_0)| \not\equiv 0. \tag{4.1.10}$$

On the assumptions 1) - 4), for any $\varkappa > 0$ it is possible to find $\varepsilon(\varkappa; H_0, \overline{\overline{H}}_1, G_0; \rho, R, C; \mu_0) > 0$ such that, when $0 < \varepsilon < \varepsilon_0$ and $0 < \mu < \varepsilon^4$, then:

I. *The domain* Re F_ε;

$$p_0 \in \mathrm{Re}\, G_0, \quad |\mathrm{Im}\, q_0| = 0, \quad 0 < \tau_i < \varepsilon,$$

consists of two sets $\mathbf{F}_\varepsilon$ and f_ε, of which one, $\mathbf{F}_\varepsilon$, is invariant with respect to the canonical equations with Hamiltonian (4.1.1) *and the other, f_ε, is small:*

$$\mathrm{mes}\, f_\varepsilon < \varkappa\, \mathrm{mes}\, F_\varepsilon. \tag{4.1.11}$$

II. *$\mathbf{F}_\varepsilon$ consists of invariant n-dimensional analytic tori T_ω, given by the parametric equations*

$$\left.\begin{aligned} p_0 &= p_{0_\omega} + f_{0_\omega}(Q), \quad q_0 = Q_0 + g_{0_\omega}(Q), \\ p_1 &= \sqrt{2(\tau_\omega + f_{1_\omega}(Q))}\cos[Q_1 + g_{1_\omega}(Q)], \\ q_1 &= \sqrt{2(\tau_\varepsilon + f_{1_\omega}(Q))}\sin[Q_1 + g_{1_\omega}(Q)], \end{aligned}\right\} \quad (4.1.12)$$

where $Q = Q_0, Q_1$ *are angular parameters and* p_{0_ω} *and* τ_ω *are constants depending on the number of the torus* ω.

III. *The invariant tori* T_ω *differ little from the tori*

$$p_0 = p_{0_\omega} = \text{const}, \quad \tau = \tau_\omega = \text{const}:$$

$$|f_{i\omega}(Q)| < \varkappa\varepsilon, \quad |g_{i_\omega}(Q)| < \varkappa\varepsilon. \quad (4.1.13)$$

IV. *The motion determined by the Hamiltonian* (4.1.1) *on the torus* T_ω *is conditionally periodic with n frequencies* ω:

$$\dot{Q}_0 = \omega_0, \quad \dot{Q}_1 = \omega_1 \quad \left(\text{where } \omega_0 = \frac{\partial H_0}{\partial p_{0_\omega}}, \quad \omega_1 = \mu\frac{\partial \bar{\bar{H}}_1}{\partial \tau_\omega}\right). \quad (4.1.14)$$

The fundamental theorem is proved in §§9-14 on the basis of an inductive process given by the following theorem.

§2. The inductive theorem

Suppose that the function $H(p, q)$, *the domains* G, Ξ, *and the positive numbers* D, Θ, θ, ρ, $\varkappa$; β, γ, δ, ε, μ, M, K *have the following properties*:

1) *In the domain* $F: p \in G$, $|\text{Im } q| \leqslant \rho$ *the function*

$$H(p, q) = H_0(p_0) + H_1(p) + H_2(p, q) \quad (4.2.1)$$

(*where* $p = p_0, p_1$; $q = q_0, q_1$ *are angular variables*, $\dim p_0 = n_0$, $\dim p_1 = n_1$, $n_0 + n_1 = n$) *is analytic*.

2) *The mapping* A *of* G *onto* Ξ: .

$$p \to Ap = \frac{\partial}{\partial p_0}(H_0 + H_1), \quad \frac{1}{\mu}\frac{\partial H_1}{\partial p_1}, \quad (4.2.2)$$

is diffeomorphic, where

$$\theta|dp| \leqslant |dA| \leqslant \Theta|dp|, \ \left|\frac{\partial^2 H_0}{\partial p_0^2}\right| \leqslant \Theta, \ \left|\frac{\partial^2 H_1}{\partial p^2}\right| \leqslant \mu\Theta \quad (0 \leqslant \theta \leqslant 1 \leqslant \Theta < \infty). \quad (4.2.3)$$

3) *In* F

$$|H_2(p, q)| \leqslant \mu M. \quad (4.2.4)$$

4) *The following inequalities are satisfied*:

$$\delta \leqslant \delta^{(4)}(n, \Theta, \theta, \varrho, \varkappa, D) =$$

$$= \min\left\{\delta^{(1)}\left(n, 2\Theta, \frac{\theta}{2}\right);\ \delta^{(2)}(\varrho, n);\ \delta^{(3)}(n, \Theta, \theta, D, \varkappa)\right\}, \quad (4.2.5)$$

$$\mu \leqslant K, \quad (4.2.6)$$

where $\delta^{(1)}$ *is defined in* §3,

$$\delta^{(2)} = \min\{10^{-4n}\varrho^{4n},\ 2^{-58n}\},$$

$$\delta^{(3)} = \min\{e^{2n}[2^8 n(n+4)]^{-2n};\ (6 + 14\Theta)^{-1};\ 4^{-n-3}\theta^n\Theta^{-n}D^{-1}n^{-1}\varkappa\}.$$

5) *Let*

$$\beta=\delta^3\varepsilon,\quad \gamma=\delta^{\frac{1}{4n}},\quad \varepsilon=\delta^T,\quad K=\delta\varepsilon,\quad M=\varepsilon^{7/2}\delta^{-1},\quad T=16(n+4). \quad (4.2.7)$$

6) *In the notation of* Ch. V, §2, 5., Ξ *is of the form*

$$\Xi=\Xi^*{}_{K_0N_0d_0},\quad N_0<\frac{1}{\gamma}\ln\frac{1}{2M},\quad K_0\geqslant K, \qquad (4.2.8)$$

where Ξ^* *is a domain of type* $\frac{D}{\varepsilon}$.

We introduce the numbers $\delta_s (s\geqslant 1)$ *by means of the relations* $\delta_1=\delta$, $\delta_{s+1}=\delta_s^{15/14}$. *For* $s\geqslant 1$ *we put*

$$\beta_s=\delta_s^3\varepsilon_s,\quad \gamma_s=\delta_s^{\frac{1}{4n}},\quad \varepsilon_s=\delta_s^T,\quad M_s=\varepsilon_s^{7/2}\delta_s^{-1}\ (\textit{then}\quad M_{s+1}=M_s^{15/14}). \quad (4.2.9)$$

On the assumptions 1) - 6) *there exist a sequence of domains* $F^{(0)}=F$, $F^{(1)}\ldots$ *of the form* $P^{(s)}$, $Q^{(s)}=X^{(s)}\in F^{(s)}$: $P^{(s)}\in G^{(s)}$, $|\operatorname{Im} Q^{(s)}|\leqslant\rho_s$ *and a sequence of canonical diffeomorphisms* $B_s: P^{(s)}, Q^{(s)}\to P^{(s-1)}, Q^{(s-1)}$ *of the domains* $F^{(s)}$ *into* $F^{(s-1)}$ *such that*:

I. *For all* $s\geqslant 1$

$$|B_s-E|<\beta_s,\ |dB_s|<2\,|dX^{(s)}|,\ F^{(s)}\subset F^{(s-1)}-2\beta_s,\qquad \rho_s>\frac{\rho}{3}\cdot \quad (4.2.10)$$

II. *For* $p, q=B_1B_2\ldots B_s\,(P^{(s)}, Q^{(s)})$ *and* $P^{(s)}, Q^{(s)}=X^{(s)}\in F^{(s)}$, $P^{(s)}=P_0^{(s)}, P_1^{(s)}$, *we have*

$$H(p,q)=H^{(s)}(P^{(s)},Q^{(s)})=H_0(P^{(s)})+H_1^{(s)}(P^{(s)})+H_2^{(s)}(P^{(s)},Q^{(s)}), \quad (4.2.11)$$

$$|H_2^{(s)}(P^{(s)},Q^{(s)})|<\mu M_{s+1},\quad \left|\frac{\partial H_2^{(s)}}{\partial X^{(s)}}\right|<\delta_s\beta_{s+1},\quad \left|\frac{\partial^2 H_2^{(s)}}{\partial X^{(s)2}}\right|<\delta_s. \quad (4.2.12)$$

III. *The mapping* $A^{(s)}$

$$P^{(s)}\longrightarrow A^{(s)}P^{(s)}=\frac{\partial}{\partial P_0^{(s)}}[H_0(P_0^{(s)})+H_1^{(s)}(P^{(s)})],\ \frac{1}{\mu}\frac{\partial}{\partial P_1^{(s)}}H_1^{(s)}(P^{(s)}) \quad (4.2.13)$$

is a diffeomorphism of the domain $G^{(s)}$ *for which*

$$\theta\,|dP^{(s)}|<|dA^{(s)}|<\bar\Theta\,|dP^{(s)}|,\qquad |A^{(s)}-A^{(s-1)}|<\beta_s\delta_s,\ \theta=\frac{1}{2}\theta,\ \bar\Theta=2\Theta. \quad (4.2.14)$$

IV. *For each* $s\geqslant 1$ *we have*

$$\operatorname{mes}(G-G^{(s)})\leqslant\frac{\varkappa}{2}\Theta^{-n}\operatorname{mes}\Xi^*. \quad (4.2.15)$$

The inductive theorem is proved by induction in §8.
Each step in the proof is based on the use of the following lemma.

§3. The inductive lemma

Suppose that the function $H(p, q)$, *the domains* G, Ξ, *and the positive numbers* Θ, θ, ρ; β, γ, δ, M; K, μ *have the following properties*:

1) *In the domain* $F: p\in G$, $|\operatorname{Im} q|\leqslant\rho$ *the function*

$$H(p,q)=H_0(p_0)+H_1(p_1)+H_2(p,q) \qquad (4.3.1)$$

(*where* $p=p_0, p_1$; $q=q_0, q_1$ *are angular variables*, $\dim p_0=n_0$, dim $p_1 = n_1$: $n_0 + n_1 = n$) *is analytic.*

2) *The mapping A of G onto* Ξ:

$$p \to Ap = \frac{\partial}{\partial p_0}(H_0 + H_1), \qquad \frac{1}{\mu}\frac{\partial H_1}{\partial p_1}, \tag{4.3.2}$$

is diffeomorphic, where

$$\theta\, dp| \leqslant |dA| \leqslant \Theta |dp|, \quad \left|\frac{\partial^2 H_0}{\partial p_0^2}\right| \leqslant \Theta, \quad \left|\frac{\partial^2 H_1}{\partial p^2}\right| \leqslant \mu\Theta \qquad (0 < 0 \leqslant 1 \leqslant \Theta < \infty). \tag{4.3.3}$$

3) *In F*

$$|H_2(p, q)| \leqslant \mu M, \qquad M \leqslant \delta^{2n+3} K\beta^2. \tag{4.3.4}$$

4) *The following inequalities are satisfied:*

$$15\delta \leqslant 3\gamma \leqslant \varrho \leqslant 1, \quad 3\beta \leqslant 2\delta, \quad \mu \leqslant K \leqslant 1, \tag{4.3.5}$$

$$\delta \leqslant \delta^{(1)}(n, \Theta, \theta) = \min\left\{\delta^{(0)}(n, \Theta); \frac{\theta}{2n}\right\}, \tag{4.3.6}$$

where $\delta^{(0)}(n, \Theta)$ *is defined in the fundamental lemma* (§4).

On the assumptions 1) - 4) *there exist a domain* $F'(X = (P, Q) \in F'$: $P \in G' \subset G$, $|\mathrm{Im}\, Q| \leqslant \rho' < \rho - 3\gamma$) *and a canonical diffeomorphism* $B: P, Q \to p, q$ *of* F' *into* F *such that:*

I. $|B - E| < \beta$, $|dB| < 2|dX|$. $F' \subset F - 2\beta$.

II. *For* $(p, q) = x = BX$; $X = (P, Q \in F')$; $P = (P_0, P_1)$ *we have*

$$H(p, q) = H_0(P_0) + H_1'(P) + H_2'(P, Q), \tag{4.3.7}$$

$$|H_2'(P, Q)| < \mu M', \qquad \left|\frac{\partial H_2'}{\partial X}\right| < \mu \frac{M'}{\beta}, \qquad \left|\frac{\partial^2 H_2'}{\partial X^2}\right| < 2\mu \frac{M'}{\beta^2}, \tag{4.3.8}$$

$$M' = \frac{M^2}{K\beta^2}\delta^{-\nu}, \qquad \nu = 4n + 7. \tag{4.3.9}$$

III. *The mapping* A'

$$P \to A'P = \frac{\partial}{\partial P_0}(H_0 + H_1'), \qquad \frac{1}{\mu}\frac{\partial H_1'}{\partial P_1} \tag{4.3.10}$$

is a diffeomorphism of G' *onto* Ξ', *for which*

$$\theta' |dP| < |dA'| < \Theta' dP|, \qquad \left|\frac{\partial^2 H_0}{\partial P_0^2}\right| < \Theta, \qquad \left|\frac{\partial^2 H_1'}{\partial P_1^2}\right| < \mu\Theta', \tag{4.3.11}$$

where $\theta' = (1 - \delta)\theta$, $\Theta' = (1 + \delta)\Theta$.

Here, in the notation of Ch. V, §2,

$$\Xi' = \Xi_{\mu K N} - d, \qquad d = (5 + 7\Theta)\beta, \qquad N = \frac{1}{\gamma}\ln\frac{1}{2M}. \tag{4.3.12}$$

IV. mes $(G - G') < \theta^{-n}$ mes$(\Xi - \bar{\Xi}')$, *where* $\bar{\Xi}' = \Xi' - \beta = \Xi_{\mu KN} - \bar{d}$ *and* $\bar{d} = (6 + 7\Theta)\beta$.

The inductive lemma is proved in §7. The key point in the proof is the use of the following lemma.

§4. Fundamental lemma

Suppose that the function $H(p, q)$, *the domains* G, Ξ, *and the positive* *numbers* θ, Θ, ρ; β, γ, δ; μ, M, K *have the following properties*:

1) *In the domain* $F: p \in G$, $|\mathrm{Im}\ q| \leqslant \rho$ ($p = p_0, p_1$; $q = q_0, q_1$ *are angular variables*, $\dim p_0 = n_0$, $\dim p_1 = n_1$, $n_0 + n_1 = n$) *the function*

$$H(p, q) = H_0(p_0) + H_1(p) + \widetilde{H}_2(p, q) \qquad \left(\oint \widetilde{H}_2\, dq_0 \equiv 0\right) \tag{4.4.1}$$

is analytic.

2) *The mapping* A *of* G *onto* Ξ:

$$p \to A_p = \frac{\partial}{\partial p_0}(H_0 + H_1), \qquad \frac{1}{\mu}\frac{\partial H_1}{\partial p_1}, \tag{4.4.2}$$

is diffeomorphic, where

$$\theta\,|dp| \leqslant |dA| \leqslant \Theta\,|dp|, \qquad \left|\frac{\partial^2 H_0}{\partial p_0^2}\right| \leqslant \Theta, \qquad \left|\frac{\partial^2 H_1}{\partial p^2}\right| \leqslant \mu\Theta \qquad (0 < \theta \leqslant 1 \leqslant \Theta < \infty). \tag{4.4.3}$$

3) *In* F

$$|\widetilde{H}_2(p, q)| \leqslant \mu M, \qquad M \leqslant \delta^{\nu} K \beta^2 \qquad (\nu = 2n + 3). \tag{4.4.4}$$

4) *The following inequalities are satisfied*

$$10\delta \leqslant 2\gamma \leqslant \varrho \leqslant 1, \qquad 3\beta \leqslant 2\delta, \qquad \mu \leqslant K \leqslant 1, \tag{4.4.5}$$

$$\delta \leqslant \delta^{(0)}(n, \Theta) = 4^{-2n} n^{-1} (n+1)^{-(2n+2)} e^{2n+2} \Theta^{-1}. \tag{4.4.6}$$

We put $N = \frac{1}{\gamma} \ln \frac{1}{M}$ *and* $G_{\mu KN} = A^{-1} \Xi_{\mu KN}$, *where* $\Xi_{\mu KN}$ *consists of those* $\xi \in \Xi$, $\xi = \xi_0, \xi_1$, *for which*

$$|(\xi_0, k_0) + \mu(\xi_1, k_1)| \geqslant \begin{cases} K|k|^{-(n+1)}, & if \quad k_0 \neq 0, \\ \mu K |k|^{-(n+1)}, & if \quad k \neq 0, \end{cases} \tag{4.4.7}$$

for all integral vectors k, $0 < |k| < N$.

On the assumptions 1) - 4) *there exists a diffeomorphism* B: $P, Q \to p, q$ *of the domain* $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\ Q| \leqslant \rho - 2\gamma$ *into* F, *where*:

I. $|B - E| < \beta$, $|dB| < 2|dX|$ $(X = P, Q)$.

II. *For* $(p, q) = x = BX$ *and* $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\ Q| \leqslant \rho - 2\gamma$,

$$H(p, q) = H_0(P_0) + H_1(P) + H_2'(P, Q), \tag{4.4.8}$$

where

$$|H_2'| < \mu M', \qquad M' = \frac{M^2}{K\beta}\delta^{-2\nu} \qquad (2\nu = 4n + 6). \tag{4.4.9}$$

The first step (see §10) of the proof of the fundamental theorem is the use of the following lemma.

§5. Lemma on averaging over rapid variables

Suppose that the function $H(p, q)$, *the domains* G_0, F_1, Ω, *and the positive numbers* θ, Θ, ρ; β, γ, δ; K, M, $\overline{M}$ *have the following properties*:

1) *In the domain* F: $p_0 \in G_0$, $|\mathrm{Im}\ q_0| \leqslant \rho$, $x_1 \in F_1$ ($p = p_0, p_1$;

$q = q_0, q_1$; $x_1 = p_1, q_1$; the variables q_0 angular[1]; $\dim p_0 = n_0$, $\dim p_1 = n_1$, $n_0 + n_1 = n$), *the function*

$$H(p, q) = H_0(p_0) + \overline{H}_1(p_0, p_1, q_1) + \widetilde{H}_1(p_0, p_1, q_0, q_1) \qquad \left(\oint \widetilde{H}_1 \, dq_0 = 0\right) \tag{4.5.1}$$

is analytic.

2) *The mapping A_0 of G_0 onto Ω:*

$$p_0 \to A_0 p_0 = \omega_0 = \frac{\partial H_0}{\partial p_0}, \tag{4.5.2}$$

is diffeomorphic, where

$$\theta \, |dp_0| \leqslant |dA_0| \leqslant \Theta \, |dp_0|, \qquad \left|\frac{\partial^2 H_0}{\partial p_0^2}\right| \leqslant \Theta \qquad (0 < \theta \leqslant 1 \leqslant \Theta < \infty). \tag{4.5.3}$$

3) *In F*

$$|\overline{H}_1| \leqslant M, \qquad |\widetilde{H}_1| \leqslant M, \qquad M \leqslant \delta^\nu \beta^2 \qquad (\nu = 2n + 3). \tag{4.5.4}$$

4) *The following inequalities are valid:*

$$10\delta \leqslant 2\gamma \leqslant \varrho \leqslant 1, \qquad 3\beta \leqslant \delta, \qquad \delta \leqslant K, \qquad M \leqslant \overline{M}, \tag{4.5.5}$$

$$\delta \leqslant \delta^{(0)}(n, \Theta) = 4^{-2n} n^{-2} (n+1)^{-(2n+2)} e^{2n+2} \Theta^{-1} \tag{4.5.6}$$

We put $N = \frac{1}{\gamma} \ln \frac{1}{M}$ and introduce the notation $(G_0)_{KN} = A_0^{-1} \Omega_{KN}$, where Ω_{KN} consists of those $\omega_0 \in \Omega$ for which

$$|(\omega_0, k_0)| \geqslant K \, |k_0|^{-(n+1)} \tag{4.5.7}$$

for all integral vectors k_0, $0 < |k_0| < N$.

On the assumptions 1) - 4) *there exists a diffeomorphism $B : P, Q \to p, q$ of the domain F':*

$$P_0 \in G_0', \qquad |\operatorname{Im} Q_0| \leqslant \varrho - 2\gamma, \qquad X_1 \in F_1 - 3\beta \qquad (X_1 = P_1, Q_1),$$

into F, where:

I. $|B - E| < \dfrac{M}{K \beta \delta^{2n+2}}, \qquad |dB| < 2 \, |dX| \qquad (X = P, Q)$

II. *For $p, q = x = BX$, where $X \in F'$,*

$$H(p, q) = H_0(P_0) + \overline{H}_1(P_0, P_1, Q_1) + H_2'(P_0, P_1, Q_0, Q_1), \tag{4.5.8}$$

where

$$|H_2'| < M', \qquad M' = \frac{M \overline{M}}{\beta^2} \delta^{-2\nu}, \qquad 2\nu = 4n + 6. \tag{4.5.9}$$

III. $G_0' = (G_0)_{KN} - 2\beta = A_0^{-1} \Omega_{KN} - 2\beta$.

The proof of this lemma in analogous to that of the fundamental lemma (see §6, 8.).

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1 In this lemma, in contrast to all the others, it is not assured that q_1 is an angular variable.

§6. Proof of the fundamental lemma

1. The construction. The canonical transformation

$$p = P + \frac{\partial S}{\partial q}, \quad Q = q + \frac{\partial S}{\partial P} \tag{4.6.1}$$

with generating function $Pq + S(P, q)$, $S = \sum_{|k|>0} S_k(P) e^{i(k,q)}$, reduces $H(p, q)$ to the form

$$H(p, q) = H_0(P_0) + H_1(P) + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5'(P, Q), \tag{4.6.2}$$

where

$$\Sigma_1 = \left[\left(\omega(P) \frac{\partial S}{\partial q} \right) + [\widetilde{H}_2(P, q)]_N \right],$$
$$\Sigma_2 = \left[H_0(P_0 + \Delta P_0) - H_0(P_0) - \left(\frac{\partial H_0}{\partial P_0} \right) \Delta P_0 \right],$$
$$\Sigma_3 = \left[H_1(P + \Delta P) - H_1(P) - \left(\frac{\partial H_1}{\partial P} \Delta P \right) \right],$$
$$\Sigma_4 = [\widetilde{H}_2(P + \Delta P, q) - \widetilde{H}_2(P, q)],$$
$$\Sigma_5 = [\widetilde{H}_2(P, q) - [\widetilde{H}_2(P, q)]_N]$$

and

$$\widetilde{H}_2(P, q) = \sum_{|k|>0} h_k(P) e^{i(k,q)}, \qquad [\widetilde{H}_2(P, q)]_N = \sum_{0<|k|<N} h_k(P) e^{i(k,q)},$$
$$\omega(P) = \frac{\partial}{\partial P} [H_0(P_0) + H_1(P)],$$
$$\Delta P = p - P = \frac{\partial S}{\partial q}, \qquad \Delta P_0 = p_0 - P_0 = \frac{\partial S}{\partial q_0},$$

the variables p, q in (4.6.2) being replaced, after all the differentiations, by their expressions in terms of P, Q from (4.6.1).

We define S so that $\Sigma_1 \equiv 0$. For this we put

$$S_k(P) = \frac{ih_k(P)}{(k, \omega(P))} \quad (0 < |k| < N), \qquad S_k = 0 \quad (|k| \geqslant N). \tag{4.6.3}$$

2. Estimate for S. For $P \in G_{\mu KN}$ we have

$$|(\omega(P), k)| = |(\xi_0, k_0) + \mu(\xi_1, k_1)| \geqslant \begin{cases} K|k|^{-(n+1)}, & \text{if } k_0 \neq 0, \\ \mu K|k|^{-(n+1)}, & \text{if } k \neq 0, \end{cases} \tag{4.6.4}$$

where $\xi = \xi_0, \xi_1$ is the frequency vector introduced, connected with the frequency vector $\omega = \omega_0, \omega_1$ by the relationships $\omega_0 = \xi_0$, $\omega_1 = \mu\xi_1$.

In accordance with 1) of Ch. V, §3, 2., it follows from condition 3) that $|h_k| \leqslant \mu M e^{-|k|\rho}$. By comparison with (4.6.3) and (4.6.4) we find, in view of (5.3.10), that

$$\left.\begin{aligned} |S_k| &< \frac{\mu M}{\mu K} \frac{L_1}{\delta^{\nu_1}} e^{-|k|(\rho-\delta)} \qquad (k \neq 0), \\ |S_k| &< \frac{\mu M}{K} \frac{L_1}{\delta^{\nu_1}} e^{-k(\rho-\delta)} \qquad (k_0 \neq 0), \end{aligned}\right\} \tag{4.6.5}$$

where $L_1 = \left(\frac{\nu_1}{e} \right)^{\nu_1}$, $\nu_1 = n + 1$.

Therefore, in accordance with 2) of Ch. V, §3, 2. for $|\mathrm{Im}\, q| \leqslant \rho - 2\delta$ we have

$$|S| < \frac{M}{K}\frac{L_2}{\delta^{\nu_2}} \qquad (L_2 = 4^n L_1,\ \nu_2 = 2n+1). \tag{4.6.6}$$

3. Estimate for B. In accordance with condition 3) it follows from (4.6.6) that

$$|S| < \frac{\beta^2}{16n} \qquad \text{for } \delta \leqslant L_3^{-1}, \qquad L_3 = 16nL_2. \tag{4.6.7}$$

Therefore the lemma of Ch. V, §4, 3. on the canonical transformation is applicable. According to this lemma the equations (4.6.1) define a canonical diffeomorphism B: $P, Q \to p, q$, mapping the domain $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 5\delta$ ($\leqslant \rho - 2\delta - 3\beta$, since $3\beta \leqslant 2\delta$ in accordance with (4.4.5)) into F, where, in view of (4.6.7) and (4.6.5),

$$\left.\begin{aligned} &|B - E| < \beta, \quad |dB| < 2|dX|, \quad |\Delta P| < \frac{M}{K}\frac{L_2}{\delta^{\nu_2+1}}, \\ &|\Delta P_0| < \mu \frac{M}{K}\frac{L_2}{\delta^{\nu_2+1}}. \end{aligned}\right\} \tag{4.6.8}$$

4. Estimates for Σ_2, Σ_3. We shall apply Taylor's formula (5.3.14). If $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 5\delta$, then for $|\lambda| \leqslant 1$ we shall have, in view of (4.6.8), $P + \lambda\Delta P \in G_{\mu KN} - \beta$ and it therefore follows from (4.6.8), condition 2) and $\delta \leqslant L_4^{-1}\Theta^{-1}$ that

$$|\Sigma_2| < \mu^2 \frac{M^2}{K^2}\frac{\Theta n^2 L_2^2}{2\delta^{2\nu_2+2}} < \mu^2 \frac{M^2}{K^2}\frac{1}{\delta^{\nu_3}} \qquad (\nu_3 = 2\nu_2 + 3,\ L_4 = n^2 L_2^2). \tag{4.6.9}$$

By the same reasoning and in the same domain

$$|\Sigma_3| < \frac{M^2}{K^2}\mu \frac{\Theta n^2 L_2^2}{2\delta^{2\nu_2+2}} < \mu \frac{M^2}{K^2}\frac{1}{\delta^{\nu_3}}. \tag{4.6.10}$$

5. Estimate for Σ_4. We use formula (5.3.4). For $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 5\delta$, $|\lambda| < 1$, we have $P + \lambda\Delta P \in G_{\mu KN} - \beta$ and by Cauchy's formula (5.3.3) $\left|\frac{\partial \tilde{H}}{\partial P}\right| < \frac{\mu M}{\beta}$. Therefore, for $\delta \leqslant L_3^{-1} < L_2^{-1} n^{-1}$

$$|\Sigma_4| < \mu \frac{M^2}{K\beta}\frac{L_2 n}{\delta^{\nu_2+1}} \leqslant \mu \frac{M^2}{K\beta}\frac{1}{\delta^{\nu_4}} \qquad (\nu_4 = \nu_2 + 2). \tag{4.6.11}$$

6. Estimate for Σ_5. We shall apply 3) of Ch. V, §3, 2. For $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 5\delta - \gamma$, in accordance with (4.6.8) $P \in G$, $|\mathrm{Im}\, q| < \rho - \gamma - \delta$. For $N = \frac{1}{\gamma}\ln\frac{1}{M}$, $\delta \leqslant L_5^{-1}$, in view of (4.4.5) we find from 3) of Ch. V, §3, 2. that

$$|\Sigma_5| < \mu \frac{M^2 L_5}{\delta^{n+1}} < \mu \frac{M^2}{\delta^{\nu_5}} \qquad \left(\nu_5 = n + 2,\ L_5 = 2\left(\frac{2n}{e}\right)^n\right). \tag{4.6.12}$$

7. Estimate for H'. By condition 4) we have $\delta \leqslant \delta^{(0)}(n, \theta) = L_4^{-1}\Theta^{-1}$. It is easy to see that $L_3 \leqslant L_4$, $L_5 \leqslant L_4$. Since $\Theta > 1$, then for $\delta < \delta^{(0)}(n, \Theta)$ we shall have $\delta \leqslant L_3^{-1}$, $\delta \leqslant L_5^{-1}$ and the estimates (4.6.9)-(4.6.12) are valid for $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 2\gamma \leqslant \rho - \gamma - 5\delta$. In this domain (4.6.2) takes the form $H(p, q) = H_0(P_0) + H_1(P) + H_2'(P, Q)$, where, in view of 4.-6. and condition 4),

$$|H_2'(P, Q)| \leqslant \left(\mu^2 \frac{M^2}{K^2} + \mu \frac{M^2}{K^2} + \mu \frac{M^2}{K\beta} + \mu M^2\right) \delta^{-\nu_3} < \mu \frac{M^2}{K\beta} \delta^{-(4n+6)},$$

which proves the fundamental lemma.

8. **Proof of the lemma on averaging.** This proof is similar to that given above and we shall indicate only the necessary changes.

The generating function in 1. is taken to be of the form $P_0 q_0 + P_1 q_1 + S$, where

$$S(P_0, P_1, q_0, q_1) = \sum_{k_0 \neq 0} S_{k_0}(P_0, P_1, q_1) e^{i(k_0, q_0)},$$

and, in place of (4.6.3), we have

$$S_{k_0}(P_0, P_1, q_1) = \frac{ih_{k_0}(P_0, P_1, q_1)}{(\omega_0(P_0), k_0)} \qquad (0 < |k_0| < N).$$

In consequence of this Σ_3 is of the form

$$\Sigma_3 = \bar{H}_1(p_0, p_1, q_1) - \bar{H}_1(P_0, P_1, Q_1). \tag{4.6.13}$$

For $X \in F'$ in place of (4.6.8) we have

$$|\Delta P_0| < \frac{ML_2}{K\delta^{\nu_2+1}}, \quad |\Delta Q_0| < \frac{ML_2}{K\beta\delta^{\nu_2}}, \quad |\Delta X_1| < \frac{ML_2}{K\beta\delta^{\nu_2}} \qquad (\nu_2 = 2n+1). \tag{4.6.14}$$

The quantity $|\Sigma_3|$ is estimated by Lagrange's formula (5.3.4).

From (4.6.13) and (4.6.14) we obtain the estimate

$$|\Sigma_3| < \frac{M^2 2nL_2}{K\beta^2\delta^{\nu_2}} \leqslant \frac{M^2}{\beta^2\delta^{\nu_4}}. \tag{4.6.15}$$

Further, with $N = \frac{1}{\gamma} \ln \frac{1}{\bar{M}}$ the following estimate is obtained for Σ_5:

$$|\Sigma_5| \leqslant \frac{M\bar{M}}{\delta^{\nu_5}}. \tag{4.6.16}$$

Since $M \leqslant \bar{M}$, the estimates (4.6.15) and (4.6.16) reduce to

$$|H_2'| < \frac{M\bar{M}}{\beta^2} \delta^{-2\nu},$$

which proves the lemma of §5.

§7. Proof of the inductive lemma

1. We put $H_2(p, q) = \bar{H}_2(p) + \tilde{H}_2(p, q)$, where $\oint \tilde{H}_2(p, q)dq \equiv 0$. Then $H(p, q) = H_0(p_0) + H_1'(p) + \tilde{H}_2(p, q)$, where $H_1' = H_1(p) + \bar{H}_2(p)$. We consider the mapping A':

$$p \to A'p = \frac{\partial}{\partial p_0}(H_0 + H_1'), \quad \frac{1}{\mu}\frac{\partial H_1'}{\partial p_1} = Ap + \Delta(p); \quad \Delta = \frac{\partial \bar{H}_2}{\partial p_0}, \ \frac{1}{\mu}\frac{\partial \bar{H}_2}{\partial p_1}.$$

We apply the lemma on the variation of frequency (Ch. V, §4, 5.). From conditions 3), 4) it follows that $M < \frac{\delta\theta}{2n}\beta^2$ and therefore, by Cauchy's formula (5.3.3), for $p \in G - \beta$ we have $|\Delta| < \frac{M}{\beta} < \delta\beta$, $|d\Delta| < \delta\theta|dp|$. Putting $\Omega_0 = \Xi_{\mu KN}$, $b = 3\mu$ in the lemma of Ch. V, §4, 5. we find that A' maps the domain $G' = A'^{-1}$ diffeomorphically onto $\Xi' = \Xi_{\mu KN} - d$, $d = (5 + 7\Theta)\beta$, and $G' + 3\beta$ into $\Xi_{\mu KN}$, the conclusions III and IV of the inductive lemma thereby being fulfilled.

2. We apply the fundamental lemma to the function

$$H(p, q) = H_0(p_0) + H_1'(p) + \widetilde{H}_2(p, q)$$

in the domain F. Since, in accordance with 1., $|\widetilde{H}_2| < 2\mu M$, $\theta'|dp| < |dA'| < \Theta'|dp|$ and the numbers θ', Θ', ρ, β, γ, δ, μ, $2M$, K satisfy the conditions of the fundamental lemma (in view of conditions 1)-4) of the inductive lemma), the fundamental theorem is applicable. It furnishes in the domain $P \in G_{\mu KN} - 2\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 2\gamma$ the diffeomorphism B: $P, Q \to p, q$ and the inequality $|H_2'(P, Q)| < \mu M'$. Hence, by Cauchy's formula, for $P \in G_{\mu KN} - 3\beta$, $|\mathrm{Im}\, Q| \leqslant \rho - 3\gamma$ we obtain the estimates of conclusion II of the inductive lemma.

3. In accordance with 1. it follows from $P, Q \in F'$: $P \in G'$, $|\mathrm{Im}\, Q| \leqslant \rho - 3\gamma$ that $P \in G_{\mu KN} - 3\beta$. But then, in accordance with 2., conclusions I and II of the inductive lemma are valid and the inductive lemma is thereby proved completely.

§8. Proof of the inductive theorem

1. We shall show that *under the conditions of the inductive theorem the inductive lemma is applicable*. For conditions 1), 2) in §§2 and 3 are common. From $\delta \leqslant \delta^{(4)}$ it follows that $\delta \leqslant \delta^{(1)}(n, \Theta, \theta)$ from 4) of §3. From $\delta \leqslant \delta^{(2)}$ it follows that $\gamma \leqslant 0, 1\rho$; $\gamma \leqslant 2^{-14}$. Also, for $\delta \leqslant \delta^{(1)}$, $\delta^{(2)}$,

$$15\delta < 15\delta^{\frac{1}{2}}\delta^{\frac{1}{4n}} \leqslant 3\delta^{\frac{1}{4n}} = 3\gamma, \qquad 3\beta = 3\delta^3\varepsilon < 2\delta^2\varepsilon < 2\delta.$$

Thus 4) of §3, follows from 4) and 5) of §2. Finally, $M < \delta^{2n+3}K\beta^2$ in condition 3) of §3, follows from the inequality $-3T - 1 > 3T + 2n + 10$ (in fact, $T = 16n + 24$). Thus, all the conditions of the inductive lemma are fulfilled.

2. From $\delta \leqslant \delta^{(2)}$ it follows, in accordance with (5.3.2), that, for $s \geqslant 1$,

$$\delta_s + \delta_{s+1} + \ldots < 2\delta_s, \; 3(\gamma_s + \gamma_{s+1} + \ldots) < 6\gamma_s \leqslant 6\gamma_1 < 2\frac{\varrho}{3}. \qquad (4.8.1)$$

From (4.8.1) we see that *if, for $s \geqslant 1$, we put*

$$\theta_s = \theta_{s-1}(1 - \delta_s), \; \Theta_s = \Theta_{s-1}(1 + \delta_s), \quad \varrho_s = \varrho_{s-1} - 3\gamma_s \quad (\theta_0 = \theta; \; \Theta_0 = \Theta, \; \varrho_0 = \varrho),$$

the following inequalities will be satisfied:

$$\theta_s > \theta = \frac{\theta}{2}, \; \Theta_s < \overline{\Theta} = 2\Theta, \; \varrho_s > \varrho_\infty = \frac{1}{3}\varrho. \qquad (4.8.2)$$

It is easy to verify that, for any $s \geqslant 1$, the numbers β_s, γ_s, δ_s, ε_s, M_s; K, μ satisfy the inequalities of conditions 3), 4) of the inductive lemma with constants θ_{s-1}, Θ_{s-1}, ρ_{s-1}. For $s = 1$ this is established in paragraph 4.

3. From $\delta \leqslant \delta^{(3)}$ it follows, in view of inequality (5.3.10), that for $N_s = \frac{1}{\gamma_s} \ln \frac{1}{2M_s} < \frac{1}{\gamma_s} \ln \frac{1}{\delta_s^{4T}}$ the following inequalities are valid:

$$\delta_s N_s^n = \delta_s \left(\delta_s^{-\frac{1}{4n}} \ln \delta_s^{-4T}\right)^n \leqslant \delta_s \left(\frac{16nT}{e} \delta_s^{-\frac{1}{2n}}\right)^n \leqslant \delta_s^{\frac{1}{2}} \left(\frac{16nT}{e}\right)^n < 1. \quad (4.8.3)$$

Further we put $\sigma_1 = \sum_{1 \leqslant m < N_1} m^{-2}$ and $\sigma_s = \sum_{N_{s-1} \leqslant m < N_s} m^{-2}$ for $s > 1$. Since $\sum_{s=1}^{\infty} \sigma_s < 2$, from $\delta < \delta^{(3)}$, (4.8.1) and (4.8.3) we derive the following inequalities:

$$\sum_{s=1}^{\infty} [K\sigma_s + (6 + 7\Theta_s)\beta_s N_s^n] \leqslant \sum_{s=1}^{\infty} [\delta_1\sigma_s + \delta_s]\,\varepsilon < 4\delta_1\varepsilon < \frac{\varkappa\varepsilon}{2\bar{D}}, \quad (4.8.4)$$

where

$$\bar{D} = \left(\frac{2\Theta}{\theta}\right)^n LD, \qquad L = n2^{n+2}.$$

4. Suppose that the terms

$$A^{(s-1)};\ F^{(s-1)},\ G^{(s-1)};\ H^{(s-1)}(P^{(s-1)},\ Q^{(s-1)});\ \Xi^{(s-1)} \qquad (s \geqslant 1) \quad (4.8.5)_{s-1}$$

satisfy the conditions of the inductive lemma with constants

$$\theta_{s-1},\ \Theta_{s-1};\ \varrho_{s-1};\ \beta_s,\ \gamma_s,\ \delta_s,\ M_s;\ K,\ \mu, \quad (4.8.6)_{s-1}$$

defined as above. Then the inductive lemma defines

$$A',\ B;\ F',\ G';\ H_1'(P),\ H_2'(P, Q);\ \Xi',\ \bar{\Xi}';\ \theta',\ \Theta',\ \varrho',$$

which we denote respectively by

$$A^{(s)},\ B_s;\ F^{(s)},\ G^{(s)};\ H_1^{(s)}(P^{(s)}),\ H_2^{(s)}(P^{(s)},\ Q^{(s)});\ \Xi^{(s)},\ \bar{\Xi}^{(s)};\ \theta_s,\ \Theta_s,\ \varrho_s. \quad (4.8.7)_s$$

From conclusion II of the inductive lemma we obtain in $F^{(s)}$, since $T = 16(n + 4)$,

$$|H_2^{(s)}| \leqslant \mu \frac{M_s^2}{K\beta_s^2} \delta_s^{-\nu} < \mu \delta_s^{(7T-3T-4n-16)} < \mu\delta^{15/14(7/2\,T-1)} = \mu M_{s+1}. \quad (4.8.8)$$

From conclusions I, II, III of the inductive lemma, bearing in mind 2. and (4.8.8), we reach the conclusion that *if the terms of* $(4.8.5)_{s-1}$ *satisfy the conditions of the inductive lemma with constants* $(4.8.6)_{s-1}$ *then the terms of* $(4.8.5)_s$ *will satisfy the conditions of the inductive lemma with constants* $(4.8.6)_s$.

5. But, in accordance with 1., the terms of $(4.8.5)_0$ (where $A^{(0)} = A$, $F^{(0)} = F$, $G^{(0)} = G$, $H^{(0)} = H$, $P^{(0)} = p$, $Q^{(0)} = q$, $\Xi^{(0)} = \Xi$) satisfy the conditions of the inductive lemma with constants $(4.8.6)_0$. Hence the inductive lemma can be applied an indefinite number of times; as a result we obtain the terms of $(4.8.7)_s$ for $s = 1, 2, \ldots$

In view of (4.8.2) and (4.8.8) conclusions I, II, III of the inductive theorem follow from conclusions I, II, III of the inductive lemma. We have not yet proved that $F^{(s)}$ is non-empty. This follows from conclusion IV of the inductive theorem which we shall now prove.

6. From conclusions III and IV of the inductive lemma we find that

$$\operatorname{mes}(G^{(s-1)}\setminus G^{(s)}) \leqslant \underline{\theta}^{-n} \operatorname{mes}(\Xi^{(s-1)}\setminus\bar{\Xi}^{(s)}), \tag{4.8.9}$$

where $\bar{\Xi}^{(s)}=\Xi^{(s-1)}_{\mu K N_s}-\bar{d}_s,\ \bar{d}_s=(6+7\Theta_s)\beta_s$ and $\Xi^{(s-1)}$ is obtained from $\Xi^{(0)}$ by the formulae

$$\Xi^{(k)}=\Xi^{(k-1)}_{\mu K N_k}-d_k,\quad d_k=(5+7\Theta_k)\beta_k,\qquad N_k=\frac{1}{\gamma_k}\ln\frac{1}{2M_k}\qquad (k=1,\ \dots,\ s-1)$$

Here $d_k > 0$, $N_1 < N_2 < \dots$ and the domain $\Xi = \Xi^{(0)}$ is, in accordance with condition 6) of the inductive theorem, of the form

$$\Xi^{(0)}=\Xi^{*}_{K_0N_0d_0},\ N_0<N_1,\ K_0>K,\ d_0>0,$$

where Ξ^* is a domain of type $\frac{D}{\varepsilon}$.

From the arithmetic lemma of Ch. V, §2, 5. we find that

$$\operatorname{mes}(\Xi^{(s-1)}\setminus\Xi^{(s)}) \leqslant \frac{DL}{\varepsilon}\,[K\sigma_s+(6+7\Theta_s)\beta_s N_s^n]\operatorname{mes}\Xi^*. \tag{4.8.10}$$

From (4.8.9), (4.8.10) and (4.8.4) it follows that

$$\operatorname{mes}(G\setminus G^{(s)})=\sum_{k=1}^{s}{}'\operatorname{mes}(G^{(k-1)}-G^{(k)})\leqslant$$

$$\leqslant\sum_{k=1}^{s}[K\sigma_k+(6+7\Theta_k)\beta_k N_k^n]\frac{DL}{\varepsilon\theta^n}\operatorname{mes}\Xi^*\leqslant\frac{\varkappa\varepsilon}{2}\,\frac{DL\theta^n}{DL\underline{\Theta}^n\varepsilon\underline{\theta}^n}\operatorname{mes}\Xi^*=\frac{\varkappa}{2}\,\Theta^{-n}\operatorname{mes}\Xi^*. \tag{4.8.11}$$

Thus conclusion IV of the inductive theorem is valid and the theorem is therefore completely proved.

§9. Lemma on the non-degeneracy of diffeomorphisms

For the proof of the fundamental theorem we give more precision to conditions 4) of §1 which express the non-degeneracy of certain mappings.

1. LEMMA. *In the conditions of the fundamental theorem for any* $\varkappa > 0$ *there exist positive numbers* θ, Θ, D, m, r, $\bar{\mu}$ *depending only on* $\varkappa$, G_0, ρ, R, H_0 *and* $\bar{H}_1$, *such that the domain* G_0 *can be split up into* $m+1$ *parts*:

$$G_0=G_{01}\cup G_{02}\cup\dots\cup G_{0m}\cup\bar{G}_0,$$

where

$$\operatorname{mes}\bar{G}_0\leqslant\frac{\varkappa}{2}\operatorname{mes}G_0, \tag{4.9.1}$$

and each of the domains G_{0i} *for any* $0<\mu<\bar{\mu}$ *has the following three properties*:

1) *The mapping* $A_0 : p_0 \to Ap_0 = \omega_0 = \dfrac{\partial H_0}{\partial p_0}$ *is a diffeomorphism of* G_{0i} *onto the domain* Ω_i *of type* D *in the notation of* Ch. V, §1, *and at each point of* G_{0i}

$$\theta\,|dp_0|\leqslant|dA_0|\leqslant\Theta\,|dp_0|,\quad \left|\frac{\partial^2 H_0}{\partial p_0^2}\right|\leqslant\Theta. \tag{4.9.2}$$

2) *The mapping A:* $p \to Ap = \xi$, *where*

$$p = p_0, \tau;\ \xi = \xi_0, \xi_1;\ \xi_0 = \frac{\partial}{\partial p_0}(H_0 + \mu \overline{\overline{H}}_1), \qquad \xi_1 = \frac{\partial \overline{\overline{H}}_1}{\partial \tau},$$

is a diffeomorphism of the domain $G_i(r)$ ($p_0 \in G_{0i}$, $|\tau| \leqslant r$) *onto the domain* $\Xi_i(r)$, *and at each point of* $G_i(r)$

$$\theta |dp| \leqslant |dA| \leqslant \Theta |dp|; \qquad \left|\frac{\partial \overline{\overline{H}}_1}{\partial p}\right|, \ \left|\frac{\partial^2 \overline{\overline{H}}_1}{\partial p^2}\right| \leqslant \Theta. \tag{4.9.3}$$

3) *We denote by* $G_i^*(\varepsilon)$ *the domain of points* $p = p_0$, τ *for which*

$$p_0 \in G_{0i}, \qquad \left|\tau_j - \frac{\varepsilon}{2}\right| \leqslant \frac{\varepsilon}{2}.$$

By $\Xi_i^*(\varepsilon, d)$ *we denote the set of points* Ap, $p \in G_i^*(\varepsilon)$, *for which* $\xi_0 \in \Omega_i - d$.

Property 3) *is that for any* $0 < \varepsilon < r$, $0 < d < r$ *the domain* $\Xi_i^*(\varepsilon, d)$ *is of type* $\frac{D}{\varepsilon}$ (see Ch. V, §1, 3).

2. Proof of the lemma. It is necessary to take as $\overline{G}_0$ a sufficiently small neighbourhood of the analytic manifolds where the determinants (4.1.9), (4.1.10) vanish. Then outside $\overline{G}_0$ the inequalities

$$\theta^* E < \left\|\frac{\partial^2 H_0}{\partial p_0^2}\right\| < \Theta^* E, \qquad \theta^* E < \|\lambda_{ij}\| < \Theta^* E, \qquad \left|\frac{\partial \lambda_i}{\partial p_j}\right| \leqslant \Theta^*$$

are valid for sufficiently small θ^* and large Θ^*. From these inequalities, with the help of the lemma on distortion (Ch. V, §4, 6), it is easy to derive inequalities (4.9.2) and (4.9.3) for certain θ, Θ, $0 < \theta \leqslant 1 \leqslant \Theta < \infty$ and sufficiently small r, $\overline{\mu}$. Then r, $\overline{\mu}$ and G_{0i} are chosen so that A_0 and A are diffeomorphisms. Finally, for sufficiently small r and $\overline{\mu}$ and sufficiently large D, condition 3) is also satisfied (in accordance with III of Ch. V, §1, 3.; this choice of D completes the proof of the lemma.

3. *REMARK.* We shall assume that the fundamental theorem is proved for each domain G_{0i} separately, i.e. that we have found

$$\varepsilon_{0i} = \varepsilon_0(\varkappa;\ H_0, \overline{\overline{H}}_1;\ \ G_{0i}, R_0, C, \varrho).$$

We put

$$\varepsilon_0 = \varepsilon_0(\varkappa;\ H_0, \overline{\overline{H}}_1;\ G_0, R_0, C, \varrho) = \min_{1 \leqslant i \leqslant m} \varepsilon_0\left(\frac{\varkappa}{2m};\ H_0, \overline{\overline{H}}_1;\ G_{0i}, R_0, C, \varrho\right).$$

In view of (4.9.1) this value ε_0 satisfies all the requirements of the fundamental theorem.

Bearing in mind the above remark *we shall henceforth suppose that in the domains*

$$p_0 \in G_0;\ p \in G(r) \qquad (\text{i.e. } p_0 \in G_0, \quad |\tau| \leqslant r); \qquad p \in G^*(\varepsilon)$$
$$\left(\text{i.e. } p_0 \in G_0, \ \left|\tau - \frac{\varepsilon}{2}\right| \leqslant \frac{\varepsilon}{2}\right) \quad (0 < \varepsilon < r)$$

conditions 1), 21, 3) *of* 1. *are fulfilled.*

§10. Averaging over rapid variables

The proof of the fundamental theorem begins with the application of the lemma of §5 to the function

$$H_0(p_0)+\mu\overline{H}_1(p_0,\ p_1,\ q_1)+\mu\widetilde{H}_1(p_0,\ q_0,\ p_1,\ q_1). \qquad (4.10.1)$$

We perform this first step of the proof below.

1. Assumptions. We shall assume that the function $H(p,\ q)$, the domains F, G_0 and the positive numbers ρ, $R=\sqrt{r}$, C; θ, Θ, D, m, r; $\varkappa$ satisfy conditions 1)-4) of the fundamental theorem and conditions 1)-3) of §9, 1. In addition we suppose that

$$\mu\leqslant\varepsilon^4,\quad \varepsilon=\delta^T,\quad T>n+2,\quad \varrho\leqslant1,\quad D>1, \qquad (4.10.2)$$

and introduce the quantities

$$\beta=\delta^3,\quad \gamma=2\delta^{\frac{1}{4n}},\quad K_0=\delta,\quad M=\mu\delta^{-1},\quad \overline{M}=\varepsilon^4\delta^{-1}. \qquad (4.10.3)$$

Finally we assume that δ is sufficiently small:

$$\delta\leqslant\delta^{(5)}(n,\ \theta,\ \Theta,\ C,\ T,\ R,\ \varrho,\ \varkappa,\ D), \qquad (4.10.4)$$

where

$$\delta^{(5)}=\min\{\delta^{(0)}(n,\Theta);\ \delta^{(6)}(n,\ \varrho,\ R,\ C);\ \delta^{(7)}(n,\ \theta,\ \Theta,\ T,\ \varkappa,\ D)\}.$$

Here $\delta^{(0)}$ is defined in §5, $\delta^{(6)}=\min\left\{0.04;\ \rho^n 8^{-4n};\ \frac{R}{2},\ \frac{1}{C}\right\}$,

$$\delta^{(7)}=\min\left\{\frac{1}{4\Theta},\ \left(\frac{e}{16nT}\right)^{2n},\ \frac{\varkappa\theta^n}{6\Theta^n Dn2^{n+2}}\right\}.$$

2. Assertions. On the assumptions of 1. in the domain F' of the space P, Q:

$$P_0\in G_0',\ |\operatorname{Im}Q_0|\leqslant\varrho-2\gamma\leqslant\varrho'=0{,}5\varrho,\ |X_1|\leqslant R-3\beta\leqslant R'=0{,}5R, \quad (4.10.5)$$

there exists a canonical diffeomorphism B: $P,\ Q\to p,\ q$, where

I. $|B-E|<\mu\delta^{-(2n+7)}$, $|dB|<2|dX|$.

II. For $p,\ q=BX$, where $X\in F'$, the function $H(p,\ q)$ from (4.1.1) is of the form

$$H(p,\ q)=H_0(P_0)+\mu\overline{H}_1(P_0,\ P_1,\ Q_1)+H_2(P_0,\ P_1,\ Q_0,\ Q_1), \qquad (4.10.6)$$

where H_0 and $\overline{H}_1$ are the same as in (4.1.1)-(4.1.3) and

$$|H_2|<\mu\varepsilon^4\delta^{-(4n+15)}. \qquad (4.10.7)$$

III. $G_0'=A_0^{-1}\Omega_{K_0N_0}-2\delta^3$ $\left(\text{where } N_0=\frac{1}{2}\delta^{-\frac{1}{4n}}\text{for}\frac{\delta}{\varepsilon^4}\right)$.

IV. If G_0'' is the domain in the space P_0 for which $A_0G_0''\supseteq\Omega_{K_0N_0}-4\Theta\delta^3$, then mes $(G_0\setminus G_0'')\leqslant\frac{\varkappa}{2}$ mes G_0.

3. Proof. From $\delta\leqslant\delta^{(6)}$, (4.10.3) and (4.10.2) it follows that

$$10\delta\leqslant2\gamma\leqslant\frac{\varrho}{2}\leqslant\frac{1}{2},\ 3\beta\leqslant\delta\leqslant\frac{R}{2},\ \delta\leqslant K,\ \mu C\leqslant M\leqslant\overline{M}\leqslant\delta^{2n+3}\beta^2. \quad (4.10.8)$$

In accordance with (4.1.7), (4.10.8) and (4.9.2) and the conditions (4.10.4) and (4.10.2), the function (4.10.1), the domains F, G_0, G_1: $|x_1| \leqslant R$ and the numbers θ, Θ, ρ; β, γ, δ; K_0, M, $\overline{M}$ satisfy conditions 1)-4) of the lemma on averaging in §5. This lemma furnishes the domain F' and the mapping B.

The inequalities (4.10.5) follow from (4.10.8). In view of (4.10.3) assertion I follows from §5, 1. Assertion II follows from §5, II and (4.1.6) of §1, since $H_2 = (\mu^2)H_2 + H_2'$ and hence, in view of (4.10.3), (4.10.2),

$$|H_2| \leqslant \mu^2\delta^{-1} + \mu\varepsilon^4\delta^{-(4n+14)} \leqslant \mu\varepsilon^4\delta^{-(4n+15)}.$$

Assertion III is the same as §5, III.

We shall prove assertion IV. We have, in view of $\delta \leqslant \delta^{(7)}$ in (4.10.4),

$$\operatorname{mes}(G_0\diagdown G_0'') \leqslant \theta^{-n}\operatorname{mes}[\Omega\diagdown(\Omega_{K_0N_0} - 4\Theta\delta^3)] \leqslant \theta^{-n}(2K_0 + \delta^2N_0^n)DL\operatorname{mes}\Omega \leqslant$$
$$\leqslant \theta^{-n}3\delta DL\operatorname{mes}\Omega \leqslant 3\delta\Theta^n\theta^{-n}DL\operatorname{mes}G_0 \leqslant \frac{\varkappa}{2}\operatorname{mes}G_0.$$

We have used successively the arithmetic lemma (Ch. V, §2, 2.) and the inequality $\delta \leqslant \left\{\frac{1}{4\Theta}, \frac{1}{N_0^n}, \frac{\varkappa\theta^n}{6\Theta^nLD}\right\}$.

The inequality $\delta < N_0^{-n}$ follows from $\delta < \left(\frac{e}{16nT}\right)^{2n}$, $N_0 < \delta^{-\frac{1}{4n}}\ln\frac{1}{\delta^{4T}}$.

In fact, in view of (5.3.1), $\ln\frac{1}{\delta^{4T}} \leqslant 4T\frac{4n}{e}\left(\frac{1}{\delta}\right)^{\frac{1}{4n}}$, and hence

$N_n^0 \leqslant \delta^{-\frac{1}{2}}\left(\frac{16nT}{e}\right)^n < \delta^{-1}$. Assertions I-IV are proved.

§11. Polar coordinates

1. Notation. In order to carry out the second and subsequent approximations we now change from the Cartesian coordinates P_1, Q_1 of §10 to polar coordinates τ, φ by means of the canonical transformation

$$P_1 = \sqrt{2\tau}\cos\varphi, \quad Q_1 = \sqrt{2\tau}\sin\varphi. \tag{4.11.1}$$

We shall denote the variables P_0, Q_0, τ, φ by the letters P_0^*, Q_0^*, P_1^*, Q_1^* and the transformation (4.11.1) by the letter B^*: $B^*X^* = X$, where $X^* = P_0^*, P_1^*, Q_0^*, Q_1^* = P_0, \tau, Q_0, \varphi$; $X = P_0, P_1, Q_0, Q_1$. We note that all the variables Q^* have the sense of angles.

2. LEMMA. *Suppose that in the conditions of* §10, 1. *the following inequalities are valid:*

$$T > 8n + 30, \quad \delta \leqslant \delta^{(8)} = 2^{-15}C^{-1}. \tag{4.11.2}$$

Consider the domain $X^* \in F^{*\prime}(\varepsilon)$, *where* $P_0^* \in G_0'$ *as in* §10, 2.; $P_1^* \in G_1(\varepsilon)$, $\left[\left|P_1^* - \frac{\varepsilon}{2}\right| < \frac{\varepsilon}{2}\right]$, $|\operatorname{Im} Q^*| \leqslant 0.5\rho$. *In this domain the function* (4.1.1) *of* p, $q = x = BX = BB^*X^*$ *takes the form*

$$H(p, q) = H_0(P_0^*) + \overline{\overline{\mu}}H_1(P_0^*, P_1^*) + H_3(P_0^*, P_1^*, Q_0^*, Q_1^*) = H^*(P^*, Q^*), \tag{4.11.3}$$

where H_0 *and* $\overline{\overline{H}}_1$ *are the same functions as in* (4.1.1)-(4.1.4), *and* H_3 *is an analytic function of period* 2π *with respect to* Q^* *and such that in* $F^{*\prime}(\varepsilon)$

$$|H_3| < \frac{\mu\varepsilon^{7/2}}{\delta}. \tag{4.11.4}$$

3. Proof. Since $\delta < \delta^{(8)}$ we have $\frac{\varepsilon}{2} < 2^{-8}R^2 = \left(\frac{R^1}{4}\right)^2$. Therefore the lemma on polar coordinates (Ch. V, §4, 7.) is applicable and in accordance with this (4.10.6) takes the form

$$H(p, q) = H_0(P_0^*) + \mu\overline{\overline{H}}_1(P_0^*, P_1^*) + \mu\widetilde{\widetilde{H}}_1 + H_2, \tag{4.11.5}$$

where, in accordance with (4.1.8), (5.4.4) and (4.10.7),

$$|\widetilde{\widetilde{H}}_1| \leqslant C|8P_1^*|^{7/2} \qquad |H_2| < \mu\varepsilon^4\delta^{-(4n+15)}.$$

It follows from $\delta \leqslant \delta^{(8)}$ (4.11.2) that in the domain $F^{*\prime}(\varepsilon)$

$$|\mu\widetilde{\widetilde{H}}_1| \leqslant \mu C 8^{7/2}\varepsilon^{7/2} < \frac{\mu}{2\delta}\varepsilon^{7/2}, \tag{4.11.6}$$

and from $T > 8n + 30$ (4.11.2) we find that

$$|H_2| < \mu\varepsilon^{7/2}\varepsilon^{1/2}\delta^{-(4n+15)} < \frac{\mu\varepsilon^{7/2}}{2\delta}. \tag{4.11.7}$$

On comparing (4.11.5)-(4.11.7) we obtain the relationships (4.11.3) and (4.11.4).

§12. The applicability of the inductive theorem

We shall show below that for sufficiently small δ the function (4.11.3) satisfies conditions 1)-6) of the inductive theorem of §2.

1. Assumptions. Let

$$\delta \leqslant \delta^{(9)}(n, \theta, \Theta, C, R, \varrho, \varkappa, D),$$

where

$$\delta^{(9)} = \min\left\{\delta^{(4)}\left(n, \Theta, \theta, \frac{\varrho}{2}, \frac{\varkappa}{2}, D\right), \delta^{(5)}(n, \Theta, \theta, C, 16(n+4), \varrho, \varkappa, D); \delta^{(8)}(C)\right\},$$

$\delta^{(4)}$ being defined in §2, $\delta^{(5)}$ in §10 and $\delta^{(8)}$ in §11.

We assume that conditions 1)-4) of the fundamental theorem and conditions 1)-3) of §9, 1. are fulfilled. In addition we assume that

$$\mu \leqslant \varepsilon^4, \qquad \varrho \leqslant 1.$$

Then for $T = 16(n + 4)$ the assumptions of §10 and §11 are satisfied; thus (4.1.1) takes the form (4.11.3) in the variables P^*, $Q^* \in F^{*\prime}$.

2. The domains Ξ, G. We call to mind the notation of certain domains introduced above.

G_0 is the original domain in the space p_0 (see §9).

$\Omega = A_0 G_0$ is a domain in the space ω_0.

$G_1(\varepsilon)$ consists of p_1 for which $\left|p_1 - \frac{\varepsilon}{2}\right| \leqslant \frac{\varepsilon}{2}$.

$G^*(\varepsilon) = G^*(\varepsilon) = G_0 \times G_1(\varepsilon)$ consists of $p = p_0, p_1$ such that $p_0 \in G_0$, $p_1 \in G_1(\varepsilon)$.

$\Xi^*(\varepsilon, d_0)$ consists of $\xi = \xi_0, \xi_1$ such that $\xi = Ap$, where $p \in G^*(\varepsilon)$ and $\xi_0 \in \Omega - d_0$.

We introduce a new domain Ξ, consisting of $\xi = \xi_0, \xi_1$ such that $\xi = Ap$, where $p \in G^*(\varepsilon)$ and

$$\xi_0 \in \Omega_{K_0 N_0} - d_0,$$

where $K_0 = \delta$, $N_0 = \frac{1}{2}\delta^{-\frac{1}{4n}} \ln \frac{\delta}{\varepsilon^4}$, $d_0 = 3\Theta\delta^3$ (for the notation $\Omega K_0 N_0$ see Ch. V, §2, 2.). We also introduce the domain

$$G = A^{-1}\Xi.$$

3. Properties of the domains Ξ, G. We shall prove the following assertions (see Ch. V, §§1 and 2).

I. *The domain $\Xi^*(\varepsilon, d_0)$ is of type $\frac{D}{\varepsilon}$ and*

$$\operatorname{mes} \Xi^*(\varepsilon, d_0) \leqslant \Theta^n \operatorname{mes} G^*(\varepsilon).$$

II. *In $\Xi^*(\varepsilon, d_0)$ the domain Ξ is of the form $[\Xi^*(\varepsilon, d_0)]_{K_0 N_0 d_0}$.*

III. $G \subseteq G_0' \times G_1(\varepsilon)$, *where* $G_0' = A_0^{-1}\Omega_{K_0 N_0} - 2\delta^3$.

IV. $G \supseteq G_0'' \times G_1(\varepsilon)$, *where* $G_0'' = A_0^{-1}(\Omega_{K_0 N_0} - 4\Theta\delta^3)$.

I follows from (4.9.3) and 3) of §9 since, in accordance with (4.10.4) and (4.10.2).

$$0 < \varepsilon < r = R^2, \quad 0 < d_0 < r = R^2.$$

II is obvious from the definition of Ξ (see 2.).

The proofs of III and IV are based on the fact that, in view of (4.9.3), if $p = p_0, p_1$; $\xi = \xi_0, \xi_1$; $\omega_0 = A_0 p_0$, $\xi = Ap$, then $|\xi_0 - \omega_0| < \mu\Theta < \delta^3\Theta$.

By definition of G it follows from $p \in G$ that $\xi_0 \in \Omega_{K_0 N_0} - 3\Theta\delta^3$. This means that $\omega_0 \in \Omega_{K_0 N_0} - 2\Theta\delta^3$ and $p \in A_0^{-1}(\Omega_{K_0 N_0} - 2\Theta\delta^3)$. It therefore follows, in accordance with Ch. V, §4, 4. that $p_0 \in G_0'$, which proves III.

If, however, $p_0 \in G_0''$, i.e. $\omega_0 \in \Omega_{K_0 N_0} - 4\Theta\delta^3$, then $\xi_0 \in \Omega_{K_0 N_0} - 3\Theta\delta^3$, and for $p_1 \in G_1(\varepsilon)$ we have $p \in G$, which proves IV.

4. Verification of conditions 1)-6) of the inductive theorem. *We shall show that, on the assumptions of* 1., *the function $H^*(p^*, q^*)$ from* (4.11.3), *the domains G, Ξ, constructed in* 3., *and the numbers D, Θ, θ, $\frac{\rho'}{2}$, κ of* §10, *together with*

$$\beta = \delta^3\varepsilon, \ \varepsilon = \delta^T, \quad T = 16(n+4), \quad \gamma = \delta^{\frac{1}{4n}}, \ \mu \leqslant \varepsilon^4, \ M = \varepsilon^{7/2}\delta^{-1}, \ K = \delta\varepsilon \quad (4.12.1)$$

satisfy conditions 1)-6) *of the inductive theorem.*

Condition 1) (where $\mu\overline{\overline{H}}_1(P^*)$ plays the role of $H_1(p)$ and $H_3(P^*, Q^*)$ that of $H_2(p, q)$) follows from §11, 2. and 3., III.

Condition 2) follows from 2), §9.
Condition 3) follows from (4.11.4) and (4.12.1).
Condition 4) follows from the fact that, in 1., $\delta \leqslant \delta^{(4)}$.
Condition 5) follows from (4,12.1).
Condition 6) follows from 3., I and II, $\Xi^*(\varepsilon, d_0)$ playing the role of Ξ^*.

In fact $K_0 = \delta > \varepsilon\delta = K$, $N_0 = \frac{1}{2}\delta^{-\frac{1}{4n}} \ln \frac{\delta}{2\varepsilon^{7/2}} = \frac{1}{\gamma} \ln \frac{1}{2M}$.

Thus all the conditions of the inductive theorem are satisfied. Its conclusions I-IV are therefore valid.

§13. Passage to the limit

We construct here the invariant tori of the canonical equations with Hamiltonian $H^*(P^*, Q^*)$ (4.11.3).

1. Convergence. We shall denote by $F = F^{(0)}$ the domain $P_0^* \in G = G^{(0)}$, $|\mathrm{Im}\ Q^*| \leqslant \frac{\rho}{2}$ in the space P^*, Q^* (the domain G is defined in §12, 2.). In accordance with §12, 4. the inductive theorem of §2 furnishes a sequence of domains $F^{(s)}$ and diffeomorphisms B_s connecting $P^{(s)}$, $Q^{(s)}$ with P^*, $Q^* = S_s(P^{(s)}, Q^{(s)})$: $S_s = B_1 B_2 \dots B_s$. In the coordinates $P^{(s)}$, $Q^{(s)}$ the Hamiltonian $H^*(P^*, Q^*)$ (4.11-3) in $F^{(s)}$ takes the form $H^{(s)}(P^{(s)}, Q^{(s)})$ of §2.

In accordance with §2, I, the convergence lemma of Ch. V, §5, 1. with $d_s = \beta_s < 4^{-s}$ is applicable to the sequences B_s and $F^{(s)}$. By virtue of this lemma the sequence of diffeomorphisms S_s on the set $F^{(\infty)} = \bigcap_{s \geqslant 0} F^{(s)}$ converges uniformly to a certain mapping S_∞:

$$|S_\infty - E| < 2\beta_1. \tag{4.13.1}$$

The set $F^{(\infty)}$ is of the form $P^{(\infty)} \in G^{(\infty)}$, $|\mathrm{Im}\ Q^{(\infty)}| \leqslant \rho_\infty$

$$\left(\rho_\infty \geqslant \frac{\rho}{6},\ G^{(\infty)} = \bigcap_{s \geqslant 0} G^{(s)}\right).$$

2. Invariance. All variables in this subsection are taken to be real.

We shall show that the set $S_\infty F^{(\infty)}$ is invariant with respect to motions determined by $H^*(P^*, Q^*)$. We shall write the canonical equations with Hamiltonian $H^{(s)}(P^{(s)}, Q^{(s)}) = H^*(P^*, Q^*)$ in the form

$$\dot{X}^{(s)} = Y^{(s)}(X^{(s)}) \quad (\text{where } X^{(s)} = P^{(s)},\ Q^{(s)};\ Y^{(s)} = -H^{(s)}_{Q^{(s)}},\ H^{(s)}_{P^{(s)}}). \tag{4.13.2$_s$}$$

The transformations S_s are canonical. Therefore, if $X^{(s)}(t)$ satisfies (4.13.2)$_s$, $X^*(t) = S_s X^s(t)$ will satisfy (4.13.2)$_*$.

In exactly the same way the equations with Hamiltonian $H_0(P_0^{(s)}) + H_1^{(s)}(P^{(s)})$ define a vector field $\overline{Y}^{(s)} = (0;\ A_0^{(s)},\ \mu A_1^{(s)})$, where $A_0^{(s)}$, $A_1^{(s)}$ are the first n_0 and the last n_1 components of the vector $A^{(s)}(P^{(s)})$ from §2, III.

In accordance with §2, III, as $s \to \infty$ the sequence of diffeomorphisms $A^{(s)}$ on $G^{(\infty)}$ converges to the limit $A^{(\infty)}$, where (cf. (4.8.1))

$$|A^{(s)}-A^{(\infty)}|\leqslant\sum_{k=s+1}^{\infty}\beta_k\delta_k<\frac{1}{2}\beta_{s+1}. \qquad (4.13.3)$$

If $A^{(\infty)}$ is written in the form $A_0^{(\infty)}$, $A_1^{(\infty)}$, the vector field $(0;\ A_0^{(\infty)},\ \mu A_1^{(\infty)})$ on $F^{(\infty)}$ will be denoted by $\overline{Y}^{(\infty)}=Y^{(\infty)}$ and the solution of the equations $(4.13.2)_\infty$ (a straight line) by $X^{(\infty)}(t)$. It follows from §2, II, that $|Y^{(s)}-\overline{Y}^{(s)}|<\frac{1}{2}\beta_{s+1}$ in $F^{(s)}$. In view of (4.13.3) we have $|Y^{(s)}-Y^{(\infty)}|<\beta_{s+1}$ in $F^{(\infty)}$. From §2, II, we obtain $\left|\frac{\partial Y^{(s)}}{\partial X^{(s)}}\right|<2n\delta_s+2\Theta<3\Theta$. Therefore the lemma of Ch. V, §5, 3. is applicable and we find that for $X^{(\infty)}(0)\in F^{(\infty)}$ the point $X^*(t)=S_\infty(X^{(\infty)}(t))$ belongs to F and satisfies $(4.13.2)_*$.

Thus the set $F^*=S_\infty F^{(\infty)}$ is invariant.

§14. Proof of the fundamental theorem

We shall complete here the proof of the fundamental theorem begun in §10. All the variables are taken to be real. We use the construction and notation of §§10-13.

1. The construction of F_ε. We put

$$\mathbf{F}_\varepsilon=BB^*\mathbf{F}^*=BB^*S_\infty F^{(\infty)}, \qquad (4.14.1)$$

where B is defined in §10, B^* in §11, S_∞ and $F^{(\infty)}$ in §13. We shall prove the assertions I-IV of the fundamental theorem.

We first prove that

$$\mathbf{F}^*\subseteq F^{*\prime}(\varepsilon)-\beta_1, \qquad (4.14.2)$$

where $P^*,\ Q^*\in F^{*\prime}(\varepsilon)$ denotes $P^*\in G_0'\times G_1(\varepsilon)$ (see §11, 2.). From §2, I, it follows that $F^*\subseteq B_1F^{(1)}\subseteq F-\beta_1$ and from §12, 3., III, that $G\subseteq G_0'\times G_1(\varepsilon)$. Therefore (4.14.2) is valid, BB^* is defined on F^* and (4.14.1) has a meaning.

Further, let

$$\begin{array}{llll} x^*=B^{*-1}x, & x=BX, & X=B^*X^*, & X^*\in\mathbf{F}^*,\\ x^*=p_0^*,p_1^*,q_0^*,q_1^*, & x=p_0,p_1,q_0,q_1, & X=P_0,P_1,Q_0,Q_1, & X^*=P_0^*,P_1^*,Q_0^*,Q_1^*,\\ x_0^*=p_0^*,q_0^*, & x_0=p_0,q_0, & X_0=P_0,Q_0, & X_0^*=P_0^*,Q_0^*,\\ x_1^*=p_1^*,q_1^*, & x_1=p_1,q_1, & X_1=P_1,Q_1, & X_1^*=P_1^*,Q_1^*. \end{array}$$

By definition of B^* we have $x_0^*=x_0$, $X_0^*=X_0$. In view of §10, I,

$$|x_0^*-X_0^*|<\mu\delta_1^{-(2n+7)}<\varepsilon^3<\beta_1,\qquad |x_1-X_1|<\varepsilon^3<\sqrt{\delta_1^3\varepsilon}. \qquad (4.14.3)$$

Also it is obvious that $|X_1|<\sqrt{2\varepsilon}$, $|x_1|<\frac{1}{2}$. In view of (4.14.2), $|P_1^*|\geqslant\beta_1=\delta_1^3\varepsilon$ and, in accordance with (4.14.3). $|x_1-X_1|<\sqrt{P_1^*}$. Therefore by the lemma of Ch. V, §4, 8., $|x_1^*-X_1^*|\leqslant\max\left\{\varepsilon^3,\ 2\frac{\varepsilon^3}{\sqrt{\delta_1^3\varepsilon}}\right\}<\varepsilon^2$.

Together with (4.13.3) and (4.14.2) this gives

$$|x^* - X^*| < \varepsilon^2. \qquad (4.14.4)$$

2. Invariant tori. We fix $P^{(\infty)} \in G^{(\infty)}$ and consider in the space of x the torus

$$x = BB^*S_\infty(P^{(\infty)}, Q^{(\infty)}) \qquad (4.14.5)$$

with coordinates $Q^{(\infty)}$ on it. In accordance with §13, 2. and since the transformations B, B^* are canonical, the equations with Hamiltonian (4.1.1) on this torus have the form

$$\dot{Q}^{(\infty)} = \omega \text{ (where } \omega = \omega_0,\ \omega_1 \text{ and } \xi = \omega_0,\ \mu\omega_1 = A^{(\infty)}(P^{(\infty)})). \qquad (4.14.6)$$

We introduce the notation:

$$p^*_\omega = A^{-1}\xi = A^{-1}A^{(\infty)}(P^{(\infty)}), \qquad (4.14.7)$$

where A is the diffeomorphism from §9. In accordance with (4.13.3), $|A - A^{(\infty)}| < \beta_1$. Therefore $|AP^{(\infty)} - Ap^*_\omega| < \beta_1$ and by the lemma of Ch. V, §4, 4. we have

$$|P^{(\infty)} - p^*_\omega| < \theta^{-1}\beta_1. \qquad (4.14.8)$$

We now denote the torus (4.14.5) by T_ω, put $p^*_\omega = p_{0\omega}$, τ_ω, $Q^{(\infty)} = Q$ and prove assertions II-IV of the fundamental theorem.

Assertion IV follows immediately from (4.14.6) and (4.14.7).

Assertion II is obtained from (4.14.5) by noting that $x = B^*x^*$ and putting

$$f_{0_\omega} = p_0 - p_{0_\omega}, \qquad g_{0_\omega} = q_0 - Q_0^{(\infty)},$$
$$f_{1_\omega} = p_1^* - \tau_\omega, \qquad g_{1_\omega} = q_1^* - Q_1^{(\infty)}.$$

The analyticity of $f_{i_\omega}(Q)$, $g_{i_\omega}(Q)$ follows from the convergence of S_s in the complex domain $F^{(\infty)}$ (§13, 1.).

We now prove assertion III. In accordance with (4.14.4), (4.13.1), (4.14.8) and since $\delta < \delta^{(7)}$ (§10, 1.), we have

$$|p^* - p^*_\omega| \leqslant |p^* - P^*| + |P^* - P^{(\infty)}| + |P^{(\infty)} - p^*_\omega| \leqslant \varepsilon^2 + 2\beta_1 + \theta^{-1}\beta_1 < \varkappa\varepsilon,$$
$$|q^* - Q^{(\infty)}| \leqslant |q^* - Q^*| + |Q^* - Q^{(\infty)}| < \varepsilon^2 + 2\beta_1 < \varkappa\varepsilon.$$

3. An estimate of the measure of F_ε. In view of §2, IV, and §12, I, we have

$$\operatorname{mes}(G \setminus G^{(\infty)}) \leqslant \frac{\varkappa}{2}\Theta^{-n}\operatorname{mes}\Xi^*(\varepsilon, d_0) \leqslant \frac{\varkappa}{2}\operatorname{mes}G^*(\varepsilon). \qquad (4.14.9)$$

Further, in accordance with §2, 3., IV, and §10, 2., IV, we have

$$\operatorname{mes}[G^*(\varepsilon) \setminus G] = \operatorname{mes}[(G_0 \times G_1(\varepsilon)) \setminus G] \leqslant$$
$$\leqslant \operatorname{mes}[(G_0 \setminus G_0''] \times G_1(\varepsilon)] \leqslant \frac{\varkappa}{2}\operatorname{mes}[G_0 \times G_1(\varepsilon)] = \frac{\varkappa}{2}\operatorname{mes}G^*(\varepsilon). \quad (4.14.10)$$

By the lemma of Ch. V, §5, 4., in view of (4.14.9) and (4.14.10),

$$\operatorname{mes}S_\infty F^{(\infty)} \geqslant \operatorname{mes}F^{(\infty)} = (2\pi)^n \operatorname{mes}G^\infty \geqslant$$
$$\geqslant (2\pi)^n(1-\varkappa)\operatorname{mes}G^*(\varepsilon) = (1-\varkappa)\operatorname{mes}F(\varepsilon).$$

But, since the canonical transformation BB^* conserves measure, we obtain §1, I :

$$\text{mes}\,\mathbf{F}_\varepsilon = \text{mes}\,BB^* S_\infty F^{(\infty)} = \text{mes}\,S_\infty F^{(\infty)} \geqslant (1-\varkappa)\,\text{mes}\,F(\varepsilon). \quad (4.14.11)$$

The estimate (4.14.11) shows that the set $\mathbf{F}_\varepsilon$ of invariant tori we have constructed is non-empty, and this completes the proof of the fundamental theorem.

Chapter V

TECHNICAL LEMMAS

The lemmas used in Chapter IV are collected together in this chapter. §2, 2. - 5. and §3, 2. play an important part.

§1. Domains of type *D*

To each domain is associated a number characterising its surface to volume ratio.

1. Let Ω be a domain in the space ω bounded by a piecewise smooth surface. It is easy to see that

I. *There exists a constant* $D > 0$ *such that, for any* $d_2 > d_1 > 0$,

$$\text{mes}\,[(\Omega - d_1)\setminus(\Omega - d_2)] \leqslant D\,(d_2 - d_1)\,\text{mes}\,\Omega. \quad (5.1.1)$$

In this case we shall say that Ω is of type D. Obviously the domain $\varepsilon\Omega$ homothetic to Ω is of type D/ε.

2. In Ch. IV, §9, we use an estimate of the type of domains constructed in a special way. Let $0 < \varepsilon < \varepsilon_0$, $0 < \mu < \mu_0$;

G_0 is a domain with a piecewise smooth boundary in the space p_0;

G is a domain with a piecewise smooth boundary in the space p;

$p = p_0,\ p_1$;

$G_1(\varepsilon) = \dfrac{\varepsilon}{2}$ is the neighbourhood of a point of the space p_1;

$G^*(\varepsilon) = G_0 \times G_1(\varepsilon)$ is a domain in the space p: $p_0 \in G_0$, $p_1 \in G_1(\varepsilon)$;

A_0 is a diffeomorphism of G_0 onto Ω, $A_0 p_0 = \omega_0 \in \Omega$;

A is a diffeomorphism of G onto Ξ, $Ap = \xi = \xi_0,\ \xi_1 \in \Xi$ depending on μ so that, if $\mu \to 0$, then $|A_0 p_0 - (Ap)_0| = |\omega_0 - \xi_0| \to 0$ uniformly wich respect to $p \in G$ together with derivatives.

3. We shall be interested in the domain $AG^*(\varepsilon)$. The following generalisations of I are easily proved:

II. *There exists a constant* $D > 0$, *not depending on the direction of* ε *and the centre* p_1 *of the domain* $G_1(\varepsilon)$, *such that all domains* $AG^*(\varepsilon)$ *for which* $G^*(\varepsilon) \subset G$ *are of type* D/ε.

III. *There exist constants* $D > 0$, $\bar{\mu} > 0$, $r > 0$, *not depending on* p_1 *and* ε, *such that for* $0 < \mu < \bar{\mu}$, $0 < d < r$ *the domain* $\Xi^*(\varepsilon, d)$ *defined below is of type* D/ε.

Here $\Xi^*(\varepsilon, d)$ consists of points $\xi = \xi_0,\ \xi_1 = Ap$ for which $p \in G^*(\varepsilon) \subset G$, $\xi_0 \subset \Omega - d$.

The proofs of assertions I, II, III are omitted in view of their elementary and cumbersome character.

4. **Strips.** The $\frac{h}{2}$-neighbourhood of the hyperplane will be called a strip Γ of width h. For example, the inequality $|(k, \omega)| < a$ defines a strip of width not greater than $2a$ in the space ω if $|k| \geqslant 1$. It is easy to calculate that the intersection of a strip of width h with one of the n coordinate axes of ω is not longer than nh. Hence, in view of (5.1.1), we obtain

$$\operatorname{mes}(\Omega \cap \Gamma) \leqslant Dnh \operatorname{mes} \Omega. \tag{5.1.2}$$

Let $\Omega' \subset \Omega$. We shall say that Ω' *is of type N in Ω if* $\Omega' = (\Omega - d) \setminus (\bigcup \Gamma_i)$, where $d > 0$ and $\bigcup \Gamma_i$ is the union of not more than N strips. Obviously, for $d_2 > d_1 > 0$,

$$(\Omega' - d_1) \setminus (\Omega' - d_2) \subseteq [(\Omega - d - d_1) \setminus (\Omega - d - d_2)] \cup \bigcup \{[(\bigcup \Gamma_i + d_2) \setminus (\bigcup \Gamma_i + d_1)] \cap \Omega\},$$

and it therefore follows from (5.1.1) and (5.1.2) that

$$\operatorname{mes}[(\Omega' - d_1) \setminus (\Omega' - d_2)] \leqslant D(1 + 2nN)(d_2 - d_1) \operatorname{mes} \Omega. \tag{5.1.3}$$

§2. Arithmetic lemmas

These lemmas express the incommensurability of numbers taken at random.

1. **Integral points.** Vectors with integral components $k_i = 0, \pm 1, \pm 2, \ldots$ will be denoted by $k = k_1, \ldots, k_n$, and the number $= |k_1| + \ldots + |k_n|$ by $|k|$. It is easy to calculate that *the number of different vectors k with $|k| = m \geqslant 1$ does not exceed $2^n m^{n-1}$, and when $|k| \leqslant m$ the number does not exceed $2^n m^n$.*

2. Let Ω be a domain of type D. We denote by Ω_{KN} (where $K > 0$, $N > 0$) the set of points ω from Ω for which

$$|(k, \omega)| \geqslant K |k|^{-\nu} \qquad (\nu = n + 1) \tag{5.2.1}$$

for any integral vectors k, $0 < |k| < N$. Let $d_1, d_2, \ldots > 0$ and $0 < N_1 < N_2 < \ldots$ We introduce the domains $\Omega^{(s)}$ by means of the relationships $\Omega^{(0)} = \Omega$, $\Omega^{(s)} = \Omega^{(s-1)}_{KN_s} - d_s$.

LEMMA. *For any $s \geqslant 1$ and any $d > 0$ the inequality*

$$\operatorname{mes}[\Omega^{(s-1)} \setminus (\Omega^{(s-1)}_{KN_s} - d)] \leqslant LD[K\sigma_s + dN_s^n] \operatorname{mes} \Omega, \tag{5.2.2}$$

is valid, where

$$\sigma_s = \sum_{N_{s-1} \leqslant m < N_s} m^{-2}, \qquad N_0 = 1, \qquad L = n2^{n+2}.$$

3. *PROOF.* We first of all satisfy ourselves that

$$\operatorname{mes}[\Omega^{(s-1)} \setminus \Omega^{(s-1)}_{KN_s}] \leqslant LDK\sigma_s \operatorname{mes} \Omega. \tag{5.2.3}$$

In fact (5.2.1) does not hold in a strip Γ_k of width not greater than

$2K|k|^{-\nu}$. There are not more than $2^n m^{n-1}$ different k's with $|k| = m$. It therefore follows from (5.1.2) that

$$\sum_{|k|=m} \operatorname{mes}(\Omega \cap \Gamma_k) \leqslant \frac{LDK}{m^2} \operatorname{mes}\Omega.$$

Summation over m, $N_{s-1} \leqslant m < N_s$ gives (5.2.3).

We now observe that there are not more than $2^n m^n$ different k's with $|k| \leqslant m$. Consequently $\Omega^{(s-1)}$ is of type $2^n N_s^n$ in Ω (see §1, 4.). From (5.1.3) we obtain

$$\operatorname{mes}[\Omega_{KN_s}^{(s-1)} \setminus (\Omega_{KN_s}^{(s-1)} - d)] \leqslant d\,(1 + 2n2^n N_s^n)\, D \operatorname{mes}\Omega. \tag{5.2.4}$$

(5.2.2) follows at once from (5.2.3) and (5.2.4).

4. The domain Ξ. Let Ξ be a domain of type D in the $n = n_0 + n_1$ dimensional space of points $\xi = \xi_0, \xi_1$. We denote by $\Xi_{\mu KN}$ (where $0 < \mu < 1$, $K > 0$, $N > 0$) the set of points ξ from Ξ for which

$$|(k_0, \xi_0) + \mu(k_1, \xi_1)| \geqslant \begin{cases} K|k|^{-\nu}, & \text{if } |k_0| \neq 0, \\ \mu K|k|^{-\nu}, & \text{if } |k| \neq 0 \end{cases} \quad (\nu = n+1), \tag{5.2.5}$$

for all integral vectors k, $0 < |k| < N$. Let $d_1, d_2, \ldots > 0$ and $0 < N_1 < N_2 \ldots$ We introduce the domains $\Xi^{(s)}$ by means of the relationships

$$\Xi^{(0)} = \Xi, \quad \Xi^{(s)} = \Xi_{\mu KN_s}^{(s-1)} - d_s. \tag{5.2.6}$$

LEMMA. *For any $s \geqslant 1$ and any $d > 0$ the inequality*

$$\operatorname{mes}[\Xi^{(s-1)} \setminus (\Xi_{\mu KN_s}^{(s-1)} - d_s)] \leqslant LD\,[K\sigma_s + dN_s^n] \operatorname{mes}\Xi,$$

is valid, where

$$\sigma_s = \sum_{N_{s-1} \leqslant m < N_s} m^{-2}, \qquad N_0 = 1, \qquad L = n2^{n+2}.$$

The proof is based on the fact that (5.2.5) is violated only in a strip Γ_k of width not greater than $2K|k|^{-\nu}$. In fact, if $|k_0| \neq 0$ the width of Γ_k along one of the first n_0 directions does not exceed $2K|k|^{-\nu}$. If, however, $k_0 = 0$ then μ can be cancelled in (5.2.5) and the width of Γ_k along one of the latter directions will not exceed $2K|k|^{-\nu}$. The subsequent proof proceeds as in 3.

5. The domain $\Xi_{K_0 N_0 d_0}$. In the domain Ξ of 4. we single out the part $\Xi_{K_0 N_0 d_0}$ (where $K_0 \geqslant K$, $0 < N_0 \leqslant N_1$, $d_0 \geqslant 0$) as follows. The point $\xi_0, \xi_1 = \xi \in \Xi$ belongs to $\Xi_{K_0 N_0 d_0}$ if, for any ω_0 for which $|\xi_0 - \omega_0| \leqslant d_0$, we have

$$|(k_0, \omega_0)| \geqslant K_0 |k_0|^{-\nu} \qquad (\nu = n+1)$$

for all integral vectors k_0, $0 < |k_0| < N_0$.

It is easy to verify that *the lemma of 4. remains valid if $\Xi^{(0)} = \Xi_{K_0 N_0 d_0}$ is put in place of $\Xi^{(0)} = \Xi$ in (5.2.6).*

§3. Analytic lemmas

These lemmas enable estimates to be made of the Fourier coefficients and the derivatives of analytic functions in terms of the functions themselves and conversely.

1. Inequalities. *For any* $m > 0$, $\nu > 0$, $\delta > 0$

$$m^{\nu} \leqslant \left(\frac{\nu}{e}\right)^{\nu} \frac{e^{m\delta}}{\delta^{\nu}} \qquad \ln\frac{1}{\delta} \leqslant \frac{\nu}{e}\left(\frac{1}{\delta}\right)^{\frac{1}{\nu}}. \tag{5.3.1}$$

In fact, $f(x) = x - \nu \ln x$ has a minimum at $x = \nu$. Therefore $\frac{e^x}{x^{\nu}} \geqslant \frac{e^{\nu}}{\nu^{\nu}}$. For $x = m\delta$ and $x = |\ln \delta|$ we obtain (5.3.1).

Further, *suppose that, for* $s \geqslant 1$, $\delta_{s+1} = \delta_s^{1+\alpha}$, $\alpha > 0$, *and* $\delta_1 \leqslant 2^{-\frac{1}{\alpha}}$. *Then*

$$\delta_{s+1} \leqslant 2^{-s}\delta_1, \quad \sum_{s=1}^{\infty} \delta_s < 2\delta_1. \tag{5.3.2}$$

In fact, for $\delta_1 \leqslant 2^{-\frac{1}{\alpha}}$, we shall have $\delta_{s+1} = \delta_s^{\alpha}\delta_s \leqslant \delta_1^{\alpha}\delta_s \leqslant 2^{-1}\delta_s$ whence (5.3.2) easily follows.

2. Fourier coefficients. *Let* $f(q) = \sum_k f_k e^{i(k,q)}$ $(q = q_1, \ldots, q_n)$. *Then*:

1) *If* $|f(q)| \leqslant M$ *always holds for* $|\operatorname{Im} q| \leqslant \rho$, *then* $|f_k| \leqslant Me^{-|k|\rho}$.

2) *If* $|f_k| \leqslant Me^{-|k|\rho}$, *then for* $|\operatorname{Im} q| \leqslant \rho - \delta$ *we shall have* $|f| < 4^n\delta^{-n}M$ *(if* $0 < \delta \leqslant \rho \leqslant 1$*)*.

The first result is obtained by displacing the contour of integration in the formula $f_k = (2\pi)^{-n}\int f(q)e^{-i(k,q)}dq$ by an amount $\pm i\rho$. For $\delta \leqslant 1$ we have $(1 + e^{-\delta})(1 - e^{-\delta})^{-1} < 4\delta^{-1}$. Therefore

$$|f| \leqslant \sum Me^{-|k|\delta} = M\Big(1 + 2\sum_{m>0} e^{-m\delta}\Big)^n = M(1 + e^{-\delta})^n(1 - e^{-\delta})^{-n} < 4^n\delta^{-n}M.$$

We introduce the notation $R_N f = \sum_{|k| \geqslant N} f_k e^{i(k,q)}$.

3) *If* $|f_k| \leqslant Me^{-|k|\rho}$, *then, for* $|\operatorname{Im} q| \leqslant \rho - \delta - \gamma$ *(where* $4\delta \leqslant 2\gamma \leqslant \rho \leqslant 1$*) we have* $|R_N f| < \left(\frac{2n}{e}\right)^n \frac{M}{\delta^{n+1}} e^{-N\gamma}$.

For taking account of §2, 1. and (5.3.1), we have

$$|R_N f| \leqslant M \sum_{m \geqslant N} (2m)^n e^{-m(\delta+\gamma)} \leqslant M\left(\frac{2n}{e\delta}\right)^n \sum_{m \geqslant N} e^{-m\gamma} \leqslant \left(\frac{2n}{e}\right)^n \frac{M}{\delta^n} \frac{e^{-N\gamma}}{1 - e^{-\gamma}} <$$
$$< \left(\frac{2n}{e}\right)^n \frac{M}{\delta^{n+1}} e^{-N\gamma},$$

since $1 - e^{-\gamma} > \delta$ for $4\delta \leqslant 2\gamma \leqslant 1$.

3. Cauchy's estimates. *If for* $x \in U$ *the function* $f(x)$ *is analytic and* $|f(x)| \leqslant M$, *then for* $x \in U - \delta$

$$\left|\frac{\partial f}{\partial x}\right| \leqslant \frac{M}{\delta}, \quad \left|\frac{\partial^2 f}{\partial x^2}\right| \leqslant \frac{2M}{\delta^2}. \tag{5.3.3}$$

The proof is obtained from the Cauchy integral $f(x) = \frac{1}{2\pi i}\oint \frac{f(\xi)\,d\xi}{\xi - x}$.

4. Formulae of Lagrange and Taylor. *If $f(x_1, \ldots, x_n)$ is a continuously differentiable* (generally speaking, vectorial: $f = f_1, \ldots, f_m$) *function in the neighbourhood of the segment (a, b) of the space x and $|df| \leqslant C|dx|$ on ab, then $|f(b) - f(a)| \leqslant C|b - a|$. In particular, if $\left|\frac{\partial f_i}{\partial x_j}\right| \leqslant C$, then $|df| \leqslant Cn|dx|$ and*

$$|f(b) - f(a)| \leqslant Cn|b - a|. \tag{5.3.4}$$

For a function $f(x)$ which is twice continuously differentiable in the domain $|x_i - a_i| < |b_i - a_i|$ and for which $\left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| \leqslant \Theta$, the following inequality is valid:

$$\left|f(b) - f(a) - \left(\frac{\partial f}{\partial x}\Big|_a, (b - a)\right)\right| \leqslant \frac{\Theta n^2}{2}|b - a|^2. \tag{5.3.5}$$

Inequalities (5.3.4) and (5.3.5) are equally valid in real and complex domains. For their proof it is sufficient to write the increment in the form of an integral of the derivative.

§4. Geometric lemmas

These lemmas guarantee the single-valued reversibility of change of variables.

1. ε-displacement. *Let U be a closed domain in the Euclidean space R and A a continuous mapping of U in R, with $|Ax - x| \leqslant \varepsilon$. Then the image AU contains $U - \varepsilon$.*

PROOF. Let $x_0 \in (U - \varepsilon)\setminus AU$. Then the mapping $A^*x = x_0 + \varepsilon\frac{Ax - x_0}{|Ax - x_0|}$ is continuous for $|x - x_0| \leqslant \varepsilon$. Therefore the degree $d(t)$ (see [55]) of the mapping A^* of the sphere $S_{t\varepsilon} : |x - \varepsilon| = t\varepsilon$ $(0 < t \leqslant 1)$ onto the sphere S_ε does not depend on t. But $d(1) = 1$ and $d \to 0$ as $t \to 0$, i.e. $X \in AU$.

2. Inversion of ε-displacement. *Suppose that, in the conditions of 1., $|dA| \neq 0$ for $|dx| \neq 0$. Then A is a diffeomorphism of the domain $U - 4\varepsilon$.*

PROOF. Let $x, y \in U - 4\varepsilon$, $Ax = Ay = z$. The sphere D of radius 2ε with centre at the middle of the interval xy lies in $U - \varepsilon$. The image $A_{xy} \subseteq D$ of the interval xy is a closed arc in z which in D shrinks to the point z. It follows from 1. and $|dA| \neq 0$ that the interval xy can shrink into a point leaving its ends in place. Therefore $x = y$.

3. Canonical transformation. *Let G and U be domains of the n-dimensional spaces P and q. If the function $S(P, q)$ in $G \times U$ is analytic and $|S| \leqslant M \leqslant 16^{-1}n^{-1}\beta^2$, the relationships $p = P + S_q$, $Q = q + Sp$ define* *a canonical diffeomorphism* B: $P, Q \to p, q$ *of the domain* $P \in G - 2\beta$, $Q \in U - 3\beta$, *where* $|B - E| \leqslant M\beta^{-1}$, $|dB| < 2|dX|$ $(X = P, Q)$ *and, for* $Q \in U - 3\beta - \delta$, *we have* $|P - p| \leqslant M\delta^{-1}$.

PROOF. For each $P \in G - \beta$ for the mapping B: $q \to q + S_p = Q$ we have by Cauchy (§3, 3.) $|S_p, S_q| \leqslant M\beta^{-1} < 0.2\beta$ $|S_{PP}, S_{Pq}, S_{qq}| \leqslant 2M\beta^{-2} < 4^{-1}n^{-1}$. Therefore $|B_P - E| \leqslant M\beta^{-1} < 0.2\beta$ and $|dB_P| \neq 0$; in accordance with 2., B_P is a diffeomorphism of the domain $q \in U - 1.8\beta$ and by 1. its image contains the domain $Q \in U - 2\beta$. Therefore for $P \in G - \beta$, $Q \in U - 2\beta$ the following mapping B is determined: $P, Q \to p, q = P + S_q, B_P^{-1}(Q)$ (where in S_q after differentiation we put $q = B_p^{-1}(Q)$). For $P \in G - 2\beta$, $Q \in U - 3\beta$, we have $|B - E| < MB^{-1} < 0.2\beta$ and $|dB - dx| < 0.5|dx|$. 3. now follows from 2. and Cauchy's estimates.

4. *Let A be a mapping of the sphere* $U_\varepsilon(x_0)$: $|x - x_0| \leqslant \varepsilon$ *and* $\theta|dx| \leqslant |dA| \leqslant \Theta|dx|$. *Then*

$$U_{\theta\varepsilon}(Ax_0) \subseteq AU_\varepsilon(x_0) \subseteq U_{\varepsilon\Theta}(Ax_0).$$

PROOF. The right-hand inequality follows from Lagrange's formula (§2, 4.). Let $y(t) = y_0 + ty$, where $y_0 = Ax_0$, $0 \leqslant t < \infty$. For small t there exists a continuous branch $A^{-1}(y(t)) \subset U_\varepsilon x_0$, where $A^{-1}y_0 = x_0$. Let $\bar{t}$ be the greatest t for which this is so; then for $\bar{y} = y(\bar{t})$ we shall have $|A^{-1}(\bar{y}) - x_0| = \varepsilon$. But by Lagrange's formula $|A^{-1}(\bar{y}) - A^{-1}(y_0)| \leqslant \theta^{-1}|\bar{y} - y_0|$, i.e. $|\bar{y} - y_0| \geqslant \theta\varepsilon$, which is the result required.

5. **Lemma on the variation of frequency.** *Let A be a diffeomorphism of the domain G of the space p onto the domain* Ω *of the space* ω *and let* $\theta|dp| \leqslant |dA| \leqslant \Theta|dp|$. *Also let* A' *be a mapping of the domain* $G - \beta$: $p \to Ap + \Delta(p)$, *where* $|\Delta(p)| < \beta$ *and* $|d\Delta| < \delta\theta|dp|$. *Let* Ω_0 *be a subdomain of* Ω, $\beta > 0$, $b > 0$, $0 < \delta < 1$. *Then there exist domains* G_1 *and* G' *such that*:

1) $G \supseteq G - \beta \supseteq G_1 \supseteq G' + b \supseteq G'$;

2) A' *is a diffeomorphism of* G_1 *and* $\theta'|dp| < |dA'| < \Theta'|dp|$, *where* $\theta' = \theta(1 - \delta)$, $\Theta' = \Theta(1 + \delta)$;

3) $A'G' = \Omega' = \Omega_0 - d$, *where* $d = 2\Theta b + (5 + \Theta)\beta$; $A'(G' + b) \subset \Omega_0$;

4) $\mathrm{mes}\,(G \setminus G') \leqslant \theta^{-n}\,\mathrm{mes}\,(\Omega \setminus \bar{\Omega}')$, *where* $\bar{\Omega}' = \Omega_0 - \bar{d}$, $\bar{d} = 2\Theta b + (6 + \Theta)\beta$.

PROOF. For $\omega \in A(G - \beta)$ it is obvious that $|A'A^{-1} - E| < \beta$ and $|dA'A^{-1}| > (1 - \delta)|d\omega|$. In accordance with 2., for $\omega \in A(G - \beta) - 4\beta$, the mapping $A'A^{-1}$ is a diffeomorphism and, in accordance with 1., $A'A^{-1}(A(G - \beta) - 4\beta) \supseteq A(G - \beta) - 5\beta$. Therefore A' is a diffeomorphism of $G'' = A^{-1}(A(G - \beta) - 4\beta)$ and $A'G'' \supseteq A(G - \beta) - 5\beta$. But, in accordance with 4., $A(G - \beta) \supseteq \Omega - \Theta\beta$ and therefore $A'G'' \supseteq \Omega = \Omega - (5 + \Theta)\beta$. We now put $G_1 = A'^{-1}\Omega_1$, $G' = A'^{-1}\Omega'$, where $\Omega' = \Omega_1 - d$, $d = 2\Theta b + (5 + \Theta)\beta$. Then $\Omega' + 2\Theta b \subseteq \Omega_1$ and, according to 4. we have $G' + b \subseteq G_1$. The validity of 1), 2), 3) is obvious. Also, in accordance with 1.,

$AG' = AA'^{-1}\Omega' \supseteq \Omega' - \beta = \overline{\Omega}'$; therefore

$$\text{mes}\,(G\setminus G') = \int\limits_{\Omega\setminus AG'} \left(\det\left|\frac{\partial A}{\partial p}\right|\right)^{-1} d\omega \leqslant \theta^{-n}\,\text{mes}\,(\Omega\setminus AG') \leqslant \theta^{-n}\,\text{mes}\,(\Omega\setminus\overline{\Omega}').$$

6. Lemma on distortion. *Let the linear transformation D*

$$z = x,\; y \to x',\; y' = z, \textit{where } x' = Ax, \quad y' = Bx + Cy, \quad z' = Dz,$$

be composed of transformations A, B, C, for which

$$\theta_A|x| \leqslant |Ax| \leqslant \Theta_A|x|, \qquad |Bx| \leqslant \Theta_B|x|, \qquad \theta_C|x| \leqslant |Cx| \leqslant \Theta_C|x|.$$

Then

$$\theta_D|z| \leqslant |Dz| \leqslant \Theta_D|z|, \tag{5.4.1}$$

where $\Theta_D = \Theta_A + \Theta_B + \Theta_C$ *and* $\theta_D^{-1} = \theta_A^{-1} + \theta_C^{-1} + \theta_A^{-1}\theta_C^{-1}\Theta_B$.

PROOF. The right-hand inequality in (5.4.1) is obvious and the left-hand inequality follows from the fact that D^{-1} is of the form

$$z' = x', y' \to x, y = z, \quad \text{where} \quad x = A^{-1}x', \quad y = C^{-1}y' - C^{-1}BA^{-1}x', \quad z = D^{-1}z'.$$

7. Lemma on polar coordinates. *Let*

$$p = \sqrt{2\tau}\cos\varphi, \quad q = \sqrt{2\tau}\sin\varphi. \tag{5.4.2}$$

If $f(p, q)$ is analytic for $|x| \leqslant R$ ($x = p, q$; $|x| = \max\{|p|, |q|\}$), then $f[p(\tau, \varphi), q(\tau, \varphi)]$ is analytic for

$$|\tau - \tau_0| < \tau_0, \quad |\operatorname{Im}\varphi| \leqslant 1 \quad \left(\tau_0 = \left(\frac{R}{4}\right)^2\right) \tag{5.4.3}$$

and furthermore, for $|\operatorname{Im}\varphi| \leqslant 1$, we have

$$\sqrt{|\tau|} \leqslant |x| \leqslant \sqrt{8|\tau|}. \tag{5.4.4}$$

PROOF. In the domain (5.4.3) the functions (5.4.2) are single-valued and $|x| \leqslant \sqrt{|2\tau|}\,\text{ch}\,1 < \sqrt{8|\tau|} \leqslant \sqrt{16\tau_0} = R$. Furthermore $2\tau = p^2 + q^2$ and therefore $\sqrt{|\tau|} \leqslant |x|$.

8. Further note on polar coordinates. *Suppose that, in a real domain,*

$$p_i = \sqrt{2\tau_i}\cos\varphi_i, \quad q_i = \sqrt{2\tau_i}\sin\varphi_i,$$
$$x_i = p_i, q_i \ (i = 1, 2), \quad \Delta z = z_1 - z_2 \ (z = p, q, \tau, \varphi \textit{ or } x).$$

If $|x_i| \leqslant \frac{1}{2}$, $|\Delta x| \leqslant \sqrt{\tau_i}$, *then* $|\Delta\tau| \leqslant |\Delta x|$, $|\Delta\varphi| \leqslant \frac{2|\Delta x|}{\sqrt{\tau_1}}$.

PROOF. We put $r_i = \sqrt{2\tau_i}$. For $|x_i| \leqslant \frac{1}{2}$, it is obvious that $|r_i| \leqslant \frac{\sqrt{2}}{2}$ and $|\Delta\tau| = \left|\Delta\frac{r^2}{2}\right| \leqslant \frac{\sqrt{2}}{2}|\Delta r| \leqslant |\Delta x|$. Also, for $\sqrt{2}|\Delta x| \leqslant \sqrt{2\tau_1} \leqslant r_1$ we have $|\Delta\varphi| \leqslant \arcsin\frac{\sqrt{2}|\Delta x|}{r_1} \leqslant \frac{2|\Delta x|}{\sqrt{\tau_1}}$, as required.

§5. Convergence lemmas

1. *Let a sequence of domains $F^{(s)}$ and diffeomorphisms*

B_s: $F^{(s)} \to F^{(s-1)}$ $(s = 1, 2, \ldots)$ *be given. We suppose that:*
1) $|B_s - E| < d_s$, 2) $F^{(s)} \subseteq F^{(s-1)} - d_s$, 3) $|dB_s| < 2|dx|$, 4) $d_s < C4^{-s}$;
Then the sequence $S_s = B_1B_2\ldots B_s$ $(s = 1, 2, \ldots)$ *converges uniformly on* $F^{(\infty)} = \bigcap F^{(s)}$ *to a continuous mapping* S_∞ *for which* $|S_\infty - E| < C$.

PROOF. Let $x \in F_s$. From 1) it follows that $|B_s x - x| < d_s$. From 2) it follows that Lagrange's formula (§3, 4.) is applicable to the interval x, $B_s x$ and to the mapping S_{s-1}. From 3) it follows that $|dS_{s-1}| < 2^s|dx|$. In accordance with 4) $|S_s x - S_{s-1}x| = |S_{s-1}B_s x - S_{s-1}x| < 2^s d_s < C2^{-s}$, which it was required to prove.

2. *Let* $F = d$ *be a neighbourhood of the interval* $x = x_0 + vt$, $0 \leqslant t \leqslant \frac{d}{\varepsilon}$. *Let* $Y(x)$ *be a smooth vector field in* F *and* $|Y - v| \leqslant \varepsilon$. *We denote by* $x(t)$ *the solution of the equation* $\frac{dx}{dt} = Y(x)$ *with initial condition* $x(0) = x_0$. *Then* $|x(t) - (x_0 + vt)| \leqslant d$ *for* $0 \leqslant t \leqslant \frac{d}{\varepsilon}$.

PROOF. We consider $y(t) = x(t) - (x_0 + vt)$. Suppose that for $t < t_0$ we always have $|y(t)| < d$ and $|y(t_0)| = d$. Since for $t < t_0$ we have $\left|\frac{dy}{dt}\right| \leqslant \varepsilon$ and $y(0) = 0$, then by Lagrange's formula $|y(t_0)| \leqslant \varepsilon t_0$ and thence $t_0 \geqslant \frac{d}{\varepsilon}$, which it was required to prove.

3. Suppose that in the conditions of 1. in $F^{(0)}$ a smooth vector field $Y^{(0)}(x)$ is given, defining the motion $S^t_{(0)}(x)$: $\frac{d}{dt}S^t_0(x) = Y^{(0)}(S^t_0(x))$, $S^0_0 x = x$. There naturally arise motions $S^t_s = S_s^{-1}S^t_0 S_s$ and corresponding fields $Y^{(s)}$ on $F^{(s)}$: $Y^{(s)}(X^{(s)}) = \frac{d}{dt}(S^t_s X^{(s)})\Big|_{t=0}$

We suppose that: 5) *the sequence* $Y^{(s)}(X)$ *converges as* $s \to \infty$, $X \in F^{(\infty)}$ *to* $Y^{(\infty)}(X)$ *and on* $F^{(\infty)}$ *we have* $|Y^{(s)} - Y^{(\infty)}| < d_{s+1}$; 6) *the interval* $x = x_0 + vt$, $0 \leqslant t \leqslant 1$, *belongs to* $F^{(\infty)}$ *and on this interval* $Y^{(\infty)} = v$; 7) *on* $F^{(s)}$ *we have* $\frac{\partial Y^{(s)}}{\partial X^{(s)}} \leqslant \Theta$, *where the constant* Θ *does not depend on* s.

Then for $0 \leqslant t \leqslant \frac{1}{1+\Theta}$ *we have* $S^t_0(S_\infty x_0) = S_\infty(x_0 + vt) \subset F^{(0)}$.

PROOF. We shall show that

$$|S^t_s x_0 - (x_0 + vt)| \leqslant d_{s+1} \text{ for } 0 \leqslant t \leqslant \frac{1}{1+\theta}. \tag{5.5.1}$$

On the interval $x_0 + vt$ $0 \leqslant t \leqslant \frac{1}{1+\Theta}$, in accordance with 5) and 6), $|Y^{(s)} - v| \leqslant d_{s+1}$. In the d_{s+1}-neighbourhood of this interval (which belongs to $F^{(s)}$ by 2) of 1.) by Lagrange's formula we find from 7) that $|Y^{(s)} - v| \leqslant (1 + \Theta)d_{s+1}$. Putting $d = d_{s+1}$ and $\varepsilon = (1 + \Theta)d_{s+1}$ in 2., we obtain (5.5.1).

In view of (5.5.1) and 2) the interval with ends $x_0 + vt$, $S^t_s x_0$ belongs to the domain $F^{(s)}$. By Lagrange's formula, in view of conditions 3) and

4), we have $|S_s S_s^t - S_s(x_0 + vt)| \to 0$ as $s \to \infty$, $0 \leqslant t \leqslant \frac{1}{1+\Theta}$, which it was required to prove.

4. Measure of the limit. *Let F be a compactum in the Euclidean space R and S_s $(s = 1, 2, \ldots)$ be a sequence of continuous mappings of F on $F^{(s)} \subset R$, converging uniformly to the mapping S_∞ on $F^{(\infty)}$. Then* $\operatorname{mes} F^{(\infty)} \geqslant \overline{\lim} \operatorname{mes} F^{(s)}$.

PROOF. For $\delta < \delta(\varepsilon)$ and any $\varepsilon > 0$, $\operatorname{mes}(F^{(\infty)} + \delta) < \operatorname{mes} F^{(\infty)} + \varepsilon$. By virtue of uniform convergence, for sufficiently large $s(\delta) F^{(s)} \subseteq F^{(\infty)} + \delta$, we obtain

$$\operatorname{mes} F^{(s)} < \operatorname{mes}(F^{(\infty)} + \delta) < \operatorname{mes} F^{(\infty)} + \varepsilon,$$

as required.

§6. Notation

The basic notation, systematically used in Chapters IV and V, is listed below.

1. Functions. All the functions considered are assumed to be complex-analytic and real for real values of the arguments. We consider n-dimensional complex spaces of *canonically conjugate variables* $p = p_1, \ldots, p_n$ and $q = q_1, \ldots, q_n$, denoted also by $x = p_1, \ldots, q_n = x_1, \ldots, x_{2n}$. These variables are split into two groups: n_0 *"rapid" variables* $p_1, \ldots, p_{n_0}$ comprise the vector p_0, and the remaining n_1 *"slow" variables* $p_{n_0+1}, \ldots, p_n$ the vector p_1. Similarly we define q_0, q_1 and $x_0 = p_0, q_0$; $x_1 = p_1, q_1$ (the use of p_1 both as a vector and as a component does not lead to confusion since components are scarcely ever used). We also consider n-dimensional *spaces of frequencies* $\omega = \omega_0, \omega_1$ and of reduced frequencies $\xi = \xi_0, \xi_1$, where $\omega_0 = \xi_0$, $\omega_1 = \mu \xi_1$ and μ is a small parameter. The maximum of the modulus of coordinates $|x| = \max_j |x_j|$ serves as *the norm* in all these spaces.

The functions considered are of period 2π with respect to the variables q (or q_0) and are expressible in terms of *a Fourier series* as follows:

$$f(q) = \sum_k f_k e^{i(k,q)} = f_0 + \sum' f_k e^{i(k,q)} = \bar{f} + \tilde{f}(q) = [f(q)]_N + R_N f(q),$$

where $\bar{f} = f_0$; $(k, q) = \sum_{j=1}^{n} k_j q_j$, $\Sigma' = \sum_{k \neq 0}$, $[\,]_N = \sum_{|k| < N}$, and k is a vector with integral coordinates k_j. The modulus $|k| = \sum_{j=1}^{n} |k_j|$ serves as the norm in the space conjugate to q of numbers of harmonics of k. We use abbreviated notation of the type

$$\frac{\partial f}{\partial x} = f_x = \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_m},$$

where f is a numerical or vector function $f(x_1, \ldots, x_m)$.

2. **Domains.** Let U be a compact complex domain, i.e. a bounded domain in a complex numerical space considered together with its boundary. If $d > 0$, we denote by $U + d$, $U - d$ the d-neighbourhood of U and the set of points belonging to U and the d-neighbourhood. If U_1 and U_2 are two domains then $U_1 \bigcup U_2$ denotes their union, $U_1 \bigcup U_2$ their common part and $U_1 \quad U_2$ that part of U_1 not contained in U_2; $U_1 \subseteq U_2$ denotes that U_1 is contained in U_2, and $u \in U$ that the point u belongs to U; $U_1 \times U_2$ denotes the set of pairs u_1, u_2 where $u_1 \in U_1$, $u_2 \in U_2$.

Re U denotes the intersection of U with the real space, Im the imaginary part, and mes U *the Lebesgue measure of* Re U *even when* U *is complex.*

The letters G, G_0, G_1; Ω; Ξ, Ξ_0, Ξ_1 denote compact domains in the complex spaces p, p_0, p_1; ω_0; ξ, ξ_0, ξ_1, respectively. The letter F denotes domains in the space x specified by conditions of the type $p \in G$, $|\text{Im } q| \leqslant \rho$. *The points* q *and* $q + 2\pi k$ *are* thereby *identified so that* mes $F = (2\pi)^n$ mes G.

A list of special notation for the domains is given in Ch. IV, §12, 2. (see also Ch. V, §1, 2.).

3. **Mappings.** The mappings considered are given by analytic functions. A one-to-one mapping which is continuously differentiable together with its inverse at each point is called a diffeomorphic mapping or diffeomorphism of the compact domain U_1 on U_2. The differential of the mapping A at the point x is a linear operator $dx \to dA = \dfrac{\partial A}{\partial x} dx$.

Diffeomorphisms of the domain G of the form $p \to \omega$ and $p \to \xi$ are denoted by the letter A. The letters B and S denote diffeomorphisms of the domains F that are canonical transformations (see, for example, [14]). E denotes the identity mapping. $A \leqslant E$ denotes that $|Ax| \leqslant |x|$ for any x.

By means of the substitutions B and S we introduce new variables into the space x denoted by X, x^*, X^*, $X^{(s)}$ and so on. The variables X_0, X_1; P_0, P_1; Q_0, Q_1 then correspond to x_0, x_1; p_0, p_1; q_0, q_1 etc.

4. **Constants.** The numbers ρ, θ, Θ, C, x are positive constants. The numbers β, γ, δ, ε, μ are very small in comparison with these positive constants and $\gamma \gg \delta \gg \varepsilon \gg \beta \gg \mu$. The numbers N are large and positive.

L and ν denote constants that are large, positive and absolute (i.e. depending only on the number n of degrees of freedom).

The index s enumerates approximations.

Chapter VI

APPENDIX

This chapter consists of remarks concerning a number of solved and unsolved problems. The five sections are independent of each other.

In §1 it is shown why the motion in integrable problems of dynamics is always conditionally periodic.

Certain unsolved problems are considered in §2: the possibility of topological instability of many-dimensional systems (zones of instability) is discussed and also two model problems in which the perturbations are large (the problem of mappings of a circle onto itself and the reducibility of equations with conditionally periodic coefficients). The results of C.L. Siegel and E.G. Belaga on the linear normal form to which a system of differential equations in the neighbourhood of a position of equilibrium, periodic solution or conditionally periodic motion can be reduced are reported on in §3.

Certain mechanisms giving rise to intermixing are discussed in §4. Finally, in §5 a short description is given of the smoothing techniques which enabled J. Moser to weaken the requirement that the Hamiltonian should be analytic.

§1. Integrable systems

We give here an explanation why conditionally periodic motions always arise in integrable problems of dynamics.

J. Liouville proved (see [4]) that if, in the system with n degrees of freedom,

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \qquad (p = p_1, \ldots, p_n;\ q = q_1, \ldots, q_n) \tag{6.1.1}$$

n first integrals in involution (see 1.),

$$H = F_1, F_2, \ldots, F_n, \quad (F_i, F_j) = 0, \tag{6.1.2}$$

are known, then the system is integrable by quadratures. Many examples of integrable problems are known. In all these examples the integrals (6.1.2) can be found.

It was pointed out long ago that the manifolds in these examples, specified by the equations $F_i = f_i$ = const, turn out to be tori, and motion along them is conditionally periodic. We shall prove, following [20], that such a situation is unavoidable in any problem admitting single-valued integrals. (6.1.2). The proof is based on simple topological arguments.

1. Notation. A point of the $2n$-dimensional Euclidean space p, q will be denoted by $x = x_1, \ldots, x_{2n}$. We shall denote by grad F the vector gradient $F_{x_1}, \ldots, F_{x_{2n}}$ of the function $F(x)$. The Hamiltonian equations (6.1.1) then take the form

$$\dot{x} = I \operatorname{grad} H, \quad \text{where } I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \tag{6.1.3}$$

and E is the unit matrix of order n.

We introduce the *skew-scalar product* of two vectors x, y:

$$[x, y] = (Ix, y) = -[y, x], \tag{6.1.4}$$

which, as can easily be verified, expresses the sum of the areas of the projections of the parallelogram with sides x, y onto the coordinate planes $p_i q_i$ ($i = 1, \ldots, n$).

Linear transformations S which preserve the skew-scalar product (so that $[Sx, Sy] = [x, y]$ for all x, y) are called *simplicial*. For example, the transformation with matrix I is simplicial.

The skew-scalar product of the gradients [grad F, grad G] is called the *Poisson bracket* (F, G) of the functions F and G. Obviously, F is a first integral of the system (6.1.3) if and only if its Poisson bracket (F, H) with the Hamiltonian vanishes identically. If the Poisson bracket of two functions vanishes identically, the functions are said *to be in involution.*

2. THEOREM. *Let the Hamiltonian system with n degrees of freedom (6.1.3) have n single-valued first integrals (6.1.2) which are pairwise in involution. Let the equations* $F_i = f_i = \text{const}$ $(i = 1, \dots, n)$ *define in the 2n-dimensional space x an n-dimensional compact manifold* $M = M_f$ *at each point of which the gradients* grad F_i $(i = 1, \dots, n)$ *are linearly independent.*

Then M is an n-dimensional torus and the point $x(t)$ *representing a solution of the equations (6.1.3) has a conditionally periodic motion along it.*

PROOF. A) *M is parallelizable*, i.e. it has n tangential vector fields linearly independent at each point.

For, let us consider the system (6.1.3) with Hamiltonian F_i. Since $(F_i, F_j) = 0$, all the functions F_j are first integrals and every trajectory lies wholly on M. Therefore the velocity field I grad F_i touches M (Fig. 17). On account of the non-degeneracy of I the vectors I grad F_i $(i = 1, \dots, n)$ are at each point linearly independent.

B) *Let D be a surface in M and* Γ *its boundary. Then* $\oint pdq$ *(i.e. the sum of the areas of the projections of D onto the coordinate planes* p_i, q_i $(i = 1, \dots, n)$*) is zero.*

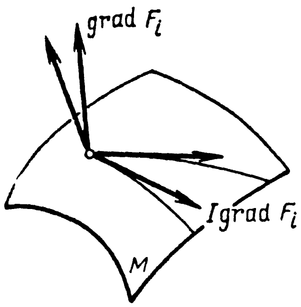

Fig. 17.

It is sufficient to prove this result for infinitely small parallelograms lying in M. If Γ is a parallelogram with sides ξ, η, then the sum of the areas of the projections is the skew-scalar product of ξ and η:

$$\oint_\Gamma p\,dq = [\xi, \eta].$$

Suppose now that ξ and η touch M at a certain point. In accordance with A) any vector tangential to M is a linear combination of the n vectors I grad F_i. But these vectors are skew-orthogonal since, in accordance with (6.4.2),

$$[\operatorname{grad} F_i, \operatorname{grad} F_j] = 0,$$

and hence, since Γ is simplicial,

$$[I \operatorname{grad} F_i, I \operatorname{grad} F_j] = 0.$$

Therefore $[\xi, \eta] = 0$, as required.

C) *The vector fields I* grad F_i *are irrotational* (they are gradients of many-valued functions).

In accordance with B) $\int_{q_0}^{q} p\,dq$ does not depend on the path of integration lying on M_f in the neighbourhood of the point q_0. The integral can therefore be regarded as a many-valued function $S(q;\ f)$. The equations $p = \frac{\partial S}{\partial q}$, $g = \frac{\partial S}{\partial f}$ in each neighbourhood define a canonical transformation $f,\ g \to p,\ q$ with generating function S (see [4]). In the variables $f,\ g$ the Hamiltonian F_i is f_i and the Hamiltonian equations reduce to

$$\dot{g}_i = 1, \quad \dot{g}_j = \dot{f} = 0 \quad (j \neq i). \tag{6.1.5}$$

In view of the linear independence of the velocity fields I grad F_i the differentials dg_i are at each point of M linearly independent. We shall consider the g_i as local coordinates on M. At the intersection of two neighbourhoods they differ by constants so that the differentials dg_i are defined in the large. The functions g_i, however, are many-valued on M.

In g coordinates the vector fields I grad F_i on M are, like the gradients of g_i, irrotational. The manifold M is thus parallelizable by means of irrotational fields. Hence it is easy to deduce that M is a torus.

D) LEMMA. *Suppose that on the n-dimensional manifold M there exist n differentials dg_i linearly independent at each point* (closed differential forms of the first degree). *Then this manifold is a direct product of n circles and straight lines.*

For, let O be a point of M and N a universal covering space of M. To each path OA on M there corresponds a path $O'A'$ on N. The functions $g_i(A') = \int_{O'A'} dg_i$ transform N into a Euclidean space $g_1, \ldots, g_n$. It is easy to verify (cf. [60]) that in consequence of the independence of the dg_i:

α) this mapping is onto the whole space g;

β) each point of g corresponds to only one point A' of N;

γ) if O' and O'' of N cover O and $g(O''') = g(O') + g(O'')$, then O''' covers O.

In view of α) and β) N can be identified with the Euclidean space $g_1, \ldots, g_n$ and in view of γ) points covering O form a lattice in N (the aggregate of integral linear combinations of k linearly independent vectors). Obviously points A' and A'' of N cover a certain point A if and only if $A' - A''$ is a vector of the lattice. On identifying these points in the Euclidean space N we obtain the direct product of k circles and $n - k$ straight lines. The lemma is now proved.

E) *Completion of the proof of the theorem.* In accordance with C) the manifold M_f satisfies the conditions of the lemma. Being compact it is an n-dimensional torus. Since, in accordance with (6.3.5), the coordinates g vary uniformly in the motion (6.3.3), this motion is conditionally periodic. The theorem is now proved.

3. Remarks. A) *Action-angle variables.* The set of points of the $2n$-dimensional space x, where the n vectors grad F_i are linearly dependent, is in general of dimension $n - 1$. The n-dimensional manifolds M_f, on which the grad F_i are linearly dependent, are therefore an exception (the n-dimensional and $(n-1)$-dimensional manifolds in $2n$-dimensional space do not, in general, intersect). Thus, in the "general case" a $(2n-1)$-dimensional singular set divides the $2n$-dimensional phase space of the integrable system into domains which are a direct product of the n-dimensional torus and part of the n-dimensional Euclidean space. It is known that action-angle variables $I - \omega$ (denoted by p, q in Ch. I, §1) can be introduced without difficulty into such a domain.

For, let us choose the torus M_f and consider the n integrals

$$\Delta_i S(f) = \oint_{\gamma_i} p\, dq \qquad (i = 1, \ldots, n)$$

over the basic cycles γ_i of the torus M_f. It is obvious that the quantities

$$I_i(x) = \frac{1}{2\pi} \Delta_i \ S(F_1(x), \ldots, F_n(x)) \qquad (i = 1, \ldots, n),$$

being functions of F_j, are themselves first integrals in involution.

Let us assume that the $I_i(x)$ are functionally independent. Then the arguments of 2. can be applied. In C) of 2. a canonical transformation p, $q \to I$, w is constructed (the variables I, w being denoted in 2. by f, g). A circuit of the cycle γ_i gives to the variable w_j an increment

$$\Delta_i w_j = \Delta_i \frac{\partial S}{\partial I_j} = \frac{\partial}{\partial I_j} \Delta_i S = \frac{\partial 2\pi I_i}{\partial I_j} = 2\pi \delta_{ij}.$$

Therefore the variables w are angular coordinates on the torus.

B) *Canonical structures.* For simplicity we restricted ourselves in 1. and 2. to the case in which the phase space p, q is Euclidean. It is, however, easy to verify that all the conclusions remain valid in the more general case when the phase space is a differentiable manifold on which a canonical structure is given (i.e. a non-degenerate closed 2-form is chosen which plays the role of $dp \wedge dq$).

C) *"Dynamical systems" and classical dynamics.* The theorem of 2. has a simple group-theoretical basis. The Poisson brackets form a Lie algebra and the motions described by the Hamiltonian equations the corresponding Lie group. The commutativity of the motions with Hamiltonians F_i follows from (6.4.2). Compact uniform spaces where a Lie commutative group acts transitively are tori.

It would be interesting to consider from this point of view the more general case when there exists a closed set of m single-valued first integrals $F_1, \ldots, F_m$ not in involution:

$$(F_i, F_j) = \varphi_{ij}(F_1, \ldots, F_m).$$

The functions φ_{ij} define a Lie algebra. *What kind of algebra is this and what restrictions does it impose on the topology of the invariant manifold* M_f ($F_i = f_i =$ const) *and on the motions* (6.3.3) *on it*? It can be shown that these motions preserve a certain measure on M_f. A one-parameter group of smooth transformations of the manifold preserving measure is called a "dynamical system".

What "dynamical systems", defined by differential equations on a *smooth manifold, can arise in problems of mechanics?*

We know that geodesic flows (see, for example, [3], [33]) and conditionally periodic motions can be obtained. A number of other dynamical systems (see [34]) have also been investigated recently. Is it possible that they are encountered in mechanics and in particular in "natural systems" with Hamiltonian $H = T + U$ (where T is the kinetic and U the potential energy)?

§2. Unsolved problems

There are many problems to which methods already worked out are applicable. We note, for example, G.D. Birkhoff's "billiards problem" [3], the establishment of non-ergodicity of the geodesic flow on a convex analytic manifold [3], the problem of "magnetic surfaces" ([35], [36]), various problems of the dynamics of a solid body [37], and numerous problems of celestial mechanics. We shall not dwell on these problems, but consider more fundamental questions.

1. Zones of instability. The fundamental and primary problem arising in connection with the contents of the preceding chapters is as follows:

PROBLEM I. *Does there exist a real instability in many-dimensional problems of perturbation theory when the invariant tori do not divide the phase space?*

We have already indicated (Ch. I, §10) that the first uninvestigated case is the problem of the stability of a fixed point of a canonical mapping of a four-dimensional space onto itself. In many-dimensional problems the invariant tori are of at least two dimensions fewer than the phase space: the tori lie in the phase space as lines in a three-dimensional space (Fig. 18).

It can be supposed that topological instability is a typical case: certain trajectories beginning in the gaps between invariant tori can go a long way, since all the gaps merge into a connected set extending to infinity (see Fig. 18). The topological instability of planetary motions would, in particular, follow from the validity of this hypothesis. The results of Chapter III do not exclude the possibility that an arbitrarily small change in the initial conditions can completely change the character of the motion for an infinite time. In Chapter III we proved only that such a change in the initial conditions must be of a highly special form: the majority of changes in the initial conditions leave the motion conditionally periodic. We can say that we have proved the "metric stability" of motion, i.e. stability for all initial conditions except for a set of small measure. We can now formulate our hypothesis as follows:

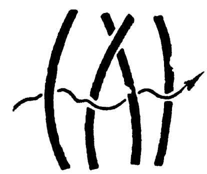

Fig. 18.

A typical case in many-dimensional problems of perturbation theory is the combination of topological instability with metric stability of conditionally periodic motions.

The verification of this hypothesis requires a more detailed consideration of the asymptotic methods adapted for the case of resonance.

2. Invariant tori divide the phase space. This case also merits a detailed study of the zones of instability. Let us for definiteness consider Birkhoff's problem (Fig. 6). Since the time of Poincaré it has been known that the general case of the behaviour of the separatrices of two neighbouring hyperbolic points is the intricate network depicted[1] in Fig. 19. Strictly speaking, the proofs of the non-existence of the first integrals and the divergence of the series of perturbation theory [10], [38] are based on this idea, but the presence in an analytic system of a general type of hyperbolic non-singular points in any neighbourhood of O has not yet been rigorously proved. The existence of zones of instability is thereby not proved, nor the fact the separatrices intersect in the general case. I do not doubt that this is so, but it should be stated that no strict proof exists in the literature.

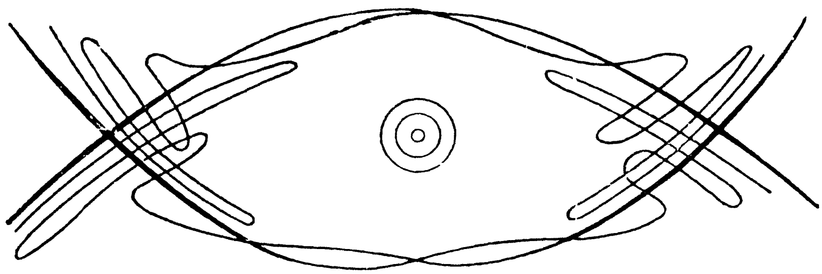

Fig. 19.

A study of the behaviour of a typical trajectory in a zone of instability from the point of view of ergodic theory is also of interest. As the simplest model we can consider the rearrangement of the parts $\Delta_1 = [0, a)$, $\Delta_2 = [a, b)$, $\Delta_3 = [b, 1)$ of the interval $[0, 1)$ in the order Δ_3, Δ_2, Δ_1.

3. Large perturbations. The destiny of invariant tori is traced out in the preceding chapters for very small values of the perturbation parameter μ. Are conditionally periodic motions retained also for large perturbations?

PROBLEM II. *Do there exist bounded motions filling a set of positive measure in the three-body (and n-body) problem with arbitrary mass values and mutual distances comparable with one another*? Is there a critical value of μ at which the invariant torus is destroyed? No answers have yet been found to these questions even in model problems, two of which we shall briefly consider here.

[1] On discovering this Poincaré wrote: "The intersections form a kind of lattice, web or network with infinitely tight loops; neither of the two curves must ever intersect itself, but it must bend in such a complex fashion that it intersects all the loops of the network infinitely many times.

One is struck by the complexity of this figure which I am not even attempting to draw. Nothing can give us a better idea of the complexity of the three-body problem and of all problems of dynamics where there is no holomorphic integral and Bolin's series diverge" ([1], vol. III, p. 389).

We have borrowed Fig. 19 from the work of V.K. Mel'nikov [35].

4. Analytic mappings of a circle onto itself. On the circle we introduce an angular coordinate φ (mod 2π). Then a rotation through an angle $2\pi\omega$ is written in the form

$$T_0: \quad \varphi \longrightarrow \varphi + 2\pi\omega. \tag{6.2.1}$$

Let us consider the more general mapping of a circle onto itself – a rotation through a variable angle $f(\varphi)$:

$$T: \quad \varphi \longrightarrow \varphi + f(\varphi), \tag{6.2.2}$$

where $f(\varphi)$ is a function of period 2π. If $\varphi_1 + f(\varphi_1) < \varphi_2 + f(\varphi_2)$ always holds for $\varphi_1 < \varphi_2$, the transformation (6.2.2) is one-to-one.

We are interested in the sequence of points on the circle φ, $T\varphi$, $T^2\varphi, \ldots$, obtained from φ by iterations of the transformation T. Poincaré [2] proved that there always exists a time average of the angle of rotation, the so-called *rotation number*

$$2\pi\omega = \lim_{n\to\infty} \frac{f(\varphi) + f(T\varphi) + \ldots + f(T^{n-1}\varphi)}{n}.$$

The arithmetic properties of the number ω have an essential influence on the behaviour of the points $T^n\varphi$. We assume that ω satisfies the usual requirements of irrationality: for a certain $K > 0$,

$$|n\omega + m| \geqslant K|n|^{-2}. \tag{6.2.3}$$

PROBLEM III. *Let the mapping T be analytic. Is it possible to convert T into a rotation T_0 through an angle $2\pi\omega$ by an analytic change of variable S:*

$$STS^{-1} = T_0 \ ? \tag{6.2.4}$$

In (6.2.4) S is the mapping $\varphi \to \varphi + g(\varphi)$, the function $g(\varphi)$ is analytic, of period 2π and $|g'| < 1$. A. Denjoy [39] proved that a change of variable S, satisfying (6.2.4), exists and is continuous. A. Finzi [40] established that S is continuously differentiable. The analyticity of S can be proved only on the assumption that T differs little from a rotation. In this connection see [18].

5. Generalization of Floquet's theory. Suppose that we are given a matrix $A(q)$ analytically dependent on the point $q = q_1, \ldots, q_k$ of a k-dimensional torus (so that $A(q + 2\pi) = A(q)$). Let the point $q(t)$ have conditionally periodic motion over the torus with frequencies $\dot{q} = \omega$ ($\omega = \omega_1, \ldots, \omega_k$).

Let us consider the system of linear differential equations with conditionally periodic coefficients

$$\dot{x} = A[q(t)]x \qquad (x = x_1, \ldots, x_n). \tag{6.2.5}$$

The system (6.2.5) is called *reducible* to the system with constant coefficients

$$\dot{y} = By \qquad (y = y_1, \ldots, y_n), \tag{6.2.6}$$

if there exists a matrix $C(q)$ that is analytic on the torus and such that the substitution $x = C(q)y$ turns (6.2.5) into (6.2.6).

In the case of periodic coefficients ($k = 1$) any system is reducible in accordance with the Floquet-Lyapunov theorem (at any rate with complex $C(q)$; in the real case $C(q)$ can be of period 4π).

If $k \geqslant 2$, the coefficients of (6.2.5) are conditionally periodic and the system can turn out to be irreducible even for $n = 1$ (through the fault of the small denominators). But if the frequencies ω satisfy the usual arithmetic requirements of the type (6.2.3), then the system (6.2.5) is reducible for $n = 1$. We shall assume that the arithmetic requirements are satisfied.

PROBLEM IV.[1] *Is the system always reducible for k, $n > 1$?*

A.E. Gel'man [41] has proved that for $n = 2$ the reducible $A(q)$ fill a certain domain in the functional space of all matrices on the torus. L.Ya. Adrianova [42] has extended this result to the case $n > 2$. If domains of irreducible $A(q)$ exist, the question of their normal form arises. It would also be of interest to investigate the more general question of the normal form of the linear system (6.2.5) in which $q = q(t)$ is the phase point of a certain dynamical system. Such a problem arises naturally in the study of the neighbourhood of an invariant manifold as is seen below.

§3. Neighbourhood of an invariant manifold

A qualitative investigation of the behaviour of solutions of a system of ordinary differential equations usually begins with the finding of individual, particularly simple solutions: positions of equilibrium and periodic trajectories. After that the distribution of integral curves in the neighbourhood of these solutions is investigated which sometimes leads to important conclusions regarding the behaviour of the solutions as a whole.

It is also of interest to consider the behaviour of integral curves in the neighbourhood of invariant manifolds of a more complex structure than that of fixed points and periodic trajectories. The next case in order of complexity is the torus filled with conditionally periodic trajectories. We shall set forth here the results obtained by E.G. Belaga [22] in the study of the neighbourhood of such a torus. Analogous results for positions of equilibrium and periodic trajectories were obtained earlier by C.L. Siegel [9].

1. The neighbourhood of positions of equilibrium. Let us consider an analytic system of ordinary differential equations with the point O as its position of equilibrium. In the first approximation the equations are linear:

$$x = Ax + O(x^2) \qquad (x = x_1, \ldots, x_n), \tag{6.3.1}$$

where A is a constant matrix. If the eigenvalues λ_i of the matrix A are distinct, then by a linear transformation $y = Bx$ the matrix A can be reduced to diagonal form

$$y = \Lambda y + O(y^2) \qquad (y = y_1, \ldots, y_n), \tag{6.3.2}$$

---

[1] D.G. Mackie kindly informed me of A.M. Gleason's example which leads to a negative answer if smoothness of $A(q)$ is not assumed. We are interested in the case in which the matrix $A(q)$ depends analytically on q.

where Λ is the diagonal matrix formed by the eigenvalues.

For the linear system

$$\dot{z} = \Lambda z \tag{6.3.3}$$

an investigation of the structure of the neighbourhood of O presents no difficulty. But will the term $O(y^2)$ in (6.3.2) change perhaps the picture obtained?

In his dissertation Poincaré showed that this does not happen if the following assumptions are made:

1) the convex envelope of points λ_i in the complex plane does not contain O;

2) none of the numbers λ_i is a linear combination of all the λ with integral non-negative coefficients.

On the assumptions 1), 2), Poincaré constructed a non-linear change of variables:

$$z = y + O(y^2), \tag{6.3.4}$$

which reduces (6.3.2) to the form (6.3.3) in a certain neighbourhood of O. The substitution (6.3.4) obviously solves the problem of constructing a neighbourhood of O: the integral curves of the systems (6.3.1) and (6.3.3) behave identically.

Condition 1) is not satisfied in an integral domain in the space Λ. It turns out, however, that systems which are not reducible to the form (6.3.3) are an exception; *for almost all* Λ *(excluding a set of Lebesgue measure zero), whatever the analytic component* $O(y^2)$ *in* (6.3.2) *may be, there exists an analytic transformation of* (6.3.4) *to the form* (6.3.3). This result is due to C.L. Siegel [9].

The substitution (6.3.4) can be sought in the form of a Taylor series for y; the coefficients are calculated successively by formulae containing small denominators (6.3.5). The proof of convergence given by Siegel is based on the argument that among these denominators small denominators are encountered only occasionally.

If Newton's method is employed in determining the substitution (6.3.4), the same result is obtained without the use of the above argument.

The Newtonian approximations not only converge but even become stable: each coefficient of the Taylor series is determined exactly after a finite number of approximations (after the s-th approximation terms of degree $2^{s-1} + 1$ become stable).

For the existence of the substitution (6.3.4) it is sufficient that the small denominators arising should, for a certain $K > 0$, allow the estimate

$$|\lambda_i - (\lambda, k)| \geqslant K|k|^{-(n+1)} \qquad \left(|k| = \sum_{j=1}^{n} |k_j| > 1;\ i = 1, \ldots, n\right) \tag{6.3.5}$$

for any vector k with integral non-negative components k_j. Condition (6.3.5) is automatically satisfied if conditions 1) and 2) are fulfilled (see above); it is satisfied for almost all Λ, but not for any canonical system.

2. The neighbourhood of periodic motions. The mapping of an area normal to the trajectory at any point onto itself is naturally connected with periodic motion. We shall formulate here the results of Siegel [8] on the construction of such a mapping in the simplest case (the so-called "problem of the centre", see [53]).

Let

$$z \longrightarrow Tz = e^{2\pi i\omega}z + a_2z^2 + a_3z^3 + \dots \tag{6.3.6}$$

be a conformal mapping of the neighbourhood of O of the complex plane z onto itself and let ω satisfy condition (6.2.3). *Then the fixed point O is stable. Furthermore, there exists an analytic substitution*

$$w = z + b_2z^2 + b_3z^3 + \dots,$$

transforming T into a rotation through an angle $2\pi\omega$:

$$w \longrightarrow e^{2\pi i\omega} w.$$

Consequently the neighbourhood of O in the z plane is divided into analytic invariant curves $|w|$ = const. The trajectory $T^n z$ $(n = 1, 2, \dots)$ fills this curve everywhere densely.

It is instructive to compare the conformal mappings with the canonical (see Introduction, §4).

An analogous result is obtained for the many-dimensional case. On combining Floquet's theory (see §2, 3.) with this result we can convince ourselves that a study of the neighbourhood of a periodic trajectory in the "general case" reduces to the study of a certain linear system with constant coefficients. We shall not dwell on this result in detail, since it is contained in the more general theorem of the following subsection.

3. **The neighbourhood of a conditionally periodic motion.** We suppose that a system of $n + m$ differential equations has an m-dimensional invariant torus T filled with trajectories of conditionally periodic motion with frequencies $\omega_1, \dots, \omega_m$. We assume that coordinates $q_1, \dots, q_m$; $x_1, \dots, x_n$ can be introduced into the neighbourhood of this torus, where the q are angular coordinates on the torus T, the equation of which now has the form $x = 0$. Our differential equations will take the form

$$\left.\begin{aligned} \dot{x} &= A(q)x + f(x, q) \qquad (x = x_1, \dots, x_n), \\ \dot{q} &= \omega + g(x, q) \qquad (q = q_1, \dots, q_m), \end{aligned}\right\} \tag{6.3.7}$$

where $f = O(x^2)$, $g = O(x)$.

We shall *assume* that a linear system with matrix $A(q)$ is reducible (see 5.). Therefore in place of (6.3.7) we shall consider the system

$$\dot{y} = \Lambda y + f(y, g), \qquad \dot{q} = \omega + g(y, g) \tag{6.3.8}$$

with a diagonal constant matrix Λ of eigenvalues $\lambda_1, \dots, \lambda_n$ and with analytic $f = O(y^2)$, $g = O(y)$ of period 2π in q.

Using the same Newton's method, E.G. Belaga [22] proved that, *under the assumption* (6.3.11) *there exists an analytic change of variables*

$$Y = y + \varphi(y, q), \qquad Q = q + \psi(y, g), \tag{6.3.9}$$

reducing (6.3.8) *to the linear system*

$$\dot{Y} = \Lambda Y, \quad \dot{Q} = \omega. \tag{6.3.10}$$

The variables of (6.3.9) (*where* $\varphi = O(y^2)$, $\psi = O(y)$ *are of period* 2π *in* q) *can be introduced if the following arithmetic condition is satisfied: for a certain* $K \geqslant 0$

$$|(k, \lambda) - \varepsilon\lambda_j + i(l, \omega)| \geqslant K(|k| + |l|)^{-(m+n+1)}$$
$$(\varepsilon = 0, 1; \ j = 1, \ldots, n; \ i^2 = -1) \qquad (6.3.11)$$

for all integral vectors $k = k_1, \ldots, k_n$; $l = l_1, \ldots, l_m$; *in* (6.3.11)

$$|k| = \sum_{\alpha=1}^{n} k_\alpha > 1 + \varepsilon, \qquad k_\alpha \geqslant 0, \qquad |l| = \sum_{\beta=1}^{m} |l_\beta|.$$

The non-linear system (6.3.8) is thereby reduced to the easily integrable linear system (6.3.10). We note that condition (6.3.11) imposes a restriction only on the eigenvalues λ_i defined by the matrix A in (6.3.7) and on the frequencies ω of the conditionally periodic motion; the functions f and g in (6.3.7) are arbitrary. Condition (6.3.11) is violated only on a set of Lebesgue measure zero in the λ, ω-space. Unfortunately all canonical systems are in this exceptional set.

It would be interesting to construct a theory for the neighbourhood of conditionally periodic motion of a canonical system analogous to the theory constructed by Birkhoff (Ch. I, §9) for the neighbourhood of a position of equilibrium and of a periodic trajectory. Such a theory would help us to understand the structure of the zones of instability in the many-dimensional case (see §2, 1.). But it must be based on the corresponding linear theory which we do not yet have at our disposal (see §2, 5.).

4. More complex cases. The structure of the neighbourhood of motions more complex than conditionally periodic has not been investigated at all.[1]

The question of reducing a system to linear form in the case of complex time also seems to me to be of interest. If the time is complex, the "integral curves" are two-dimensional surfaces (in the real sense). The study of the behaviour of these surfaces in the large is only beginning [43], [44].

§4. Intermixing

The "ergodic hypothesis" of statistical mechanics is concerned with the idea that in a dynamical system of "general form" motion on a surface of constant energy $H = h$ has properties of ergodicity and intermixing. The results of the preceding chapters show that ergodicity and intermixing are not general phenomena, but are connected with special conditions. We here consider certain mechanisms that can cause intermixing.

1. Collisions. As the simplest model let us consider the motion of two perfectly elastic spherical particles on the surface of a two-dimensional torus having a Euclidean metric. For simplicity we shall first consider one of the particles as fixed. The second particle (which can now be regarded as a point) moves on a "torus billiard table" (Fig. 20),

1 *Added in proof.* Interesting results on the existence of invariant manifolds in dissipative systems have recently been obtained by N.N. Bogolyubov and J. Moser, using Newton's method.

being reflected from the fixed circumference according to the law "the angle of incidence is equal to the angle of reflection".

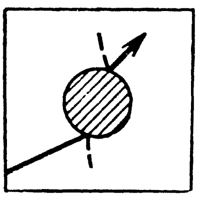

Fig. 20

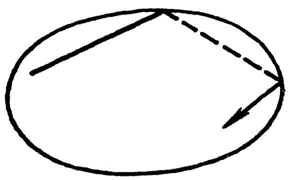

Fig. 21

Let us at the same time consider an elliptic billiard table (Fig. 21). The ellipse can be regarded as an oblate ellipsoid on which the point moves along a geodesic, passing at each reflection from one side to the other. In precisely the same way a torus billiard table (Fig. 20) can be be regarded as a two-sided torus with a hole on which the point moves along a geodesic. But if the two-sided ellipse is an oblate ellipsoid, the two-sided torus with a hole will be an oblate "Kringel" (of genus 2). Thus, motion on our torus billiard table is a limiting case of motion along a geodesic on the knot-shaped surface.

As is well-known, the character of the motion along a geodesic depends strongly on the Gaussian curvature of the surface. On surfaces of negative curvature the geodesics are sharply unstable; in this case ergodicity, a very strong intermixing, etc have been proved ([33], [46], [47]).

We now turn to our billiard tables and consider the curvature to which Fig. 20 and Fig. 21 correspond. The ellipsoid has a positive curvature to whose integral is equal to 4π (the Gauss-Bonnet Theorem). On flattening the ellipsoid all the curvature is accumulated along the boundary of an ellipse. For a "Kringel" the integral of the curvature is equal to -4π. Thus, a two-sided torus billiard table can be regarded as an oblate surface with negative curvature everywhere: on flattening, all the curvature is accumulated along the circumference.

The preceding arguments are not, of course, a proof of the ergodicity of our model. But they show that attempts can be made to apply the methods of investigating geodesics on surfaces of negative curvature to the proof of the ergodic hypothesis.[1]

2. Singular spectrum. There is another case of intermixing, this time much slower, in the following example given by A.N. Kolmogorov [11]:

$$\dot{x} = \omega_1 A(x, y), \qquad \dot{y} = \omega_2 A(x, y) \tag{6.4.1}$$

(x, y(mod 2π) are angular coordinates of a point of the torus, $A > 0$ is an analytic function on the torus, $\frac{\omega_2}{\omega_1}$ is irrational).

For certain values of $\frac{\omega_2}{\omega_1}$, which are abnormally well approximated by rational numbers, $A(x, y)$ can be chosen so that (6.4.1) is a system with

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1 The first attempts in this direction were made by N.S. Krylov [48].
Added in proof. Ya.G. Sinai has very recently proved ergodicity in the billiard table problem considered here and also in a number of problems of statistical mechanics.

intermixing. It would be interesting to clarify the character of the spectrum of the dynamical system (6.4.1) (the spectrum is, presumably, singular).

There is one further mechanism of intermixing associated with secular motions. As the simplest model I take a one-dimensional system with low friction; but analogous phenomena also exist in the conservative case (cf. [49]).

3. Evolution. Let us consider a one-dimensional system with potential energy $U(q)$ (Fig. 22) and low friction $\mu F(p, q)$, $\mu \ll 1$. It is clear that almost every point of M will in the course of time arrive at one of the potential minima A or B. But at which one?

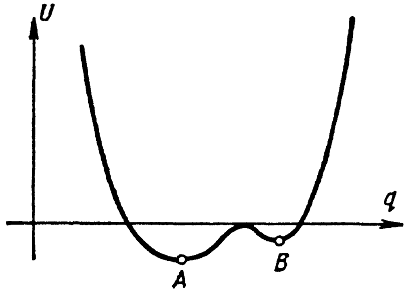

Fig. 22.

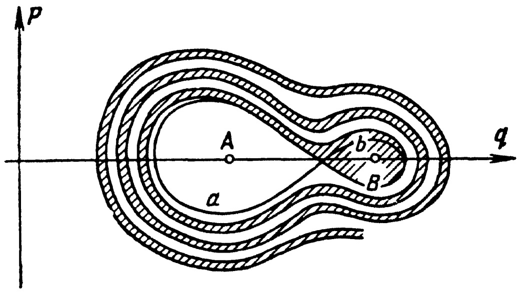

Fig. 23.

It turns out that, *if the initial energy* $h > 0$, *there exist definite probabilities* p_A *and* p_B *of arriving at A or at B*; *these probabilities are given by the formulae* (see Fig. 23)

$$\frac{p_A}{p_B} = \frac{\oint_a F\,dq}{\oint_b F\,dq}, \qquad p_A + p_B = 1. \tag{6.4.2}$$

Since the motion is determined by initial conditions, it is necessary to define the probability of moving from x to A. Let $\Omega(d) = d$ be the neighbourhood of a point x of the phase space and $\Omega_A(d, \mu)$ the set of initial conditions from $\Omega(d)$ leading to arrival at A for friction μF. By definition,

$$p_A = \lim_{d \to 0} \lim_{\mu \to 0} \frac{\operatorname{mes} \Omega_A(d, \mu)}{\operatorname{mes} \Omega(d)}.$$

Formula (6.4.2), the proof of which is left to the reader, shows that p_A does not depend on x (if $H(x) > 0$) and is determined by the value of $F(p, q)$ at the critical level of energy $H(p, q) = 0$.

There can be a similar loss of determinacy in conservative systems in the degenerate case (Ch. I, §5), when the secular variation of an n_0-dimensional invariant torus reduces it to that particular manifold where the stratification of the phase space into invariant tori of the unperturbed system has a singularity.

Another slower process of intermixing is connected with the passage of an evolving system through resonances (cf. Ch. II, §5). All these phenomena still await a rigorous mathematical investigation. Interesting non-rigorous arguments are contained in a dissertation by B.V. Chirikov (Novosibirsk, 1959).

A.M. Molchanov in [21], pp. 42-49, formulated an incorrect "theorem on the division of motions". A weaker assertion (the change of variable is many-valued when there is periodicity in the slow motion) is plausible. But A.M. Molchanov states that he has no proof of this assertion.

§5. Smoothing techniques

In this section we give an account of the basic idea of smoothing techniques which go back to J. Nash [50] and have enabled J. Moser to replace the requirement that the Hamiltonian should be analytic by the requirement that the 333-rd derivatives should be continuous. The reader will find detailed formalized proofs in the excellent papers by Moser [24], [25]; in these papers, however, one of the important ideas (the use of inequality (6.5.6)) is somewhat concealed. We shall follow the proof given by J. Moser at the Moscow State University in the autumn of 1962.

1. Technical lemmas. Let us consider the periodic functions

$$f(x) = \sum_{k \neq 0} f_k e^{i(k,x)} \qquad (x = x_1, \dots, x_n;\ k = k_1, \dots, k_n). \tag{6 5.1}$$

We introduce the following notation:

$$\sum_{|k| \leqslant N} f_k e^{i(k,x)} = f_N(x), \qquad \sum_{|k| > N} f_k e^{i(k,x)} = R_N f \qquad (|k| = |k_1| + \dots + |k_n|). \tag{6.5.2}$$

We shall find estimates for the derivatives

$$|f^{(l)}| = |f^{(l)}(x)| = \max_{0 \leqslant l_1 + \dots + l_n \leqslant l} \left| \frac{\partial^{l_1 + \dots + l_n}}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} f \right|. \tag{6.5.3}$$

Let $0 \leqslant \lambda \leqslant l$ be an integer. The following assertions are known and easily proved:

If $|f^{(l)}| \leqslant M$, *then* $|R_N f^{(\ )}| \leqslant CMN^{-(l-\ \ -\delta)}$. (6.5.4)

If $|f| \leqslant M_0$, *then* $|f_N^{(\ )}(x)| \leqslant CM_0 N^{\ \ +\delta}$. (6.5.5)

If $|f| \leqslant M_0$, $|f^{(l)}| \leqslant M$, *then* $|f^{(\lambda)}| \leqslant CM_0 \left(\frac{M}{M_0}\right)^{\frac{\lambda}{l}}$. (6.5.6)

Here $\delta > 0$ is a constant not depending on f, M_0, M, N, l, λ; $C > 0$ is a constant not depending on f, M_0, M, N (see [59]).

2. Newtonian approximations with smoothing. Suppose that there exists a process of Newtonian-type successive approximations similar to that described in Chapter I. We shall assume that, with respect to the "perturbation" $f(x)$, a "change of variable" $x \to x + g(x)$ is constructed, where the function $g(x)$ is defined, for example, by the series

$$g(x) = \sum_{k \neq 0} \frac{if_k}{(k, \omega)} e^{i(k,x)}. \tag{6.5.7}$$

We assume (on account of the usual arithmetical properties of ω) that

$$|g(x)| \leqslant K\,|f^{(\nu)}(x)|. \tag{6.5.8}$$

We further suppose that in the next approximation the role of the perturbation is played by the function f' – the sum of several quantities of the form

$$\Sigma_1 = f(x)\frac{\partial g}{\partial x}, \qquad \Sigma_2 = f(x) - f(x+g(x)) \text{ etc.} \tag{6.5.9}$$

The quantities in (6.5.9) are of order f^2 if we take g and the derivatives of f and g to be of the same order of smallness as f. This assumption is valid if $f(x)$ is analytic (see Chapters I and IV).

But if $f(x)$ is differentiable only a finite number of times, in view of the "loss of smoothness" (6.5.8) we can carry out only a finite number of approximations. It turns out that this difficulty can be overcome by means of *smoothing*. We replace $f(x)$ by an infinitely differentiable function $f_N(x)$ (see (6.5.2)), transferring the remainder $R_N f$ to the terms of the next approximation (6.5.9). For a sufficiently large (but finite) number of derivatives of $f(x)$ and sufficiently small $|f|$ the quantities N_s (s is the number of the approximation, $N_s \to \infty$) can be dealt with so that the approximations converge.

In accordance with (6.5.6) it is sufficient to estimate the function and its highest derivative. If $f(x)$ satisfies the inequalities

$$|f^{(0)}| < M_0, \qquad |f^{(l)}| < M \qquad (M_0 \ll 1 \ll M), \tag{6.5.10}$$

then in the next approximation we shall obtain for the perturbation f' the estimates

$$|f'^{(0)}| < M_0' \leqslant M_0^{1+\alpha}, \qquad |f'^{(l)}| < M' \leqslant M^{1+\alpha} \tag{6.5.11}$$

for a certain $0 < \alpha < 1$ not depending on f, M_0, M (for example, $\alpha = \frac{1}{2}$).
From the estimates (6.5.11) the convergence of the approximations is easily deduced. The derivation of the inequalities (6.5.11) is outlined below; to avoid overloading the formulae we shall not write in the constants δ, K, C. For example, (6.5.5) will be written in the form

$$|f_N^{(\lambda)}(x)| \leqslant M_0 N^{\lambda}.$$

3. An estimate for $|f'|$. In accordance with 2. the finite trigonometric sum $g(x)$ is determined by the formula (6.5.7), where $0 < |k| \leqslant N$. On estimating $|f^{(\nu)}(x)|$ in (6.5.8) with the help of (6.5.6), we find from (6.5.10) that, for $1 \leqslant \nu \leqslant l$,

$$|g^{(0)}| \leqslant M_0\left(\frac{M}{M_0}\right)^{\frac{\nu}{l}}, \qquad |g^{(l)}| \leqslant MN^{\nu} \tag{6.5.12}$$

(the estimate for $g^{(l)} = g_N^{(l)}(x)$ follows from (6.5.5)). Interpolating (6.5.10) and (6.5.12) with the help of (6.5.6) we find that, for $0 \leqslant \nu$, $\lambda \leqslant l$,

$$|f^{(\lambda)}| \leqslant M_0\left(\frac{M}{M_0}\right)^{\frac{\lambda}{l}}, \quad |g^{(\lambda)}| \leqslant M_0\left(\frac{M}{M_0}\right)^{\frac{\nu}{l}+\frac{\lambda}{l}} N^{\frac{\nu\lambda}{l}}. \tag{6.5.13}$$

On differentiating $0 \leqslant \lambda \leqslant l$ times the product $f\dfrac{\partial g}{\partial x}$ we obtain from (6.5.13) the coarse estimate

$$\left|\left(f\frac{\partial g}{\partial x}\right)^{(\lambda)}\right| \leqslant M_0^2\left(\frac{M}{M_0}\right)^{\frac{\lambda}{l}+\frac{2\nu}{l}} N^{2\nu}. \tag{6.5.14}$$

In accordance with (6.5.4) we have

$$|R_N f^{(0)}| \leqslant MN^{-l}, \qquad |R_N f^{(l)}| \leqslant M. \tag{6.5.15}$$

But, in accordance with 2., f' is made up of quantities of the form $f\dfrac{\partial g}{\partial x}$ (cf. (6.5.9)) and $R_N f$. Therefore (6.5.14), (6.5.15) give

$$|f'^{(0)}| \leqslant M_0' = M_0^2\left(\frac{M}{M_0}\right)^{\frac{2\nu}{l}} N^{\frac{\nu}{l}} + MN^{-l}, \quad |f'^{(l)}| \leqslant M' = M_0^2\left(\frac{M}{M_0}\right)^{1+\frac{2\nu}{l}} N^{2\nu} + M. \tag{6.5.16}$$

4. Convergence. We now choose N and show that, for sufficiently large l and sufficiently small M_0, the inequalities (6.5.11) follow from (6.5.16). Let $M < M_0^{-\varkappa}$, $N = M_0^{-\beta}$, where $\varkappa > 0$ and $\beta > 0$ will be chosen below. In accordance with (6.5.16) the inequalities (6.5.11) are satisfied if

$$2 - \frac{2\nu}{l}(1+\varkappa) - \frac{\beta\nu}{l} > 1+\alpha, \ \beta l - \varkappa > 1+\alpha, \left(1+\frac{2\nu}{l}\right)(1+\varkappa) + 2\beta\nu < \varkappa(1+\alpha)+2. \tag{6.5.17}$$

All the quantities in these inequalities, except for ν, are in our control. Let us first of all choose $0 < \alpha < 1$. We shall show that the inequalities (6.5.17) are compatible. For $l \gg \nu$, $\beta\nu$, $\varkappa\nu$ the inequality $(6.5.17)_1$ is satisfied; for $l \gg \dfrac{\varkappa}{\beta}$, inequality $(6.5.17)_2$ is satisfied; and for $\left(\alpha - \dfrac{2\nu}{l}\right)\varkappa \gg \beta\nu$–inequality $(6.5.17)_3$ also. We now choose $\beta > 0$, then $\varkappa$ sufficiently large so that $\alpha\varkappa \gg \beta\nu$ and, finally, l sufficiently large so that $l \gg \dfrac{\varkappa}{\beta}$, $\dfrac{\nu}{l} \ll \alpha$, $\dfrac{\nu\varkappa}{l} \ll 1$, $\dfrac{\beta\nu}{l} \ll 1$. Let M_0 be sufficiently small.

With the parameters α, β, $\varkappa$, l chosen in this way (6.5.11) follows from (6.5.10). In a completely analogous way we put $N_{s+1} = N_s^{1+\alpha} = (M_0^{(s)})^{-\beta}$ in the $(s+1)$-th approximation and obtain an estimate of the type (6.5.11) with

$$M_0^{(s+1)} \leqslant (M_0^{(s)})^{1+\alpha}, \ M^{(s+1)} \leqslant (M^{(s)})^{1+\alpha} \leqslant (M_0^{(s+1)})^{-\varkappa}. \tag{6.5.18}$$

Since $M_0 \ll 1$ the convergence of the approximations follows easily from (6.5.18).

5. Remarks. 1) The convention at the end of 2. not to write in the constants δ, C, K is harmless, provided that $l \gg \varkappa \gg \delta$, that the inequalities (6.5.17) are satisfied with something to spare, and that $M_0 \ll (1, K, C^{-1})$ is sufficiently small.

2) Newton's method often requires estimates of the derivatives f' not with respect to x but with respect to $y = x + g(x)$. It is not difficult, however, in this case also to obtain inequalities of the type (6.5.16).

3) On choosing l sufficiently large, one can obtain convergence of

the approximations with r-th derivatives ($r \ll l$). In [25] l was taken to be 333.

4) Instead of substituting a trigonometric sum for $f(x)$ or $g(x)$ it is sometimes useful to adopt a more accurate smoothing.

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UNIFORM DISTRIBUTION OF POINTS ON A SPHERE AND SOME ERGODIC PROPERTIES OF SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS IN A COMPLEX REGION*

V. I. ARNOL'D AND A. L. KRYLOV

The phenomena of dense trajectories, ergodicity, and mixing occur often in analysis. The metric theory of dynamical systems (see [1]) gives an approach to these questions, at least in the case of "one-dimensional time." In this paper we consider some problems in which a noncommutative discrete group plays the role of time. We were led to these problems by an attempt to study ergodic properties of solutions of linear differential equations in a complex region (see [2]).

1. Uniform distribution of points on the sphere.

Theorem 1. *Let A, B be two rotations of the sphere S^2, and x a point of the sphere. If the sequence of points*

$$x;\ Ax,\ Bx;\ A^2x,\ ABx,\ BAx,\ B^2x;\ \ldots \tag{1}$$

is dense on the sphere, then it is uniformly distributed.

By uniformly distributed we shall mean the following: Let Δ be a region of the sphere which is bounded by a piecewise smooth curve. Starting from x and performing n rotations A or B, we get 2^n image points

$$A^n x,\ A^{n-1}Bx,\ A^{n-2}BAx,\ \cdots,\ B^n x. \tag{2}$$

The number of points in (2) which lie in the region Δ we denote by $p_n(\Delta)$. Theorem 1 asserts that

$$\lim_{n\to\infty} \frac{p_n(\Delta)}{2^n} = \frac{\text{mes}\,\Delta}{\text{mes}\,S^2}. \tag{3}$$

For the proof of Theorem 1 we use a method of H. Weyl [3]. Consider an arbitrary continuous function $f(x)$ defined on the sphere S^2. Form $f_n(x)$, the arithmetic mean of $f(x)$ at the points (2). Following Weyl, to prove (3) it is sufficient to establish that

$$\lim_{n\to\infty} f_n(x) = \bar{f} \equiv \int_{S^2} f(x)\,dx \,/\, \text{mes}\,S^2. \tag{4}$$

For the study of the time means f_n, unitary operators arise in a natural way in $L_2(S^2)$:

$$\mathfrak{A}f(x) = f(A^{-1}x);\quad \mathfrak{B}f(x) = f(B^{-1}x).$$

Using these operators, the time means may be written in the form

$$f_n(x) = \frac{1}{2^n}\,(\mathfrak{A}+\mathfrak{B})^n f(x) = \left[\frac{\mathfrak{A}+\mathfrak{B}}{2}\right]^n f(x). \tag{5}$$

It is known (see, for example, [4]) that the space $L_2(S^2)$ can be decomposed into an orthogonal sum of subspaces R_l $(l = 0, 1, 2, \ldots)$, invariant under all rotations of the sphere. The space R_l is of dimension $2l + 1$, consists of the spherical functions of degree l, and has no proper subspace invariant under all rotations.

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It is easy to see that to establish (4) it is sufficient to consider functions $f(x)$ which belong to some invariant subspace R_l.

Lemma 1. *Let A and B be finite-dimensional unitary operators. Then either*

$$\left\| \left(\frac{A+B}{2}\right)^k \right\| < 1, \tag{6}$$

for all $k \geq 1$, or else for some vector $f \neq 0$ we have

$$Af = Bf;\ A^2 f = ABf = BAf = B^2 f;\ A^3 f = A^2 Bf = ABAf = \cdots = B^3 f;\ etc. \tag{7}$$

The proof of the lemma stems from the fact that if $\|f+g\| = \|f\| + \|g\|$ and $\|f\| = \|g\| = 1$, then $f = g$.

Now we establish formula (4) for f belonging to R_l, $l > 0$. In R_l, if the operators $\mathfrak{A}$ and $\mathfrak{B}$ in (5) satisfy (6), then $f_n \to 0$ and (4) follows. We show that (7) is impossible. From the assumption that the points of (1) are dense in S^2, it follows easily that the closure of all products of A and B is the whole group of rotations of the sphere. Therefore, from (7) it follows that the representation of this group on each subspace R_l is commutative, which is not the case for $l > 0$. This completes the proof of Theorem 1.*

2. **Generalization.** Theorem 1 may be considered as an ergodic theorem in which the role of time is played by a free semigroup with two generators. We may construct a dynamical system in which "time" is a group Γ with a finite number of generators $a_1, a_2, \cdots, a_s$. We shall thus speak of a group of measure-preserving transformations A_γ $(\gamma \in \Gamma)$ on a measure space Ω for which $A_{\gamma_1\gamma_2} = A_{\gamma_1} A_{\gamma_2}$ and $A_{\gamma^{-1}} = A_\gamma^{-1}$.

In order to define the time means, we consider the collection Γ_n of elements of Γ obtained by multiplying exactly n factors of the form $a_1, a_1^{-1}, a_2, a_2^{-1}, \cdots, a_s, a_s^{-1}$. Let the number of such products be $N(n)$. Then the "time mean" $f_n(x)$ of the function $f(x)$, $x \in \Omega$, we define by $f_n(x) = \sum_{\gamma \in \Gamma_n} f(A_\gamma x)/N(n)$.

The method of § 1 permits the investigation of the behavior of $f_n(x)$ in certain cases, approximately described by the term "discrete spectrum."

Let Ω be a homogeneous space (in § 1 the sphere S^2) on which a compact Lie group G acts transitively, and let the transformations A_γ $(\gamma \in \Gamma)$ belong to G. For a class of groups Γ we have succeeded in proving that the sequence of points $A_\gamma x$ is uniformly distributed in their closure, provided the latter is connected. In other words, the time means $f_n(x)$ of a continuous function converge to the phase mean on the closure of the trajectory $A_\gamma x (\gamma \in \Gamma)$.

As examples we consider two cases:

1) Γ = free group with two generators a, b.

2) Γ = group with generators a, b, c, and the relation $abc = e$.

It is easy to see that $f_n(x) = S_n f(x)$, where $S_0 = E$ and, respectively,

$$S_1 = {}^1/_4 (\mathfrak{A} + \mathfrak{B} + \mathfrak{A}^{-1} + \mathfrak{B}^{-1});\quad S_{n+1} = {}^4/_3 S_1 S_n - {}^1/_3 S_{n-1} \quad (n \geqslant 1), \tag{8}$$

$$S_1 = {}^1/_6 (\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{A}^{-1} + \mathfrak{B}^{-1} + \mathfrak{C}^{-1});\ S_{n+1} = {}^3/_2 S_1 S_n - {}^1/_4 S_n - {}^1/_4 S_{n-1}\ (n \geqslant 1) \tag{9}$$

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*M. Maljutov has given another proof of Theorem 1.

(here $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are the unitary operators induced by their respective groups Γ: $Af(x) = f(A_a^{-1}x)$).

Let us consider the closure of the trajectory $A_\gamma x (\gamma \in \Gamma)$. It is a homogeneous space M, acted on transitively by the closure $\overline{\Gamma}$ of the group A_γ in G. Decompose $L_2(\underline{M})$ into an orthogonal sum of finite-dimensional subspaces R_l, invariant and irreducible relative to $\overline{\Gamma}$ (see [4]).

The operators S_1 (and hence S_n, by virtue of their dependence on S_1) are self-adjoint. The study of S_n on R_l reduces to the study of its eigenvalues, for which one obtains recurrence equations from (8) or (9). Solving these equations, we see that either $S_n \to 0$ on R_l as $n \to \infty$, or else R_l is one-dimensional and for $f \in R_l$, $D \in \overline{\Gamma}$ implies $Df = \pm f$.

Let us show now that if M is connected, then f is constant on M. Let K be the component of the identity in $\overline{\Gamma}$. For D in K, obviously $Df = f$, and therefore on Kx the function f is constant. But since M is connected, it coincides with Kx, and thus f is constant on M. From this it follows easily that the time means converge to the phase mean.

Remark 1. Above we studied the time means on the "spheres" Γ_n. It is easy to establish analogous theorems about means on the "balls" $\bigcup_{k=0}^{n} \Gamma_k$.

Remark 2. If A and B are two arbitrarily chosen rotations of the sphere, then usually the sequence (1) is everywhere dense. Probably for two arbitrarily chosen elements A, B of any compact Lie group, the general case will be that in which the products of A and B are everywhere dense. But if the group G is not compact, then the subgroup formed from an arbitrary number of elements may be discrete (for example: let G be the group of motions of the Lobačevskian plane and Γ a subgroup of the discrete group associated with a surface of genus p (see [5])).

3. Equations with complex time. From a geometrical point of view the solutions of ordinary differential equations in a complex region are represented as two-dimensional surfaces stratifying the phase space. Such a surface may fill out or be everywhere dense in the phase space. In this case one would naturally except uniform distribution in some sense. Let us consider the system of linear differential equations

$$\frac{dx}{dz} = A(z)\,x, \tag{10}$$

where z is a complex variable, x is a vector $(x_1, \cdots, x_n)$ in n-dimensional complex space C_n, and A is a matrix depending analytically on z except at three singular points z_1, z_2, z_3 on the Riemann sphere.

The phase space is of real dimension $2n+2$. It is the product of the Riemann sphere minus the three points z_1, z_2, z_3 (henceforth denoted by Z) and $C_n(z)$. It is stratified by the solutions, namely surfaces of real dimension 2 locally defined by $x = x(z)$, where $x(z)$ satisfies the system (10) and $z \in Z$.

To each path on Z starting from z_0, and to each vector $x_0 \in C_n(z_0)$ corresponds a unique solution $x(z)$ with the initial value $x_0(z_0)$. This defines a linear transformation from $C_n(z_0)$ to $C_n(z)$. In particular, to each closed path γ corresponds a linear transformation A_γ of the space $C_n(z)$ into itself. The transformation A_γ depends only on the homotopy class of the path γ on Z. These transformations form an anti-representation of the fundamental group of Z. The group of transformations A_γ are called the monodromy group of the system (10).

Lemma 2. *If the monodromy group is bounded, then the system* (10) *has a unique integral*

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$(B(z)x, \bar{x}) = \text{const}$, *where* $B(z)$ *is a positive definite unitary matrix uniquely defined for each* $z \in Z$.

The proof depends on the fact that, in view of the compactness of the closure of the A_γ, the representation A_γ is equivalent to a unitary one.

From Lemma 2 it follows that every two-dimensional surface forming a solution in the $(2n+2)$-dimensional space is a level surface on the $(2n+1)$-dimensional surface $(Bx, \bar{x}) = c$, and the points on distinct branches of the solution $x(z)$ over the point z_0 lie on the sphere $(B(z_0)x, \bar{x}) = c$. According to the results of §§ 1 and 2, these points are uniformly distributed in their closure (if it is connected): for the fundamental group of Z possesses three generators a, b, c with the relation $abc = e$.

There is a case in which it is clearly possible to extract the conditions of boundedness of the monodromy group. This is the hypergeometric equation of Gauss:

$$z(1-z)\frac{d^2x}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dx}{dz} - \alpha\beta x = 0. \tag{11}$$

We assume that the parameters α, β, γ are real.

Theorem 2. *The hypergeometric equation* (11) *has a single-valued first integral*

$$b_{11}x\bar{x} + b_{12}x\bar{x}' + b_{21}x'\bar{x} + b_{22}x'\bar{x}' = \text{const}, \tag{12}$$

where $x' = dx/dz$ *and the* $b_{ij}(z)$ *are single-valued* (*but not complex-analytic*) *functions defined for* $z \neq 0, 1, \infty$ *and forming a self-adjoint matrix* $\| b_{ij}(z) \|$.

In order to find the functions b_{ij}, it suffices to notice that both generators A_a, A_b of the monodromy group, explicitly written out in [7], leave a certain inner product invariant.

In accord with Riemann and Schwartz, connected with the equation (11) is a curvilinear triangle with vertices $\lambda\pi, \mu\pi, \nu\pi$, where $\lambda = |1-\gamma|$, $\mu = |\gamma - \alpha - \beta|$, $\nu = |\alpha - \beta|$ (see [6]). *If the sum of the angles of this triangle exceeds* π, *then the matrix* $\|b_{ij}(z)\|$ *is positive definite, the monodromy group consists of unitary matrices in the metric* $\|x\| = (B(z_0)x, \bar{x})$, *and all branches of the solution over each point* z *lie on a sphere* (12) *of the space of* x, x', *and are uniformly distributed on this sphere for almost all values of the parameters* α, β, γ (the excluded values form a one-dimensional manifold).

4. In conclusion we point out some unresolved questions.

1°. Are the ergodic theorems of Birkhoff and von Neumann true for dynamical systems with noncommutative "time"?

2°. Do the results of § 2 generalize to an arbitrary group with a finite number of generators?

3°. Do the results of §§ 1 and 2 generalize to the noncompact case (e.g., if Ω = the Euclidean or Lobačevskian plane)?

4°. What sort of generalizations do §§ 1 and 2 have in the case where the role of time is taken by a Lie group, e.g., the group of motions of the Lobačevskian plane?

5°. Equation (10) can be written in the form $dx = (A(z)\,dz)x$. If we mean by $A(z)dz$ the matrix of differentials, then the arguments of § 3 are transferred to an equation on a Riemann surface. The difficulty is in showing that the monodromy group is bounded.

6°. Uniform distribution of the surface representing the solutions of (10) in the $(2n+1)$-dimensional manifold M_c: $(Bx, \bar{x}) = c$ probably occurs under the following metric: on Z introduce the metric of constant negative curvature (see [5]), and on $C_n(z)$ the metric defined by the inner product $(B(z)x, y)$.

7°. The system (10) may be regarded as a dynamical system in which the role of time is played by the universal covering Z, i.e., the Lobačevskian plane. But it is also associated with an ordinary dynamical system with continuous time. To this end we consider as a point of the new phase space the point $(z, x) \in M_c$ together with the direction ξ of the vector tangent to Z at z. The action is defined thus: the point z moves steadily along the geodesic defined by ξ, and x "follows" z according to the equation (10). The metric and the invariant measure are defined in 6°.

The indicated construction permits an "increased flow" defined on the manifold and on the group of automorphisms (constituting the representation of the fundamental group of the manifold). The resulting "product" present an interesting study.

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5

ON A THEOREM OF LIOUVILLE CONCERNING INTEGRABLE PROBLEMS OF DYNAMICS*

V. I. ARNOL'D

1. Liouville proved that if in a system with n degrees of freedom,

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

the first n integrals in the involution $H = F_1, F_2, \cdots, F_n$ are known, then the system is integrable in quadratures (see [1]). Many examples are known of integrable problems. It has often been noted that in these examples the bounded invariant manifolds determined by the equations $F_i = f_i = \text{const}\,(i = 1, \cdots, n)$ turn out to be tori and motions on them are conditionally periodic. We shall prove that this situation is necessary in any problem which is integrable in the indicated sense. The proof is based on simple topological considerations.

2. **Notation.** A point (p, q) of $2n$-dimensional Euclidean space will be denoted by $x = (x_1, \cdots, x_{2n})$. We denote the vector $(F_{x_1}, \cdots, F_{x_{2n}})$ associated with the function $F(x)$ by $\operatorname{grad} F$. Then the Hamiltonian equations take the form

$$\dot{x} = I \operatorname{grad} H, \tag{2}$$

where

$$I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

and E is the nth order unit matrix.

For the two vectors x, y we define the skew-scalar product $[x, y] = (Ix, y) = -[y, x]$. Linear transformations S preserving the skew-scalar product, so that $[Sx, Sy] = [x, y]$ for all x, y, are called *symplectic*. For example a transformation with the matrix I is symplectic.

The skew-scalar product $[\operatorname{grad} F, \operatorname{grad} G]$ is called the Poisson bracket (F, G) of the functions F and G. Evidently the function is a first integral of the system (2) if and only if its Poisson bracket with the Hamiltonian function (F, H) is equal to zero. If the Poisson bracket of two functions is equal to zero, then one says that these functions *stand in involution*.

A point of the n-dimensional torus is given by n angular coordinates $\phi_1, \cdots, \phi_n \pmod{2\pi}$. A *conditionally periodic* motion of a point on a torus is one under which the coordinates change uniformly.

3. **Theorem.** *Suppose that the Hamiltonian system* (2) *with n degrees of freedom has n single-valued integrals $H = F_1, F_2, \cdots, F_n$, standing pairwise in invo-*

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lution with one another. Suppose that the equations $F_i = f_i$ $(i = 1, \cdots, n)$ *distinguish in the* $2n$*-dimensional space* n*-dimensional compact manifolds* $M = M_f$, *at each point of which the vectors* $\operatorname{grad} F_i$ $(i = 1, \cdots, n)$ *are linearly independent. Then* M *is an* n*-dimensional torus and the point* x *depicting the solution of equations* (2) *moves along it in a conditionally periodic manner.*

We divide the proof into four parts.

1) M is *parallelizable,* i.e. it has n tangent vector fields linearly independent at each point. Indeed, consider the system (2) with the Hamiltonian function F_i. Since $(F_i, F_j) = 0$, all the functions F_j are first integrals and each trajectory lies entirely on M_f. Accordingly, the field of velocities $I \operatorname{grad} F_i$ is tangent to M. Because of the nondegeneracy of the matrix I, the vectors $I \operatorname{grad} F_i$ $(i = 1, \cdots, n)$ are linearly independent.

2) *Suppose that* D *is a surface in* M *and* Γ *its boundary. Then* $\oint_\Gamma p\,dq$ *(the sum of the areas of the projections of* D *on the planes* $p_i q_i$ $(i = 1, \cdots, n)$) *is equal to zero.*

For the proof we may restrict ourselves to infinitely small parallelograms lying in M. If Γ is a finite parallelogram with sides ξ, η, then the sum of the areas of the projections is evidently $[\xi, \eta]$. Now suppose that ξ, η are tangent to M at some point. From 1), any vector tangent to M is a linear combination of the $I \operatorname{grad} F_i$. But from $[\operatorname{grad} F_i, \operatorname{grad} F_j] = 0$, because I is symplectic, it follows that $[I \operatorname{grad} F_i, I \operatorname{grad} F_j] = 0$. Accordingly $[\xi, \eta] = 0$, as was required to be proved.

3) *The vector fields* $I \operatorname{grad} F_i$ *on* M *are irrotational (i.e. they are gradients of (multiple-valued) functions.*

From 2), $\int_{q_0}^{q} p\,dq$ does not depend on the particular path of integration lying on M_f in the neighborhood of the point q_0. Therefore this integral may be considered as a multiple-valued function $S(q, f)$. The equations $p = \partial S/\partial q$, $g = \partial S/\partial f$ define in each small region a canonical transformation $p, q \leftrightarrow f, g$ with generating function $S(q, f)$ (see [1]). In the variables f, g the Hamiltonian function F_i is f_i and the Hamiltonian equations yield $\dot{g}_i = 1$, $\dot{g}_j = \dot{f} = 0$.

Because of the linear independence of the vector fields of velocities $I \operatorname{grad} F_i$, the differentials dg_i at each point are linearly independent. We shall consider g_i as local coordinates on M. In the intersection of two neighborhoods they differ by a constant, so that the differentials dg_i are uniquely defined, and g multiply defined on M. In the g-coordinates the velocity fields $I \operatorname{grad} F_i$ on M are irrotational, as gradients of the functions g_i. Thus the manifold M is parallelizable by means of irrotational fields. Therefore it is easily deduced that it is a torus.

4) **Lemma.** *Suppose that on the* n*-dimensional compact manifold* M *there*

exist n differentials dg_i, i e. closed differential forms of degree 1, linearly independent at each point. Then this manifold is the direct product of n circumferences.

Indeed, suppose that O is a point of M and that M' is the universal covering surface of M. To each path OA on M there corresponds a path $O'A'$ on M'. The functions $g_i(A') = \int_{O'A'} dg_i$ map M' into the Euclidean space $g_1, \cdots, g_n$. It is easily verified (see for example [2]) that

a) this is a mapping onto the entire space $g_1, \cdots, g_n$;

b) at each point g there is only one image of a point A';

c) if O' and O'' cover O and $g(O'') = g(O') \pm g(O'')$, then O''' covers O.

Because of a) and b), we may identify M' with Euclidean space. In addition, from c), the points covering O form a grating (collection of integer-valued linear combinations of k independent vectors). More generally, the points A' and A'' cover one point A if and only if $A' - A''$ is a vector of the grating. Identifying in the Euclidean space all of these points, we obtain the direct product of k circumferences and $n - k$ straight lines. Because of the compactness of M, $k = n$. The lemma is proved.

From 3), the manifold M_f satisfies the conditions of the lemma. Accordingly it is an n-dimensional torus. Since the coordinates g change continuously, the motion on M_f is conditionally periodic. The theorem is proved.

Remark 1. The set of points of the $2n$-dimensional space x where n of the vectors $\operatorname{grad} F_i$ are linearly dependent generally speaking has dimension $n - 1$. Therefore n-dimensional manifolds M on which $\operatorname{grad} F_i$ are linearly dependent are exceptional (generally speaking, an n-dimensional manifold and an $(n - 1)$-dimensional manifold do not intersect in $2n$-dimensional space).

Remark 2. For $n = 2$ the Hamiltonian function F_1 and the first integral F_2 automatically stand in involution. In this case the theorem is almost trivial, since the torus is the only compact oriented manifold admitting a vector field without singular points. Therefore even in a non-Hamiltonian system of the fourth order with two first integrals the level surfaces $F_i = f_i$ are tori (see [3]).

Remark 3. In the formulation of the theorem we have restricted ourselves for simplicity to the case when the initial phase space was Euclidean. Nothing will change if this is an arbitrary manifold with a canonical structure, i.e. with canonical transformations in the form of admissible changes of variables. For simplicity we have not formulated the corresponding theorem in the case when M_f is not compact, either.

Remark 4. The question remains open as to the topological character of the level manifolds M_f when the number k of first integrals F_i is less than the number n of degrees of freedom, and also when the integrals at hand do not stand in involution. Further, although all known systems integrable in quadratures have n first integrals in involution, the necessity of this has not been proved.

4. The consideration of problems of dynamics naturally leads to the question as to how many linearly independent irrotational fields a manifold may have. For example the three-dimensional sphere is parallelizable, but does not have even one irrotational tangent field without singularities. Indeed, the gradient of a local function on the sphere is the gradient of a unique function, since the sphere is simply connected. At a maximum point of the function the gradient vanishes.

In general it is not difficult to show that *the compact manifold M admits k irrotational tangent vector fields* $\operatorname{grad} F_j$, *linearly independent at each point, if and only if it is a skew product with a k-dimensional field as basis.* Indeed, under small changes the vectors $\operatorname{grad} F_j$ remain linearly independent. By such changes one may arrange it so that for any one-dimensional cycle γ_i we will have

$$\oint_{\gamma_i} dF_j = \frac{M_{ij}}{N},$$

where M_{ij} and N are integers. The univalent functions $\phi_j = \exp(2\pi i N F_j)$, $s \le j \le k$, map M onto the k-dimensional torus T. Because of the linear independence of the $\operatorname{grad} F_j$ at each point M is representable in the form of a skew product with basis T.

The question as to the existence of nondegenerate closed forms of degree > 1 and also of planes and frames, is not so simple. Some necessary conditions, in terms of obstructions and characteristic classes, are known, but they are certainly far from being sufficient.

The following question on fiberings is connected with the above question. Suppose that the neighborhood of each point of the n-dimensional manifold M may be diffeomorphically mapped onto the Euclidean cube. Consider the preimages of the horizontal planes $x_n = \text{const}$. If the mappings of intersecting neighborhoods are so in agreement that the preimages in the intersection either do not intersect or else coincide, then we say that there is given a fibering devoid of singularities of M into $(n-1)$-dimensional surfaces. A typical example: the fibering of the two-dimensional torus into the trajectories of a conditionally-periodic motion.

Analogously one defines fiberings into $(n-k)$-dimensional surfaces.

Problem. *Determine whether there exists a fibering of a given n-dimensional manifold into $(n-k)$-dimensional surfaces. Investigate the behavior of the surfaces of the fibering in the large.*

The study of fiberings is the natural generalization of the qualitative theory of ordinary differential equations. The fibering of an n-dimensional manifold into $(n-1)$-dimensional ones calls into mind more the qualitative theory on two-dimensional manifolds than the fibering of an n-dimensional space into curves.

Added in proof. While this paper was being put into type A. S. Švarc kindly called my attention to papers of French topologists devoted to the problem posed above. See for example [4].

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INSTABILITY OF DYNAMICAL SYSTEMS WITH SEVERAL DEGREES OF FREEDOM*

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1. Recent progress in perturbation enables us to find many conditionally periodic motions in every nonlinear dynamical system which is close to an integrable system (see [1,2]. The stability of all the motions of the system follows from these results only when the dimension of the phase space is ≤ 4. The purpose of the present note is to give *an example* (3) *of a system with a* 5*-dimensional phase space which satisfies all the conditions of* [1,2] *but is nonstable.** The secular changes I_2 in the system (3) have the velocity $\exp(-1/\sqrt{\epsilon})$ and consequently cannot be dealt with by any approximation of the classical theory of perturbations.

We first introduce some definitions.

2. **The whiskered torus.** By a *torus* T^k we shall mean a direct product of k circumferences, which can be described by the k angular coordinates $\phi = \phi_1, \cdots, \phi_k$ (mod 2π). *A conditionally periodic motion with frequencies* ω is defined by the equations $\phi = \omega =$ const (where $\Sigma\, n_i\omega_i \neq 0$ for integral n_i, $\Sigma\, n_i^2 \neq 0$). Assume that in the phase space of the dynamical system there is an invariant torus T and on it a conditionally periodic motion. We shall call T a *whiskered torus* if T is a component of the intersection of two invariant open manifolds Y^-, Y^+ where all the trajectories on the *arriving whisker* Y^- approach T as $t \to +\infty$, and on the *departing whisker* Y^+ all the trajectories approach T as $t \to -\infty$.

Example 1. By the *standard* whiskered torus we shall mean the torus $x = y = z = 0$ in the system

$$\dot{x} = \lambda x, \quad \dot{y} = -\mu y, \quad \dot{z} = 0, \quad \dot{\phi} = \omega \tag{1}$$

defined in the $(l_+ + l_- + l_0 + k)$-dimensional space x, y, z, ϕ (ϕ taken mod 2π).

Of essential importance below is the concept of an *obstructing set.* Let M be a smooth submanifold of the space X. The tangent plane to M at the point x will be denoted by TM_x. The manifold *N complements M at the point $x \in M \cap N$,* if $TM_x + TN_x = TX_x$. We shall say that the set Ω *obstructs the manifold M at the point $x \in M$* if every manifold N which *complements M* at x is intersected by Ω.

Example 2. A spiral Ω which winds onto a closed curve M obstructs the curve.**

Another example is given by the standard whiskered torus (1). Let U be a neighborhood of the point ξ of an arriving whisker $x = z = 0$. By $\Omega = \bigcup_{t>0} U(t)$ we denote the set of all points of all the trajectories which begin in U. Then it is easy to prove the following theorem.

Theorem 1. *The set Ω obstructs the departing whisker $y = z = 0$ at an arbitrary point η of the whisker.*

3. **The transition chain.** If the whiskered torus T has the property that the images of an arbitrary

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* In contradistinction to stability, nonstability is itself stable. I believe that the mechanism of "transition chains" which guarantees that nonstability in our example is also applicable to the general case (for example, to the problem of three bodies).

** The articles [3,4] are based on this fact.

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* Editor's note: translation into English published in Soviet. Math. Dokl. 5 (1964). Translation of V.I. Arnol'd: Instability of dynamical systems with many degrees of freedom. Dokl. Akad. Nauk SSSR 156:1 (1964), 9–12

neighborhood of an arbitrary point ξ of one of its arriving whiskers obstruct the departing whisker at an arbitrary point η of the latter, then the torus will be said to be a *transition torus*. By Theorem 1, *the standard torus* (1) *is a transition torus.*

Assume that the dynamical system with phase space X has certain transition toruses $T_1, \cdots, T_s, \cdots$. These toruses will be said to form a *transition chain* if the departing whisker Y_s^+ of every preceding torus T_s complements the arriving whisker of the following torus Y_{s+1}^- at some point of their intersection $x_s \in Y_s^+ \cap Y_{s+1}^-$.

Let $T_1, \cdots, T_s, \cdots$ be a transition chain. Then the following theorem is easily proved.

Theorem 2. *An arbitrary neighborhood of the torus* T_1 *is connected with an arbitrary neighborhood of the torus* T_s *of trajectories of the given dynamical system.*

Consequently, for the proof of nonstability it is sufficient to find a transition chain which connects distant toruses T_1, T_s. The search for whiskered toruses and especially the investigation of their intersections in the general problem of the theory of perturbations demands very complicated calculations. We will confine ourselves here to an example in which a specially chosen perturbation vanishes on the toruses T_s.

4. **A nonstable system.** We consider a system with two degrees of freedom which is periodic in the time t with period 2π. "The phase space" $I_1, I_2; \phi_1, \phi_2; t$ is the direct product of the planes I_1, I_2 with the three-dimensional torus ϕ_1, ϕ_2, t (mod 2π). *The Hamiltonian, depending on the parameters* ϵ, μ, *will have the form* $H = H_0 + \epsilon H_1$, *where* *

$$H_0 = \tfrac{1}{2}(I_1^2 + I_2^2), \quad \varepsilon H_1 = \varepsilon(\cos\varphi_1 - 1)\,[1 + \mu\beta], \quad B = \sin\varphi_2 + \cos t. \tag{2}$$

In other words, we consider the system of differential equations

$$\dot\varphi_1 = I_1, \dot\varphi_2 = I_2; \quad \dot I_1 = \varepsilon\sin\varphi_1\,[1 + \mu B], \; \dot I_2 = \varepsilon(1 - \cos\varphi_1)\,\mu\cos\varphi_2; \quad B = \sin\varphi_2 + \cos t. \tag{3}$$

We first investigate the nonperturbed system ($\epsilon = 0$). Every three-dimensional torus $I_1 = \omega_1$, $I_2 = \omega_2$ is invariant. On it the three-frequency motion $\dot\phi_1 = \omega_1, \dot\phi_2 = \omega_2, \dot t = 1$ takes place. A torus is said to be *nonresonant*, if the frequencies on it are independent of one another (i.e., $n_1\omega_1 + n_2\omega_2 + n_0 \neq 0$ for integers in $n \neq 0$). The equation $I_1 = 0$ determines a family of resonant toruses (since $\omega_1 = 0$).

We now consider a perturbed system: assume $0 < \epsilon\mu \ll \epsilon \ll 1$. In [1,2] it is proved that for the majority of nonresonant initial conditions the quantities $I_1(t), I_2(t)$ will change little in the course of the whole infinite interval of time $-\infty < t < +\infty$. It turns out, however, that *close to the resonant manifold* $I_1 = 0$ *there is a zone of nonstability*. More precisely, we have the following theorem.

Theorem 3. *Assume* $0 < A < B$. *For every* $\epsilon > 0$ *we can find a* $\mu_0 > 0$ *such that for* $0 < \mu < \mu_0$ *the system* (3) *is nonstable: there exists a trajectory which connects the region* $I_2 < A$ *with the region* $I_2 > B$.

5. **A proof of nonstability.** Choose a fixed $\epsilon > 0$.

A. First assume $\mu = 0$. Then the variables can be separated:

$$H = H^{(1)} + H^{(2)}, \quad H^{(1)} = \tfrac{1}{2} I_1^2 + \varepsilon(\cos\varphi_1 - 1), \quad H^{(2)} = \tfrac{1}{2} I_2^2. \tag{4}$$

Thus $\dot I_2 = 0$, $\dot\phi_2 = I_2 = \omega = \text{const}$, and the change of I_1, ϕ_1 with time will be described by the

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* It is easy to construct an actual mechanical system with the Hamiltonian (2).

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Hamiltonian of the ordinary pendulum $H^{(1)}$. Let the number ω be irrational. Then the following assertion is easily proved.

Assertion A. *The manifold T_ω defined by the equations $I_1 = \phi_1 = I_2 - \omega = 0$ is a two-dimensional whiskered torus of the system* (3). *The whiskers are three-dimensional and have the equations*

$$H^{(1)} = 0, \qquad H^{(2)} = {}^1/_2\,\omega^2 \qquad or \quad I_1 = \pm 2\sqrt{\varepsilon} \sin\frac{\varphi_1}{2}, \qquad I_2 = \omega. \tag{5}$$

The whiskers are supplemented by the asymptotic trajectories

$$I_1(t) = \pm 2\sqrt{\varepsilon}\, \mathrm{ch}^{-1}\tau, \;\; \varphi_1(t) = \pm \operatorname{arc\,ctg}(-\operatorname{sh}\tau), \; \varphi_2(t) = \varphi_2^0 + \omega(t - t^0), \tag{6}$$

where $\tau = \sqrt{\epsilon}\,(t - t^0)$, $I_1(t^0) = \pm 2\sqrt{\epsilon}$, $\phi_1(t^0) = \pm\pi$, $\phi_2(t^0) = \phi_2^0$.

Thus for $t \to +\infty$ the point of the departing whisker of the torus T_ω falls again on the same torus T_ω. In other words, the departing whisker forms one manifold together with the arriving whisker. Of course, for $\mu \neq 0$ this manifold splits into two whiskers, which intersect each other.* We shall see that (in distinction to the separatrices of systems with a phase space of dimension ≤ 4, considered in [5,6] *these whiskers also intersect the whiskers of neighboring toruses $T_{\omega'}$.*

B. Now suppose $\mu \neq 0$. From (3) it is evident that the toruses T_ω remain invariant for all μ. Assume that ω is irrational. By the standard method of contractive mappings we can prove the following assertion.**

Assertion B. *The manifold T_ω is a whiskered transition torus of the system* (3), *if μ is sufficiently small.*

Assume $\omega_1 < A < B < \omega_s$. For the proof of Theorem 3 it is sufficient to construct a transition chain of toruses $T_{\omega_1}, \cdots, T_{\omega_s}$ and to make use of Theorem 2. The construction of such a chain is based on a study of the perturbation of the whiskers (5) for small μ. It turns out that the following lemma holds.

Lemma 1. *Assume $A < \omega < B$. Then the departing whisker Y_ω^+ of the torus T_ω intersects with the arriving whiskers $Y_{\omega'}^-$ of all toruses $T_{\omega'}$ which are sufficiently close that $|\omega - \omega'| \leq \kappa$ (where $\kappa = \kappa(\epsilon, \mu, A, B) > 0$).*

The proof of Lemma 1 requires certain calculations. The nonperturbed whiskers have the equation (5): $H^{(1)} = 0$, $H^{(2)} = \omega^2/2$, where $H^{(k)}$ are the function of (4). Assume $\alpha > 0$ (for example, $\alpha = \pi/2$). It is easy to show that for $|\phi_1| < 2\pi - \alpha$ the equations of the perturbed departing whisker Y_ω^+ can be written in the form

$$H^{(1)} = \Delta_1^+(\varphi_1; \varphi_2, t; \omega); \qquad H^{(2)} = {}^1/_2\,\omega^2 + \Delta_2^+(\varphi_1; \varphi_2, t; \omega), \tag{7}$$

where the functions $\Delta_k^+ = O(\mu)$ have the period 2π with respect to ϕ_2, t and are equal to 0 for $\phi_1 = 0$. In exactly the same way the arriving whisker $Y_{\omega'}^-$ for $|\phi_1 - 2\pi| < 2\pi - \alpha$ has equations

$$H^{(1)} = \Delta_1^-(\varphi_1; \varphi_2, t; \omega'), \qquad H^{(2)} = {}^1/_2\omega'^2 + \Delta_2^-(\varphi_1; \varphi_2, t; \omega'). \tag{8}$$

The intersection of the whisker Y_ω^+ and $Y_{\omega'}^-$ will be sought on the plane $\phi_1 = \pi$. In the notations of (7) and (8) Lemma 1 is an assertion concerning the solvability with respect to ϕ_2, t of a system of

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*The splitting of separatrices was studied by Poincaré in the final chapter of "New methods" [5]. The investigations of Poincaré were recently continued by Mel'nikov [6].

**It is convenient to use the conical metric $\|f(x)\| = \max |x^{-1} f(x)|$.

equations

$$\Delta_1^+(\pi;\varphi_2,t;\omega)=\Delta_1^-(\pi;\varphi_2,t;\omega'),$$
$$\tfrac{1}{2}\omega^2+\Delta_2^+(\pi;\varphi_2,t;\omega)=\Delta_2^-(\pi;\varphi_2,t;\omega')+\tfrac{1}{2}\omega'^2. \tag{9}$$

The solvability of the system (9) can be deduced from the following approximate expressions for $\Delta_k^\pm$

Lemma 2 (compare [6]). *The perturbations of the whiskers are* $\Delta_k^\pm=\mu\delta_k^\pm+O(\mu^2)$, *where*

$$\mu\delta_k^\pm(\pi;\varphi_2^0,t^0;\omega)=\int_{\mp\infty}^{0}\{H,H^{(k)}\}\,d(t-t^0)|_{(6)} \tag{10}$$

(the Poisson bracket is integrated along the nonperturbed trajectory (6)).

For in fact, in accordance with the definitions (7), (8), the quantities $\Delta_k^\pm$ represent the increments of $H^{(k)}$ in the perturbed motion (3). The derivative of the function $H^{(k)}$, in view of the system of equations (3), is exactly the Poisson bracket $\{H,H^{(k)}\}$. Consequently, $\Delta_k^\pm$ *is exactly equal to the integrals* (10) *extended over the perturbed trajectories.* Thus we easily derive the estimate $\Delta_\mu^\pm-\mu\delta_k^\pm=O(\mu^2)$, which completes the proof of Lemma 2.

From Lemma 2 it is obvious that the solvability of system (9) depends basically on the solvability with respect to ϕ_2^0, t^0 of the approximate system

$$\delta_1=0,\quad \mu\delta_2=\tfrac{1}{2}(\omega^2-\omega'^2), \tag{11}$$

where

$$\delta_k=\delta_k^+(\pi;\varphi_2^0,t^0,\omega)-\delta_k^-(\pi;\varphi_2^0,t^0,\omega)=\int_{-\infty}^{+\infty}\{H,H^{(k)}\}\,d(t-t^0)|_{(6)}. \tag{12}$$

An easy calculation, based on formulas (2)–(6), gives the result

$$\delta_1=-2\varepsilon\int_{-\infty}^{+\infty}u\frac{\partial B}{\partial t}\,dt,\qquad \delta_2=2\varepsilon\omega\int_{-\infty}^{+\infty}u\frac{\partial B}{\partial\varphi_2}\,dt, \tag{13}$$

where $u=\mathrm{ch}^{-2}\tau$, $\tau=\sqrt{\epsilon}\,(t-t^0)$, $B=B(\phi_2,t)$, $\phi_2=\phi_2^0+\omega(t-t^0)$. For $B=\sin\phi_2+\cos t$ the integrals (13) involve * the residues

$$\delta_1=2\pi\left(\mathrm{sh}^{-1}\frac{\pi}{2\sqrt{\varepsilon}}\right)\sin t^0,\quad \delta_2=2\pi\omega^2\left(\mathrm{sh}^{-1}\frac{\omega\pi}{2\sqrt{\varepsilon}}\right)\cos\varphi_2^0. \tag{14}$$

Setting $t^0=0$ in (14) we see that the system (11) is solvable for

$$|\omega^2-\omega'^2|<4\pi\mu\omega^2\,\mathrm{sh}^{-1}\frac{\omega\pi}{2\sqrt{\varepsilon}}\asymp\mu e^{-1/\sqrt{\varepsilon}} \tag{15}$$

From Lemma 2 it now follows that for sufficiently small μ the system (9) is also solvable. From the inequality (15) it is easy to obtain, uniformly for $A<\omega<B$, the estimate $\max_{\omega'}|\omega-\omega'|=\kappa(\omega)$ from below, as required in Lemma 1. Thus the proof of Lemma 1 is complete. It allows us to construct a chain of transitional toruses $T_{\omega_1},\cdots,T_{\omega_s}$ $(\omega_1<A<B<\omega_s)$. From the formula (14) it is clear that for sufficiently small μ this chain can be so chosen that the consecutive intersecting whiskers lie in general position and complement one another in the sense of §2. Then the chain $T_{\omega_1},\cdots,T_{\omega_s}$ will be

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* The analogous integrals in [6], on page 32, are wrongly calculated

a transition chain. The application of Theorem 2 to the transition chain $T_{\omega_1}, \cdots, T_{\omega_s}$ completes the proof of Theorem 3.

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УСПЕХИ МАТЕМАТИЧЕСКИХ НАУК

ЗАСЕДАНИЯ МОСКОВСКОГО МАТЕМАТИЧЕСКОГО ОБЩЕСТВА*

Заседание 10 марта 1964 г.

Заседание было посвящено памяти крупнейшего французского математика Жака Адамара. Выступавшие П. С. Александров, А. О. Гельфонд, Г. Е. Шилов, О. А. Олейник, А. Н. Колмогоров говорили о значении работ Адамара для развития современной математики и о личных впечатлениях от встреч и бесед с замечательным ученым.

Заседание 17 марта 1964 г.

1. В. И. Арнольд «О неустойчивости динамических систем со многими степенями свободы».

В последнее время показано, что каждая динамическая система, достаточно близкая к интегрируемой, имеет много инвариантных торов. В случае, когда размерность фазового пространства не превосходит четырех, из этих результатов вытекает устойчивость системы. В работе указан пример неустойчивой гамильтоновой системы с пятимерным фазовым пространством I_1, I_2; φ_1, φ_2, t:

$$\dot{I}_1=\varepsilon \sin \varphi_1 [1+\mu B], \quad \dot{I}_2=\varepsilon [\cos \varphi_1-1]\, \mu \frac{\partial B}{\partial \varphi_2}, \quad \dot{\varphi}_k=I_k, \quad B=\sin \varphi_2+\cos t,$$

где φ_1, φ_2, t — угловые переменные, а ε, μ — малые параметры. Величины I_1 и I_2 не испытывают вековых возмущений во всех приближениях теории возмущений. Более того, для большинства начальных условий они действительно мало меняются в течение бесконечного промежутка времени. Однако при сколь угодно малых ε, μ существуют и такие начальные условия, при которых величина I_2 меняется очень сильно. Доказательства основаны на изучении пересечений сепаратрис двумерных инвариантных торов $I_1=\varphi_1=I_2-\omega=0$.

2. На заседании Общества происходило обсуждение планов редакции издательства «Мир» и сборника переводов «Математика». В обсуждении приняли участие В. И. Арнольд, В. И. Битюцков, А. О. Гельфонд, А. Н. Колмогоров, А. Г. Курош, Б. В. Шабат, Л. Э. Эльсгольц.

3. Председательствующий А. Н. Колмогоров объявляет о принятии Правлением Общества резюме следующих докладов:

1°. П. П. Забрейко и М. А. Красносельский «Об L-характеристиках операторов».

* Editor's note: V.I. Arnol'd: On the instability of dynamical systems with several degrees of freedom. Published in Uspekhi Mat. Nauk 19:5 (1964), 181

2°. Л. С. Франк «Разностные методы решения задачи Коши для некорректных систем первого порядка».

Заседание 24 марта 1964 г.

1. А. М. Олевский «Об ортогональных рядах по полным системам».

Рассматриваются ряды

$$\sum_{n=1}^{\infty} c_n \varphi_n(x), \tag{1}$$

где $\{\varphi_n(x)\}$ — полная ортонормированная в L^2 [0, 1] система функций. В докладе изложены результаты автора, относящиеся к следующим вопросам.

1°. Сходимость ортогональных рядов и поведение коэффициентов. Установлено существование по любой полной ортонормированной системе расходящихся рядов с коэффициентами, достаточно быстро стремящимися к нулю.

Далее, сформулировано необходимое и достаточное (в классе полных систем) условие на коэффициенты сходящихся ортогональных рядов (рядов Фурье).

Подробное изложение результатов этого пункта см. в [5] и [6].

2°. Расходящиеся ряды Фурье. Здесь основной является следующая теорема, опубликованная в [7]: *для любой полной ортонормальной системы существует непрерывная функция, ряд Фурье которой после некоторой перестановки членов расходится почти всюду*.

3°. Коэффициенты Фурье непрерывных функций. Общий смысл результатов этого пункта состоит в следующем: какова бы ни была полная ортонормальная система $\{\varphi_n(x)\}$, о коэффициентах Фурье произвольной непрерывной функции по этой системе нельзя сказать ничего большего, чем выполнение условия

$$\sum_{n=1}^{\infty} c_n^2 < \infty. \tag{2}$$

Постановка вопроса восходит к Карлеману, который доказал существование непрерывной функции, коэффициенты Фурье которой по тригонометрической системе удовлетворяют условию

$$\sum_{n=1}^{\infty} |c_n|^p = \infty \tag{3}$$

при всех $p < 2$. В связи с этим возникло следующее

Определение. Непрерывная функция $f(x)$ *обладает особенностью Карлемана по отношению к системе* $\{\varphi_n(x)\}$, если коэффициенты Фурье

$$c_n = \int_0^1 f(x)\, \varphi_n(x)\, dx \tag{4}$$

удовлетворяют условию (3) при всех $p < 2$.

Теорема Карлемана в дальнейшем подвергалась обобщению различными авторами (Палей, Банах, Орлич, С. Б. Стечкин), работы которых относятся к тригонометрической системе и к более общим системам, ограниченным в совокупности.

Для произвольных полных систем аналогичный вопрос был поставлен в [5], где было доказано, что для любой полной ортонормальной системы существует функция, обладающая особенностью Карлемана.

Аналогичный результат справедлив и для особенностей вида

$$\sum_{n=1}^{\infty} c_n^2 \omega(n) = \infty. \tag{5}$$

Именно для любой полной ортонормальной системы и для любой последовательности $\omega(n) \uparrow \infty$ существует непрерывная функция $f(x)$, коэффициенты Фурье которой (4) удовлетворяют условию (5).

Таким образом, нельзя построить полную систему, которая позволила бы различать по поведению коэффициентов Фурье классы функций C и L^2 посредством шкалы l^p или весовых шкал l_ω^2. Наиболее общая формулировка результата содержится в следующей теореме, из которой обе предыдущие могут быть весьма просто выведены.

Теорема 1. *Пусть $\{\varphi_n(x)\}$ — произвольная полная ортонормированная система. Тогда существует фиксированная подпоследовательность номеров $\{n_k\}$ такая, что для любой последовательности $\{b_k\}$ с $\Sigma\, b_k^2 < \infty$ найдется непрерывная функция $f(x)$, коэффициенты Фурье которой* (4) *удовлетворяют условию*

$$\sum_{n=n_k+1}^{n_{k+1}} c_n^2 \geqslant b_k^2 \quad (k > k_0).$$

Отметим, что положить здесь $n_k = k$, вообще говоря, нельзя.

Мы укажем теперь на возможность усиления приведенных результатов в двух направлениях. Прежде всего, особенности могут быть локализованы. Кроме того, условие непрерывности может быть заменено некоторой гладкостью.

Для краткости мы сформулируем последующие результаты лишь для карлемановских особенностей.

Теорема 2. *Пусть $\{\varphi_n(x)\}$ — произвольная полная ортонормальная система и $E \subset [0,1]$ — произвольное множество положительной меры. Тогда существует непрерывная функция $F(x)$ такая, что всякая непрерывная функция $f(x)$, совпадающая с $F(x)$ на E, обладает особенностью Карлемана по отношению к системе $\{\varphi_n(x)\}$.*

Теорема 3. *Пусть $\{\varphi_n(x)\}$ — произвольная полная ортонормальная система и x_0 — произвольная точка отрезка* $[0,1]$. *Тогда существует дифференцируемая на* $[0,1]$ *функция $F(x)$ такая, что всякая непрерывная функция $f(x)$, совпадающая с $F(x)$ в некоторой окрестности точки x_0, обладает особенностью Карлемана.*

Отметим, что теоремы такого же типа могут быть доказаны для характеристических функций множеств. Например, справедлива

Теорема 4. *Пусть $\{\varphi_n(x)\}$ — произвольная полная ортонормальная система. Тогда существует множество $E \subset [0,1]$, характеристическая функция которого $\chi(E)$ обладает особенностью Карлемана.*

Эта теорема также допускает локализацию.

Теорема 1 и следствия из нее показывают, что нельзя построить полную систему, для которой коэффициенты Фурье всех непрерывных функций были бы достаточно хорошими. Вместе с тем оказывается, что существуют такие полные системы, для которых коэффициенты Фурье *всех* непрерывных функций ведут себя плохо. Например, имеет место

Теорема 5. *Существует полная ортонормированная система $\{\varphi_n(x)\}$ такая, что всякая непрерывная функция $f(x) \not\equiv 0$ обладает особенностью Карлемана по отношению к этой системе.*

Заметим, что система, о которой здесь идет речь, вовсе не должна состоять из очень плохих функций. Разумеется, ни одна из них не может быть непрерывной, но можно сделать так, чтобы каждая функция $\varphi_n(x)$ имела лишь конечное число точек разрыва, а в остальном была бы сколь угодно гладкой.

4°. Существование полной ограниченной системы сходимости. Как известно, Д. Е. Меньшовым [1] был впервые установлен принципиально важный факт существования всюду расходящегося ортогонального ряда (1) с коэффициентами, удовлетворяющими условию (2). В связи с этим возникло следующее

Определение. Ортонормированная система $\{\varphi_n(x)\}$ называется *системой сходимости*, если ряд (1) сходится почти всюду как только выполнено условие (2).

Иными словами, $\{\varphi_n\}$ — система сходимости, если для всякой функции $f(x) \in L^2$ ряд Фурье сходится почти всюду.

Известным примером полной ортонормированной системы сходимости является система Хаара. Для других классических полных систем (тригонометрической и системы Уолша) вопрос о том, являются ли они системами сходимости, не решен и поныне. Для тригонометрической системы — это известная проблема Лузина.

Следует заметить, что тригонометрическая система и система Уолша состоят из функций, ограниченных в совокупности, т. е. удовлетворяют условию

$$|\varphi_n(x)| < M.$$

В то же время система Хаара, напротив, устроена из высоких и узких «пиков» (см. [4], стр. 57). В связи с этим возникла следующая проблема: существует ли полная ортонормированная система функций, ограниченных в совокупности, являющаяся системой сходимости?

Этот вопрос сформулирован П. Л. Ульяновым в обзорных статьях [2], [3] в числе других нерешенных задач теории ортогональных рядов.

Мы сформулируем теорему, дающую положительный ответ на этот вопрос.

Теорема 6. *Существует полная ортонормированная система $\{\theta_n(x)\}$, состоящая из функций, ограниченных в совокупности, и являющаяся системой сходимости.*

Доказательство теоремы осуществляется конструктивно: дается построение системы $\{\theta_n\}$.

Отметим, что остается открытым вопрос о существовании полной ограниченной в совокупности ортонормальной системы функций, по которой ряды Фурье из L^p при $p < 2$ сходятся почти всюду. Система $\{\theta_n(x)\}$ во всяком случае этим свойством не обладает. Более того, с ее помощью может быть доказана следующая

Теорема 7. *Существует полная в L $[0,1]$ ортонормированная ограниченная в совокупности система $\{\varphi_n(x)\}$ и функция $f(x)$, принадлежащая всем классам L^p $[0,1]$ при $1 \leqslant p < 2$, такие, что частные суммы $s_{n_k}(x)$ ряда Фурье (1) этой функции расходятся почти всюду на $[0,1]$, какова бы ни была последовательность $n_k \uparrow \infty$.*

Такой результат был впервые получен Марцинкевичем (см. [4], стр. 358), но лишь для $f(x) \in L^p$, $p < \frac{6}{5}$. Очевидно, что наша теорема в этом отношении окончательна. Полезно отметить, что система $\{\varphi_n\}$ в теореме 7 получается из системы $\{\theta_n\}$ теоремы 6 при помощи некоторой перестановки элементов, не нарушающей впрочем ее основное свойство: быть системой сходимости. Таким образом, мы можем сделать следующий вывод: для полных и даже ограниченных в совокупности ортонормальных систем сходимость почти всюду рядов Фурье из L^2 ничего не влечет за собой относительно сходимости рядов из L^p при $p < 2$.

Следует заметить, что конструкция, используемая нами при построении системы $\{\theta_n(x)\}$, может быть применена и при исследовании некоторых других вопросов. В качестве примера приведем одну задачу из теории лакунарных систем. Именно, хорошо известна теорема Сидона, состоящая в том, что если ограниченная функция имеет лакунарный тригонометрический ряд Фурье, то $\sum |c_n| < \infty$, где c_n — коэффициенты Фурье. Аналогичный результат известен и для системы Радемахера. Это и некоторые другие свойства, характерные, вообще, для лакунарных систем, образуют группу банаховых эквивалентностей (см. [4], стр. 295). В книге [4] (стр. 297) поставлен вопрос: обладает ли этим свойством любая лакунарная система? Оказывается, что *на этот вопрос следует дать отрицательный ответ.*

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2. С. П. Новиков «Двумерные слоения».

3. Происходило избрание новых членов Общества. В результате голосования членами Общества избраны:

Арнольд Владимир Игоревич,

Гапошкин Владимир Федорович,

Михайлов Валентин Петрович,

Постникова Людмила Петровна,

Сирота Александр Исаакович.

4. Президент Общества П. С. Александров объявил о принятии Правлением Общества резюме следующих докладов:

1°. А. Г. Рамм «Об аналитическом продолжении решения уравнения Шрёдингера и поведении решения нестационарной задачи при $t \to \infty$».

2°. А. И. Перов «Об одном многомерном обобщении определителя Вронского».

Заседание 31 марта 1964 г.

1. А. Н. Тихонов «О методах решения нестационарных задач».

2. Президент Общества П. С. Александров объявляет о принятии Правлением Общества резюме следующих докладов:

1°. Е. Г. Дьяконов «Метод мажорирующего оператора для решения разностных аналогов некоторых сильно эллиптических систем».

2°. А. Ахмедов «Об одном способе моделирования работы некоторого класса систем массового обслуживания».

Заседание 7 апреля 1964 г.

1. Ю. И. Любич «Абстрактная задача Коши».

Заседание 14 апреля 1964 г.

1. В. И. Арнольд «Периодичность Ботта (применение вариационного исчисления к топологии групп Ли)».

2. Председательствующий Г. Е. Шилов объявляет о принятии Правлением Общества резюме следующих докладов:

1°. Я. В. Хион «Ω-кольцоиды, Ω-кольца и их представления».

2°. Линь Цзун-чи «Асимптотика решений линейных дифференциальных уравнений при сочетании возмущения границы с возмущением оператора».

ERRATA TO V. ARNOL'D'S PAPER "SMALL DENOMINATORS. I"*

V.I. Arnol'd
translated by Gerald Gould

G.A. Merman and N.N. Bogolyubov have kindly pointed out to me that in formula (6) of my paper "Small denominators. I" (see Izv. Akad. Nauk SSSR Ser. Mat. **25**, 21–86 (1961)) a term was omitted. Nevertheless, the estimates in §5 remain valid (even taking account of this term) if one makes the following corrections.

1. Replace $z(\varphi, \varepsilon)$ by $z(\varphi, \varepsilon, \Delta_1)$ in the following places: p. 40, lines 1, 6, 7, 10, 13 from the bottom; p. 42, lines 13 and 14 from the bottom; p. 43, lines 12, 13 and 15 from the bottom.

2. Replace $\varphi(z, \varepsilon)$ by $\varphi(z, \varepsilon, \Delta_1)$ in the following places: p. 42, formula (1); p. 43, formulae (6), (15); p. 44, line 13.

3. Replace $g(z, \varepsilon)$ by $g^*_\sharp(z, \varepsilon, \Delta(\Delta_1, \varepsilon))$ in the following places: p. 42, formula (1); p. 44, lines 8, 10, 12, 13.

4. On p. 43 add the following to formula (5):

$$g^*(z, \varepsilon, \Delta) = g(z, \varepsilon) + g_2(z, \varepsilon, \Delta)\,,$$
$$\varphi(z, \varepsilon) = \varphi^*(z, \varepsilon, \Delta_0^*), \quad z(\varphi, \varepsilon) = z(\varphi, \varepsilon, 0)\,.$$

5. On p. 43 add the following term to the right of formula (8):

$$g_2(z_{III}, \varepsilon, \Delta) - g_2(z_{II}, \varepsilon, \Delta)\,.$$

6. On p. 43 add the following term to the right of formula (14):

$$\hat{F}_1(z(\varphi, \varepsilon, \Delta_1), \varepsilon) - \hat{F}_1(z(\varphi, \varepsilon), \varepsilon)\,.$$

7. On pp. 44–45 subsection 4 must be before subsection 2, and in line 8 of p. 45 1/2 must be replaced by 1/12; and on p. 40, in formula (10) 1/36 must be replaced by 1/48.

8. on p. 44, line 2, replace $\Delta = \Delta_0^*$ by $|\Delta_1| \leqslant C$, and in line 4 replace $2C + 2\delta \cdot 2C$ by $2C + 2\delta \cdot 4C$.

* Izv. Akad. Nauk SSSR Ser. Mat. **28**, 479–480 (1964)

9. On. p. 45, subsection 5, the following should come after formula (22):

Since, in accordance with (2) and (5), the function $g_2(z, \varepsilon, \Delta(\Delta_1, \varepsilon))$ satisfies the equation

$$g_2(z+2\pi\mu,\ \varepsilon,\ \Delta(\Delta_1,\varepsilon)) - g_2(z,\ \varepsilon,\ \Delta(\Delta_1,\varepsilon)) = \widetilde{\Phi}(z,\ \varepsilon,\ \Delta_0^*) - \widetilde{\Phi}(z,\ \varepsilon,\ \Delta(\Delta_1,\varepsilon))\,,$$

and, in accordance with subsection 4, for $|\operatorname{Im} z| \leqslant R_0$ the right-hand side of this equation is less than $\Delta_1/6$, it follows by Theorem 1 that

$$|g_2| \leqslant \frac{\Delta_1}{24\delta^4}, \quad \left|\frac{\partial g_2}{\partial z}\right| \leqslant \frac{\Delta_1}{24\delta^5} \qquad (|\operatorname{Im} z| \leqslant R_0 - 2\delta)\,.$$

Consequently,

$$\begin{aligned} |g_2(z_{III}, \varepsilon, \Delta) - g_2(z_{II}, \varepsilon, \Delta)| &\leqslant |z_{III} - z_{II}| \frac{\Delta_1}{24\delta^5} \\ &< (2C + 2\delta \cdot 4C + 4C)\frac{\Delta_1}{24\delta^5} < \frac{\Delta_1 C}{3\delta^5}\,. \end{aligned} \tag{24}$$

Next, we have

$$|\hat{F}_1(z(\varphi,\varepsilon,\Delta),\ \varepsilon) - \hat{F}_1(z(\varphi,\varepsilon),\ \varepsilon)| < \frac{16C^2\Delta_1}{\delta^{10}}\,. \tag{25}$$

Taking (24) and (25) into account and applying the lemma. . . (and then continue as on p. 45).

10. On pp. 46–47 one must introduce the argument Δ_k in the function $\varphi_{k-1}(\varphi_k, \varepsilon)$.

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МАЛЫЕ ЗНАМЕНАТЕЛИ И ПРОБЛЕМА УСТОЙЧИВОСТИ В КЛАССИЧЕСКОЙ И НЕБЕСНОЙ МЕХАНИКЕ*

§ 1. Введение

Еще Лаплас заметил, что приближенная соизмеримость средних движений Юпитера ($n_{\text{Ю}} = 299'',1$) и Сатурна ($n_{\text{С}} = 120'',5$) влечет за собой большое возмущение их движения вокруг Солнца. Причиной является *малый знаменатель* $2n_{\text{Ю}} - 5n_{\text{С}}$, входящий в один из коэффициентов тригонометрического ряда, выражающего координаты планет. Кажется полезным собрать сделанное в этой области со времени ставших уже классическими работ Пуанкаре, Ляпунова и Биркгофа.

Простейшей задачей с малыми знаменателями является уравнение

$$f(x + \alpha) - f(x) = g(x) \tag{1}$$

с неизвестной периодической функцией $f(x) = f(x + 2\pi)$. Если $f = \sum_{n \neq 0} f_n e^{inx}$ и $g = \sum_{n \neq 0} g_n e^{inx}$, то формальное сравнение коэффициентов Фурье дает

$$f_n = \frac{g_n}{e^{in\alpha} - 1} .$$

При $\alpha = 2\pi \frac{m}{n}$ знаменатель $e^{in\alpha} - 1$ обращается в нуль и может быть сколь угодно малым при любом α_0. Из теории диофантовых приближений известно, что для почти каждого (в смысле меры Лебега) α существует $c(\alpha) > 0$, такое что

$$\left| \frac{\alpha}{2\pi} - \frac{m}{n} \right| > \frac{c}{|n|^3} \tag{2}$$

при всех целых m и $n \neq 0$ (см. [1]).

Если функция $g(x)$ аналитична, то g_n убывают в геометрической прогрессии. При условии (2) коэффициенты f_n также убывают в геометрической прогрессии, и, следовательно, ряд $\sum f_n e^{inx}$ сходится к аналитическому решению уравнения (1) при почти всех α.

Как и в этом простом примере, во многих задачах с малыми знаменателями (например, в проблеме трех тел) известны формальные ряды, удовлетворяющие условию задачи при почленной подстановке. Эти ряды выражают условно-периодические движения с частотами $\omega_1, \ldots, \omega_k$ и имеют вид

$$\sum_n \frac{a_{n_1 n_2 \ldots n_k}}{n_1\omega_1 + \ldots + n_k\omega_k} \cos \left[(n_1\omega_1 + \ldots + n_k\omega_k) t + \theta_{n_1 \ldots n_k}\right]. \tag{3}$$

Но вопрос о сходимости представляет обычно гораздо большие трудности, чем в простейшей задаче. Появляясь в каждом приближении теории возмущений, малые знаменатели $n_1\omega_1 + \ldots + n_k\omega_k$ приводят к тому, что формальные ряды, как правило, расходятся ([2], [3]).

Мне известны следующие решенные в настоящее время задачи с малыми знаменателями:

1. В 1942 г. К. Л. Зигель [4] решил *теоретико-функциональную проблему центра*, доказав устойчивость аналитического отображения окрестности нуля комплексной плоскости на себя

$$z \to az + a_2 z^2 + a_3 z^3 + \ldots$$

при почти всех a, равных по модулю единице.

2. В 1952 г. К. Л. Зигель доказал при весьма общих предположениях *приводимость нелинейной системы* обыкновенных аналитических дифференциальных

уравнений в окрестности положения равновесия к линейной системе подходящей аналитической заменой переменных [5].

3. В 1954 г. А. Н. Колмогоров, применив метод Ньютона, сумел найти инвариантные торы, несущие условно-периодические траектории, в так называемой *невырожденной задаче теории возмущений* ([6], [7]).

4. На прошлом съезде А. Е. Гельман доложил свою работу о *приводимости системы двух линейных дифференциальных уравнений с условно-периодическими коэффициентами к* системе с постоянными коэффициентами [8]. Студент МГУ Э. Г. Белага использовал этот результат для исследования окрестности условно-периодического решения. С помощью аналитической замены переменных он привел аналитическую нелинейную систему в окрестности инвариантного тора к линейной системе с постоянными коэффициентами.

5. Соединяя асимптотические ряды Биргофа и метод Ньютона, можно доказать *устойчивость периодического решения аналитической гамильтоновой системы в общем эллиптическом случае* (см. ниже § 2). В частности, студент МГУ А. М. Леонтович показал недавно, что лагранжево периодическое решение ограниченной задачи трех тел (ср. [4], [9]) устойчиво при почти всех отношениях масс тел, когда оно устойчиво по первому приближению.

6. Некоторая модификация метода Ньютона позволяет построить сходящийся вариант теории возмущений и в случае вырождения (см. ниже § 3). Особенно полно исследован случай, когда имеется только одна «быстрая» переменная. На этом пути удается, в частности, доказать, что при медленном периодическом изменении функции Гамильтона аналитической системы с одной степенью свободы *адиабатический инвариант мало меняется за бесконечный промежуток времени*, если только система не является линейной.

§ 2. Проблема устойчивости

Рассмотрим простейший, но содержащий уже основную трудность случай этой проблемы. *Дано сохраняющее площадь аналитическое отображение T окрестности нуля плоскости x, y на себя. Пусть нуль — неподвижная точка. Устойчива ли она?*

Для линейных отображений вопрос решается вычислением собственных значений λ_1, λ_2. Ввиду сохранения площади $\lambda_1\lambda_2=1$. Если λ_1, λ_2 не действительны, то $\bar{\lambda}_1=\lambda_2$, $|\lambda|=1$. Отображение является эллиптическим поворотом и устойчиво.

Пример 1. Обыкновенный поворот на угол $\varphi=\arg\lambda$.

Если линейная часть данного отображения есть эллиптический поворот, оно называется отображением *эллиптического типа*.

Пример 2. Рассмотрим отображение A, при котором каждая окружность $x^2+y^2=2r$ поворачивается на свой угол

$$\varphi(r)=\arg\lambda+c_1r+c_2r^2+\ldots.$$

Это отображение — эллиптического типа, и оно устойчиво.

Пример 3. Если записать только что определенное отображение A в системе координат $(p,\ q)$, связанной с $(x,\ y)$ аналитическим преобразованием, сохраняющим площадь и оставляющим O на месте, S: $(x,\ y)\to(p,\ q)$, то получим на плоскости $(p,\ q)$ устойчивое отображение $T=SAS^{-1}$.

Следовательно, если при подходящем выборе S данное отображение T можно получить с помощью конструкции примера 3, то оно устойчиво.

Со времен Бирккгофа [10] известно, что если не заботиться о сходимости рядов, то при условии $\lambda^n\neq1$ $(n=1,2,3,\ldots)$ отображение эллиптического типа T всегда можно формальной заменой переменных привести к виду A примера 2. Но ряды, определяющие S, в общем случае расходятся, и из указанных алгебраических результатов нельзя сделать вывода об устойчивости ([4], [11]).

Можно, однако, сходящейся заменой привести преобразование к виду, отличающемуся от A лишь членами сколь угодно большого порядка относительно r. В новых координатах вблизи O наше преобразование можно тогда рассматривать как поворот A семейства окружностей на зависящий от радиуса угол, возмущенный весьма маленькими дополнительными членами.

Применяя метод Ньютона, можно показать (см. [12]), что те из инвариантных окружностей, на которых угол поворота «достаточно иррационален» [как α в (2) § 1], не исчезают при малом возмущении. Поэтому неподвижная точка окружена сколь угодно малыми инвариантными аналитическими замкнутыми кривыми и, следовательно, устойчива. Можно показать далее, что эти кривые заполняют множество положительной меры, имеющее O точкой плотности.

Между инвариантными кривыми остаются «зоны неустойчивости». Действительно, рассмотрим инвариантную окружность «невозмущенного» преобразования A, поворачивающуюся на угол $\varphi(r) = 2\pi \frac{m}{n}$. При n-кратной итерации поворота каждая точка вернется на место. Это свойство при малом возмущении вообще говоря не сохраняется, и инвариантная окружность рассыпается. Но по известной теореме Пуанкаре—Биркгофа о неподвижной точке (см. [10], [4]), существуют возвращающиеся на место после n-кратной итерации T точки, причем в «общем случае» среди них будут и эллиптические, и гиперболические относительно T^n. Как мы видели выше, точки эллиптического типа «вообще говоря» устойчивы и окружены аналитическими инвариантными кривыми, очевидно не охватывающими O. Следовательно, в общем случае окрестность O не расслаивается на инвариантные замкнутые кривые. Отсюда вытекает, в частности, упоминавшаяся выше расходимость рядов Биркгофа.

Инвариантные кривые в окрестности неподвижной точки в общем эллиптическом случае.

В окрестности каждой из найденных устойчивых точек повторяется та же картина с инвариантными кривыми и зонами неустойчивости и т. д. (см. рисунок). Форма границ зон неустойчивости мне неизвестна. Было бы также важно изучить поведение при итерациях отображения T «общей» точки зоны неустойчивости с точки зрения эргодической теории (найти эргодические компоненты, перемешивание, спектр, энтропию).

Результаты, аналогичные указанным выше, получены для окрестности периодического решения и положения равновесия автономной гамильтоновой системы с двумя степенями свободы и неавтономной — с одной степенью свободы [12]. В общем случае системы с n степенями свободы удается найти только n-мерные инвариантные торы, заполненные условно-периодическими траекториями. Эти торы образуют множество положительной меры, но они не разделяют $2n$-мерные фазовое пространство, и вопрос об устойчивости остается открытым.

§ 3. Вырождение. Адиабатические инварианты

Классическая теория возмущений ([13], [2]) рассматривает условно-периодические движения, т. е. колебания с несколькими частотами $\omega_1, \ldots, \omega_n$ (ср. (3) § 1). Есть много задач, в которых в первом приближении движение *вырождается*, т. е. некоторые частоты равны 0. Например, в проблеме трех тел перигелии и узлы в первом приближении неподвижны.

В этих случаях возмущение порядка ε вызывает — в дополнение к «быстрым» колебаниям с частотами ω_i, имевшимся и при $\varepsilon = 0$, — медленные колебания. Их частоты λ_j будут порядка ε. Соответствующие малые знаменатели

$$\sum_{i=1}^{k} m_i\omega_i + \sum_{j=k+1}^{n} n_j\lambda_j$$

опасны не только из-за приближенной соизмеримости всех частот ω_i, λ_j, но и (при $m_i=0$) из-за малости частот λ_j. Появление этих знаменателей порядка ε существенно усложняет картину теории возмущений по сравнению с невырожденным случаем, рассмотренным в [6].

Метод Ньютона позволяет найти инвариантные торы с условно-периодическими траекториями, если в функции Гамильтона вырожденной системы

$$H=H_0(p_1, \ldots, p_k)+\varepsilon H_1(p_1, \ldots, p_n; q_1, \ldots, q_n)+\ldots \tag{4}$$

можно выделить секулярную часть порядка ε, не зависящую от фаз Q_j медленных движений, и периодическую часть порядка ε^2

$$H=H_0(P_1, \ldots, P_k)+\varepsilon \bar{H}_1(P_1, \ldots, P_n)+ \\ +\varepsilon^2 H_2(P_1, \ldots, P_n; Q_1, \ldots, Q_n)+\ldots \tag{5}$$

(где все функции переменных q, Q имеют по ним период 2π). Переход от (4) к (5) (усреднение по быстрым переменным) легко осуществим, если имеется только одна быстрая переменная ($k=1$). Эту переменную тогда удобно принять за время.

Простейший случай — дифференциальное уравнение на торе

$$\frac{dy}{dx}=\varepsilon f(x, y)$$

— рассмотрен в заметке [14]. Аналогичные соображения применимы в обсуждаемой ниже задаче о поведении адиабатического инварианта при медленном периодическом изменении аналитической функции Гамильтона $H(p, q; \lambda)$ ($\lambda=\varepsilon t$ — медленное время). Интересно, что малые знаменатели в конечном счете выпадают, и получается простой результат, не содержащий характерного для задач с малыми знаменателями ограничения «почти всюду».

Рассмотрим фазовую плоскость (p, q) при фиксированном значении параметра λ. Проходящая через точку (p_0, q_0) линия уровня энергии $H(p, q; \lambda)=H(p_0, q_0; \lambda)$ ограничивает в случае колебательной системы некоторую область. Обозначим через $I(p_0, q_0; \lambda)$ величину площади этой области. Как известно (см. [13], [15]), I является адиабатическим инвариантом, т. е. изменение

$$I[p(t), q(t); \varepsilon t]-I(p_0, q_0; 0)$$

за большое время t, $0<t<\frac{1}{\varepsilon}$, имеет порядок ε.

Из адиабатической инвариантности I не следует, вообще говоря, что I мало меняется за неограниченное время $-\infty<t<\infty$ при малом ε. Это связано с возможностью накоплений малых изменений адиабатического инварианта. Рассмотрим, например, линейную колебательную систему

$$\ddot{x}=-\omega^2 x(1+\alpha\cos\varepsilon t).$$

Как известно, при некоторых ε (а именно, $\varepsilon\approx\frac{2\omega}{k}$; $k=1,2,\ldots$) возможен параметрический резонанс, и $I(t)\to\infty$, когда $t\to\infty$. Здесь скорость изменения параметров системы ε может быть, очевидно, сколь угодно малой.

Оказывается, однако, что в *нелинейной* системе с медленно периодически меняющейся функцией Гамильтона $H(p, q; \lambda)=H(p, q; \lambda+2\pi)$ *адиабатический инвариант сохраняется вечно: для любого* $\eta>0$ *найдется* $\varepsilon_0(\eta)>0$, *такое что из* $\varepsilon<\varepsilon_0$ *вытекает*

$$|I(t)-I(0)|<\eta$$

при всех $-\infty<t<\infty$.

Линейная система занимает исключительное положение потому, что частота ее колебаний не зависит от амплитуды. В нелинейной же системе при увеличении амплитуды частота меняется, и колебания не успевают нарасти, как нарушается условие резонанса $\varepsilon\approx\frac{2\omega}{k}$.

Вечно сохраняется также адиабатический инвариант I_y автономной системы с функцией Гамильтона

$$H=\frac{\dot{x}^2+\dot{y}^2+U(\varepsilon x, y)}{2}.$$

Нужно только, чтобы (в первом приближении) отношение частот ω_x/ω_y зависело при фиксированной общей энергии $H=h$ от амплитуды колебаний y. В частности, поле с потенциалом

$$U=\omega^2(\lambda)\,y^2 \quad (\omega=1+\lambda^2,\ \lambda=\varepsilon x)$$

является при $\varepsilon \ll 1$ «ловушкой», способной вечно удерживать частицы с x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ порядка 1. Это вытекает из вечной адиабатической инвариантности величины

$$I_y=\frac{\dot{y}^2+U}{2\omega}.$$

§ 4. Нерешенные задачи

Несмотря на имеющееся продвижение, ситуация остается довольно сложной. Кроме старых нерешенных задач, подобных проблеме устойчивости солнечной системы, возникает много новых вопросов. Я хочу обсудить лишь очень небольшое число проблем, кажущихся мне важными по разным причинам.

I. *Имеется ли действительная неустойчивость в многомерных задачах теории возмущений, когда инвариантные торы не делят фазовое пространство* (см. § 2)?

Первыми шагами при рассмотрении этого вопроса могли бы быть исследование движения в зоне неустойчивости и перенесение относящейся сюда теории Биркгофа [10] на многомерный случай окрестности условно-периодического движения. Однако даже более простая задача об уравнениях с условно-периодическими коэффициентами не решена из-за отсутствия нужного обобщения теории Флоке. Рассмотрим этот вопрос подробнее.

II. *Являются ли линейные дифференциальные уравнения с условно-периодическими коэффициентами в общем случае приводимыми?*

Пусть $A(p)$ — матрица, аналитически зависящая от точки тора $p=(p_1, \ldots, p_k)$, так что $A(p+2\pi)=A(p)$. Пусть точка p движется по тору условно-периодически, с частотами

$$\dot{p}=\lambda \quad (\lambda=(\lambda_1, \ldots, \lambda_k)).$$

Тогда система линейных дифференциальных уравнений

$$\dot{x}=A(p)\,x \quad (x=(x_1, \ldots, x_n)) \tag{6}$$

будет иметь условно-периодические коэффициенты $A(p(t))$. Система (6) называется *приводимой* к системе с постоянными коэффициентами

$$\dot{y}=By, \tag{7}$$

если существует аналитическая на торе матрица $C(p)$, такая что замена переменной $x=C(p)\,y$ превращает (6) в (7).

В случае периодических коэффициентов, т. е. когда $k=1$, всякая система, согласно классической теории Флоке [16], приводима.

При $k>1$ коэффициенты условно-периодичны, и даже одно уравнение ($n=1$) может быть неприводимым из-за малых знаменателей. Однако, если λ удовлетворяет обычным арифметическим требованиям типа (2) § 1, одно уравнение приводимо. Предположим, что эти арифметические требования на λ выполнены (а не выполняются они только для множества точек λ меры нуль). Спрашивается, *при любой ли аналитической матрице $A(p)$ система (6) приводима*, в случае $k>1$, $n>1$?

Известно только, что при $n=2$ приводимые матрицы $A(p)$ заполняют некоторую область в функциональном пространстве всех матриц [8]. Если существуют области, заполненные неприводимыми $A(p)$, то интересна проблема нормальной формы для них. Было бы также интересно исследовать более общий вопрос о нормальной форме линейной системы (6), в которой p — фазовая точка динамической системы, например геодезического потока. Такая задача естественно возникает при изучении уравнений в вариациях.

Может представлять интерес также задача о перенесении теории Флоке на случай, когда комплексное время t пробегает замкнутую риманову поверхность (одномерное комплексное замкнутое многообразие).

III. Задачи с большими возмущениями. Отыскание условно-периодических движений в случае, когда система не близка к интегрируемой, связано со значительными трудностями. Простейшая модельная задача, в которой эти трудности уже имеются, состоит в следующем. Пусть дано аналитическое обратимое и не меняющее ориентации отображение окружности на себя (примером такого отображения является поворот окружности). Пуанкаре [17] определил *число вращения* — средний угол поворота — для любого такого отображения. Данжуа [18] доказал, что если число вращения несоизмеримо с 1, то отображение можно превратить в поворот непрерывной заменой переменной. Предположим, что число вращения удовлетворяет арифметическим требованиям (2) § 1.

Будет ли тогда указанная замена переменной аналитической?

Дифференцируемость этой замены доказана недавно А. Финчи [19]. Можно показать, что она аналитична для отображений, мало отличающихся от поворота [20].

IV. Вырождение. Задача трех тел. В случае, когда имеется несколько быстрых частот, вырождение плохо изучено даже в рамках асимптотической теории. Например, отсутствует строгое исследование поведения при $t \sim \frac{1}{\varepsilon}$ переменных действия системы с двумя степенями свободы (см. [13], [15]). В задачах, где отношение частот быстрых движений зависит от фаз медленных, вероятно, возникают интересные неизученные эффекты. С другой стороны, интересно узнать, типично ли условно-периодическое поведение в задачах, где такой зависимости нет, например в плоской задаче трех тел.

Пусть массы планет m, m' малы по сравнению с массой M центрального тела

$$m = \mu\alpha M, \quad m' = \mu\alpha' M,$$

где μ — малый параметр, α, α', M — фиксированные константы. Рассмотрим в фазовом пространстве область, ограниченную условиями (8)

$$c < a < C; \quad c' < a' < C'; \quad e, \; e' < \varepsilon,$$

где a, a' — большие полуоси, e, e' — эксцентриситеты и $0 < c < C < c' < C'$ — постоянные. Обозначим через $V(\varepsilon, \mu)$ объем этой области (зависящий еще и от постоянных α, α', M, c, C, c', C). Пусть $W(\varepsilon, \mu)$ — мера множества условно-периодических траекторий, лежащих в области (8).

Верно ли, что при всех $\varepsilon < \varepsilon_0$, $\mu < \mu_0$ *имеем*

$$W(\varepsilon, \mu) > \theta(\varepsilon_0, \mu_0)\, V(\varepsilon, \mu),$$

где $\theta(\varepsilon_0, \mu_0) > 0$ *не зависит от* ε, μ *и стремится к* 1 *при* $\varepsilon_0, \mu_0 \to 0$?

То, что $W(\varepsilon_0, \mu_0) > 0$, можно, вероятно, доказать, рассматривая окрестность подходящего периодического решения (см. § 2). При исследовании типичности условно-периодических движений в общей проблеме трех и n тел, по-видимому, не встретится новых принципиальных трудностей по сравнению с задачей, сформулированной выше.

V. Магнитные ловушки. Имеющиеся в настоящее время общие теоремы применимы ко многим классическим неинтегрируемым проблемам механики. Я хотел бы указать здесь еще на одну задачу, важную для современной физики. Речь идет о движении заряженной частицы в магнитном поле [21].

Хотя движение и описывается гамильтоновыми уравнениями, разделение его на быстрое ларморовское вращение и медленный дрейф выполнено сейчас (в первом приближении) методом усреднения в неканонической форме [22]. Чтобы применить общую теорию § 3, следовало бы *ввести каноническим преобразованием быстрые и медленные переменные*, аналогичные элементам Делоне в астрономии, и провести разделение движений в рамках первого приближения классической теории возмущений (т. е. усреднением функции Гамильтона). Если бы это было сделано, можно было бы (ср. § 3) строго доказать, например, *вечную адиабатическую инвариантность магнитного момента в аксиально-симметричной ловушке.* Тем самым было бы доказано, что такая ловушка удерживает частицы вечно.

В качестве простейшей математической модели можно рассмотреть *поведение линий постоянной геодезической кривизны k на данной поверхности* [23]. Следует ожидать, что линия достаточно большой кривизны k постоянно остается в кольце между двумя линиями уровня гауссовой кривизны поверхности.

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* Editor's note: V.I. Arnol'd: Small denominators and the problem of stability in classical and celectial mechanics, Proceedings of the Fourth All-Union Mathematics Congress (Leningrad, 3–12 July 1961), vol.2, Nauka, Leningrad, 1964, pp. 403–409

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УСТОЙЧИВОСТЬ И НЕУСТОЙЧИВОСТЬ В КЛАССИЧЕСКОЙ МЕХАНИКЕ*

Предлагаемые дальше лекции служат введением в цикл работ, посвященных строгому качественному исследованию нелинейных уравнений классической механики, начатый работой А. Н. Колмогорова в 1954 г. Речь идет главным образом о проблеме устойчивости движения консервативных систем (задача трех тел и т. п.).

Полученные результаты дают следующую картину поведения многочастотных «условно-периодических» движений при возмущении:

1. Для большинства начальных условий возмущенное движение условно-периодично.

2. Если размерность фазового пространства не более 4, то возмущенное движение устойчиво.

3. Если размерность фазового пространства 5 и более, то «общим случаем» является топологическая неустойчивость возмущенного движения.

Основное внимание уделено общим принципам и методам, приложения которых далеко не исчерпываются полученными к настоящему времени конкретными результатами. Несколько таких результатов перечислено в первой лекции.

I. ВВЕДЕНИЕ

§ 1. Результаты

Трудность качественных вопросов классической механики хорошо известна. Несмотря на длительные усилия многих математиков, большая часть этих вопросов все

* Editor's note: V.I. Arnol'd: Stability and instability in classical mechanics, Second Math. Summer School, Part II (Russian), Naukova Dumka, Kiev, 1965, pp. 85–119

еще ожидает решения. Лишь в последнее время, начиная с работ К. Л. Зигеля (1942 г.) и А. Н. Колмогорова (1954 г.), наметился некоторый прогресс в решении проблем устойчивости движения динамических систем. В частности:

1. Доказана устойчивость положений равновесия и периодических решений консервативных систем с двумя степенями свободы в так называемом общем эллиптическом случае.

2. Доказана вечная адиабатическая инвариантность переменной действия при медленном периодическом изменении параметров нелинейной колебательной системы с одной степенью свободы. Установлено, что «магнитная ловушка» с аксиально симметричным магнитным полем способна вечно удерживать заряженные частицы[1].

3. Найдены условно-периодические движения в задаче многих тел. Если массы n «планет» достаточно малы по сравнению с массой центрального тела, то движение условно-периодично для большинства начальных условий, при которых эксцентриситеты и наклонения кеплеровых эллипсов малы. При этом большие полуоси вечно остаются вблизи своих начальных значений, а эксцентриситеты и наклонения — малыми.

4. Доказана устойчивость быстрого вращения тяжелого несимметричного твердого тела, закрепленного в произвольной точке.

Здесь изложим только основные идеи[2].

В следующих двух параграфах приведены некоторые сведения из механики и математики, мало известные математикам и механикам.

§ 2. Предварительные сведения из механики

1. Малые знаменатели. Астрономы давно заметили, что резонансные явления, связанные с соизмеримостью частот взаимодействующих движений, приводят к «ма-

[1]) Хотя мы употребляем термины «частицы», «планеты», речь всюду идет о математических теоремах, касающихся поведения решений определенных дифференциальных уравнений. Применимость этих теорем к реальным системам должна быть особо исследована в каждом отдельном случае.

[2]) Подробные доказательства и список литературы опубликованы в УМН, т. 18, 5 и 6, 1963. Здесь мы не будем воспроизводить их в деталях.

лым знаменателям» и большим математическим трудностям[1]).

Пример 1. Юпитер за день проходит по своей орбите $\omega_1 = 299'',1$, а Сатурн — $\omega_2 = 120'',5$.

Частоты ω_1, ω_2 почти соизмеримы:

$$2\omega_1 - 5\omega_2 \approx 0.$$

Выражение $m\omega_1 + n\omega_2$ входит знаменателем в ряды теории возмущений, имеющие вид

$$\sum_{m,\, n \neq 0} a_{mn} \frac{e^{i(m\omega_1 + n\omega_2)\, t}}{m\omega_1 + n\omega_2}. \qquad (1)$$

Со времен Лапласа известно большое долгопериодическое возмущение движения планет вокруг Солнца, связанное с малым знаменателем $2\omega_1 - 5\omega_2$.

2. Проблемы устойчивости. Первой и весьма стимулирующей исследования задачей этого рода (не решенной и поныне) был вопрос об устойчивости планетных орбит. Не вызовут ли малые возмущения планет друг другом через достаточно большое время столкновений или ухода в бесконечность?

Теория устойчивости движения, разработанная в известных трудах А. Пуанкаре и А. М. Ляпунова, позволяет обнаруживать асимптотическую устойчивость. Но проблемы устойчивости классической механики относятся всегда к «нейтральному случаю» чисто мнимых характеристических показателей: асимптотически устойчивые движения в них невозможны из-за сохранения объема в фазовом пространстве (теорема Лиувилля). Поэтому указанные методы не дают ничего для исследования устойчивости движения нелинейных консервативных систем.

Основная трудность, встречающаяся в этих исследованиях, связана с расходимостью рядов теории возмущений (1) из-за малых знаменателей $m\omega_1 + n\omega_2$. А. Пуанкаре, занимаясь плоской ограниченной задачей трех тел, показал, что эти трудности встречаются уже в

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[1]) А. Пуанкаре писал: «Трудности, встречающиеся в небесной механике вследствие существования малых делителей и приблизительных соизмеримостей средних движений, связаны с самой природой вещей и не могут быть обойдены».

модельных задачах, допускающих совсем простую математическую формулировку. Дж. Д. Биркгоф подробно исследовал одну из таких задач.

Пример 2 («проблема Биркгофа»). Пусть дано сохраняющее площадь аналитическое отображение T окрестности нуля плоскости p, q на себя. Пусть нуль — неподвижная точка.

Устойчива ли она? Предполагается, что линейная часть T в нуле есть поворот плоскости p, q.

К настоящему времени на этот вопрос получен положительный ответ; обсудим его в § 4.

3. Замечания. В следующих разделах будем пользоваться «устрашающим формальным аппаратом динамики». Канонический вид уравнений движения не обязателен для применения излагаемых методов, но он облегчает многие выкладки. Предполагаются известными понятия конфигурационного и фазового пространства, уравнения Лагранжа и Гамильтона, циклические координаты и законы сохранения, канонические преобразования, скобки Пуассона, интегральные инварианты и переменные действие — угол в объеме учебника Ландау и Лифшица «Механика». В качестве контроля полезно решить задачу.

Задача. Пусть точка движется по инерции по поверхности (сила тяжести отсутствует). Найти инвариантные двумерные многообразия в фазовом пространстве и исследовать движение фазовой точки по ним. Рассмотреть случаи, когда S есть: а) тор, б) эллипсоид вращения.

§ 3. Предварительные сведения из математики

1. Что такое условно-периодическое движение. Рассмотрим поверхность тора (баранки, рис. 1) и введем на ней «географические» координаты: долготу q_1 и широту q_2. Углы q_1 и q_2 будем выражать в радианах и рассматривать с точностью до целого кратного 2π. Картой тора может служить, например, квадрат $0 \leqslant q_{1,2} \leqslant 2\pi$ на плоскости q_1, q_2. Удобно пользоваться также всей плоскостью q_1, q_2, разделенной на квадраты со стороной 2π. Каждая точка тора имеет изображение в каждом квадрате такой карты (рис. 2).

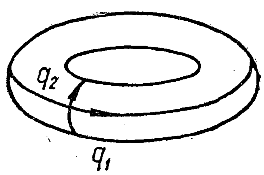

Рис. 1.

Рассмотрим точку $q_1(t)$, $q_2(t)$, движущуюся по тору так, что ее координаты меняются равномерно:

$$\frac{dq_1}{dt} = \omega_1, \quad \frac{dq_2}{dt} = \omega_2. \tag{2}$$

На карте q_1, q_2 это движение изобразится прямой линией.

Если $\omega_1/\omega_2 = m/n$, где m и n целые числа, то через время $t = 2\pi \frac{m}{\omega_1} = 2\pi \frac{n}{\omega_2}$ точка вернется на прежнее место, сделав m оборотов по параллели и n по меридиану (на рис. 2 $m = 2$, $n = 3$). В этом случае уравнение (2) определяет периодическое движение.

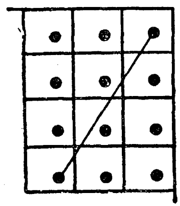

Рис. 2.

Если же ω_1/ω_2 — иррациональное число, то движущаяся точка никогда не придет на прежнее место. В этом случае движение (2) называется условно-периодическим с двумя частотами ω_1, ω_2. Траекторию $q_1(t)$, $q_2(t)$ называют обмоткой тора.

С условно-периодическими движениями тесно связаны условно-периодические функции. Если $F(q_1, q_2)$ — функция на торе, разлагающаяся в ряд Фурье

$$F(q_1, q_2) = \sum_{m, n=-\infty}^{+\infty} f_{mn} e^{i(mq_1 + nq_2)},$$

то ее изменение со временем при движении (2) будет иметь вид

$$f(t) = F[q_1(t), q_2(t)] = \sum_{m, n=-\infty}^{+\infty} f_{mn} e^{i[(m\omega_1 + n\omega_2)t + \varphi_{mn}]}. \tag{3}$$

Функции (3) называют условно-периодическими. Примером может служить $f(t) = \cos t + \cos \sqrt{2}\, t$. Появление рядов вида (3) в какой-нибудь задаче всегда указывает на условно-периодическое движение (2).

2. Некоторые свойства условно-периодических движений. 1. Траектория условно-периодического движения всюду плотна на торе.

Это значит, что рано или поздно движущаяся точка $p(t)$, $q(t)$ побывает в любой области Δ. Свойство 1 легко вытекает из следующего факта:

1a. Пусть α — иррациональное число и Δ — дуга окружности $|z| = 1$. Тогда среди точек $e^{2\pi i n\alpha}$ $(n = 1, 2, 3, \ldots)$ есть точки Δ [1].

Отметим также, что траектория условно-периодического движения равномерно распределена: доля времени от $t = 0$ до $t = T$, которое движущаяся точка проводит в области Δ (рис. 3), пропорционально площади этой области, если T велико [2].

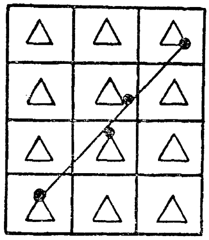

Рис. 3.

2. Для любой интегрируемой по Риману функции $F(q_1, q_2)$ среднее по времени равно среднему по пространству:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\omega_1 t, \omega_2 t)\, dt = \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} F(q_1, q_2)\, dq_1\, dq_2.$$

Вот пример к свойству 2.

3. Задача Лагранжа о среднем движении перигелиев. Пусть вектор $a(t)$ на плоскости x, y есть сумма трех векторов

$$a(t) = a_1(t) + a_2(t) + a_3(t)$$

длин a_1, a_2, a_3, вращающихся равномерно с независимыми [3] угловыми скоростями $\omega_1, \omega_2, \omega_3$. Обозначим через $\varphi(t)$ угол вектора $a(t)$ с осью x (рис. 4).

Задача. Найти среднюю угловую скорость вектора a.

$$\omega = \lim_{T\to\infty} \frac{\varphi(T)}{T}.$$

Ответ: $\omega = \dfrac{\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3}{\alpha_1 + \alpha_2 + \alpha_3}$,

где $\alpha_1, \alpha_2, \alpha_3$ — углы треугольника со сторонами a_1, a_2, a_3 (Боль, Серпинский, Г. Вейль, 1909).

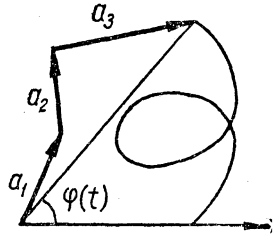

Рис. 4.

Прежде чем переходить к общей теории, рассмотрим простейший пример, в котором уже проявляются многие существенные черты изучаемых явлений.

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[1] Может ли число 2^n начинаться с цифры 7? Согласно 1а, число 2^n может начинаться с любой комбинации цифр.

[2] С какой цифры чаще начинается 2^n: с 7 или 8?

[3] Числа $\omega_1, \omega_2, \omega_3$ независимы, если из $k_1\omega_1 + k_2\omega_2 + k_3\omega_3 = 0$ с целыми k_i вытекает $k_i = 0$.

§ 4. Простейшая проблема устойчивости

Вернемся к примеру 2 из § 2: дано сохраняющее площадь аналитическое отображение T окрестности нуля плоскости p, q на себя. Устойчива ли неподвижная точка 0?

Кратко изложим результат применения общих методов к этому случаю.

1. Три примера. Для линейных преобразований вопрос решается вычислением собственных значений λ_1, λ_2. Ввиду сохранения площади, $\lambda_1\lambda_2 = 1$. Если λ_1 и λ_2 не действительны, то $\bar{\lambda}_1 = \lambda_2$, $|\lambda_{1,2}| = 1$, $\lambda_{1,2} = e^{\pm i\omega}$.

Пример 1. Рассмотрим обыкновенный поворот A плоскости p, q на угол ω вокруг точки 0. Каждая окружность $p^2 + q^2 = \text{const}$ инвариантна, т. е. переходит в себя: она поворачивается как целое на угол ω. Траектория всюду плотна на окружности, если $\omega \neq 2\pi\, m/n$.

Всякое линейное преобразование T с $\lambda_{1,2} = e^{\pm i\omega}$ может быть приведено к виду A линейным изменением S системы координат:

$$\begin{array}{ccc} p,\ q & \xrightarrow{T} & p,\ q \\ S \downarrow & & \downarrow S \\ p',\ q' & \xrightarrow{A} & p',\ q'. \end{array}$$

Такое отображение T называется эллиптическим поворотом.

Перейдем к нелинейным отображениям. Если линейная часть данного отображения T в нуле есть эллиптический поворот, то T называется отображением *эллиптического типа*.

Пример 2. Рассмотрим отображение B (рис. 5), при котором каждая окружность $p^2 + q^2 = 2\tau$ поворачивается на свой угол

$$\omega(\tau) = \omega_0 + \omega_1\tau + \ldots . \tag{4}$$

Это отображение эллиптического типа, и оно устойчиво.

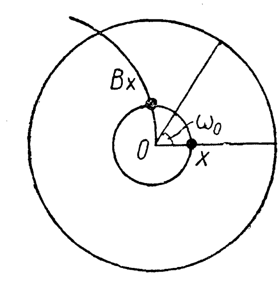

Рис. 5.

Рассмотрим систему координат p', q', связанную с p, q аналитическим преобразованием S, сохраняющим площадь и оставляющим 0 на месте. На плоскости p', q' рассмотрим отображение B (пример 2).

Пример 3. Запишем отображение B в координатах p, q. Получится отображение $C = S^{-1}BS$.

Это отображение эллиптического типа, и оно устойчиво, так как заменой переменных S превращается в устойчивое отображение B.

Нельзя ли получить любое отображение эллиптического типа конструкций примера 3? Это дало бы, в частности, и положительное решение проблемы устойчивости.

2. *Формальное решение.* Со времен Биркгофа известно, что если не заботиться о сходимости рядов, то при условии

$$\omega \neq 2\pi \frac{m}{n} (m = 0, \pm 1, \pm 2, \ldots; n = 1, 2, 3, \ldots) \tag{5}$$

отображение эллиптического типа T всегда можно формальной заменой переменных S привести к виду $STS^{-1} = B$ примера 2. Замена S определяется с помощью «рядов Биркгофа», аналогичных рядам теории возмущений. Эти ряды, в общем случае, расходятся. Из существования формальных рядов S не вытекает устойчивости отображения T.

Тем не менее, можно оборвать ряд S, и сходящейся заменой переменных $S^{(s)}$ привести T к виду, лишь малыми сколь угодно высокого порядка $0(\tau^s)$ отличающемуся от B. Получающиеся при этом коэффициенты ω_0, ω_1, ... в (4) не зависят от способа $S^{(s)}$ приведения T к виду B; они являются инвариантами T относительно сохраняющих площадь аналитических преобразований. Если $\omega_1 \neq 0$, то говорят, что отображение T — общего эллиптического типа.

В этом случае угол $\omega(\tau)$, на который поворачивается окружность $\tau = \text{const}$ при отображении B, меняется с τ (см. (4)). Поэтому некоторые окружности поворачиваются на угол, соизмеримый с 2π, другие же — на несоизмеримый.

В надлежащих переменных отображение T вблизи 0 можно рассматривать как поворот B на переменный угол $\omega(\tau)$, возмущенный весьма малыми дополнительными членами. Поэтому наша задача свелась к изучению T, отличающихся от B лишь малыми по сравнению с τ^s возмущениями.

3. *Инвариантные кривые.* Если бы ряды Биркгофа S сходились, то окрестность точки 0 вся состояла бы из близких к окружностям $\tau = \text{const}$ инвариантных кривых отображения T.

Оказывается, в действительности большинство инвариантных окружностей отображения B, на которых угол $\omega(\tau)$ несоизмерим с 2π, не исчезает при малом возмущении B, а лишь немного деформируется. Поэтому неподвижная точка 0 окружена сколь угодно малыми инвариантными относительно T аналитическими замкнутыми кривыми и, следовательно, устойчива. Можно показать, что эти кривые заполняют множество положительной меры, имеющее 0 точкой плотности.

Но эти кривые не заполняют окрестности точки 0 целиком и вообще не заполняют никакой области: между ними остаются еще «зоны неустойчивости», возникающие при возмущении из окружностей $\tau = \mathrm{const}$, где $\omega(\tau)$ соизмеримо с 2π. На каждом луче, выходящем из 0, инвариантные кривые высекают след вроде канторова совершенного множества, но положительной меры.

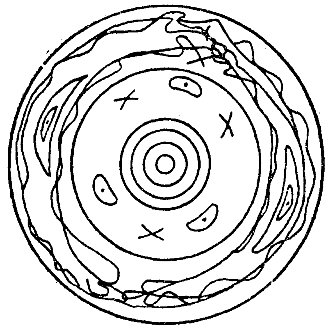

Рис. 6.

4. *Зоны неустойчивости*. Рассмотрим инвариантную окружность «невозмущенного» преобразования B, поворачивающуюся на угол $\omega(\tau) = 2\pi \frac{m}{n}$. При n — кратной итерации B каждая точка окружности вернется на свое место. Это свойство B при малом возмущении, вообще говоря, не сохраняется, и такая инвариантная окружность «рассыпается». Дж. Д. Биркгоф доказал, что вместо целой окружности неподвижных относительно B^n точек T^n имеет, вообще говоря, конечное четное число неподвижных точек вблизи этой окружности. Среди этих точек половина имеет эллиптический и половина «гиперболический» тип [1)].

Как указано в *3*, точки эллиптического типа, вообще говоря, устойчивы и окружены инвариантными кривыми, не охватывающими 0 (рис. 6). Следовательно, в общем случае окрестность 0 не расслаивается на инвариантные замкнутые кривые. Отсюда уже вытекает упоминавшаяся выше расходимость рядов Биркгофа (см. *2*, § 4).

Сепаратриссы гиперболических точек, пересекаясь друг с другом, создают в «зонах неустойчивости» запутанную сеть. В окрестности каждой эллиптической точки имеет

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1) Строение последних легко понять, рассмотрев гиперболический поворот $p \to 2p$, $q \to \frac{1}{2} q$.

место та же картина с инвариантными кривыми, зонами неустойчивости и т. д.

5. Условия устойчивости. Укажем здесь предположения, в которых доказано существование инвариантных кривых (см. *3*) и устойчивость T. В первоначальном доказательстве использовались иррациональность $\omega_0/2\pi$, условие $\omega_1^2 + \omega_2^2 + \ldots \neq 0$ и аналитичность T. Эти условия ослаблены Ю. Мозером, который вместо иррациональности $\omega_0/2\pi$ требует $\omega_0/2\pi \neq \frac{m}{3}, \frac{m}{4}$, а вместо аналитичности T — непрерывность 333 производных.

При $\omega_0/2\pi = m/3$ возможна неустойчивость, как это установил еще Т. Леви-Чивита.

II. КЛАССИЧЕСКАЯ ТЕОРИЯ ВОЗМУЩЕНИЙ. МАЛЫЕ ЗНАМЕНАТЕЛИ

§ 1. Интегрируемые и неинтегрируемые проблемы динамики

Будем рассматривать консервативные динамические системы с n степенями свободы, определяемые каноническими уравнениями движения

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad (p = p_1, \ldots, p_n;\ q = q_1, \ldots, q_n) \quad (1)$$

с аналитической функцией Гамильтона $H(p, q)$. Классические методы динамики позволяют исследовать лишь так называемые интегрируемые случаи.

Пример 1. Предположим, что фазовое пространство p, q является прямым произведением n-мерного тора на область n-мерного эвклидова пространства. Пусть q_i (mod 2π) — угловые координаты на торе, а p_i — в пространстве и функция Гамильтона зависит только от $p : H = H(p)$. Уравнения Гамильтона (1) принимают вид

$$\dot{p} = 0, \ \dot{q} = \omega(p) \quad \left(\omega = \frac{\partial H}{\partial p} = \omega_1, \ldots, \omega_n\right)$$

и тотчас интегрируются. Каждый тор $p = \text{const}$ инвариантен; если частоты ω несоизмеримы (из $\omega_1 k_1 + \ldots + \omega_n k_n = 0$ с целыми k_i следует $k_i = 0$), то движение называется условно-периодическим с n частотами $\omega_1, \ldots, \omega_n$; легко доказать, что траектория $p(t)$, $q(t)$ заполняет тор всюду плотно. Переменные p, q из примера 1 называются переменными действие — угол.

К настоящему времени найдено много интегрируемых задач. Решение всех таких задач с n степенями свободы основано на том, что существуют (и найдены) n однозначных первых интегралов в инволюции[1].

Можно показать, что существование таких интегралов влечет за собой следующую картину поведения траекторий в $2n$-мерном фазовом пространстве p, q. Некоторое особое $2n-1$-мерное множество разбивает фазовое пространство на инвариантные области. Каждая из них расслоена на инвариантные n-мерные многообразия. Если область ограничена, то эти многообразия суть торы, несущие условно-периодические движения. В такой области можно ввести координаты действие — угол примера 1. Если n первых интегралов в инволюции уже найдены, то каноническое преобразование, вводящее переменные действие — угол, дается квадратурой.

Пример 2. Интегрируемые задачи: задача двух тел; задача о притяжении двумя неподвижными центрами; движение свободной точки по геодезической на поверхности трехосного эллипсоида; тяжелое симметричное твердое тело, закрепленное в точке на оси; несимметричное твердое тело, закрепленное в центре тяжести; линейные колебания.

Неинтегрируемые[2] задачи: задача n тел, в том числе так называемая плоская ограниченная круговая задача трех тел; движение свободной точки по геодезической на выпуклой поверхности; тяжелое несимметричное твердое тело; нелинейные колебания с $n > 1$ степенями свободы.

Интегрируемые случаи в основном были найдены в XIX в. (Якоби, Лиувилль, Ковалевская и др.). Но после работ Пуанкаре стало ясно, что динамическая система общего вида неинтегрируема; интегралы не только неизвестны, но не существуют вовсе, так как траектории в целом не ложатся на инвариантные n-мерные многообразия.

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[1] Функции $f(p, q)$ и $g(p, q)$ находятся в инволюции, если их скобка Пуассона $\frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p}$ тождественно равна нулю.

[2] Осторожнее сказать непроинтегрированные, так как доказательства неинтегрируемости сложны и проведены строго лишь в отдельных случаях.

§ 2. Классическая теория возмущений

Предположим, что система отличается от интегрируемой малыми «возмущениями»; в обозначениях примера 1

$$H(p, q) = H_0(p) + \mu H_1(p, q) + \cdots, \tag{2}$$

где μ мало и $H_1 + \cdots$ имеет период 2π по q. Согласно Пуанкаре, исследование этого случая есть «основная проблема динамики». Как влияет возмущение μH_1 на поведение траекторий при $t \to \infty$? Сохраняются ли инвариантные торы? Остается ли траектория по крайней мере вблизи тора $p = \text{const}$?

Сравнение интегрируемых и неинтегрируемых задач примера 2 показывает, какое значение эти вопросы имеют для механики. Полный ответ на них содержал бы, в частности, решение проблемы устойчивости планетной системы.

Для приближенного исследования траекторий при больших t в астрономии давно возник специальный аппарат теории возмущений. Если удается каноническим преобразованием $p, q \to p', q'$ привести H к виду

$$H(p, q) = H_0'(p') + \mu^2(p', q') + \cdots, \tag{3}$$

то в течение времени $t \sim \frac{1}{\mu}$ движение $p'(t)$, $q'(t)$ на величину $\sim \mu$ будет отличаться от условно-периодического, описываемого $H_0'(p')$. Возвращаясь к p, q, получим для $p(t)$, $q(t)$ приближенные выражения с ошибкой порядка μ при $t \sim \frac{1}{\mu}$. Если требуется большая точность, можно сделать следующее приближение $p', q' \to p'', q''$, приводящее H к виду

$$H(p, q) = H_0''(p'') + \mu^3 H_1''(p'', q'') + \cdots.$$

Теперь ошибка будет $\sim \mu^3 t$. Если последовательные приближения сходятся, то в пределе получится $H(p, q) = H_0^{(\infty)}(p^{(\infty)})$, т. е. система интегрируема: торы $p^{(\infty)}(p, q) = \text{const}$ инвариантны и заполнены траекториями условно-периодических движений.

При проведении указанной программы наталкиваемся на две трудности.

1. Малые знаменатели. Будем искать каноническое преобразование $p,\ q \to p',\ q'$ в виде $p' = p + \mu \frac{\partial S}{\partial q}$; $q' = q + \mu \frac{\partial S}{\partial p'}$,

$$S(p', q) = \sum_{k \neq 0} S_k(p')\, e^{i(k, q)}.$$

Функция $H(p, q)$ в координатах p', q' запишется в виде[1)]

$$H_0(p) + \mu \overline{H}_1(p) + \mu \widetilde{H}_1(p, q) + \cdots = H_0(p') + \mu \overline{H}_1(p') + \\ + \mu \left[\frac{\partial H_0}{\partial p} \frac{\partial S}{\partial q} + \widetilde{H}_1 \right] + \mu^2 \cdots.$$

Чтобы получить (3), нужно уничтожить зависящие от q члены порядка μ, т. е. нужно, чтобы $\left(\omega, \frac{\partial S}{\partial q}\right) + \widetilde{H}_1 = 0$ или

$$S_k(p') = \frac{i h_k(p')}{(\omega, k)}, \text{ где } \widetilde{H}_1 = \sum_{k \neq 0} h_k e^{i(kq)}. \tag{4}$$

Знаменатель (ω, k) при некоторых «резонансных» значениях k обращается в нуль каковы бы ни были ω сколь угодно малые при подходящих k. Эти малые знаменатели (ω, k) ставят под сомнение законность наших формальных преобразований при $n > 1$.

2. Расходимость приближений. Есть случаи, когда ряды каждого приближения обрываются и потому сходятся. Такие случаи подробно исследовал Биркгоф. Однако Зигель показал, что все приближения вместе в этом случае, как правило, расходятся. Из сходимости следовала бы описанная в примере 1 структура траекторий. В действительности же траектории возмущенной системы могут не лежать на инвариантных торах.

Предположим, что $\det \left| \frac{\partial \omega}{\partial p} \right| \neq 0$. Тогда в любой окрестности любого инвариантного тора невозмущенной системы есть n-мерный тор, на котором все траектории через одно и то же время замыкаются. При малом возмущении это n-мерное многообразие замкнутых траекторий, вообще говоря, разрушается. Следовательно, ряды теории возмущений не будут сходиться ни в какой области фазового пространства.

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1) Черта обозначает среднее по $q : \overline{H}_1(p) = (2\pi)^{-n} \int H_1(p, q)\, dq$.

Указанные соображения не исключают возможности существования у возмущенной системы инвариантных торов, на которых $(\omega, k) \neq 0$. Эти торы не могут заполнять никакой области.

§ 3. Малые знаменатели

При исследовании влияния малых знаменателей (ω, k) астрономы давно уже применяли некоторые арифметические соображения. Простейшее из них состоит в том, что иррациональных чисел больше, чем рациональных. Далее, компоненты наугад взятого вектора ω несоизмеримы. Поэтому при почти всех[1] векторах ω имеем $(\omega, k) \neq 0$ при всех целых $k \neq 0$.

Более точно эту мысль выражает следующая теорема из теории диофантовых приближений.

Теорема. *Почти каждый вектор* $\omega = \omega_1, \ldots, \omega_n$ *удовлетворяет неравенствам*

$$|(\omega, k)| \geqslant K |k|^{-\nu} \quad (|k| = |k_1| + \cdots + |k_n|; \ \nu = n + 1) \quad (5)$$

для всех целочисленных $k \neq 0$ *при некотором* $K(\omega) > 0$.

Доказательство. Рассмотрим ограниченную область Ω, зафиксируем $K > 0$ и целое k. Тогда неравенство (5) нарушено лишь в «резонансной зоне» ширины меньше $2K|k|^{-\nu}$, объем этой зоны не превосходит $K|k|^{-\nu}D$, где постоянная $D > 0$ зависит лишь от Ω.

Всех k с $|k| = m$ не более Lm^{n-1} (постоянная $L > 0$ зависит лишь от n). Поэтому мера всех резонансных зон с $|k| = m$ не превосходит $Km^{-2}DL$, а всех с $|k| > 0$ — $\sum_{m=1}^{\infty} K_m^{-2} DL \leqslant K\bar{D}(\Omega), \bar{D} = 2LD$. При $K \to 0$ суммарная мера резонансных зон стремится к 0, откуда непосредственно вытекает доказываемое утверждение.

Таким образом, для большинства ω малые знаменатели (ω, k) не только не равны нулю, но могут быть оценены снизу степенью $|k|$. В связи с этим возникает надежда на сходимость рядов теории возмущений (4) для большинства ω: ведь коэффициенты Фурье h_k аналитической функции H_1 убывают в геометрической прогрессии.

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[1] Всех, за исключением множества лебеговой меры нуль.

Действительно, пусть функция H_1 аналитична в полосе $|\operatorname{Im} q| \leqslant \rho$, и в этой полосе $|H_1| \leqslant M$. Сдвигая в формуле для коэффициентов Фурье

$$h_k = (2\pi)^{-n} \int_0^{2\pi} H_1 e^{-i(k, q)} dq$$

контур интегрирования на $\pm i\rho$, получаем

$$|h_k| \leqslant M e^{-|k|\rho}. \tag{6}$$

При условии, что малые знаменатели допускают оценку

$$|(\omega, k)| \geqslant K |k|^{-\nu}, \tag{5}$$

коэффициенты Фурье S_k функции S убывают в геометрической прогрессии почти так же быстро, как и коэффициенты h_k: при любом $\delta > 0$ имеем в виду (5), (6):

$$|S_k| \leqslant \frac{ML}{K\delta^\nu} e^{-|k|(\rho-\delta)},$$

где ν, L — постоянные, зависящие только от размерности.

Следовательно, ряд S сходится при $|\operatorname{Im} q| < \rho$, а в несколько более узкой полосе $|\operatorname{Im} q| \leqslant \rho - 2\delta$ сумма допускает оценку вида

$$|S| \leqslant \frac{ML}{K\delta^\nu}. \tag{7}$$

Таким образом, ряд (4) сходится при почти всех ω. Однако: 1) полученные функции S всюду разрывно зависят от p, поэтому S, строго говоря, не определяет никакого преобразования $p, q \to p', q'$;

2) тем более сомнительна сходимость приближений $p^{(s)}$, $q^{(s)}$ при $s \to \infty$.

III. МЕТОД НЬЮТОНА. ТЕОРЕМА КОЛМОГОРОВА

Система последовательных приближений теории возмущений наталкивается на существенные трудности, связанные с «малыми знаменателями». Эти трудности были преодолены А. Н. Колмогоровым с помощью иной схемы последовательных приближений: с помощью метода Ньютона.

§ 1. Метод Ньютона

«Метод касательных» Ньютона служит для нахождения последовательных приближений $x_1, x_2, \ldots$ к корню x уравнения $f(x)=0$ (рис. 7).

Пусть x_1 — хорошее приближение к корню: $f(x_1) = \varepsilon \ll 1$ (следовательно, $|x-x_1| \sim \varepsilon$). Так как

$$f(x) = f(x_1) + f'(x_1)(x-x_1) + 0(x-x_1)^2,$$

то, пренебрегая $0(x-x_1)^2 \sim \varepsilon^2$, получаем уравнение

$$\varepsilon + f'(x_1)(x_2 - x_1) = 0,$$

корень которого x_2 дает приближение $f(x_2) \sim \varepsilon^2$ и, следовательно, $|x-x_2| \sim \varepsilon^2$.

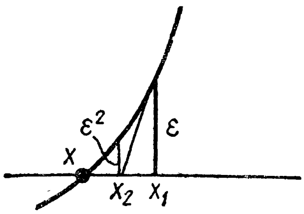

Рис. 7.

Дальнейшие приближения дадут $|x-x_3| \sim \varepsilon^4$, $|x-x_4| \sim \varepsilon^8$, так что каждый раз ошибка возводится в квадрат. Возникающая «сверхсходимость» ($|x-x_s| \sim \varepsilon^{2^{s-1}}$) является основным для нас свойством метода Ньютона.

В теории возмущений процедура, аналогичная изложенному выше построению x_2, хорошо известна: это обычная замена переменной первого приближения (см. раздел 2).

Идея Колмогорова состоит в том, чтобы и в следующих приближениях итерировать ту же процедуру и вместо обычной последовательности приближений с точностью до $\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4, \ldots$ получить быстро сходящуюся последовательность $\varepsilon, \varepsilon^2, \varepsilon^4, \varepsilon^8, \ldots$.

Более точно, речь идет о следующих оценках. Пусть в исходной системе возмущение H_1 аналитично по угловым переменным q в полосе $|\operatorname{Im} q| < \rho_1$ и допускает в этой полосе оценку $|H_1| \leqslant M_1 \ll 1$. Для упрощения письма будем записывать эту оценку в виде

$$|H_1|_{\rho_1} \leqslant M_1.$$

После первого приближения теории возмущений величина возмущения H_2 будет уже порядка M_1^2. Точнее, оценки, аналогичные проведенным в конце предыдущей лекции, дают (для нерезонансных значений p)

$$|H_2|_{\rho_2} < \frac{M_1^2}{\delta_1^{2\nu}} = M_2.$$

Здесь $\rho_2 = \rho_1 - L\delta_1$, $\delta_1 > 0$ произвольно, а L и ν — постоянные, зависящие только от числа степеней свободы, постоянных K, ν (в оценке малых знаменателей $|(k, \omega)| > K|k|^{-\nu}$).

В каждом следующем приближении получим аналогичную оценку $|H_s|_{\rho s} < M_s$ во все более и более узкой полосе ширины

$$\rho_s = \rho_1 - L(\delta_1 + \delta_2 + \cdots + \delta_{s-1}).$$

Чтобы не потерять аналитичности после бесконечного числа приближений, выберем $\delta_1, \delta_2, \ldots$ столь малыми, чтобы сумма ряда

$$L(\delta_1 + \delta_2 + \cdots)$$

не превосходила 0,5 ρ_1.

Чтобы величины

$$M_s = \frac{M_{s-1}^2}{\delta_{s-1}^{2\nu}}$$

стремились к нулю при $s \to \infty$, нужно, чтобы $\delta_s \to 0$ медленнее, чем $M_s \sim M_{s-1}^2$. Положим, например,

$$\delta_2 = \delta_1^{1\frac{1}{2}}, \quad \delta_3 = \delta_2^{1\frac{1}{2}}, \ldots, \delta_{s+1} = \delta_s^{1\frac{1}{2}}, \ \ldots \quad (\delta_1 \ll \rho_1)$$

и докажем сходимость H_s к нулю при $|\operatorname{Im} q| < 0{,}5\ \rho$.

При $\delta_1 \ll \rho_1$, очевидно, $\rho_s > 0{,}5\ \rho_1$ и, следовательно,

$$|H_s|_{0,5\rho} < |H_s|_{\rho_s} < M_s = \frac{M_{s-1}^2}{\delta_{s-1}^{2\nu}}.$$

Заметим теперь, что если $M_{s-1} < \delta_{s-1}^T$, то

$$M_s < \delta_{s-1}^{2T-2\nu} < \delta_{s-1}^{1\frac{1}{2}T} = \delta_s^T,$$

коль скоро $2T - 2\nu > 1\frac{1}{2}T$, например, $T = 4\nu + 1$. Итак, при $M_1 < \delta_1^T$ имеем $M_s < \delta_s^T \to 0$ $(s \to \infty)$, что и требовалось доказать[1].

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[1]) Распространение описанного метода на случай конечно-дифференцируемых функций принадлежит Ю. Мозеру. Для компенсации «потери гладкости» Мозер вводит в каждое приближение операцию сглаживания.

§ 2. Теорема Колмогорова

Для применения метода Ньютона к отысканию условно-периодических движений А. Н. Колмогоров предложил поступать следующим образом:

1. Будем искать только один инвариантный тор $T\omega^*$ возмущенной системы, на котором происходит условно-периодическое движение с частотами ω^*. Набор частот ω^*, удовлетворяющий неравенствам (3), фиксируем заранее. Тор $T\omega^*$ будем искать в окрестности соответствующего инвариантного тора невозмущенной системы $p = p^* + \mu \ldots$; $\partial H_0/\partial p^* = \omega^*$.

В формуле (2) вместо $\omega(p) = \partial H_0/\partial p$ поставим ω^*. Тогда в выражении $H(p, q)$ благодаря новым переменным появится дополнительный член $\mu\left[(\omega - \omega^*)\frac{\partial S}{\partial q}\right]$. При $|p - p^*| \sim \mu$ этот член будет порядка μ^2.

2. В указанной окрестности тора $p = p^*$ удается ввести аналитическим каноническим преобразованием $p, q \to p', q'$ новые переменные p', q', в которых функция Гамильтона $H(p, q) = H_0(p) + H_1(p, q)$ принимает вид $H(p, q) \equiv \equiv H^{(1)}(p', q') = H_0^{(1)}(p') + H_1^{(1)}(p', q')$, где $|H_1^{(1)}| \sim |H_1|^2$. Возникающая таким образом квадратная сходимость, типичная для ньютоновского метода касательных, позволит найти инвариантный тор $T\omega^*$.

Более точно п. 2 заключается в следующем. Пусть при $|\operatorname{Im} q| \leqslant \rho$ имеем $|H_1| \leqslant \mu_1$. С помощью оценок (4), (5), (7) второго раздела и п. 1 удается получить переменные p', q', такие, что при $|\operatorname{Im} q'| \leqslant \rho - L\delta$, $|p' - p^{*\prime}| \leqslant \leqslant \mu_1$ имеем

$$|H_1^{(1)}(p', q')| \leqslant \mu_2 = \frac{\mu_1^2}{\delta^{2\nu}}, \tag{1}$$

где $L > 0$, $\nu > 0$ — постоянные, зависящие только от числа степеней свободы n. В точке $p^{*\prime}$ имеем $\frac{\partial H_0^{(1)}}{\partial p'} = \omega^*$, $\delta > 0$ в нашей власти: оно только должно не превосходить некоторой постоянной, зависящей лишь от H_0, ω^* и ρ.

Покажем теперь, как, располагая оценкой (1), построить сходящиеся последовательные приближения к инвариантному тору $T\omega^*$. Так как $H^{(1)}(p', q')$ имеет тот же вид, что и $H(p, q)$, можно с помощью (1) построить канонические преобразования

$$p', q' \to p'', q'' \to \dots \to p^{(s)}, q^{(s)} \to \dots,$$
$$H(p, q) = H^{(s)}(p^{(s)}, q^{(s)}) = H_0^{(s)}(p^{(s)}) + H_1^{(s)}(p^{(s)}, q^{(s)}).$$

При этом в стягивающихся областях, определенных неравенствами

$$|p^{(s)} - p^{*(s)}| \leqslant \mu_s, \ |\operatorname{Im} q^{(s)}| \leqslant \rho_s \quad \left(\mu_{s+1} = \frac{\mu_s^2}{\delta_s^{2\nu}}, \ \rho_{s+1} = \right.$$
$$\left. = \rho_s - L\delta_{s+1}, \ \left.\frac{\partial H_0^{(s)}}{\partial p^{(s)}}\right|_{p^{*(s)}} = \omega^*, \ s = 1, 2, \dots\right),$$
$$(\rho_0 = \rho, \ \mu_1 = \mu),$$

ввиду (1)

$$|H_1^{(s)}| \leqslant \mu_{s+1} = \frac{\mu_s^{(2)}}{\delta_s^{2\nu}}. \tag{2}$$

Теперь распорядимся величинами δ_s. Положим $\delta_{s+1} = \delta_s^{1\frac{1}{2}}$ $(s = 1, 2, \dots)$. Если $\mu_s < \delta_s^T$ и T достаточно велико, то, ввиду (2),

$$\mu_{s+1} \leqslant \delta_s^{2T-2\nu} \leqslant \delta_s^{1\frac{1}{2}T} = \delta_{s+1}^T. \tag{3}$$

Зафиксируем такое большое T, например $T = 4\nu + 1$, и предположим, что при $|\operatorname{Im} q| \leqslant \rho$ имеем $|H_1(p, q)| \leqslant \leqslant \mu_1 = \delta_1^T$, где δ_1 достаточно мало. Тогда при всех $s = = 1, 2, \dots$ будет $|H_1^{(s)}(p^{(s)}, q^{(s)})| \leqslant \delta_s^T$ в области $|\operatorname{Im} q^{(s)}| \leqslant \leqslant \rho_s$, $|p^{(s)} - p^{*(s)}| \leqslant \mu_s$. Кроме того, при достаточно малом δ_1 будет $\rho_s > \rho/2 > 0$ $(s = 1, 2, \dots)$. Построенные области, как легко сообразить, ввиду (2), (3) стягиваются к инвариантному аналитическому тору $T\omega^*$.

Таким образом, приходим к следующей картине возмущенного движения. Предположим, что $\det \left|\frac{\partial \omega}{\partial p}\right| = = \det \left|\frac{\partial^2 H_0}{\partial p^2}\right| \neq 0$. Тогда в малой окрестности любой точки p есть точки, где частоты ω соизмеримы, и точки, для которых $\omega(p) = \omega^*$ допускает оценку (5) второго раздела. В соответствии с этим на части торов $p = \text{const}$ канонические уравнения с функцией Гамильтона $H_0(p)$

определяют всюду плотные условно-периодические траектории, а на других — нет.

Оказывается, при малом возмущении ($H = H_0(p) + \mu H_1(p, q)$, $\mu \ll 1$) большинство инвариантных торов с несоизмеримыми частотами ω^*, удовлетворяюшими (5): $|(\omega, k)| > K|k|^{-\nu}$ с фиксированным K, не исчезает, а лишь немного деформируется. Траектории возмущенного движения, начинающиеся на деформированном торе $T\omega^*$, заполняют его всюду плотно и условно-периодически. Торы $T\omega^*$ образуют замкнутое нигде не плотное множество (между ними остаются щели, заполненные остатками разрушающихся шаров с соизмеримыми ω). Но это инвариантное нигде не плотное множество имеет положительную меру, которая стремится к мере всего фазового пространства, когда $K \to 0$, $\mu \to 0$.

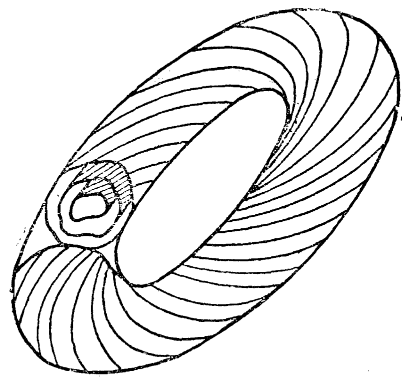

Рис. 8.

В случае $n=2$ двумерные торы $T\omega^*$ делят трехмерный инвариантный «уровень энергии» $H = \text{const}$ (рис. 8). Поэтому траектория, начинающаяся в щели между двумя торами $T\omega^*$ не может выйти из нее. Таким образом, при $n = 2$ существование инвариантных торов позволяет делать выводы об устойчивости движения.

В случае $n > 2$ n-мерные торы $T\omega^*$ не делят $2n - 1$-мерное многообразие $H = \text{const}$ и «щели» могут простираться в бесконечность. В этом случае получаем информацию о движении лишь для большинства начальных условий.

IV. ТРИ ВИДА ВЫРОЖДЕНИЯ

Мы рассматривали до сих пор случай, когда в интегрируемой системе с n степенями свободы совершается n-частотное условно-периодическое движение. Однако во многих важных задачах число независимых частот n_0 меньше числа степеней свободы n. В этих случаях говорят о «вырождении». Различают три вида вырождения — собственное, предельное и случайное.

§ 1. Собственное вырождение

В задаче о ньютоновском притягивающем центре три степени свободы ($n = 3$). Между тем кеплерово движение одночастотно ($n_0 = 1$). Эта задача — типичный пример собственного вырождения, когда в целой области число частот n_0 меньше числа степеней свободы n.

Для возмущения вырожденных (собственно) систем характерно возникновение в дополнение к невозмущенным «быстрым» движениям еще нескольких «медленных» или «эволюционных» движений, так что общее число частот становится равным числу степеней свободы.

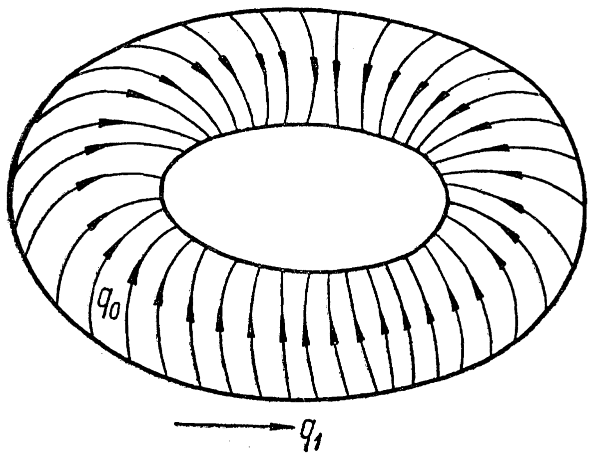

Рис. 9.

Пример. Пусть в переменных действие — угол $p_1, \ldots, p_n$ имеем

$$H = H_0(p_1, \ldots, p_{n_0}),$$

где $n_0 < n$. Будем обозначать вектор «быстрых переменных» $p_1, \ldots, p_{n_0}$ через p_0, а вектор «медленных переменных» $p_{n_0+1}, \ldots, p_n$ через p_1; аналогичный смысл имеют q_0 и q_1.

Канонические уравнения

$$\dot{q}_0 = \omega_0(p_0); \ \dot{p}_0 = 0; \ \dot{q}_1 = 0, \ \dot{p} = 0 \ \left(\omega_0(p_0) = \frac{\partial H_0}{\partial p_0}\right)$$

описывают условно-периодическое движение с n_0 частотами $\omega_0 = \omega_1, \ldots, \omega_{n_0}$ по n_0-мерному инвариантному тору $p_0 = \text{const}$, $p_1 = \text{const}$, $q_1 = \text{const}$ (рис. 9).

Предположим теперь, что имеется возмущение

$$H(p, q) = H_0(p_0) + \mu H_1(p, q) + \cdots. \qquad (1)$$

Тогда классическая теория возмущений дает следующую картину движения (с точностью $\sim \mu$ при $t \sim 1/\mu$). Выделим в H_1 «вековую часть»

$$\overline{H}_1(p, q_1) = \overline{H}_1(p_1, \ldots, p_n; q_{n_0+1}, \ldots, q_n) = \\ = (2\pi)^{-n_0} \int H_1 dq_0$$

и периодическую часть $\widetilde{H}_1(p, q)$:

$$H_1(p, q) = \overline{H}_1 + \widetilde{H}_1.$$

Оказывается, вековая и периодическая части возмущения играют совершенно разную роль. Канонические уравнения с функцией Гамильтона $\mu \overline{H}_1$

$$\dot{p}_1 = -\mu \frac{\partial \overline{H}_1}{\partial q_1} + \cdots, \qquad \dot{q}_1 = \mu \frac{\partial \overline{H}_1}{\partial p_1} + \cdots$$

определяют медленное, вековое изменение параметров p_1, q_1, определяющих инвариантный тор. Периодическая часть, $\widetilde{H}_1$, приводит лишь к дополнительному дрожанию возмущенной траектории около условно-периодического движения с медленно меняющимися параметрами, описываемого функцией Гамильтона $H_0 + \mu \overline{H}_1$.

Указанная картина движения получается при помощи преобразования $p_0, q_0 \to p_0', q_0'$ из § 2, второго раздела, если рассматривать p_1, q_1 как параметры.

Для получения более точных выводов о характере возмущенного движения нужно исследовать «усредненные» канонические уравнения с функцией Гамильтона $\overline{H}_1(p_1, q_1)$, зависящей от параметров p_0. Рассмотрим случаи, когда эти уравнения интегрируемы или близки к интегрируемым, что имеет место, например, в плоской задаче трех тел при малых массах или в общей задаче n тел при малых массах, эксцентриситетах и наклонениях.

В случае интегрируемости при надлежащем выборе переменных p_1, q_1 вековая часть $\overline{H}_1 = \overline{H}_1(p_1, \ldots, p_n)$ не будет зависеть от угловых переменных q_1, и мы приходим в качестве первого приближения к условно-периодическому движению

$$\dot{q}_0 = \omega_0(p_0), \quad \dot{p}_0 = 0; \quad \dot{q}_1 = \omega_1(p), \quad \dot{p}_1 = 0$$

с n_0 «быстрыми» частотами ω_0 и $n_1 = n - n_0$ «медленными» частотами $\omega_1 \sim \mu$. На это движение накладывается в среднем равное нулю (и потому в первом приближении пренебрежимое) возмущение, происходящее от $\mu\tilde{H}_1$.

Аналогично построениям третьего раздела, можно обосновать картину, описанную выше, следующим образом. Возмущенная система действительно имеет инвариантные торы; их можно найти надлежащим обобщением ньютоновского процесса. Техника становится более сложной, но все трудности преодолимы, и мы не будем здесь на них останавливаться. Основной результат: при достаточно малых возмущениях μ истинное движение условно-периодично и близко к описанному выше первому приближению при всех $-\infty < t < +\infty$ (для большинства начальных условий).

§ 2. Предельное вырождение

Начнем с примера колебательной системы с одной степенью свободы (маятник, малые колебания):

$$H = \frac{1}{2}\left(x^2 + \dot{x}^2\right).$$

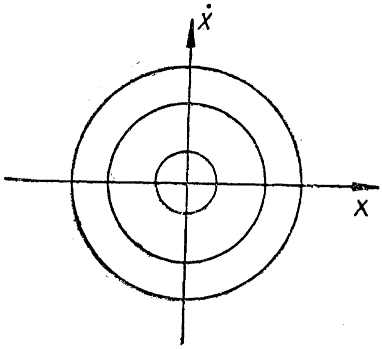

Рис. 10.

Фазовые траектории — концентрические окружности на плоскости x, $\dot{x}$, а также положение равновесия $x = \dot{x} = 0$ (рис. 10). Таким образом, среди инвариантных торов n-мерных $p = \text{const}$, на которые распадается фазовое пространство интегрируемой системы, могут встречаться в качестве «предельных случаев» отдельные торы размерности $k < n$ (для маятника $n = 1$, $k = 0$). В этих случаях говорят о «предельном вырождении». Особенно часто (теория колебаний) встречаются два случая предельного вырождения: положения равновесия ($k = 0$) и периодические движения ($k = 1$).

Асимптотическая теория, соответствующая теории возмущений, была развита применительно к указанным случаям Дж. Д. Биркгофом. Роль малого параметра в его теории играет расстояние до неподвижной точки (до периодического решения).

Точные результаты о поведении траектории в целом опять можно получить ньютоновскими приближениями. При большинстве начальных условий, близких к равновесному, движение оказывается условно периодичным. При этом, конечно, предполагается, что в линейном приближении система нейтрально устойчива; интересно отметить, что вторым существенным требованием является нелинейность системы — линейная система более подвержена резонансу, так как в ней амплитуда не зависит от частоты.

Настоящая устойчивость (в смысле Ляпунова) получается из указанных результатов лишь в первых размерностях; пример был подробно разобран в первом разделе. Другой пример — классическая задача об устойчивости лагранжевых периодических решений плоской круговой ограниченной задачи трех тел — разобрал А. М. Леонтович.

Более сложные случаи вырождения ($k > 1$) совершенно не изучены. Главная причина — неудовлетворительность первого (линейного) приближения, несовершенство теории Флоке для линейных систем с условно-периодическими коэффициентами. Первые шаги в этом направлении сделаны А. Е. Гельманом, Л. Я. Андриановой и Э. Г. Белагой.

§ 3. Случайное вырождение

В отличие от двух предыдущих видов вырождения, случайное вырождение встречается всюду. В следующем разделе увидим, как оно разрушает топологическую устойчивость многомерных систем. Случайное вырождение заключается в том, что при некоторых «резонансных» начальных условиях число частот невозмущенного условно-периодического движения меньше максимального числа частот. Один такой случай уже рассматривался в конце первого раздела (рождение зон неустойчивости, см. рис. 6).

В наших обычных обозначениях начальные условия случайного вырождения определяются из уравнения

$$(\omega, k) = 0,$$

где $\omega(p)$ — вектор частот, а k — целочисленный вектор. Таким образом, в общем случае начальные условия случайно вырожденных движений образуют всюду плотное множество. Заметим еще, что при $n \leqslant 2$, $\omega \neq 0$ зоны случайного вырождения с разными k не пересекаются, а при $n > 2$ образуют все вместе связанное множество.

Поведение случайно вырожденного движения при возмущении рассмотрим в следующем разделе.

V. ЗОНЫ НЕУСТОЙЧИВОСТИ

В предыдущих лекциях мы искали в том или ином смысле устойчивые движения. Теперь займемся неустойчивыми.

Рассмотрим снова систему с функцией Гамильтона

$$H = H_0(p) + \mu H_1(p, q) \quad (q \bmod 2\pi, \ \mu \ll 1)$$

и предположим, что невозмущенные частоты $\omega = \partial H_0/\partial p$ меняются от тора к тору:

$$\det\left|\frac{\partial\omega}{\partial p}\right| = \det\left|\frac{\partial^2 H_0}{\partial p^2}\right| \neq 0.$$

Условие $\omega_1 = 0$ определяет резонансную зону — одну из зон случайного вырождения. Эта зона — $(n-1)$-мерное многообразие в n-мерном пространстве p (каждая точка p соответствует целому n-мерному тору фазового пространства). В большинстве точек рассматриваемой зоны вырождение «однократное», т. е. остальные частоты $\omega_2, \ldots, \omega_n$ целочисленно независимы. Пересечение зон однократного вырождения —$n-2$-мерные многообразия «двукратного вырождения», и т. д.

Рассмотрим однократное вырождение средствами классической теории возмущений, не заботясь о строгости.

§ 1. Первое приближение. Либрация и $\sqrt{\mu}$

Пусть p^* — точка однократного вырождения: $\omega_1(p^*) = 0$. Для невозмущенной системы ($\mu = 0$) в рассматриваемом случае сохраняются не только первые интегралы $p_1 = p_1^*$, $\ldots, p_n = p_n^*$, но и q_1. Поэтому в соответствии с духом теории возмущений, H_1 надлежит усреднять по переменным $q_2, \ldots, q_n$; получается

$$H = H_0(p) + \mu \overline{H}_1(q_1; p_1, p_2, \ldots, p_n) + \ldots, \tag{1}$$

где точками означены члены порядка μ, в среднем равные нулю и потому «не существенные». Если их отбросить, то останется гамильтонова система с одной степенью свободы p_1, q_1, зависящая от параметров—констант $p_2^*, \ldots, p_n^*$.

Полученная система «первого приближения» для p_1, q_1 легко интегрируется. Для того чтобы наглядно представить результат, удобно ввести переменное $p = p - p_1^*$ вместо p_1. Тогда функция $H_0 + \mu \overline{H}_1$ принимает вид

$$H(p, Q) = \frac{a}{2} p^2 + \mu U(Q) + \cdots, \tag{2}$$

где

$$a = \left.\frac{\partial^2 H_0}{\partial p_1^2}\right|_{p^*}, \quad U(Q) = \overline{H}_1(Q; p^*).$$

Точками обозначены члены, малые по сравнению с p^2 и с μ; постоянный член $H_0(p^*)$ также опущен, ибо он не влияет на уравнения Гамильтона для $p, Q = q_1$; линейный по p член равен 0, так как $\left.\frac{\partial H_0}{\partial p_1}\right|_{p^*} = \omega_1(p^*) = 0$.

Гамильтониан (2) описывает одномерную систему с малым периодическим потенциалом $U(Q)$ (например, маятник в слабом поле). Поведение решений ясно из рис. 11, имеющего характерный «либрационный» вид. Наклон сепаратрис седел, расстояние между ними, а также частота колебаний в потенциальной яме — все эти величины пропорциональны $\sqrt{\mu}$.

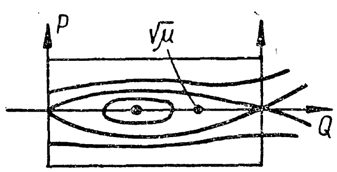

Рис. 11.

Возвращаясь к исходной многомерной системе, увидим, что в рамках теории возмущений найденную картину следует просто умножить на условно-периодическое движение по остальным координатам $q_2, \ldots, q_n$. Истинное положение вещей, конечно, сложнее.

Из проведенных рассуждений во всяком случае видно, что подмногообразие случайного вырождения $\omega_1 = 0$ при возмущении разрастается в область, размер которой пропорционален $\sqrt{\mu}$ и в которой резонанс $\omega_1 = 0$ играет существенную роль.

Отметим одну характерную особенность системы (2): ввиду периодичности потенциала $U(Q)$ всегда существуют сепаратриссы, идущие из седла в седло. Для дальнейшего будет полезно изучить поведение такой сепаратриссы при возмущении.

Начнем с модельной задачи, в которой седло присутствует уже до возмущения. Как обнаружил еще Пуанкаре, при возмущении сепаратрисса расщепляется. Эффект расщепления сепаратриссы недавно подробно изучен В. К. Мельниковым.

§ 2. Расщепление сепаратриссы

Сущность этого явления состоит в следующем. Рассмотрим динамическую систему с одной степенью свободы, имеющую неустойчивое положение равновесия (например, маятник, перевернутый вверх ногами). На фазовой плоскости p, q такому положению равновесия отвечает неподвижная точка — седло S. В седло входит сепаратрисса Г, отделяющая друг от друга траектории разных типов (в случае маятника — качание от вращения). Сама сепаратрисса Г есть траектория, изображающая асимптотически приближающееся к неустойчивому равновесию движение.

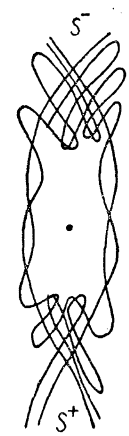

Рис. 12.

Предположим теперь, что наша система подвергается периодическим по времени малым (пропорциональным $\mu \ll 1$) возмущениям. Допустимо, что эти возмущения таковы, что седло S остается положением равновесия и при $\mu \neq 0$. Рассмотрим в фиксированный момент времени t_0 плоскость начальных условий p, q. Для возмущенной системы на каждой такой плоскости роль сепаратриссы играет своя кривая $\Gamma_\mu(t_0)$. Точка принадлежит $\Gamma_\mu(t_0)$, если решение с начальным условием p, $q \in \Gamma_\mu(t_0)$ стремится к седлу S при $t \to +\infty$. Расстояние между $\Gamma_\mu(t_0)$ и невозмущенной сепаратриссой Г разлагается в ряд по степеням μ.

Явление расщепления сепаратриссы возникает в случае, когда невозмущенная сепаратрисса Г идет из седла в седло (пример: маятник без трения). В этом случае на плоскости $p, q\,(t = t_0)$ возникает пара кривых $\Gamma_\varepsilon^+(t_0)$, $\Gamma_\varepsilon^-(t_0)$, близких к Г, но, вообще говоря, различных. Точка $x^+(t_0)$ $(x^-(t_0))$ принадлежит Γ^+ (Γ^-), если проходящая через нее траектория $x^+(t)$ $(x^-(t))$ стремится при $t \to +\infty$ $(t \to -\infty)$ к седлу S^+ (S^-) (рис. 12).

Еще Пуанкаре заметил, что эти две кривые на плоскости $t = t_0$ могут пересекаться, но не дал ни одного примера и (как видно из 33-й главы «Новых методов небесной механики») не сумел нарисовать образованную их пересечениями запутанную сеть (чертеж 12 принадлежит Мельникову).

Первый член ряда, выражающего расстояние между Γ^+ и Γ^-, имеет очень прозрачный смысл. Для упрощения некоторых выкладок ограничимся случаем гамильтоновой системы

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad H = H_0(p, q) + \varepsilon H_1(p, q, t),$$
$$\varepsilon \ll 1,$$

где ε — малый параметр, ω — частота возмущения, $H_1\left(t + \frac{2\pi}{\omega}\right) = H_1(t)$. Например, для маятника

$$H_0(p, q) = \frac{p^2}{2} + A^2 \cos q$$

и уравнение сепаратриссы Γ принимает вид

$$p = 2A \sin\frac{q}{2}; \qquad q(t) = 2 \operatorname{arcctg}(-\operatorname{sh} At), \qquad p(t) = \frac{2A}{\operatorname{ch} At}.$$

Седла S^- и S^+ в этом примере имеют координаты $p = 0$, $q = 0$ и $q = 2\pi$.

Расстояние между точкой x_ε^+, лежащей на $\Gamma_\varepsilon^+(t_0)$, и ближайшей к ней точкой x_0 невозмущенной сепаратриссы Γ удобно измерять величиной приращения энергии невозмущенного движения (рис. 13):

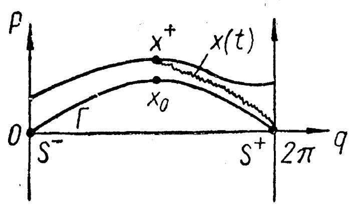

Рис. 13.

$$\Delta^+ H_0 = H_0(x^+) - H_0(x_0).$$

Так как в точках x_0 и S^+ функция H_0 принимает одинаковые значения, $\Delta^+ H_0$ есть приращение $-H_0$ вдоль траектории возмущенного движения, а потому выражается через скобку Пуассона $\left\{H_0 + \varepsilon H_1, H_0\right\} = \frac{dH_0}{dt}$:

$$\Delta^+ H_0 = \varepsilon \int_{t_0}^{+\infty} \left\{H_0, H_1\right\} dt,$$

где интегрирование происходит по траектории возмущенного движения. Поэтому естественно, что величина $\Delta^+ H_0$

разлагается в сходящийся ряд $\Delta^+ H_0 = \varepsilon\Delta_1^+ + \varepsilon^2\Delta_2^+ + \cdots$, первый член которого дается тем же интегралом

$$\Delta_1^+ = \int_{t_0}^{+\infty} \{H_0, H_1\}\, dt,$$

распространенным на траекторию невозмущенного движения.

Щель между возмущенными сепаратриссами Γ^+ и Γ^- характеризуется разностью $\Delta H_0 = \Delta^+ H_0 - \Delta^- H_0$, где, аналогично $\Delta^+ H_0$,

$$\Delta^- H_0 = H_0(x^-) - H_0(x_0) = \varepsilon \int_{t_0}^{-\infty} \{H_0, H_1\}\, dt.$$

Таким образом, для величины ΔH_0 получается ряд

$$\Delta H_0 = \varepsilon\Delta_1 + \varepsilon^2\Delta_2 + \cdots,$$

первый член которого дается интегралом

$$\Delta_1 = \int_{-\infty}^{+\infty} \{H_0, H_1\}\, dt,$$

распространенным на всю невозмущенную сепаратриссу Γ от S^- до S^+.

В примере с маятником, принимая за x_0 середину сепаратриссы $p = 2A \sin\frac{q}{2}$, получим для возмущения

$$\Delta_1 = \left(\int_{-\infty}^{\infty} \frac{4A \operatorname{sh} At}{\operatorname{ch}^3 At} \sin \omega t\, dt\right) \sin \omega t_0 = \frac{\pi\nu^2}{2 \operatorname{sh} \frac{\pi}{2}\nu}, \text{ где } \nu = \frac{\omega}{A}.$$

VI. НЕУСТОЙЧИВОСТЬ ДИНАМИЧЕСКИХ СИСТЕМ СО МНОГИМИ СТЕПЕНЯМИ СВОБОДЫ

Прогресс теории возмущений позволил найти много условно-периодических движений в каждой нелинейной динамической системе, близкой к интегрируемой. Устойчивость всех движений системы вытекает из этих результатов лишь в случаях, когда размерность фазового пространства $\leqslant 4$. Теперь укажем пример (3) системы с

5-мерным фазовым пространством, удовлетворяющей всем условиям первых трех разделов, но неустойчивой[1].

Вековое изменение величины I_z в системе (3) имеет скорость порядка $\exp(-1/\sqrt{\varepsilon})$ и потому не улавливается ни в каком приближении теории возмущений.

Сначала введем некоторые определения.

1. Усатый тор. Тором T^k называется прямое произведение k окружностей, допускающее k угловых координат $\varphi_1, \ldots, \varphi_k \pmod{2\pi}$. Условно-периодическое движение с частотами ω определяется уравнениями $\dot{\varphi} = \omega = \text{const}$, где $\Sigma n_i \omega_i \neq 0$ при целых n_i, $\Sigma n_i^2 = 0$. Пусть в фазовом пространстве динамической системы имеется инвариантный тор T^k, и на нем движение условно-периодично. Назовем T^k усатым тором, если он является компонентой пересечения двух инвариантных открытых многообразий Y^-, Y^+, причем все траектории на входящем усе Y^- стремятся к T^k при $t \to +\infty$, а на выходящем усе Y^+ — при $t \to -\infty$.

Пример 1. Стандартным усатым тором назовем тор $x = y = z = 0$ в системе

$$\dot{x} = \lambda x, \ \dot{y} = -\mu y, \ z = 0, \ \dot{\varphi} = \omega, \tag{1}$$

определенной в $l_+ + l_- + l_0 + k$-мерном пространстве x, y, z, φ ($\varphi \bmod 2\pi$) (рис. 14).

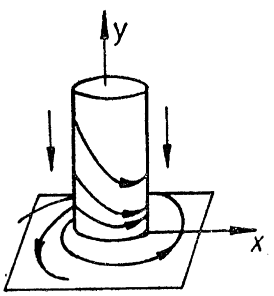

Рис. 14.

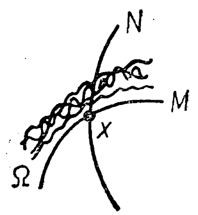

Рис. 15.

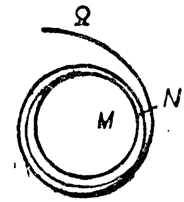

Рис. 16.

Для дальнейшего существенно понятие загораживающего множества. Пусть M—гладкое подмногообразие пространства X. Касательную плоскость к M в точке x будем обозначать через TM_x. Многообразие N дополняет M в точке $x \in M \cap N$, если $TM_x + TN_x = TX_x$. Скажем, что

[1] В отличие от устойчивости, неустойчивость устойчива. Нам кажется, что механизм «переходных цепочек», обеспечивающий неустойчивость в нашем примере, действует и в общем случае (например, в задаче трех тел).

множество Ω загораживает многообразие M в точке $x \in M$, если каждое многообразие N, дополняющее M в x, пересекается с Ω (рис. 15).

Пример 2. Спираль Ω, навивающаяся на замкнутую кривую M, загораживает ее[1] (рис. 16).

Другой пример доставляется стандартным усатым тором (1). Пусть U — окрестность точки — ξ входящего уса $x = z = 0$. Обозначим через $\Omega = \bigcup_{t>0} U(t)$ множество всех точек всех траекторий, начинающихся в U. Легко доказывается такая теорема.

Теорема 1. *Множество Ω загораживает выходящий ус $y = z = 0$ в любой его точке η.*

2. Переходная цепочка. Если усатый тор T обладает тем свойством, что образы любой окрестности любой точки ξ его входящего уса загораживают выходящий ус в любой точке последнего η, то такой тор назовем переходным. Согласно теореме 1, стандартный тор — переходный.

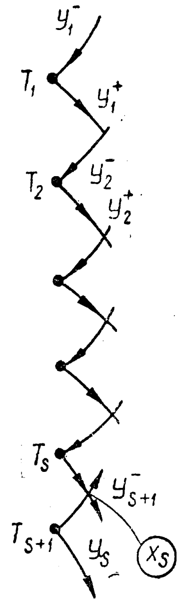

Рис. 17.

Пусть динамическая система с фазовым пространством X имеет несколько переходных торов $T_1, \ldots, T_s, \ldots$. Назовем эти торы переходной цепочкой, если выходящий ус Y_s^+ каждого предыдущего тора T_s дополняет входящий ус Y_{s+1}^- следующего тора T_{s+1} в некоторой точке их пересечения $x_s \in Y_s^+ \cap Y_{s+1}^-$ (рис. 17). Пусть $T_1, \ldots, T_s, \ldots$ — переходная цепочка, тогда можно доказать следующую теорему.

Теорема 2. *Любая окрестность тора T_1 соединена с любой окрестностью тора T_s траекторией рассматриваемой динамической системы.*

Таким образом, для доказательства неустойчивости достаточно найти переходную цепочку, соединяющую далекие торы T_1, T_s. Отыскание усатых торов и, особенно, изучение их пересечений в общей задаче теории возмущений требует громоздких вычислений. Ограничимся примером, в котором специально подобранное возмущение исчезает на торах T_s.

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[1] На этом обстоятельстве основаны работы К. Ситникова и А. М. Леонтовича об осциллирующих движениях.

3. **Неустойчивая система.** Рассмотрим систему с двумя степенями свободы, периодическую по времени t с периодом 2π. «Фазовое пространство» $I_1, I_2; \varphi_1, \varphi_2; t$ — прямое произведение плоскости I_1, I_2 на трехмерный тор $\varphi_1, \varphi_2, t \pmod{2\pi}$. Функция Гамильтона [1], зависящая от параметров ε, μ, будет иметь вид $H_0 + \varepsilon H_1$, где

$$H_0 = \frac{1}{2}(I_1^2 + I_2^2),\ \varepsilon H_1 = \varepsilon(\cos\varphi_1 - 1)[1 + \mu B],$$
$$B = \cos t + \sin\varphi_2. \quad (2)$$

Иначе говоря, рассматривается система дифференциальных уравнений

$$\dot\varphi_1 = I_1,\ \dot\varphi_2 = I_2, \dot I_1 = \varepsilon \sin\varphi_1[1 + \mu B],$$
$$\dot I_2 = -\varepsilon(\cos\varphi_1 - 1)\mu\cos\varphi_2, \quad (3)$$

где $B = \cos t + \sin\varphi_2$.

Рассмотрим сперва невозмущенную систему ($\varepsilon = 0$). Каждый трехмерный тор $I_1 = \text{const}$, $I_2 = \text{const}$ инвариантен. На нем происходит трехчастотное движение $\dot\varphi_1 = I_1$, $\dot\varphi_2 = I_2$, $\dot t = 1$. Тор называется невырожденным, если на нем частоты независимы (т. е. $n_1 I_1 + n_2 I_2 + n_0 \neq 0$ при целых $n \neq 0$). Уравнение $I_1 = 0$ определяет семейство вырожденных торов.

Теперь рассмотрим возмущенную систему: пусть $0 < \varepsilon\mu \ll \varepsilon \ll 1$. Для большинства начальных условий величины $I_1(t)$, $I_2(t)$ будут мало меняться в течение всего бесконечного промежутка времени $-\infty < t < +\infty$. Оказывается, однако, что вблизи резонансного многообразия $I_1 = 0$ появляется зона неустойчивости. Точнее, справедлива теорема.

Теорема 3. *Пусть $0 < A < B$. Для всякого $\varepsilon > 0$ найдется $\mu_0 > 0$ такое, что при $0 < \mu < \mu_0$ система неустойчива: существует траектория системы (3), соединяющая область $I_2 < A$ с областью $I_2 > B$.*

4. **Доказательство неустойчивости.** Зафиксируем $\varepsilon > 0$.

А. Пусть сначала $\mu = 0$. Тогда переменные разделяются:

$$H = H^{(1)} + H^{(2)},\ H^{(1)} = \frac{1}{2}I_1^2 + \varepsilon(\cos\varphi_1 - 1),$$
$$H^{(2)} = \frac{1}{2}I_2^2. \quad (4)$$

[1] Нетрудно построить реальную механическую систему с функцией Гамильтона (2).

Таким образом, $\dot{I}_2 = 0$, $\dot{\varphi}_2 = I_2 = \omega = \mathrm{const}$, а изменение I_1, φ_1 со временем описывается гамильтонианом обычного маятника $H^{(1)}$. Пусть число ω иррационально. Легко доказывается такое утверждение.

Многообразие $T\omega$, определенное уравнениями $I_1 = \varphi_1 = I_2 - \omega = 0$, есть двумерный усатый тор системы (3). Усы трехмерны и имеют уравнения

$$I_1 = \pm 2\sqrt{\varepsilon}\sin\frac{\varphi_1}{2},\ I_2 = \omega \quad \text{или} \quad H^{(1)} = 0,\ \ H^{(2)} = \frac{1}{2}\omega^2. \quad (5)$$

Усы заполнены асимптотическими траекториями

$$I_1(t) = \pm 2\sqrt{\varepsilon}\,\mathrm{ch}^{-1}\tau,\ \ \varphi_1(t) = \pm 2\,\mathrm{arcctg}\,(-\mathrm{sh}\,\tau),$$
$$\varphi_2 = \overset{0}{\varphi}_2 + \omega(t - t^0), \quad (6)$$

где

$$\tau = \sqrt{\varepsilon}(t - t_0) \ \text{и}\ \varphi_1(t^0) = \pi,\ I_1(t^0) = 2\sqrt{\varepsilon_0},\ \varphi_2(t^0) = \overset{0}{\varphi}_2.$$

Таким образом, точка выходящего уса тора $T\omega$ попадает при $t \to +\infty$ снова к тому же тору $T\omega$. Иначе говоря, выходящий ус составляет одно многообразие со входящим. Естественно, при $\mu \neq 0$ это многообразие расщепится на два уса, пересекающихся друг с другом. Увидим, что (в отличие от сепаратрис систем с фазовым пространством размерности $\leqslant 4$, рассмотренных в пятом разделе) эти усы пересекаются также и с усами соседних торов $T\omega'$.

Б. Пусть теперь $\mu \neq 0$. Из (3) видно, что торы $T\omega$ остаются инвариантными при всех μ. Пусть ω иррационально. Стандартным методом сжатых отображений доказывается и другое утверждение.

Многообразие $T\omega$ есть усатый переходный тор системы (3), если μ достаточно мало.

Пусть $\omega_1 < A$, $\omega_s > B$. Для доказательства теоремы 3 достаточно построить переходную цепочку торов $T\omega_1, \ldots, T\omega_s$ и воспользоваться теоремой 2. Построение такой цепочки основано на исследовании возмущения усов (5) при малых $\mu \neq 0$. Оказывается, справедлива следующая лемма.

Лемма 1. *Пусть $A < \omega < B$. Тогда выходящий ус $Y^+T\omega$ тора $T\omega$ пересекается со входящими усами $Y^-T\omega'$ всех близких торов $T\omega'$, если $|\omega - \omega'| < \varkappa$,* где $\varkappa = \varkappa(\varepsilon, \mu, A, B) > 0$.

Д о к а з а т е л ь с т в о леммы 1 требует некоторых вычислений. Невозмущенные усы имеют уравнения $H^{(1)} = 0$,

$H^{(2)} = \frac{1}{2}\omega^2$, где $H^{(k)}$ — функции (4). Пусть $\alpha > 0$ (например, $\alpha = \frac{\pi}{2}$). Легко показать, что при $|\varphi_1| < 2\pi - \alpha$ уравнение возмущенного выходящего уса $Y^+T\omega$ можно записать в виде

$$H^{(1)} = \Delta_1^+(\varphi_1;\ \varphi_2, t; \omega); H^{(2)} = \frac{1}{2}\omega^2 + \Delta_2^+(\varphi_1;\ \varphi_2,\ t;\ \omega), \quad (7)$$

где функции $\Delta_k^+ = 0(\mu)$ имеют период 2π по φ_2, t и равны 0 при $\varphi_1 = 0$. Точно так же входящий ус $Y^-T\omega'$ при $|\varphi_1 - 2\pi| < 2\pi - \alpha$ имеет уравнения

$$H^{(1)} = \Delta_1^-(\varphi_1;\ \varphi_2, t;\ \omega'); \quad H^{(2)} = \frac{1}{2}\omega'^2 + \Delta_2^-(\varphi_1; \varphi_2, t; \omega). \quad (8)$$

Пересечение усов $Y^+T\omega$ и $Y^-T\omega$ будем искать в плоскости $\varphi_1 = \pi$. В обозначениях (7), (8) лемма 1 есть утверждение о разрешимости относительно φ_2, t системы уравнений

$$\begin{cases} \Delta_1^+(\pi;\ \varphi_2, t;\ \omega) = \Delta_1^-(\pi;\ \varphi_2,\ t;\ \omega'), \\ \frac{1}{2}\omega^2 + \Delta_2^+(\pi;\ \varphi_2, t;\ \omega) = \Delta_2^-(\pi;\ \varphi_2,\ t;\ \omega') = \frac{1}{2}\omega'^2. \end{cases} \quad (9)$$

Разрешимость системы (9) выводится из следующих приближенных выражений для $\Delta_k^\pm$.

Лемма 2. *Возмущения усов суть* $\Delta_k^\pm = \mu\delta_k^\pm + 0(\mu^2)$, *где*

$$\mu\delta_k^\pm(\pi;\ \varphi_2^0, t^0, \omega) = \int_{\mp\infty}^{0} \left\{H, H^{(k)}\right\}\Big| d(t - t_0) \quad (10)$$

(скобка Пуассона интегрируется вдоль невозмущенной траектории (6)).

Действительно, согласно определениям (7), (8), величины Δ_k^+ суть приращения $H^{(k)}$ в возмущенном движении (3). Производная функции $H^{(k)}$ в силу системы уравнений (3) есть как раз скобка Пуассона $\{H, H^{(k)}\}$. Поэтому $\Delta_k^\pm$ в точности равны интегралам (10), распространенным на возмущенные траектории. Отсюда легко получить оценку $\Delta_k^\pm - \mu\delta_k^\pm = 0(\mu^2)$, доказывающую лемму 2.

Из леммы 2 видно, что разрешимость системы (9) зависит, в основном, от разрешимости относительно φ_2^0, t^0 приближенной системы

$$\delta_1 = 0, \ \mu\delta_2 = \frac{1}{2}(\omega^2 - \omega'^2), \quad (11)$$

где

$$\delta_k = \delta_k^+ (\pi;\ \varphi_2^0,\ t^0;\ \omega) - \delta_k^- (\pi,\ \varphi_2^0,\ t^0,\ \omega) =$$

$$= \int_{-\infty}^{+\infty} \{H,\ H^k\}\, d\,(t - t^0). \qquad (12)$$

Несложные вычисления, основанные на формулах (2) — (6), дают

$$\delta_1 = -2\varepsilon \int_{-\infty}^{+\infty} u \frac{\partial B}{\partial t}\, dt, \quad \delta_2 = 2\varepsilon\omega \int_{-\infty}^{+\infty} u \frac{\partial B}{\partial \varphi_2}\, dt, \qquad (13)$$

где $u = \mathrm{ch}^{-2}\tau$, $\tau = \sqrt{\varepsilon}\,(t - t^0)$, $B = B(\varphi_2, t)$, $\varphi_2 = \varphi_2^0 + \omega(t - t^0)$. При $B = \cos t + \sin\varphi_2$ интегралы (13) берутся вычетами:

$$\delta_1 = 2\pi \left(\mathrm{sh}^{-1} \frac{T_1}{2\sqrt{\varepsilon}}\right) \sin t^0, \quad \delta_2 = 2\pi\omega^2 \left(\mathrm{sh}^{-1} \frac{\omega\pi}{2\sqrt{\varepsilon}}\right) \cos\varphi_2^0. \quad (14)$$

Полагая в (14) $t^0 = 0$, убеждаемся в разрешимости системы (11) при

$$|\,\omega^2 - \omega'^2\,| < 4\pi\mu\omega^2\, \mathrm{sh}^{-1} \frac{\omega\pi}{2\sqrt{\varepsilon}} \asymp \mu e^{-1/\sqrt{\varepsilon}}\,. \qquad (15)$$

Из леммы 2 следует теперь, что при достаточно малых μ разрешима и система (9). Из неравенства (15) легко получается равномерная при $A < \omega < B$ оценка $|\omega - \omega'|_{\max}$ снизу, требуемая в лемме 1. Таким образом, лемма 1 доказана. Она позволяет построить цепочку переходных торов $T\omega_1, \ldots, T\omega_s$. Из формул (14) видно, что при достаточно малом μ эту цепочку можно выбрать так, чтобы последовательные пересекающиеся усы лежали в общем положении и дополняли друг друга в смысле § 2. Тогда цепочка $T_1, \ldots, T_s$ будет переходной. Применение теоремы 2 к переходной цепочке $T_1, \ldots, T_s$ завершает доказательство теоремы 3.

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CONDITIONS FOR THE APPLICABILITY, AND ESTIMATE OF THE ERROR, OF AN AVERAGING METHOD FOR SYSTEMS WHICH PASS THROUGH STATES OF RESONANCE IN THE COURSE OF THEIR EVOLUTION*

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1. The behavior of solutions of systems of the form

$$\dot\varphi = \omega(I;\varepsilon) + \varepsilon f(I,\varphi;\varepsilon),$$
$$\dot I = \varepsilon F(I,\varphi;\varepsilon), \tag{1}$$
$$\varphi = \varphi_1,\dots,\varphi_k;\quad I = I_1,\dots,I_l$$

(where $\phi\,(\mathrm{mod}\,2\pi)$ are angles, $\epsilon \ll 1$, dots indicate time derivatives, the functions ω, f, F are analytic for $I \in G$, $|\mathrm{Im}\,\phi| < \rho$, $|\epsilon_0| < \epsilon$ (their dependence on ϵ is not indicated below); G is a complex compact domain) is usually studied by the "averaging method", i.e. by substituting for (1) the averaged system

$$\dot J = \varepsilon \overline{F}(J),\quad \overline{F}(J) = (2\pi)^{-k}\iint F(J,\varphi;0)\,d\varphi. \tag{2}$$

Although terms neglected by averaging $\epsilon\widetilde{F} = \epsilon F - \epsilon\overline{F}$ are of the same order of magnitude as the remaining ones, it is assumed that in time $t \sim 1/\epsilon$ the difference between exact and averaged solutions with the same initial conditions $|I(t) - J(t)|$ remains small. Indeed, in the case of one frequency ($k = 1$) it is easy [1] to obtain the estimate

$$|I(t) - J(t)| < C_1\varepsilon \quad \text{for } 0 < t < 1/\varepsilon.$$

Here and below the $C_1, \cdots, C_{22}$ are sufficiently large constants, independent of ϵ, K, N, x.

In the present note we consider a system with two frequencies ($k = 2$). We will indicate a condition sufficient for the smallness of $|I - J|$, and will obtain the estimates $C_2^{-1}\sqrt{\epsilon} < |I(t) - J(t) < C_3\sqrt{\epsilon}\,\ln^2(1/\epsilon)$.

2. We will first of all give an example which shows that without additional assumptions averaging may lead to erroneous results.

Example 1. Let us consider the system

$$\dot\varphi_1 = I_1,\quad \dot\varphi_2 = I_2,\qquad \dot I_1 = \varepsilon,\quad \dot I_2 = \varepsilon a\cos(\varphi_1 - \varphi_2)\qquad (a > 1).$$

The averaged equations are $\dot J_1 = \epsilon$, $\dot J_2 = 0$. Let $I_1(0) = I_2(0) = J_1(0) = J_2(0) = 1$, $\phi_1(0) = \phi_2(0) = \arccos(1/a)$. The exact solution $I_1(t) = I_2(t) = 1 + \epsilon t$ after the time $t = 1/\epsilon$ loses all connection with the averaged solutions $J_1(t) = 1 + \epsilon t$, $J_2(t) = 1$.

Returning to system (1), $k = 2$, we assume that $\omega_2(I) \neq 0$. Let us introduce the ratio of frequencies $\lambda(I) = \omega_1/\omega_2$.

Condition A. Assume that $C_4^{-1}\epsilon < |\dot\lambda| < C_4\epsilon$, i.e. that the quantity

$$A(I,\varphi) = \left(\frac{\partial\omega_1}{\partial I}F\right)\omega_2 - \left(\frac{\partial\omega_2}{\partial I}F\right)\omega_1$$

does not vanish for any ϕ, if $I \in G$.

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Under condition A the system can not remain fixed in any resonance. In example 1 condition A is violated: $A = I_2 - I_1 \cos(\phi_1 - \phi_2)$ changes sign at $I_1 = I_2$, if $a > 1$.*

3. **Theorem 1.** *If condition A is satisfied, we have the estimate*

$$|I(t) - J(t)| < C_3\sqrt{\varepsilon} \ln^2 (1/\varepsilon) \qquad \textit{for all} \quad 0 \leqslant t \leqslant 1/\varepsilon. \tag{3}$$

Roughly speaking, Theorem 1 estimates the difference of magnitude $\sqrt{\epsilon}$ between solutions of the exact and the average systems. The following example shows that, generally speaking, $|I(t) - J(t)| > C_2^{-1}\sqrt{\epsilon}$.

Example 2. Consider the system

$$\dot{\varphi}_1 = I_1 + I_2, \quad \dot{\varphi}_2 = I_2; \qquad \dot{I}_1 = \varepsilon, \quad \dot{I}_2 = \varepsilon \cos(\varphi_1 - \varphi_2).$$

Let $\phi_1(0) = \phi_2(0) = I_1(0) = I_2(0) - 1 = 0$. Condition A is satisfied for $I_1 < I_2$. In the averaged system $J_2(t) \equiv 1$. In the exact solution

$$I_2(T) - 1 = \varepsilon \int_0^T \cos \varepsilon \frac{t^2}{2} dt = \sqrt{2\varepsilon} \int_0^\tau \cos x^2 \, dx, \quad \tau = \sqrt{\varepsilon/2}\, T.$$

For $T = 1/\epsilon$, obviously, $I_2(T) - J_2(T) = I_2(T) - 1 > C_2^{-1}\sqrt{\epsilon}$. **

If we analyse example 2, it is easy to see that the resonance $\omega_1 = \omega_2$ disperses the bundle of trajectories, which in the beginning differ only by phases ϕ. The scattering of the quantity I_2 after going through the resonance is of the order of $\sqrt{\epsilon}$.

The idea used in proving Theorem 1 is to divide the space I into two parts: a finite number (of the order of $\ln^2(1/\epsilon)$) of resonant zones of width K and a nonresonant part. In resonant zones the dispersion $|I - J|_r \sim K$ (neglecting logarithms) accumulates. In the nonresonant domain we form new variables P, satisfying the conditions $|P - I| \sim \epsilon/K$, $|\dot{P} - \epsilon \bar{F}(P)| \sim \epsilon^2/K^2$. From these estimates we infer that $|P - J| \sim \epsilon/K$. Thus $|I - J|_{nr} \leq |I - P| + |P - J| \sim \epsilon/K$. Consequently, $|I - J| \leq |I - J|_r + |I - J|_{nr} \sim K + \epsilon/K$. For $K \sim \sqrt{\epsilon}$ we obtain (3).

4. **Estimates.** Let $N > 1 > K > 0$. Denote by G_N the set of points I of domain G, in which $(\omega_1, n) = \omega_1 n_1 + \omega_2 n_2 \neq 0$ for integral n_1, n_2; $0 < |n| = |n_1| + |n_2| < N$. By $G_{K,N}$ we denote the set of points which are in G_N together with a neighborhood of radius K.

It follows from condition A that $d(\omega_1 n) \neq 0$ for $(\omega_1 n) = 0$. Therefore in $G_{K,N}$

$$|(\omega, n)| > C_5^{-1}K, \quad 0 < |n| < N. \tag{4}$$

We denote the complement of $G_{K,N}$ by $R_{K,N} = G \backslash G_{K,N}$. Let $I(t)$, $\phi(t)$ $(0 \leq t \leq 1/\epsilon)$ be a solution of system (1), and moreover let $I(t) \in G$. The segment $0 \leq t \leq 1/\epsilon$ is decomposed into two parts: $g_{K,N}$ (where $I(t) \in G_{K,N}$) and $r_{K,N}$ (where $I(t) \in R_{K,N}$). Let $K < C_6^{-1}$. From condition A we have the following lemmas.

Lemma 1. *The set $r_{K,N}$ consists of no more than C_7N^2 segments. The length of each one of them does not exceed C_8K/ϵ.*

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* Example 1 shows, that condition A can not be replaced by the analogous condition $\bar{A}$ for the averaged system (2), as was suggested in [2]. However, it is possible that under condition $\bar{A}$ inequality (3) is valid for most initial conditions. This is exactly the case in example 1, which can easily be integrated: $\ddot{q} = -\partial U/\partial q$, $U = -\epsilon \times (q - a \sin q)$, where $q = \phi_1 - \phi_2$. See also [3,4].

** Example 2 contradicts the statement $|I - J| < C\epsilon$, formulated in [2]. Probably in the general case $|I - J| > C_2^{-1}\sqrt{\epsilon}\ln^2(1/\epsilon)$.

Lemma 2. *Let* $\alpha < t < \beta$ *be one of the segments which form* $g_{K,N}$. *If* $\alpha + x \leq t \leq \beta - x$, *then* $I(t) \in G_{K(x),N}$, *where* $K(x) = K + C_9^{-1} \epsilon x$.

Denote by $J(J_0, t_0; t)$ the solution of system (2) with $J(t_0) = J_0$. We have at once

Lemma 3. *For* $|t - t_0| < 1/\epsilon$ *we have* $|J(J_0, t_0; t) - J(J_0', t_0; t)| < C_{10}|J_0 - J_0'|$.

Let $|\dot{x}| \leq a|x| + b(t)$, $|x(0)| < C$; $a, b, c \geq 0$. It is easy to prove

Lemma 4. $|x(t)| \leq [c + \int_0^t b(t)dt]e^{at}$.

The customary methods (see §6) are used to prove the fundamental

Lemma 5. *There exist functions* $P = I + S(I, \phi)$, $S(I, \phi + 2\pi) \equiv S(I, \phi)$ *and independent of* ϵ, K *constants* $C_{11} - C_{14}$ *such that*

$$|\dot{P} - \varepsilon \bar{F}(P)| < C_{11}\varepsilon^2 / K^2, \qquad |P - I| < C_{12}\varepsilon / K \tag{5}$$

when $|\mathrm{Im}\,\phi| \leq 0,5\rho$; $I \in G_{K,N}$; $N = C_{13}\ln(1/\epsilon)$; $|\epsilon| \leq G_{14}^{-1}K^2$.

5. **Proof of Theorem 1.** Let $0 < \epsilon < C_{14}^{-1} C_6^{-2}$, $K = \sqrt{C_{14}\epsilon} < C_6^{-1}$. Applying Lemma 5, form $P(I, \phi)$ and $G_{K,N}$. Denote consecutive segments forming $g_{K,N}$ by $[t_r^L, t_r^R]$ ($r = 1, 2, \cdots$; for definiteness, $t_1^L = 0 \in g_{K,N}$, $1/\epsilon \in r_{K,N}$). Introduce the notations $\alpha = L\ R$; $I(t_r^\alpha) = I_r^\alpha$; $J(I_r^L, t_r^L, t) = J_r(t)$; $P(I(t), \phi(t)) = P(t)$. Taking into account that $J_r(t_r^L) = I_r^L$ we obtain from Lemma 3,

$$\begin{aligned} |I(t) - J(t)| &\leqslant \sum_r |J_{r+1}(t) - J_r(t)| \leqslant C_{10} \sum_r |J_{r+1}(t_{r+1}^L) - J_r(t_{r+1}^L)| \\ &\leqslant C_{10} \sum_r \{|I_{r+1}^L - I_r^R| + |I_r^R - J_r(t_r^R)| + |J_r(t_r^R) - J_r(t_{r+1}^L)|\}. \end{aligned} \tag{6}$$

It follows from Lemma 1 that

$$|I_{r+1}^L - I_r^R| + |J_r(t_{r+1}^L) - J_r(t_r^R)| < C_{15}K. \tag{7}$$

From Lemma 5 we find for $t_r^L \leq t \leq t_r^R$, $a = C_{16}\epsilon$, $b = |\dot{P} - \epsilon\bar{F}(P)|$

$$|\dot{P} - \dot{J}_r| \leqslant a|P - J_r| + b. \tag{8}$$

Quantity b will be estimated by means of Lemmas 2 and 5:

$$b(t) < C_{11}\varepsilon^2 / (K + C_9^{-1}\varepsilon x)^2 \quad \text{for } t_r^L + x \leqslant t \leqslant t_r^R - x.$$

Consequently, $\int b(t)dt < C_{17}\epsilon/K$ for $t_r^L \leq t \leq t_r^R$. According to (5)

$$|P(t_r^L) - J(t_r^L)| = |P(t_r^L) - I_r^L| < c = C_{12}\epsilon/K.$$

Now applying Lemma 4 to inequality (8), we find

$$|P(t_r^R) - J_r(t_r^R)| \leq e^{C_{18}}(C_{12} + C_{17})\epsilon/K < C_{18}\epsilon/K. \tag{9}$$

From (5) and (9) there follows

$$|I_r^R - J_r(t_r^R)| \leqslant |I_r^R - P(t_r^R)| + |P(t_r^R) - J_r(t_r^R)| < (C_{12} + C_{18})\varepsilon / K. \tag{10}$$

Combining (6), (7), (10) and Lemma 1, we find for $0 \leq t \leq 1/\epsilon$

$$|I(t) - J(t)| < C_{10}C_7(C_{13}\ln(1/\varepsilon))^2[C_{15}K + (C_{12} + C_{18})\varepsilon / K].$$

For $K = \sqrt{C_{14}\epsilon}$ the right hand side is smaller than $C_3\sqrt{\epsilon}\ln^2(1/\epsilon)$, which is what we set out to prove.

6. **Proof of Lemma 5.** Let

$$S = \sum S_n e^{i(n,\varphi)}; \quad S_n = \frac{i\varepsilon F_n}{(\omega, n)}; \quad 0 < |n| < N, \tag{11}$$

where $F(I, \phi) = \bar{F}(I) + \tilde{F}(I, \phi)$ can be expanded in a Fourier series: $\tilde{F} = \Sigma F_n e^{i(n,\phi)}(|n| > 0)$ and $[\tilde{F}]_N = \Sigma F_n e^{i(n,\phi)}$ $(0 < |n| < N)$. Then $\dot{P} = \epsilon \bar{F}(P) + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5$, where $\Sigma_1 = [\epsilon \tilde{F}]_N + (\partial S/\partial\phi)\omega \equiv 0$ (see (11)); $\Sigma_2 = \epsilon\bar{F}(I) - \epsilon\bar{F}(P)$; $\Sigma_3 = \epsilon\tilde{F} - [\epsilon\tilde{F}]_N$; $\Sigma_4 = (\partial S/\partial\phi)\epsilon F$; $\Sigma_5 = (\partial S/\partial\phi)\epsilon f$.

For $|\mathrm{Im}\,\phi| < 0.9\rho$, $I \in G_{0.1\,K,N}$, in view of (4), we have (compare [5])

$$|S| < C_{12}\varepsilon / K, \quad |\partial S / \partial\varphi| < C_{12}\varepsilon / K, \quad |\partial S / \partial I| < C_{12}\varepsilon / K^2. \tag{12}$$

If $I \in G_{K,N}$, $|\epsilon| < C_{14}^{-1} K^2$, then the whole segment $IP \subset G_{0.1K,N}$. Therefore, for $I \in G_{K,N}$ and $|\mathrm{Im}\,\phi| < 0.5\rho$ we have, from (12),

$$|\Sigma_2| < \varepsilon \left| \frac{\partial\bar{F}}{\partial I} S \right| < c_{19}\varepsilon^2/K; \quad |\Sigma_4| < C_{20}\varepsilon^2/K^2, \ |\Sigma_5| < C_{21}\varepsilon^2/K.$$

In view of the analyticity of $\tilde{F}$ for $|\mathrm{Im}\,\phi| < \rho$, from $N = C_{13}\ln(1/\epsilon)$ for sufficiently large $C_{13} = C_{13}(\rho)$ there follows $|\Sigma_3| < C_{22}\epsilon^2$. Thus $|\dot{P} - \epsilon\bar{F}(P)| < C_{11}\,\epsilon^2/K^2$ (where $C_{11} = C_{19} + C_{20} + C_{21} + C_{22}$), which is what we set out to prove.

The reason for writing this note is the mistake in [2]. The author is grateful to A. M. Molčanov for pointing out this mistake.

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Translated by:
V. I. Filippenko

On a topological property of globally canonical maps in classical mechanics.

(8 novembre 1965)**

Translated by Alain Chenciner and Jaques Fejoz

Summary. The inequalities of M. Morse on the number of critical points of a function on a manifold are used in order to find the periodic solutions of the problems of mechanics.

1. H. Poincaré has clarified the importance of area-preserving transformations of an annulus in the restricted three-body problem and in all problems of mechanics with two degrees of freedom. Such applications share a remarkable topological property:

Poincaré's lemma. *Let A be an area-preserving diffeomorphism of a planar annulus. Let γ be a simple curve in the annulus which is not homologous to* 0. *The curves γ and $A\gamma$ then have at least two points in common.*

Indeed $A\gamma$ is neither inside nor outside γ, given that the areas circumscribed by γ and $A\gamma$ agree.

It is Poincaré's lemma which underlies Birkhoff's theorem on the existence of periodic orbits, the "last theorem of Poincaré" and so on [4, 2, 5].

Problems in mechanics with more than two degrees of freedom lead [1] to globally canonical maps (see definition below) of a toric annulus $\Omega = \mathrm{T}^n \times \mathrm{B}^n$, $\mathrm{B}^n \subset \mathbb{R}^n$.

In this Note Poincaré's lemma and its consequences are extended to systems with several degrees of freedom. The argument involving areas does not hold anymore; the torus T^n, $n > 1$, does not bound any domain in Ω. Another topological argument is used –the theory of M. Morse.

2. *Globally canonical maps.* Consider a toric annulus $\Omega = \mathrm{T}^n \times \mathrm{B}^n$, where

$$\mathrm{B}^n \subset \mathbb{R}^n = \{p\}, \quad (p = p_1, ..., p_n); \quad \mathrm{T}^n = \{q \bmod 2\pi\}, \quad (q = q_1, ..., q_n).$$

Definition. The map $\mathbf{A} : \Omega \to \Omega$ is *globally canonical* if it is homotopic to the identity and

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$$\oint_\gamma p\,dq = \oint_{\mathbf{A}\gamma} p\,dq, \quad (p\,dq = p_1\,dq_1 + ... + p_n\,dq_n), \tag{1}$$

for every (possibly non homotopic to 0) closed curve $\gamma \subset \Omega$.

Let x be a point in the annulus Ω. We shall denote by $p(x), q(x)$ its coordinates and write

$$P(x) = p(\mathbf{A}x), \quad Q(x) = q(\mathbf{A}x) \tag{2}$$

as usual.

Lemma. *The map* **A** *is globally canonical if and only if the integral*

$$A(x) = \int_{x_0}^{x} (Q - q)\,dP + (p - P)\,dq \tag{3}$$

defines a single-valued function $A(x)$.

The function A is called the *generating function* of the map **A**.

Proof of the lemma. Let γ be a closed curve in Ω, **A** a globally canonical map. We shall show that

$$\oint_\gamma (Q - q)\,dP + (p - P)\,dq = 0. \tag{4}$$

Indeed, given (2) one can write (1) as

$$\oint_\gamma p\,dq = \oint_\gamma P\,dQ. \tag{5}$$

Hence, $\oint_\gamma (Q-q)\,dP + (p-P)\,dq = \oint_\gamma d(P(Q-q))$. This latter integral represents the increase of $P(Q-q)$ along γ.

But **A** is homotopic to the identity, hence the increase of $P(Q-q)$ is equal to zero. As a consequence

$$\oint dP\,(Q - q) = 0. \tag{6}$$

Conversely, (4) and (6) yield (5), hence (1) Q.E.D.

3. *Intersecting tori.* Let T be the torus $p = 0$ in Ω and **A**T be the image of T under a globally canonical map **A**.

Theorem 1. *The tori* T *and* **A**T *have at least* 2^n *points (counted with their multiplicities) in common provided that* **A**T *has equation*

$$p = p(q), \quad \left|\frac{\partial p}{\partial q}\right| < \infty. \tag{7}$$

Proof. Consider a function defined on **A**T:

$$f(x) = \int_{x_0}^{x} p\,dq \quad \text{(the curve } x_0x \text{ lies on } \mathbf{A}\text{T)}. \tag{8}$$

Integral (8) does not depend on the path described in **A**T.

Indeed for a closed curve γ in **A**T we have

$$\oint_\gamma p\,dq = \oint_{\mathbf{A}^{-1}\gamma} p\,dq = 0$$

because $\mathbf{A}^{-1}$ is globally canonical, $\mathbf{A}^{-1}\gamma \subset \mathrm{T}$ and on T we have $p = 0$. Thus $f(x)$ is a differentiable function on the torus **A**T. According to the inequalities of M. Morse (see for instance [3]), $f(x)$ has at least 2^n critical points.[3] It follows from definition (8) that

$$df = p\,dq. \tag{9}$$

If x is a critical point of f on **A**T, i.e. $df(x) = 0$, then it results from (7) and (9) that $p(x) = 0$, $x \in \mathrm{T}$, Q.E.D.

Corollary. *The assertion of theorem 1 holds true if the tori* T *and* **A**T *have equations*

$$p = p'(q),\ p = p''(q) \quad \left(\left|\frac{\partial p'}{\partial q}\right| < \infty,\ \left|\frac{\partial p''}{\partial q}\right| < \infty,\right), \tag{10}$$

and if on T *the* 2*-form* $dp \wedge dq = dp_1 \wedge dq_1 + ... + dp_n \wedge dq_n$ *is identically* 0.

Indeed, the change of canonical variables $p, q \mapsto p - p'(q), q$ transforms (10) into (7) with $p(q) = p''(q) - p'(q)$.

4. *The fixed points.* Let $\mathbf{A}_0$ be some globally canonical map

$$\mathbf{A}_0 : p, q \mapsto p, q + \omega(p), \tag{11}$$

where $\omega : \mathrm{B}^n \to \mathbb{R}^n$, $\omega(p) = \omega_1(p), ..., \omega_n(p)$. If

$$\det\left|\frac{\partial \omega}{\partial p}\right| \not\equiv 0, \tag{12}$$

there exists a point $p_0 \in \mathrm{B}^n$ such that all the $\omega_i(p_0)$ are commensurable to 2π:

$$\omega_1(p_0) = \frac{2\pi m_1}{N}, \quad ..., \quad \omega_n(p_0) = \frac{2\pi m_n}{N}. \tag{13}$$

Obviously for the map $\mathbf{A}_0^N$ every point of the torus $p = p_0$ is a fixed point.

Theorem 2. *Let* **A** *be a globally canonical map close enough to* $\mathbf{A}_0$*. Then the map* $\mathbf{A}^N$ *has at least* 2^n *fixed points (counted with their multiplicities) in the neighborhood of the torus* $p = p_0$*.*

Proof. It follows from (11, 12, 13) that the map $\mathbf{A}_0^N$ is of the form

$$\mathbf{A}_0^N : p, q \mapsto p, q + \alpha(p), \quad \alpha(p_0) = 0, \quad \det\left|\frac{\partial \alpha}{\partial p}\right|_{p_0} \neq 0. \tag{14}$$

[3] $2^n = \sum_{i=0}^n b_i$, $b_i = \operatorname{rank} H_i(\mathrm{T}^n, \mathbb{R})$.

Hence the nearby map $\mathbf{A}^N$ is of the form

$$\mathbf{A}^N : p, q \mapsto p + \beta_1(p, q), \quad q + \alpha(p) + \beta_2(p, q), \quad \beta_{1,2} \ll 1. \tag{15}$$

We shall denote by T the torus defined by the equation

$$\alpha(p) + \beta_2(p, q) = 0. \tag{16}$$

Provided that $\mathbf{A}$ is close enough to $\mathbf{A}_0$, it follows from (14) that:

(i) the implicit function theorem applies to (16);

(ii) equation (16) defines a differentiable torus T, $p = p'(q)$, $|\partial p'/\partial q| < \infty$;

(iii) the torus $\mathbf{A}$T has equations $p = p''(q)$, $|\partial p''/\partial q| < \infty$;

(iv) the tori T and $\mathbf{A}$T are close to the torus $p = p_0$.

It follows from the lemma of paragraph 2 that the generating function A of $\mathbf{A}^N$ is a single-valued function on Ω. We will denote by $a(x)$ its restriction to the torus T. It is a differentiable function on T which, according to the theory of M. Morse, has at least 2^n critical points. It follows from (3) that

$$dA = (Q - q)\, dP + (p - P)\, dq.$$

By virtue of (15, 16), one has $Q - q = 0$ on T, hence $da = (p - P)\, dq$. Using (ii) and (iii), one shows that at critical points of $a(x)$ one has $p - P = 0$. Together with (16), this means that the 2^n critical points of $a(x)$ are fixed points of $\mathbf{A}^N$.

5. *Remark A.* By substituting in the proofs the theory of L. A. Lusternik–L. G. Schnirelman for the theory of M. Morse, in theorem 1 we get $(n + 1)$ geometrically distinct, intersection points of T and $\mathbf{A}$T. One can ask *whether there exists* $(n + 1)$ *points of intersection of* T *and* $\mathbf{A}$T *for globally canonical homeomorphisms* $\mathbf{A}$?

Remark B. Theorem 2 yields the existence of an infinite number of periodic orbits in the neighborhood of a generic elliptic periodic orbit (extension of the theorem of Birkhoff to $n > 1$).

Remark C. It seems very likely that theorem 1 holds true without hypothesis (7), provided that $\mathbf{A}$ is a diffeomorphism.[4] The proof would yield several "recurrence theorems".

For instance, consider the planar n-body problem. Assume that the initial values a_i, b_i of the axes of the Keplerian ellipses be such that the ellipses do not meet. Then for every τ there would exist some initial phases[5] l_i, g_i such that after time τ the axes would become equal to their initial values again.

Remark D. It is very likely too that the last theorem of Poincaré can be extended as follows:

Let $\mathbf{A} : \Omega \to \Omega$ ($\Omega = \mathrm{B}^n \times \mathrm{T}^n$; $B^n = \{p, |p| \leq 1\}; \mathrm{T}^n = \{q \pmod{2\pi}\}$) be a globally canonical diffeomorphism such that, for every $q \in \mathrm{T}^n$ the spheres $\mathrm{S}^{n-1}(q) = \partial \mathrm{B}^n \times q$ and $\mathbf{A}\mathrm{S}^{n-1}(q)$ are linked in $\partial \mathrm{B}^n \times \mathbb{R}^n$ ($\mathbb{R}^n$ being the universal cover of T^n). Then $\mathbf{A}$ has at least 2^n fixed points in Ω (counted with their multiplicities).

[4] If $\mathbf{A}$ is not a diffeomorphism, one constructs counter-examples with $n = 1$.

[5] The phases are the angles $l_i, g_i \mod 2\pi$, which determine the orientation of the ellipses in the plane (g_i) and the positions of the "planets" on the ellipses (l_i).

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