3 Free Profinite Groups

3.1 Profinite Topologies

Let $\mathcal{N}$ be a nonempty collection of normal subgroups of finite index of a group G and assume that $\mathcal{N}$ is filtered from below, i.e., $\mathcal{N}$ satisfies the following condition:

$$\text{whenever } N_1, N_2 \in \mathcal{N}, \text{ there exists } N \in \mathcal{N} \text{ such that } N \leq N_1 \cap N_2.$$

Then one can make G into a topological group by considering $\mathcal{N}$ as a fundamental system of neighborhoods of the identity element 1 of G (cf. Bourbaki [1989]. Ch. 3, Proposition 1). We refer to the corresponding topology on G as a *profinite topology*. If every quotient G/N ($N \in \mathcal{N}$) belongs to a certain class $\mathcal{C}$, we say more specifically that the topology above is *a pro-$\mathcal{C}$ topology*.

Let $\mathcal{C}$ be a formation of finite groups, and let G be a group. Define

$$\mathcal{N}_{\mathcal{C}}(G) = \{N \triangleleft_f G \mid G/N \in \mathcal{C}\}. \tag{1}$$

Then $\mathcal{N}_{\mathcal{C}}(G)$ is nonempty and filtered from below. The corresponding profinite topology on G is called the *pro-$\mathcal{C}$ topology of G* or, if emphasis is needed, *the full pro-$\mathcal{C}$ topology of G*. Note that the pro-$\mathcal{C}$ topology of G is Hausdorff if and only if

$$\bigcap_{N \in \mathcal{N}_{\mathcal{C}}(G)} N = 1. \tag{2}$$

A group G is called *residually $\mathcal{C}$* if it satisfies condition (2).

Remark 3.1.1 Assume that a profinite topology on G is determined by a collection $\mathcal{N}$ of normal subgroups of finite index filtered from below. Consider the set $\mathcal{C}$ of all groups G/M, where M ranges over all open normal subgroups of G. Then $\mathcal{C}$ is a formation of finite groups, and the given topology on G is a pro-$\mathcal{C}$ topology of G, although not necessarily the full pro-$\mathcal{C}$ topology of G. Indeed, consider a finite group T of order $n > 1$, and let G be the direct product of infinitely many copies of T. Let $\mathcal{N}$ be the collection of all the open normal subgroups of the profinite group G, and let $\mathcal{C}$ be as indicated above. As we shall show in Example 4.2.12, the pro-$\mathcal{C}$ topology of G is richer than its natural profinite topology.

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If $\mathcal{C}$ is the class of all finite groups (respectively, all finite p-groups, or all finite solvable groups, etc.), then, instead of residually $\mathcal{C}$, we say that G is a *residually finite group* (respectively, a *residually finite p-group* or a *residually finite solvable group*, etc.). The corresponding topology on G is called *the (full) profinite topology* on G (respectively, *the (full) pro-p topology, the (full) prosolvable topology* etc. on G). We remark that, for example, the full pronilpotent topology on a group G is a prosolvable topology on G, but it is not necessarily its full prosolvable topology (although in some cases it may be).

Next we describe some basic properties of the pro-$\mathcal{C}$ topology of a group G. Recall that the core H_G of H in G is the intersection of all conjugates of H in G. Observe that if $H \leq_f G$, then H has only finitely many conjugates; so,

$$H_G = \bigcap_{g \in G} H^g \triangleleft_f G.$$

Lemma 3.1.2 *Let $\mathcal{C}$ be a formation of finite groups. Assume that G is an abstract group and let $H \leq_f G$. Then*

(a) *H is open in the pro-$\mathcal{C}$ topology of G if and only if $G/H_G \in \mathcal{C}$.*
(b) *H is closed in the pro-$\mathcal{C}$ topology of G if and only if H is the intersection of open subgroups of G.*

Proof. (a) If $G/H_G \in \mathcal{C}$, then H_G is open; hence so is H. Conversely, if H is open, then so is every conjugate H^g of H in G; moreover, $H \leq_f G$, and so H has only finitely many conjugates. Therefore, H_G is open. Hence there exists some $N \triangleleft_f G$ with $G/N \in \mathcal{C}$ and $N \leq H_G$. Then there is an epimorphism $G/N \longrightarrow G/H_G$; thus $G/H_G \in \mathcal{C}$.

(b) Since an open subgroup has finite index, it is necessarily closed; therefore the intersection of open subgroups is closed. Conversely, assume H is a closed subgroup of G, and let $x \in G - H$. Then there exists some $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $xN \cap H = \emptyset$. Hence $x \notin HN$; so

$$H = \bigcap_{N \in \mathcal{N}_{\mathcal{C}}(G)} HN.$$

Since HN is open, the result follows. □

Example 3.1.3 Let $\mathcal{C}$ be a formation of finite groups, and assume that the group G is residually $\mathcal{C}$. If $H \leq G$, the pro-$\mathcal{C}$ topology of G induces on H a pro-$\mathcal{C}$ topology, but this is not necessarily the full pro-$\mathcal{C}$ topology of H, as the following examples show.

(1) Assume that $\mathcal{C}$ is the formation of all finite groups, $G = F$ is a free group of rank 2, and $H = F'$ the commutator subgroup of F. It is known that F' is a free group of countably infinite rank (cf. Magnus, Karras and Solitar [1966]). Let $\mathcal{I}$ be the topology induced on F' by the profinite topology of

F. It is plain that there are only countably many open subgroups in $\mathcal{I}$, while the profinite topology of F' has uncountably many open subgroups.

(2) Let $G = \langle a, b \mid b^2 = 1, bab = a^{-1}\rangle$ be the infinite dihedral group, and let $H = \langle a \rangle$. Then the pronilpotent topology of G induces on H only its pro-2 topology.

Next we indicate some cases where the induced pro-$\mathcal{C}$ topology on a subgroup coincides with the full pro-$\mathcal{C}$ topology of the subgroup.

Lemma 3.1.4

(a) *Let $\mathcal{C}$ be an extension closed variety of finite groups. Let H be a subgroup of G, open in the pro-$\mathcal{C}$ topology of G. Then the pro-$\mathcal{C}$ topology of G induces on H its full pro-$\mathcal{C}$ topology.*

(b) *Let $\mathcal{C}$ be an NE-formation of finite groups. Let H be a normal subgroup of G, open in the pro-$\mathcal{C}$ topology of G. Then the pro-$\mathcal{C}$ topology of G induces on H its full pro-$\mathcal{C}$ topology.*

Proof. (a) It suffices to show that if $N \triangleleft H$ and $H/N \in \mathcal{C}$, then there exists some $M \triangleleft G$ such that $G/M \in \mathcal{C}$ and $M \leq N$. We claim that we may take $M = N_G$, the core of N in G. Observe that if we put $K = H_G \cap N$, then $H/K \leq H/H_G \times H/N$, and hence $H/K \in \mathcal{C}$. Choose $g_1, \dots, g_r \in G$ so that $K_G = \bigcap_{i=1}^r K^{g_i}$. Then $K^{g_i} \triangleleft H_G$ and $H_G/K^{g_i} \in \mathcal{C}$. Now, $H_G/K_G \leq H_G/K^{g_1} \times \cdots \times H_G/K^{g_r}$; and hence $H_G/K_G \in \mathcal{C}$. Thus the extension G/K_G of H_G/K_G by G/H_G belongs to $\mathcal{C}$. Finally, note that $N_G = K_G$, so that we can take $M = N_G$, as asserted.

(b) Let $N \triangleleft H$ with $H/N \in \mathcal{C}$. Choose $g_1, \dots, g_r \in G$ so that $N_G = \bigcap_{i=1}^r N^{g_i}$. We claim that $H/N_G \in \mathcal{C}$. Note first that $H/N^{g_1} \cong H/N \in \mathcal{C}$. Moreover $N^{g_1}/N^{g_1} \cap N \cong N^{g_1}N/N \triangleleft H/N$; hence $N^{g_1}/N^{g_1} \cap N \in \mathcal{C}$, since $\mathcal{C}$ is closed under taking normal subgroups. It follows from the exactness of

$$1 \longrightarrow N^{g_1}/N^{g_1} \cap N \longrightarrow H/N^{g_1} \cap N \longrightarrow H/N^{g_1} \longrightarrow 1$$

that $H/N^{g_1} \cap N \in \mathcal{C}$, because $\mathcal{C}$ is also extension closed. The claim is now clear by induction. Next, observe that $G/H \in \mathcal{C}$, since H is open in the topology of G (see Lemma 3.1.2). Hence from the exactness of

$$1 \longrightarrow H/N_G \longrightarrow G/N_G \longrightarrow G/H \longrightarrow 1$$

we deduce that $G/N_G \in \mathcal{C}$. Consequently N_G, and thus N, are open in the pro-$\mathcal{C}$ topology of G. □

Lemma 3.1.5 *Let $\mathcal{C}$ be a variety of finite groups. Let $G = K \rtimes H$ be a semidirect product of the group K by the group H. Then*

(a) *The pro-$\mathcal{C}$ topology of G induces on H its full pro-$\mathcal{C}$ topology.*

(b) *Assume, in addition, that G is residually $\mathcal{C}$. Then H is closed in the pro-$\mathcal{C}$ topology of G.*

Proof. (a) Since $\mathcal{C}$ is subgroup closed, the pro-$\mathcal{C}$ topology of H is finer than the topology induced from G. Conversely, let $N \triangleleft_f H$ with $H/N \in \mathcal{C}$. Then $KN \triangleleft_f G$ and $G/KN \in \mathcal{C}$, since $G/KN \cong H/N$. Next note that $KN \cap H = N$.

(b) Consider the continuous maps

$$G \underset{\varphi}{\overset{\iota}{\rightrightarrows}} G,$$

where ι is the identity, $\varphi(kh) = h$ $(k \in K, h \in H)$, and G is assumed to have the pro-$\mathcal{C}$ topology. Then $H = \{g \in G \mid \iota(g) = \varphi(g)\}$. Hence H is closed, since the topology of G is Hausdorff. □

Corollary 3.1.6 *Let $\mathcal{C}$ be a variety of finite groups. Let $G = L * H$ be a free product of groups. Then*

(a) *The pro-$\mathcal{C}$ topology of G induces on H its full pro-$\mathcal{C}$ topology.*
(b) *Assume, in addition, that G is residually $\mathcal{C}$. Then H is closed in the pro-$\mathcal{C}$ topology of G.*

Proof. Denote by K the normal closure of L in G. Then $G = K \rtimes H$. Hence the results follow from the lemma above. □

3.2 The Pro-$\mathcal{C}$ Completion

Let G be a group and let $\mathcal{N}$ be a nonempty collection of normal subgroups of finite index of G filtered from below. Consider the topology on G determined by $\mathcal{N}$ as indicated in Section 3.1. The *completion* of G with respect to this topology is

$$K_{\mathcal{N}}(G) = \varprojlim_{N \in \mathcal{N}} G/N.$$

Then $K_{\mathcal{N}}(G)$ is a profinite group, and there exists a natural continuous homomorphism

$$\iota = \iota_{\mathcal{N}} : G \longrightarrow K_{\mathcal{N}}(G),$$

induced by the epimorphisms $G \longrightarrow G/N$ $(N \in \mathcal{N})$. Namely, $\iota(g) = (gN)_{N \in \mathcal{N}}$, for each $g \in G$. Observe that $\iota(G)$ is a dense subset of $K_{\mathcal{N}}(G)$ (see Lemma 1.1.7). The map ι is injective if and only if $\bigcap_{N \in \mathcal{N}} N = 1$.

Suppose that $\mathcal{M}$ is a subcollection of $\mathcal{N}$ which is also filtered from below. Then the epimorphisms

$$K_{\mathcal{N}}(G) \longrightarrow G/M \quad (M \in \mathcal{M})$$

induce a continuous epimorphism

$$K_{\mathcal{N}}(G) \longrightarrow K_{\mathcal{M}}(G)$$

that makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{K}_{\mathcal{N}}(G) & \longrightarrow & \mathcal{K}_{\mathcal{M}}(G) \\ {\scriptstyle \iota_{\mathcal{N}}} \nwarrow & & \nearrow {\scriptstyle \iota_{\mathcal{M}}} \\ & G & \end{array}$$

Let $\mathcal{C}$ be a formation of finite groups and let $\mathcal{N}_{\mathcal{C}}(G)$ be the collection of normal subgroups of G defined in (1). Then the completion $\mathcal{K}_{\mathcal{N}_{\mathcal{C}}(G)}(G)$ is just the pro-$\mathcal{C}$ completion of G as defined in Example 2.1.6. In this case we usually denote the completion $\mathcal{K}_{\mathcal{N}_{\mathcal{C}}(G)}(G)$ by $\mathcal{K}_{\mathcal{C}}(G)$ or by $G_{\hat{\mathcal{C}}}$. If $\mathcal{C}$ is the formation of all finite p-groups, for a fixed prime number p, then one often uses the notation $G_{\hat{p}}$ for the corresponding completion. We shall reserve the notation $\widehat{G}$ for the profinite completion of G, i.e., the completion $G_{\hat{\mathcal{C}}}$, where $\mathcal{C}$ is the formation of all finite groups.

Lemma 3.2.1 *Let $\mathcal{C}$ be a formation of finite groups and let G be a group. Then the pro-$\mathcal{C}$ completion $G_{\hat{\mathcal{C}}}$ of a group G is characterized as follows. $G_{\hat{\mathcal{C}}}$ is a pro-$\mathcal{C}$ group together with a continuous homomorphism*

$$\iota : G \longrightarrow G_{\hat{\mathcal{C}}}$$

onto a dense subgroup of $G_{\hat{\mathcal{C}}}$, where G is endowed with the pro-$\mathcal{C}$ topology, and the following universal property is satisfied:

$$\begin{array}{ccc} G_{\hat{\mathcal{C}}} & & \\ {\scriptstyle \iota}\uparrow & \searrow {\scriptstyle \bar{\varphi}} & \\ G & \xrightarrow[\varphi]{} & H \end{array}$$

whenever H is a pro-$\mathcal{C}$ group and $\varphi : G \longrightarrow H$ a continuous homomorphism, there exists a continuous homomorphism $\bar{\varphi} : G_{\hat{\mathcal{C}}} \longrightarrow H$ such that $\bar{\varphi}\iota = \varphi$. Moreover, it suffices to check this property for $H \in \mathcal{C}$.

Proof. We verify first that the completion $G_{\hat{\mathcal{C}}}$, as defined above, together with the map ι satisfy the indicated universal property. Let $\varphi : G \longrightarrow H$ be a continuous homomorphism into a pro-$\mathcal{C}$ group H. Set $\mathcal{U} = \{U \mid U \triangleleft_o H\}$ and let $U \in \mathcal{U}$. Define $N_U = \varphi^{-1}(U)$. Then there is a composition of natural continuous homomorphisms

$$\varphi_U : G_{\hat{\mathcal{C}}} \longrightarrow G/N_U \longrightarrow H/U.$$

Then the maps φ_U $(U \in \mathcal{U})$ are compatible. Hence they define a continuous homomorphism

$$\bar{\varphi} : G_{\hat{\mathcal{C}}} \longrightarrow \varprojlim_{U \in \mathcal{U}} H/U = H$$

such that $\varphi_{UV}\bar{\varphi} = \varphi_V$ whenever $U, V \in \mathcal{U}$ and $U \leq V$, where

$$\varphi_{UV} : H/U \longrightarrow H/V$$

is the canonical epimorphism. Then one verifies without difficulty that $\bar{\varphi}\iota = \varphi$.

The fact that this universal property characterizes the completion follows a standard argument that we only sketch. Say that K is a pro-$\mathcal{C}$ group and

$$\kappa : G \longrightarrow K$$

is a continuous homomorphism whose image is dense in K. Assume that the pair (K, κ) also satisfies the required universal property. Then there exist continuous homomorphisms $\bar{\iota} : K \longrightarrow G_{\hat{\mathcal{C}}}$ and $\bar{\kappa} : G_{\hat{\mathcal{C}}} \longrightarrow K$ such that $\bar{\iota}\kappa = \iota$ and $\bar{\kappa}\iota = \kappa$. Since $\iota(G)$ and $\kappa(G)$ are dense in $G_{\hat{\mathcal{C}}}$ and K, respectively, it follows that $\bar{\iota}\bar{\kappa}$ and $\bar{\kappa}\bar{\iota}$ are the identity maps on $G_{\hat{\mathcal{C}}}$ and K, respectively. Therefore $\bar{\iota}$ is a continuous isomorphism.

The last statement of the lemma is clear from the construction of $\bar{\varphi}$ in the first part of the proof. □

Proposition 3.2.2 *Let $\mathcal{C}$ be a formation and assume that G is a residually $\mathcal{C}$ group. Identify G with its image in its pro-$\mathcal{C}$ completion $G_{\hat{\mathcal{C}}}$. Let $\bar{X}$ denote the closure in $G_{\hat{\mathcal{C}}}$ of a subset X of G.*

(a) *Let*

$$\Phi : \{N \mid N \leq_o G\} \longrightarrow \{U \mid U \leq_o G_{\hat{\mathcal{C}}}\}$$

be the mapping that assigns to each open subgroup H of G its closure $\bar{H}$ in $G_{\hat{\mathcal{C}}}$. Then Φ is a one-to-one correspondence between the set of all open subgroups H in the pro-$\mathcal{C}$ topology of G and the set of all open subgroups of $G_{\hat{\mathcal{C}}}$. The inverse of this mapping is

$$U \longmapsto U \cap G;$$

in particular, $\overline{U \cap G} = U$ if $U \leq_o G_{\hat{\mathcal{C}}}$.

(b) *The map Φ sends normal subgroups to normal subgroups.*

(c) *The topology of $G_{\hat{\mathcal{C}}}$ induces on G its full pro-$\mathcal{C}$ topology.*

(d) *If $H, K \in \{N \mid N \leq_o G\}$ and $H \leq K$, then $[K : H] = [\bar{K} : \bar{H}]$; moreover, if in addition $H \triangleleft K$, then $K/H \cong \bar{K}/\bar{H}$.*

(e) *Φ is an isomorphism of lattices, i.e., if $H, K \in \{N \mid N \leq_o G\}$, then $\overline{H \cap K} = \bar{H} \cap \bar{K}$ and $\overline{\langle H, K\rangle} = \langle \bar{H}, \bar{K}\rangle$.*

Proof. Denote by $\mathcal{N}$, as usual, the collection of all open normal subgroups of G in its pro-$\mathcal{C}$ topology, i.e., the collection of those normal subgroups of G such that $G/N \in \mathcal{C}$.

(a) Let U be an open subgroup of $G_{\hat{\mathcal{C}}}$. Since G is dense in $G_{\hat{\mathcal{C}}}$, it follows that $G \cap U$ is dense in U. Hence $\overline{U \cap G} = U$. Conversely, assume that H is an open subgroup of G (in the pro-$\mathcal{C}$ topology of G). We must show that $H = G \cap \overline{H}$; plainly, $H \leq G \cap \overline{H}$. Let $g \in G \cap \overline{H}$. Recall that G is embedded in $G_{\hat{\mathcal{C}}}$ via the identification

$$g \mapsto (gN) \in G_{\hat{\mathcal{C}}} = \varprojlim_{\mathcal{N}} G/N.$$

Now, according to Corollary 1.1.8,

$$\overline{H} = \varprojlim_{N \in \mathcal{N}} HN/N.$$

So $g \in HN$ for every $N \in \mathcal{N}$. Since $H_G \in \mathcal{N}$, it follows that $g \in HH_G = H$. Thus $H \geq G \cap \overline{H}$, as desired.

(b) If $H \triangleleft G$, then $HN/N \triangleleft G/N$ for each $N \in \mathcal{N}$; hence $\bar{H} \triangleleft G_{\hat{\mathcal{C}}}$. Conversely, if $U \triangleleft_o G_{\hat{\mathcal{C}}}$ then $U \cap G \triangleleft G$; therefore the function Φ maps normal subgroups to normal subgroups.

(c) This follows from (a).

(d) It suffices to show that if $H \in \{N \mid N \leq_o G\}$, then $[G : H] = [G_{\hat{\mathcal{C}}} : \bar{H}]$. Say $n = [G_{\hat{\mathcal{C}}} : \bar{H}]$; since G is dense in $G_{\hat{\mathcal{C}}}$, we deduce that $G\bar{H} = G_{\hat{\mathcal{C}}}$. Let $t_1, \ldots, t_n \in G$ be a left transversal of $\bar{H}$ in $G_{\hat{\mathcal{C}}}$. Then we have a disjoint union

$$G_{\hat{\mathcal{C}}} = t_1\bar{H} \cup \cdots \cup t_n\bar{H}.$$

If $t \in G$, it follows from part (a) that $t\bar{H} \cap G = tH$; therefore,

$$G = (t_1\overline{H} \cup \cdots \cup t_n\overline{H}) \cap G = t_1H \cup \cdots \cup t_nH;$$

thus $n = [G : H]$.

Now, if $H \triangleleft K$ and $H, K \in \{N \mid N \leq_o G\}$, the natural homomorphism $K \longrightarrow \bar{K}/\bar{H}$ has kernel $K \cap \bar{H} = H$. From $[\bar{K} : \bar{H}] = [K : H]$, we infer that the induced homomorphism $K/H \longrightarrow \bar{K}/\bar{H}$ is an isomorphism.

(e) This follows from (a) and (d). □

The Completion Functor

Let $\varphi : G \longrightarrow H$ be a group homomorphism. We wish to define canonically a corresponding continuous homomorphism

$$G_{\hat{\mathcal{C}}} \longrightarrow H_{\hat{\mathcal{C}}},$$

whenever possible. The idea is to define compatible continuous homomorphisms $G \longrightarrow H/N$ ($N \in \mathcal{N}_{\mathcal{C}}(H)$), and then use Lemma 3.2.1. We shall do this in a completely explicit manner.

Consider the collection $\mathcal{M} = \{\varphi^{-1}(N) \mid N \in \mathcal{N}_{\mathcal{C}}(H)\}$ of normal subgroups of G. Clearly $\mathcal{M}$ is filtered from below. Assume that

$$\varphi^{-1}(N) \in \mathcal{N}_{\mathcal{C}}(G) \quad \text{for all } N \in \mathcal{N}_{\mathcal{C}}(H). \tag{3}$$

Note that this is the case if, for example, one of the following conditions is satisfied

– $\mathcal{C}$ is a variety of finite groups;
– $\mathcal{C}$ is a formation of finite groups and φ is an epimorphism;
– $\mathcal{C}$ is a formation of finite groups closed under taking normal subgroups, and $\varphi(G) \triangleleft H$.

Then $\mathcal{M}$ determines a pro-$\mathcal{C}$ topology on G. For each $N \in \mathcal{N}_{\mathcal{C}}(H)$ one has a composition of natural homomorphisms

$$\mathcal{K}_{\mathcal{M}}(G) \longrightarrow G/\varphi^{-1}(N) \longrightarrow \varphi(G)/N \cap \varphi(G) \hookrightarrow H/N.$$

These maps, in turn, induce continuous homomorphisms

$$\mathcal{K}_{\mathcal{M}}(G) \stackrel{\varphi_1}{\to} \varprojlim_{N \in \mathcal{N}} G/\varphi^{-1}(N) \stackrel{\varphi_2}{\to} \varprojlim_{N \in \mathcal{N}} \varphi(G)/N \cap \varphi(G) \stackrel{\varphi_3}{\to} \varprojlim_{N \in \mathcal{N}} H/N = H_{\hat{\mathcal{C}}},$$

where $\mathcal{N} = \mathcal{N}_{\mathcal{C}}(H)$, φ_1 is an epimorphism, φ_2 an isomorphism, and φ_3 an inclusion (see Proposition 2.2.4). On the other hand, since $\mathcal{M}$ is a subset of $\mathcal{N}_{\mathcal{C}}(G)$, there exists an epimorphism $G_{\hat{\mathcal{C}}} \longrightarrow \mathcal{K}_{\mathcal{M}}(G)$ as indicated above. Define

$$\varphi_{\hat{\mathcal{C}}} = \mathcal{K}_{\mathcal{C}}(\varphi) : G_{\hat{\mathcal{C}}} \longrightarrow H_{\hat{\mathcal{C}}}$$

to be the composition homomorphism

$$G_{\hat{\mathcal{C}}} \longrightarrow \mathcal{K}_{\mathcal{M}}(G) \longrightarrow H_{\hat{\mathcal{C}}}.$$

From now on, whenever we write $\varphi_{\hat{\mathcal{C}}}$, it is assumed that this map is defined, i.e., that condition (3) is satisfied.

It is plain that if $\mathrm{id} : G \longrightarrow G$ is the identity homomorphism, then $\mathrm{id}_{\hat{\mathcal{C}}} : G_{\hat{\mathcal{C}}} \longrightarrow G_{\hat{\mathcal{C}}}$ is the identity homomorphism. Furthermore, if $\varphi : G \longrightarrow H$ and $\psi : H \longrightarrow K$ are group homomorphisms, then $(\psi\varphi)_{\hat{\mathcal{C}}} = \psi_{\hat{\mathcal{C}}}\varphi_{\hat{\mathcal{C}}}$, whenever the maps $(\psi\varphi)_{\hat{\mathcal{C}}}$, $\psi_{\hat{\mathcal{C}}}$ and $\varphi_{\hat{\mathcal{C}}}$ are defined. Therefore we have, in particular,

Lemma 3.2.3 *Let $\mathcal{C}$ be a variety of finite groups. Then, pro-$\mathcal{C}$ completion $(-)_{\hat{\mathcal{C}}}$ is a functor from the category of abstract groups to the category of pro-$\mathcal{C}$ groups and continuous homomorphisms.*

Let $\varphi : G \longrightarrow H$ be a group homomorphism. It follows from the definition of $\varphi_{\hat{\mathcal{C}}}$ that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \downarrow{\iota} & & \downarrow{\iota} \\ G_{\hat{\mathcal{C}}} & \xrightarrow[\varphi_{\hat{\mathcal{C}}}]{} & H_{\hat{\mathcal{C}}} \end{array}$$

commutes. Since $\iota(H)$ is dense in $H_{\hat{\mathcal{C}}}$, one deduces that $(\iota\varphi)(G)$ is dense in $\varphi_{\hat{\mathcal{C}}}(G_{\hat{\mathcal{C}}})$. On the other hand $\varphi_{\hat{\mathcal{C}}}(G_{\hat{\mathcal{C}}})$ is closed by the compactness of $G_{\hat{\mathcal{C}}}$. Therefore, $\varphi_{\hat{\mathcal{C}}}(G_{\hat{\mathcal{C}}})$ is the closure of $(\iota\varphi)(G)$ in $H_{\hat{\mathcal{C}}}$. We record this in the following lemma.

Lemma 3.2.4 *Let $\mathcal{C}$ be a formation of finite groups. Let $\varphi : G \longrightarrow H$ be a homomorphism of groups and assume that $\varphi_{\hat{\mathcal{C}}} : G_{\hat{\mathcal{C}}} \longrightarrow H_{\hat{\mathcal{C}}}$ is defined. Then*

$$\varphi_{\hat{\mathcal{C}}}(G_{\hat{\mathcal{C}}}) = \overline{(\iota\varphi)(G)},$$

where $\overline{(\iota\varphi)(G)}$ denotes the closure of $(\iota\varphi)(G)$ in $H_{\hat{\mathcal{C}}}$.

Proposition 3.2.5 *Let $\mathcal{C}$ be a formation of finite groups closed under taking normal subgroups. Then the functor $(-)_{\hat{\mathcal{C}}}$ is right exact, that is, if*

$$1 \longrightarrow K \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 1$$

is an exact sequence of groups, then

$$K_{\hat{\mathcal{C}}} \xrightarrow{\varphi_{\hat{\mathcal{C}}}} G_{\hat{\mathcal{C}}} \xrightarrow{\psi_{\hat{\mathcal{C}}}} H_{\hat{\mathcal{C}}} \longrightarrow 1$$

is an exact sequence of pro-$\mathcal{C}$ groups.

Proof. Let $\mathcal{N} = \mathcal{N}_{\mathcal{C}}(G)$. Then we get in a natural way a corresponding exact sequence of inverse systems (indexed by $\mathcal{N}$)

$$\{K/\varphi^{-1}(N) \mid N \in \mathcal{N}\} \xrightarrow{\tilde{\varphi}} \{G/N \mid N \in \mathcal{N}\} \xrightarrow{\tilde{\psi}} \{H/\psi(N) \mid N \in \mathcal{N}\} \longrightarrow 1.$$

Observe that

$$\varprojlim_{N\in\mathcal{N}} G/N = G_{\hat{\mathcal{C}}}, \qquad \varprojlim_{N\in\mathcal{N}} H/\psi(N) = H_{\hat{\mathcal{C}}}, \quad \text{and} \quad \varprojlim \tilde{\psi} = \psi_{\hat{\mathcal{C}}}.$$

On the other hand, $\varphi_{\hat{\mathcal{C}}}$ is the composition of the epimorphism

$$K_{\hat{\mathcal{C}}} \longrightarrow \varprojlim_{N\in\mathcal{N}} K/\varphi^{-1}(N)$$

and $\varprojlim \tilde{\varphi}$. Our result follows now from the exactness of the functor $\varprojlim$ (see Proposition 2.2.4). □

A necessary and sufficient condition for the completion functor $(-)_{\hat{\mathcal{C}}}$ to preserve an injection $\iota : K \longrightarrow G$ is stated in the next lemma.

Lemma 3.2.6 *Let $\mathcal{C}$ be a variety (respectively, a formation closed under taking normal subgroups) of finite groups. Assume that $K \leq G$ (respectively, $K \triangleleft G$), and let $\iota : K \longrightarrow G$ denote the inclusion map. Then*

$$\iota_{\hat{\mathcal{C}}} : K_{\hat{\mathcal{C}}} \longrightarrow G_{\hat{\mathcal{C}}}$$

is injective if and only if the pro-$\mathcal{C}$ topology of G induces on K its full pro-$\mathcal{C}$ topology.

Proof. Let $N \triangleleft_f G$ be such that $G/N \in \mathcal{C}$. Then $K/K \cap N \in \mathcal{C}$. Therefore, there exists a natural epimorphism $K_{\hat{\mathcal{C}}} \longrightarrow K/K \cap N$. The map $\iota_{\hat{\mathcal{C}}}$ is the composition

$$K_{\hat{\mathcal{C}}} \longrightarrow \varprojlim_{N \in \mathcal{N}_{\mathcal{C}}(G)} K/K \cap N \longrightarrow \varprojlim_{N \in \mathcal{N}_{\mathcal{C}}(G)} G/N = G_{\hat{\mathcal{C}}}.$$

The map on the right is always an injection. Hence $\iota_{\hat{\mathcal{C}}}$ is an injection if and only if the epimorphism

$$\rho : K_{\hat{\mathcal{C}}} \longrightarrow \varprojlim_{N \in \mathcal{N}_{\mathcal{C}}(G)} K/K \cap N$$

is injective, i.e., an isomorphism. If the pro-$\mathcal{C}$ topology of G induces on K its full pro-$\mathcal{C}$ topology, then the collection of normal subgroups

$$\{K \cap N \mid N \in \mathcal{N}_{\mathcal{C}}(G)\}$$

is cofinal in $\mathcal{N}_{\mathcal{C}}(K)$; hence ρ is an isomorphism (see Lemma 1.1.9). Conversely, if ρ is an isomorphism, then $\{K \cap N \mid N \in \mathcal{N}_{\mathcal{C}}(G)\}$ is a fundamental system of neighborhoods of 1 in K (see Lemma 2.1.1); in other words, the pro-$\mathcal{C}$ topology of G induces on K its full pro-$\mathcal{C}$ topology. □

In the next result, we indicate how possibly different groups could have the same completions.

Theorem 3.2.7 *Let $\mathcal{C}$ be a formation of finite groups. Let G_1, G_2 be groups. Denote by $\mathcal{U}_i$ the collection of all normal subgroups U of G_i with $G_i/U \in \mathcal{C}$ $(i = 1, 2)$. Assume that*

(a) *For each natural number n, there exist only finitely many $U \in \mathcal{U}_i$ such that $[G_i : U] \leq n$; and*
(b) $\{G_1/U \mid U \in \mathcal{U}_1\} = \{G_2/V \mid V \in \mathcal{U}_2\}$.

Then

$$\varprojlim_{U \in \mathcal{U}_1} G_1/U \cong \varprojlim_{V \in \mathcal{U}_2} G_2/V.$$

Proof. For each $n \in \mathbf{N}$, let

$$U_n = \bigcap \{U \mid U \in \mathcal{U}_1, [G_1 : U] \leq n\} \quad \text{and}$$
$$V_n = \bigcap \{U \mid U \in \mathcal{U}_2, [G_2 : U] \leq n\}.$$

Then $U_n \in \mathcal{U}_1$ and $V_n \in \mathcal{U}_2$. So there exists some $K \in \mathcal{U}_1$ with $G_1/K \cong G_2/V_n$. It follows from (b) that K is the intersection of groups $U \in \mathcal{U}_1$ with $[G : U] \leq n$; therefore $K \geq U_n$. Hence, $|G_1/U_n| \geq |G_2/V_n|$. By symmetry $|G_1/U_n| \leq |G_2/V_n|$. Thus $G_1/U_n \cong G_2/V_n$. Let X_n be the set of all isomorphisms from G_1/U_n to G_2/V_n. Observe that if $\sigma_{n+1} \in X_{n+1}$, then $\sigma(U_n/U_{n+1}) = V_n/V_{n+1}$; hence σ_{n+1} induces an isomorphism

$$\sigma_n : G_1/U_n \longrightarrow G_2/V_n.$$

Denote by

$$\varphi_{n+1,n} : X_{n+1} \longrightarrow X_n$$

the map defined by $\sigma_{n+1} \mapsto \sigma_n$. Then $\{X_n, \varphi_{n+1,n}\}$ is an inverse system of finite nonempty sets. Hence there exists some $(\sigma_n) \in \varprojlim X_n$ (see Proposition 1.1.4). On the other hand,

$$\{G_1/U_n\}_{n=1}^{\infty} \quad \text{and} \quad \{G_2/V_n\}_{n=1}^{\infty}$$

are in a natural way inverse systems of groups; furthermore, $\{\sigma_n\}_{n=1}^{\infty}$ is an isomorphism of these systems. Finally, it follows from Lemma 1.1.9 that

$$\varprojlim_{U \in \mathcal{U}_1} G_1/U \cong \varprojlim_{n} G_1/U_n \cong \varprojlim_{n} G_2/V_n \cong \varprojlim_{V \in \mathcal{U}_2} G_2/V$$

since $\{G_1/U_n\}_{n=1}^{\infty}$ and $\{G_2/V_n\}_{n=1}^{\infty}$ are cofinal subsystems of $\{G_1/U \mid U \in \mathcal{U}_1\}$ and $\{G_2/V \mid V \in \mathcal{U}_2\}$, respectively. □

Corollary 3.2.8 *Let G_1, G_2 be finitely generated abstract groups with the same finite quotients, then $\widehat{G}_1 \cong \widehat{G}_2$.*

Using a slight variation of the argument in Theorem 3.2.7, we obtain

Theorem 3.2.9 *Let G_1 be a finitely generated profinite group and let G_2 be any profinite group. Assume that G_1 and G_2 have the same finite quotients, i.e., $\{G_1/U \mid U \triangleleft_o G_1\} = \{G_2/V \mid V \triangleleft_o G_2\}$. Then $G_1 \cong G_2$.*

3.3 Free Pro-$\mathcal{C}$ Groups

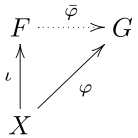
Unless otherwise specified, throughout this section $\mathcal{C}$ denotes a formation of finite groups, i.e., we assume that $\mathcal{C}$ is a class of finite groups closed under taking quotient groups and finite subdirect products; moreover, we assume that $\mathcal{C}$ contains a group of order at least two.

A topological space X with a distinguished point $*$ is called a *pointed space*. We shall denote such a space by $(X, *)$. Sometimes it is convenient to think of a profinite group as a pointed space with distinguished point 1. A mapping of pointed spaces

$$\varphi : (X, *) \longrightarrow (X', *')$$

is simply a continuous mapping from X into X' such that $\varphi(*) = *'$.

Let X be a profinite space, F a pro-$\mathcal{C}$ group and $\iota : X \longrightarrow F$ a continuous mapping such that $F = \overline{\langle \iota(X) \rangle}$. We say that (F, ι) is a *free pro-$\mathcal{C}$ group* on the profinite space X or, simply, F is a free pro-$\mathcal{C}$ group on X, if the following universal property is satisfied:



whenever $\varphi : X \longrightarrow G$ is a continuous mapping into a pro-$\mathcal{C}$ group G such that $\varphi(X)$ generates G, then there exists a (necessarily unique) continuous homomorphism $\bar{\varphi} : F \longrightarrow G$ such that the above diagram commutes: $\bar{\varphi}\iota = \varphi$.

One defines a *free pro-$\mathcal{C}$ group* on a pointed profinite space $(X, *)$ in an analogous manner: one simply assumes in the description of the universal property that the maps involved are maps of pointed spaces.

Note that if the profinite space X is empty, then a free pro-$\mathcal{C}$ group on X must be the trivial group. If X contains exactly one element and $\mathcal{C}$ does not contain nontrivial cyclic groups, then the free pro-$\mathcal{C}$ group on the profinite space X is the trivial group. Similarly, if a profinite pointed space $(X, *)$ contains exactly one point, then free pro-$\mathcal{C}$ group on the pointed space $(X, *)$ is the trivial group. If $(X, *)$ has exactly two points and $\mathcal{C}$ does not contain nontrivial cyclic groups, then a free pro-$\mathcal{C}$ group on the pointed space $(X, *)$ is the trivial group.

To avoid trivial counterexamples to some of the statements in this chapter, from now on we shall tacitly assume that if $\mathcal{C}$ does not contain nontrivial cyclic groups, then we only consider free pro-$\mathcal{C}$ groups on profinite spaces X that are either empty or of cardinality at least 2 (*respectively, we only consider free pro-$\mathcal{C}$ groups on profinite pointed spaces* $(X, *)$ *such that either* $|X| = 1$ *or* $|X| \geq 3$).

Observe that one needs to test the universal property in the definition of free pro-$\mathcal{C}$ groups only for finite groups G in $\mathcal{C}$, for then it holds automatically for any pro-$\mathcal{C}$ group G, since G is an inverse limit of groups in $\mathcal{C}$.

From the universal definition, one deduces in a standard manner that if a free pro-$\mathcal{C}$ group exists, then it is unique. We shall denote the free pro-$\mathcal{C}$ group on a profinite space X by $F_{\mathcal{C}}(X)$, and the free pro-$\mathcal{C}$ group on a pointed profinite space $(X, *)$ by $F_{\mathcal{C}}(X, *)$.

Lemma 3.3.1 *Let* (F, ι) *be a free pro-$\mathcal{C}$ group on the profinite space* X (*respectively, a free pro-$\mathcal{C}$ group on the pointed profinite space* $(X, *)$), *then the mapping* ι *is an injection and* $1 \notin \iota(X)$ (*respectively,* ι *is an injection*).

Proof. We give a proof for the nonpointed case. If $X = \{x\}$ has cardinality 1, then, by our standing assumptions, there exists a nontrivial finite cyclic group $\langle a \rangle \in \mathcal{C}$. Let $\varphi : X \longrightarrow \langle a \rangle$ be given by $\varphi(x) = a$. Let $\bar{\varphi} : F \longrightarrow \langle a \rangle$ be the continuous homomorphism such that $\varphi(\iota(x)) = a$. It follows that $\iota(x) \neq 1$. Assume now that $|X| \geq 2$. Consider the set $\mathcal{R}$ of all open equivalence relations R on X. According to Theorem 1.1.12, the clopen subsets of X form a base for the topology of X. Therefore, if $x \neq y$ are points of X, there exists $R \in \mathcal{R}$ such that $xR \neq yR$. Let $G \in \mathcal{C}$ be generated by two distinct nontrivial elements, say, a and b (such a group exists: indeed, let $H \in \mathcal{C}$ be a nontrivial group; let S be a quotient of H such that S is a simple group; if S is nonabelian, then it is a two generator group, by the classification of finite simple groups, and then put $G = S$; while if S is cyclic, take $G = S \times S$). Consider the continuous mapping

$$\psi : X \xrightarrow{\psi_R} X/R \xrightarrow{\rho} G$$

where ψ_R is the canonical quotient map, and ρ any map such that $\rho(xR) = a$ and $\rho(yR) = b$. Since ψ is continuous, there exists a corresponding continuous homomorphism $\bar{\psi} : F \longrightarrow G$ such that $\bar{\psi}\iota = \psi$. It follows that $1 \neq \iota(x) \neq \iota(y) \neq 1$, and so ι is one-to-one and $1 \notin \iota(X)$. $\square$

Next we show the existence of free pro-$\mathcal{C}$ groups.

Proposition 3.3.2 *For every profinite space X (respectively, pointed profinite space $(X, *)$), there exists a unique free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X)$ on X (respectively, there exists a unique free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X, *)$ on the pointed profinite space $(X, *)$).*

Proof. We leave the uniqueness to the reader. For the construction of $F_{\mathcal{C}}(X)$, let D be the abstract free group on the set X. Consider the following collection of subgroups of D

$$\mathcal{N} = \{N \triangleleft D \mid D/N \in \mathcal{C}; X \cap dN \text{ open in } X, \forall d \in D\}.$$

Observe that $\mathcal{N}$ is nonempty and filtered from below. Define $F_{\mathcal{C}}(X)$ to be the completion of D with respect to $\mathcal{N}$

$$F_{\mathcal{C}}(X) = \varprojlim_{N \in \mathcal{N}} D/N.$$

Let $\iota : X \longrightarrow F_{\mathcal{C}}(X)$ be the restriction to X of the natural homomorphism $D \longrightarrow F_{\mathcal{C}}(X)$. Remark that the composition of ι with each projection $F_{\mathcal{C}}(X) \longrightarrow D/N$, $N \in \mathcal{N}$, is continuous, and hence, so is ι. Next we show that $(F_{\mathcal{C}}(X), \iota)$ is a free pro-$\mathcal{C}$ group on X. Indeed, let $G \in \mathcal{C}$ and let $\varphi : X \longrightarrow G$ be a continuous map such that $G = \langle \varphi(X) \rangle$. Since D is a free abstract group on X, there exists a homomorphism (of abstract groups) $\varphi_1 : D \longrightarrow G$ that extends φ. In fact φ_1 is an epimorphism. Put $K = \mathrm{Ker}(\varphi_1)$. Then $K \in \mathcal{N}$. Therefore, we have a continuous homomorphism

$$\bar{\varphi} : F_{\mathcal{C}}(X) \longrightarrow D/K \longrightarrow G.$$

Then $\bar{\varphi}\iota = \varphi$.

The construction of $F_{\mathcal{C}}(X,*)$ is as follows: let $\tilde{D}$ be the abstract free group on the set $X - \{*\}$, and let

$$\tilde{\mathcal{N}} = \{N \triangleleft \tilde{D} \mid \tilde{D}/N \in \mathcal{C}; (X - \{*\}) \cap dN \text{ open in } X - \{*\}, \forall d \in \tilde{D}\}.$$

Put

$$F_{\mathcal{C}}(X,*) = \varprojlim_{N \in \tilde{\mathcal{N}}} \tilde{D}/N.$$

Then one checks as above that $(F_{\mathcal{C}}(X,*), \iota)$ satisfies the universal property of a free pro-$\mathcal{C}$ group on the pointed profinite space $(X,*)$. □

We shall refer to the profinite space X (respectively, $(X,*)$) as a *topological basis* of $F_{\mathcal{C}}(X)$ (respectively, of $F_{\mathcal{C}}(X,*)$).

If X is a profinite space, one can associate with it a pointed profinite space $(X \cup \{*\}, *)$, by simply adding to X a new point $*$ and endowing $X \cup \{*\}$ with the coproduct topology, i.e., $*$ is an isolated point in $X \cup \{*\}$ and a subset Y of $X \cup \{*\}$ is open if and only if $Y \cap X$ is open in X. Then one easily sees that $F_{\mathcal{C}}(X) = F_{\mathcal{C}}(X \cup \{*\}, *)$. Thus, we can think of a free pro-$\mathcal{C}$ group on a profinite space as particular instance of a free pro-$\mathcal{C}$ group on a pointed profinite space.

Exercise 3.3.3 Let $(X,*)$ be a pointed topological space, not necessarily profinite.

(a) Mimic the definition above to establish the concept of a free pro-$\mathcal{C}$ group $(F_{\mathcal{C}}(X,*), \iota)$ on the pointed space $(X,*)$. As a special case of the above definition, explain the concept of free pro-$\mathcal{C}$ group $(F_{\mathcal{C}}(X), \iota)$ on a topological space X.

(b) Define

$$(\check{X},*) = \varprojlim_{R \in \mathcal{R}} (X,*)/R,$$

where $\mathcal{R}$ is the collection of all closed equivalence relations R of X such that the quotient pointed space $(X,*)/R$ is finite and Hausdorff. Let $\tau : X \longrightarrow \check{X}$ be the natural mapping. Show that there exists a unique continuous mapping of pointed spaces $\tilde{\iota} : (\check{X},*) \longrightarrow F_{\mathcal{C}}(X,*)$ such that $\iota = \tilde{\iota}\tau$.

(c) Prove that $|\mathcal{R}| = \rho(\check{X})$, the cardinality of the collection of all clopen subsets of $\check{X}$.

(d) Show that $F_{\mathcal{C}}(X,*)$ is a free pro-$\mathcal{C}$ group on a pointed profinite space; specifically, prove that $(F_{\mathcal{C}}(X,*), \tilde{\iota})$ is the free pro-$\mathcal{C}$ group on the pointed profinite space $(\check{X},*)$.

Free Pro-$\mathcal{C}$ Group on a Set Converging to 1

If X is a set, we say that a map $\mu : X \longrightarrow G$ from X to a profinite group G *converges to* 1 if the subset $\mu(X)$ of G converges to 1, that is, if every open subgroup U of G contains all but a finite number of the elements of $\mu(X)$.

Assume now X to be a set, which we wish to view as a topological space with the discrete topology. Let $\bar{X} = X \mathbin{\dot\cup} \{*\}$ denote its one-point compactification (recall that, by definition, a subset T is open in $\bar{X}$ if either it is contained in X or $\{*\} \in T$ and $X - T$ is a finite set; see, e.g., Bourbaki [1989], I,9,8). Then $X \mathbin{\dot\cup} \{*\}$ is a profinite space. Observe that if X is a set and $X \mathbin{\dot\cup} \{*\}$ is its one-point compactification, then the map

$$X \hookrightarrow X \mathbin{\dot\cup} \{*\} \stackrel{\iota}{\longrightarrow} F_{\mathcal{C}}(X \mathbin{\dot\cup} \{*\}, *)$$

converges to 1. We shall still denote this map by ι.

To avoid trivial cases, from now on we shall assume that if $\mathcal{C}$ does not contain nontrivial cyclic groups, then $|X| \neq 2$.

Then (see Lemma 3.3.1) ι is a topological embedding, and we identify X with its image in $F_{\mathcal{C}}(X \mathbin{\dot\cup} \{*\}, *)$. The free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X \mathbin{\dot\cup} \{*\}, *)$ on this pointed space $(X \mathbin{\dot\cup} \{*\}, *)$ plays a special role because, as we shall see later (Proposition 3.5.12), every free pro-$\mathcal{C}$ group on a (pointed) topological space is in fact a free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X \mathbin{\dot\cup} \{*\}, *)$ on the one-point compactification space $(X \mathbin{\dot\cup} \{*\}, *)$ of some set X.

Let X be a set. By abuse of notation, we denote the free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X \mathbin{\dot\cup} \{*\}, *)$ on the one-point compactification space $(X \mathbin{\dot\cup} \{*\}, *)$ of X, as $F_{\mathcal{C}}(X)$ rather than $F_{\mathcal{C}}(X \mathbin{\dot\cup} \{*\}, *)$. To avoid confusion, if X is a set, we refer to $F_{\mathcal{C}}(X)$ in that case as the *free pro-$\mathcal{C}$ group on the set X converging to* 1.* If, on the other hand, X (respectively, $(X, *)$) is a profinite space (respectively, a pointed profinite space), then $F_{\mathcal{C}}(X)$ (respectively, $F_{\mathcal{C}}(X, *)$) has a unique possible meaning, and we refer to it as the free pro-$\mathcal{C}$ group on X or on the space X (respectively, the free pro-$\mathcal{C}$ group on $(X, *)$ or on the pointed space $(X, *)$). If X is a finite subset of a profinite group, then X converges to 1; so in this case the meaning of $F_{\mathcal{C}}(X)$ is unambiguous, and we refer to it as the free pro-$\mathcal{C}$ group on X.

The following lemma gives a characterization of the free group on a set converging to 1 in terms of a universal property. We leave its easy proof to the reader (it follows immediately from the definition of free pro-$\mathcal{C}$ group on a pointed space in the special case where the pointed space is the one-point compactification of a discrete space).

Lemma 3.3.4 *The following properties characterize the free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X)$ on the set X converging to* 1:

(a) *$F_{\mathcal{C}}(X)$ contains the set X as a subset converging to* 1, *and*
(b) *Whenever $\mu : X \longrightarrow G$ is a map converging to 1 of X into a pro-$\mathcal{C}$ group G and $\mu(X)$ is a set of generators of G, then there exists a unique homomorphism $\bar{\mu} : F_{\mathcal{C}}(X) \longrightarrow G$ that extends μ.*

* Some authors refer to what we call the free pro-$\mathcal{C}$ group on the set X converging to 1 as a *restricted free pro-$\mathcal{C}$ group on the set X*, and they denote it by $F^r_{\mathcal{C}}(X)$.

We shall refer to the subset X of $F_{\mathcal{C}}(X)$ as a *basis converging to* 1 or simply as a *basis* of the free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X)$. As we have indicated before, we shall prove later (see Proposition 3.5.12) that every free pro-$\mathcal{C}$ group on a topological space (or a pointed topological space) is in fact also a free pro-$\mathcal{C}$ group on a set converging to 1. So from now on in this book the word "basis" for a free pro-$\mathcal{C}$ group will be used only in the sense of being a basis converging to 1 of a free pro-$\mathcal{C}$ group. Any other type of basis will be qualified, for example "topological basis".

Lemma 3.3.5

(a) *Let $F = F_{\mathcal{C}}(X)$ be a free pro-$\mathcal{C}$ group on a set X converging to* 1. *If F is also free pro-$\mathcal{C}$ on a set Y converging to* 1, *then the bases X and Y have the same cardinality.*
(b) *Let F be a free pro-$\mathcal{C}$ group on a finite set $X = \{x_1, \dots, x_n\}$. Then, any set of generators $\{y_1, \dots, y_n\}$ of F with n elements is a basis of F.*

Proof. (a) Say X and Y are two bases of F. If both X and Y are infinite, the result follows from Proposition 2.6.2. Say that $X = \{x_1, \dots, x_n\}$ is finite and assume that $|Y| > n$. We show that this is not possible. Indeed, choose a subset $X' = \{x_1', \dots, x_n'\}$ of Y, and define a map $\mu : Y \longrightarrow F$ by $\mu(x_i') = x_i$ $(i = 1, \dots, n)$ and $\mu(y) = 1$ if $y \in Y - X'$. Since μ converges to 1, it extends to a continuous epimorphism $\bar{\mu} : F \longrightarrow F$; then, by Proposition 2.5.2, $\bar{\mu}$ is an isomorphism, a contradiction.

(b) Consider the continuous epimorphism $\psi : F \longrightarrow F$ determined by $\psi(x_i) = y_i$ $(i = 1, \dots, n)$. Then ψ is an isomorphism by Proposition 2.5.2. □

If $F = F_{\mathcal{C}}(X)$ is a free pro-$\mathcal{C}$ group on the set X converging to 1, the *rank* of F is defined to be the cardinality of X. It is denoted by $\mathrm{rank}(F)$. Given a cardinal number $\mathfrak{m}$, we denote by $F_{\mathcal{C}}(\mathfrak{m})$ or $F(\mathfrak{m})$ a free pro-$\mathcal{C}$ group (on a set converging to 1) of rank $\mathfrak{m}$.

We state the next result for easy reference. It follows immediately from the definition of rank given above and the construction of free pro-$\mathcal{C}$ groups in the proof of Proposition 3.3.2.

Proposition 3.3.6 *Let Φ be an abstract free group on a finite basis X. Then the pro-$\mathcal{C}$ completion $\Phi_{\hat{\mathcal{C}}}$ of Φ is a free pro-$\mathcal{C}$ group on X. In particular,* $\mathrm{rank}(\Phi) = \mathrm{rank}(\Phi_{\hat{\mathcal{C}}})$.

Exercise 3.3.7 Show that if $F = F_{\mathcal{C}}(X, *)$ is the free pro-$\mathcal{C}$ group on the pointed profinite space $(X, *)$ and F is finitely generated, then $|X|$ is finite, and F is the free pro-$\mathcal{C}$ group of rank $|X| - 1$.

Example 3.3.8

(a) The free profinite group of rank 1 is $\widehat{\mathbf{Z}}$. Observe that $\widehat{\mathbf{Z}}$ is the free prosolvable (or proabelian, pronilpotent, etc.) group of rank 1, as well.
(b) If p is a prime number, then $\mathbf{Z}_p$ is the free pro-p group of rank 1.

(c) Let X be any set. Then the free proabelian group on the set X converging to 1 is the direct product $\prod_X \widehat{\mathbf{Z}}$ of copies of $\widehat{\mathbf{Z}}$ indexed by X. The canonical map $\iota : X \longrightarrow \prod_X \widehat{\mathbf{Z}}$ sends $x \in X$ to the tuple $(a_y) \in \prod_X \widehat{\mathbf{Z}}$ such that $a_y = 0$ for $y \neq x$ and $a_x = 1$. One sees this easily. Indeed, if $\varphi : X \longrightarrow A$ is a map converging to 1 onto a finite abelian group A, let Y be a finite subset of X such that $\varphi(x) = 0$ for all $x \in X - Y$. Then $\prod_X \widehat{\mathbf{Z}} = (\bigoplus_Y \widehat{\mathbf{Z}}) \oplus (\prod_{X-Y} \widehat{\mathbf{Z}})$. Define the corresponding continuous homomorphism $\bar{\varphi} : \prod_X \widehat{\mathbf{Z}} \longrightarrow A$ to be 0 on $\prod_{X-Y} \widehat{\mathbf{Z}}$, and the natural extension homomorphism on the finite indexed direct sum $\bigoplus_Y \widehat{\mathbf{Z}}$.

(d) Similarly, let $\mathcal{C}$ be the class of all finite abelian groups of exponent p, where p is a prime. Then the free pro-$\mathcal{C}$ group on the set X converging to 1 is the direct product $\prod_X \mathbf{Z}/p\mathbf{Z}$ of copies of $\mathbf{Z}/p\mathbf{Z}$ indexed by X.

(e) (cf. Douady, Harbater [1964, 1995]; see also Ribes [1970], p. 70; van den Dries and Ribenboim [1986]) Let F be an algebraically closed field, and denote by $\overline{F(t)}$ the algebraic closure of the field $F(t)$, where t is an indeterminate. Then the Galois group $G_{\overline{F(t)}/F(t)}$ is a free profinite group on a set converging to 1 of rank $|F|$.

Proposition 3.3.9 *Let $(X, *)$ be a pointed profinite space.*

(a) *Assume that*

$$(X, *) = \varprojlim_{i \in I} (X_i, *),$$

where $\{(X_i, *), \psi_{ij}\}$ is an inverse system of pointed profinite spaces. Then

$$F = F_{\mathcal{C}}(X, *) = \varprojlim_{i \in I} F_{\mathcal{C}}(X_i, *).$$

(b)

$$F = F_{\mathcal{C}}(X, *) = \varprojlim_{i \in I} F_{\mathcal{C}}(Y_i),$$

where each Y_i is a finite space, and $(X, *) = \varprojlim_{i \in I} (Y_i \,\dot\cup\, \{*\}, *)$.

Proof. (a) The inverse system $\{(X_i, *), \psi_{ij}\}$ determines an inverse system of free groups $\{F_{\mathcal{C}}(X_i, *), \bar{\psi}_{ij}\}$. For each $i \in I$, denote by $\psi_i : (X, *) \longrightarrow (X_i, *)$ the canonical projection. Correspondingly, one has continuous homomorphisms of groups $\bar{\psi}_i : F_{\mathcal{C}}(X, *) \longrightarrow F_{\mathcal{C}}(X_i, *)$, which are compatible with the mappings $\bar{\psi}_{ij}$. These homomorphisms induce then a continuous homomorphism of groups

$$\psi : F_{\mathcal{C}}(X, *) \longrightarrow G = \varprojlim_{i \in I} F_{\mathcal{C}}(X_i, *).$$

Denote by ι' the restriction of ψ to X; note that ι' is a mapping of pointed spaces. We claim that $\iota'(X)$ generates G as a topological group. To see this consider an epimorphism $\rho : G \longrightarrow H$ where $H \in \mathcal{C}$. It suffices to show that $\rho\iota'(X)$ generates H. By Lemma 1.1.16, ρ factors through $F(X_{i_0})$, for some $i_0 \in I$, i.e., there exists an epimorphism $\rho' : F(X_{i_0}) \longrightarrow H$ such that $\rho = \bar{\psi}_{i_0}\rho'$. Put $Y = \rho'(X_{i_0})$. Since H is finite, i_0 can be chosen so that $Y = \rho'\bar{\psi}_{ii_0}(X_{i_0})$, whenever $i \in I$, $i \geq i_0$. Since

$$(X, *) = \varprojlim_{i \geq i_0} (X_i, *),$$

we deduce that $Y = \rho\iota'(X)$, as needed.

To prove that

$$(\varprojlim_{i \in I} F_{\mathcal{C}}(X_i, *), \iota')$$

is the free pro-$\mathcal{C}$ group on the pointed space $(X, *)$, it remains to show that this pair satisfies the required universal property. Let $\mu : X \longrightarrow H$ be a continuous mapping with $\mu(*) = 1$, where $H \in \mathcal{C}$ and $\mu(X)$ generates H. Since H is finite, there exists some $j \in I$ and a continuous mapping of pointed spaces $\mu_j : (X_j, *) \longrightarrow (H, 1)$ such that $\mu_j\psi_j = \mu$ (see Lemma 1.1.16). Now, μ_j extends to a homomorphism $\bar{\mu}_j : F_{\mathcal{C}}(X_j, *) \longrightarrow H$. Define

$$\bar{\mu} : \varprojlim_{i \in I} F_{\mathcal{C}}(X_i, *) \longrightarrow H$$

by $\bar{\mu} = \bar{\mu}_j\bar{\psi}_j$. Then clearly $\bar{\mu}\iota' = \mu$.

(b) By definition we can express $(X, *)$ as an inverse limit of finite pointed spaces

$$(X, *) = \varprojlim_{i \in I} (X_i, *).$$

Put $Y_i = X_i - \{*\}$. Clearly $F_{\mathcal{C}}(X_i, *) = F_{\mathcal{C}}(Y_i)$. The result follows then from part (a). □

Let X be a set and let $\{X_i \mid i \in I\}$ be the collection of all finite subsets of X. Make I into a poset by defining $i \preceq j$ if $X_i \subseteq X_j$. If $i \preceq j$ define $\varphi_{ji} : F_{\mathcal{C}}(X_j) \longrightarrow F_{\mathcal{C}}(X_i)$ as the epimorphism that carries x to x, if $x \in X_i$, and x to 1, if $x \in X_j - X_i$ $(x \in X)$. Observe that $\varprojlim (X_i \mathbin{\dot\cup} \{1\}, 1)$ is the one-point compactification of X. Then from Proposition 3.3.9 we deduce

Corollary 3.3.10 *Let X be a set and let $\{X_i \mid i \in I\}$ be the collection of all finite subsets X_i of X. Then*

(a) *For each $i \in I$, $F_{\mathcal{C}}(X_i)$ is a closed subgroup of the free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X)$ on the set X converging to* 1;

(b)
$$F_{\mathcal{C}}(X) = \varprojlim_{i \in I} F_{\mathcal{C}}(X_i),$$

where the canonical homomorphism

$$\varphi_i : F_{\mathcal{C}}(X) \longrightarrow F_{\mathcal{C}}(X_i)$$

is the extension of the mapping $X \longrightarrow F_{\mathcal{C}}(X_i)$ *that sends* x *to* x *for* $x \in X_i$, *and* x *to* 1 *for* $x \in X - X_i$ $(x \in X)$.

This corollary can be improved in such a way that for a given open subgroup H of $F_{\mathcal{C}}(X)$, the mappings φ_i preserve the index of H. Before we make this precise, we need the following

Lemma 3.3.11 *Let* $Y \subseteq X$ *be sets and let* $F_{\mathcal{C}}(X)$ *and* $F_{\mathcal{C}}(Y)$ *be the corresponding free pro-*$\mathcal{C}$ *groups on the sets* X *and* Y *converging to* 1, *respectively. Consider the epimorphism*

$$\varphi : F_{\mathcal{C}}(X) \longrightarrow F_{\mathcal{C}}(Y)$$

defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \in Y, \\ 1 & \text{if } x \notin Y. \end{cases}$$

Then the following is a split exact sequence

$$1 \longrightarrow N \longrightarrow F_{\mathcal{C}}(X) \xrightarrow{\varphi} F_{\mathcal{C}}(Y) \longrightarrow 1,$$

where N *is the smallest closed normal subgroup generated by* $X - Y$. (*This means that there is a continuous section of* φ *which is a homomorphism, i.e., that* $F_{\mathcal{C}}(X)$ *is a semidirect product of* N *by a closed subgroup isomorphic to* $F_{\mathcal{C}}(Y)$.)

Proof. Define a continuous homomorphism $\sigma : F_{\mathcal{C}}(Y) \longrightarrow F_{\mathcal{C}}(X)$ by $\sigma(y) = y$, for all $y \in Y$. Then σ is a section of φ. Let $K = \mathrm{Ker}(\varphi)$. After identifying $F_{\mathcal{C}}(Y)$ with $\sigma(F_{\mathcal{C}}(Y))$, we have $F = NF_{\mathcal{C}}(Y) = KF_{\mathcal{C}}(Y)$. Since

$$N \cap F_{\mathcal{C}}(Y) = K \cap F_{\mathcal{C}}(Y) = 1 \quad \text{and} \quad N \leq K,$$

it follows that $N = K$. □

Proposition 3.3.12 *Let* $F_{\mathcal{C}}(X)$ *be a free pro-*$\mathcal{C}$ *group on a set* X *converging to* 1 *and* $H \leq_o F_{\mathcal{C}}(X)$. *Then there is a collection* $\{X_j \mid j \in J\}$ *of finite subsets of* X *such that*

(a) $\{F_{\mathcal{C}}(X_j), \varphi_{jk}, J\}$ *is an inverse system of free pro-*$\mathcal{C}$ *groups, where if* $X_j \supseteq X_k$, *the epimorphism* $\varphi_{jk} : F_{\mathcal{C}}(X_j) \longrightarrow F_{\mathcal{C}}(X_k)$ *is defined by*

$$\varphi_{jk}(x) = \begin{cases} x & \text{if } x \in X_k, \\ 1 & \text{if } x \in X_j - X_k; \end{cases}$$

(b)

$$F_{\mathcal{C}}(X) = \varprojlim_{j \in J} F_{\mathcal{C}}(X_j); \quad \text{and}$$

(c)

$$[F_{\mathcal{C}}(X_j) : \varphi_j(H)] = [F_{\mathcal{C}}(X) : H],$$

for every $j \in J$, where $\varphi_j : F_{\mathcal{C}}(X) \longrightarrow F_{\mathcal{C}}(X_j)$ is the canonical projection.

Proof. Put $F = F_{\mathcal{C}}(X)$. Let $H_F = \bigcap_{f \in F} f^{-1}Hf$ (the core of H in F). Then H_F is an open normal subgroup of F contained in H. Let $\{X_i \mid i \in I\}$ be the collection of all finite subsets of X. Make I into a directed poset by defining $i \preceq j$ if $X_i \subseteq X_j$ $(i, j \in I)$. Set

$$J = \{i \in I \mid X - X_i \subseteq H_F\}.$$

Clearly J is a cofinal subset of the poset I since $X - (X \cap H_F)$ is a finite set. Statement (a) is clear. Part (b) follows from Corollary 3.3.10 and Lemma 1.1.9. To prove (c), just observe that according to Lemma 3.3.11, $\mathrm{Ker}(\varphi_j) \le H_F \le H$. □

Proposition 3.3.13 *Let $F = F_{\mathcal{C}}(X, *)$ be the free pro-$\mathcal{C}$ group on a pointed profinite space $(X, *)$. Assume that every abstract free group of finite rank is residually $\mathcal{C}$. Then the abstract subgroup of F generated by X is a free abstract group on $X - \{*\}$.*

Proof. Let $D = D(X - \{*\})$ be the abstract free group on $X - \{*\}$, and denote by $\psi : D \longrightarrow F$ the natural homomorphism induced by the canonical injection $\iota : (X, *) \longrightarrow F$. We must show that ψ is a monomorphism. Let $w = x_1^{\epsilon_1} \cdots x_r^{\epsilon_r}$ be a reduced word on $X - \{*\}$, i.e., $x_i \in X - \{*\}, \epsilon_i = \pm 1, \epsilon_i \neq -\epsilon_{i+1}$ if $x_i = x_{i+1}$ $(i = 1, \dots, r)$. Choose an open equivalence relation R of X such that if $x, y \in \{x_1, \dots, x_r\}$ and $x \neq y$, then $xR \neq yR$ in X/R. Then the corresponding element $w' = x_1^{\epsilon_1}R \cdots x_r^{\epsilon_r}R$ of the abstract free group $D = D(X/R - \{*R\})$ is also in reduced form. Hence if $w \neq 1$, then $w' \neq 1$. So, from the commutativity of the diagram

$$\begin{array}{ccc} D(X - \{*\}) & \xrightarrow{\ \psi\ } & F_{\mathcal{C}}(X, *) \\ \downarrow & & \downarrow \\ D(X/R - \{*R\}) & \xrightarrow[\psi_R]{} & F_{\mathcal{C}}(X/R, *R) \end{array}$$

we deduce that we may assume that X is a finite space. Now, from the construction of F (see the proof of Proposition 3.3.2), we get that

$$\mathrm{Ker}(\psi) = \bigcap \{N \triangleleft D \mid D/N \in \mathcal{C}\},$$

since X is finite. Therefore $\mathrm{Ker}(\psi) = 1$, for D is residually $\mathcal{C}$. □

Corollary 3.3.14 *Let $F = F_{\mathcal{C}}(X)$ be a free pro-$\mathcal{C}$ group on a set X converging to 1. Assume that every abstract free group of finite rank is residually $\mathcal{C}$. Then the abstract subgroup of F generated by X is a free abstract group on X.*

We remark that the hypotheses in Proposition 3.3.13 and Corollary 3.3.14 are valid for many classes $\mathcal{C}$ of interest, as we show in the following proposition.

Proposition 3.3.15 *Let Φ be an abstract free group and let S be a finite simple group such that the rank of Φ is at least $d(S)$.*† *Assume that $\mathcal{C}$ is a formation that contains all S-groups. Then Φ is residually $\mathcal{C}$. In particular, if $\mathcal{C}$ is a nontrivial NE-formation of finite groups, then every abstract free group is residually $\mathcal{C}$.*

Proof. The last statement is a consequence of the first part of the lemma, since a nontrivial NE-formation of finite groups contains all S-groups for some finite simple group S. To prove the first part, it suffices to show that Φ is residually a finite S-group. We may assume that Φ has finite rank.

Case 1: $S = C_p$ for some prime p.

We use the well-known fact that the matrices

$$\begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}$$

generate an abstract free subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of rank 2. Let $\Gamma(p^i)$ be the kernel of the natural map $\mathrm{SL}_2(\mathbf{Z}) \longrightarrow \mathrm{SL}_2(\mathbf{Z}/p^i\mathbf{Z})$. It follows that Φ can be embedded as a subgroup of $\Gamma(p)$. Hence, it suffices to prove that $\Gamma(p)$ is residually a finite p-group. Remark that the elements of $\Gamma(p^i)$ are those elements in $\mathrm{SL}_2(\mathbf{Z})$ the form $I + p^iA$, where I is the identity matrix and A is a 2×2 matrix over $\mathbf{Z}$. Clearly $\bigcap_{i=1}^{\infty} \Gamma(p^i) = \{I\}$ and each quotient group $\mathrm{SL}_2(\mathbf{Z})/\Gamma(p^i)$ is finite. Next, observe that for $I + p^iA \in \Gamma(p^i)$, one has

$$(I + p^iA)^p = \sum_{j=0}^{p} \binom{p}{j} (p^kA)^j \equiv I \mod \Gamma(p^{i+1}).$$

One deduces that $\Gamma(p)/\Gamma(p^k)$ is a finite p-group for all $k = 2, 3, \ldots$.

Case 2: S is a nonabelian simple group.

Set $M^0 = \Phi$, and in general, $M^{n+1} = M_S(M^n)$, the intersection of all normal subgroups N of M^n with $M^n/N \cong S$. Clearly each M^n is a proper characteristic subgroup of Φ of rank at least $d(S)$, and M^n/M^{n+1} is a finite S-group. By a result of Levi (cf. Lyndon and Schupp [1977], Proposition I.3.3), $\bigcap_{n=0}^{\infty} M^n = 1$. Thus Φ is residually a finite S-group. □

Theorem 3.3.16 *Let G be a pro-$\mathcal{C}$ group. Then there exists a free pro-$\mathcal{C}$ group F on a set converging to 1 and a continuous epimorphism $F \longrightarrow G$. Furthermore, if G is generated by a finite set with n elements, then F can be chosen to have rank n; while if G is not finitely generated, then F can be chosen to have rank equal to $\omega_0(G)$, the smallest cardinal of a fundamental system of neighborhoods of 1 in G.*

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† By the classification theorem of finite simple groups $d(S) = 2$ for a nonabelian finite simple group S.

Proof. By Proposition 2.4.4, G admits a set of generators X converging to 1. Consider the free pro-$\mathcal{C}$ group $F = F_{\mathcal{C}}(\tilde{X})$ on the set $\tilde{X}$ converging to 1, where $\tilde{X}$ is a set with the same cardinality as X. Say that $\varphi : \tilde{X} \longrightarrow X$ is a bijection. Then the composite

$$\tilde{X} \xrightarrow{\varphi} X \hookrightarrow G$$

is a mapping converging to 1, and so it extends to an epimorphism

$$\bar{\varphi} : F(\tilde{X}) \longrightarrow G.$$

If X is infinite, then $|X| = \omega_0(G)$ by Proposition 2.6.1, and therefore, $\mathrm{rank}(F(\tilde{X})) = \omega_0(G)$. □

3.4 Maximal Pro-$\mathcal{C}$ Quotient Groups

In this section we establish a relationship between free groups over the same space when the formation $\mathcal{C}$ changes. First we define a subgroup of a profinite group associated with the class $\mathcal{C}$.

Let $\mathcal{C}$ be a formation of finite groups. For a profinite group G, define

$$R_{\mathcal{C}}(G) = \bigcap \{N \mid N \triangleleft_o G, G/N \in \mathcal{C}\}.$$

Remark that $R_{\mathcal{C}}(G)$ is a characteristic subgroup of G. If p is a fixed prime number and $\mathcal{C}$ consists of all finite p-groups, we write $R_p(G)$ rather than $R_{\mathcal{C}}(G)$. The subgroups $R_{\mathcal{C}}(G)$ play a role similar to verbal subgroups in the theory of abstract groups.

Lemma 3.4.1 *Let G and H be profinite groups. Let $\mathcal{C}$ be a formation of finite groups.*

(a) *$G/R_{\mathcal{C}}(G)$ is the largest pro-$\mathcal{C}$ quotient group of G, i.e., if $K \triangleleft_c G$ and G/K is a pro-$\mathcal{C}$ group, then $K \geq R_{\mathcal{C}}(G)$.*
(b) *If $\varphi : G \longrightarrow H$ is a continuous epimorphism, then $\varphi(R_{\mathcal{C}}(G)) = R_{\mathcal{C}}(H)$.*
(c) *Assume that $\mathcal{C}$ is, in addition, closed under taking subgroups, i.e., $\mathcal{C}$ a variety of finite groups. Then, if $\varphi : G \longrightarrow H$ is a continuous homomorphism, then $\varphi(R_{\mathcal{C}}(G)) \leq R_{\mathcal{C}}(H)$.*
(d) *Suppose that the formation $\mathcal{C}$ is closed under taking normal subgroups and extensions (i.e., $\mathcal{C}$ is an NE-formation). Then, if $R_{\mathcal{C}}(G) \leq K \triangleleft_c G$, one has $R_{\mathcal{C}}(G) = R_{\mathcal{C}}(K)$.*
(e) *Suppose that $\mathcal{C}$ is an NE-formation of finite groups. If $L \triangleleft_c R_{\mathcal{C}}(G)$ and $R_{\mathcal{C}}(G)/L$ is a pro-$\mathcal{C}$ group, then $L = R_{\mathcal{C}}(G)$.*

Proof. Part (a) is plain.

(b) Since $\mathcal{C}$ is a formation, the collection of all closed normal subgroups N of G such that G/N is a pro-$\mathcal{C}$ group is filtered from below. Hence part (b) follows from Proposition 2.1.4(b).

(c) Put $B = \varphi(G)$. Note that

$$B/B \cap R_{\mathcal{C}}(H) \cong BR_{\mathcal{C}}(H)/R_{\mathcal{C}}(H) \hookrightarrow H/R_{\mathcal{C}}(H).$$

Since $\mathcal{C}$ is a variety, we have that $B/B \cap R_{\mathcal{C}}(H)$ is a pro-$\mathcal{C}$ group. Hence, $R_{\mathcal{C}}(B) \leq B \cap R_{\mathcal{C}}(H)$. By part (b), $R_{\mathcal{C}}(G) = R_{\mathcal{C}}(B)$. Thus, $R_{\mathcal{C}}(G) \leq R_{\mathcal{C}}(H)$.

(d) Put $R = R_{\mathcal{C}}(G)$. Observe that $K/R \triangleleft G/R$. Hence K/R is a pro-$\mathcal{C}$ group. Therefore, $R_{\mathcal{C}}(K) \leq R$. Since $R_{\mathcal{C}}(K)$ is a characteristic subgroup of K and K is normal in G, it follows that $R_{\mathcal{C}}(K) \triangleleft G$. Since $\mathcal{C}$ is extension closed, $G/R_{\mathcal{C}}(K)$ is a pro-$\mathcal{C}$ group. Thus $R_{\mathcal{C}}(K) = R$.

(e) This is clear from part (d) since $R_{\mathcal{C}}(R_{\mathcal{C}}(G)) = R_{\mathcal{C}}(G)$. □

Proposition 3.4.2 *Let $\mathcal{C}'$ and $\mathcal{C}$ be formations of finite groups with $\mathcal{C}' \subseteq \mathcal{C}$. Let $F = F_{\mathcal{C}}(X, *)$ be a free pro-$\mathcal{C}$ group on the pointed space $(X, *)$. Then*

$$F_{\mathcal{C}}(X, *)/R_{\mathcal{C}'}(F_{\mathcal{C}}(X, *)) \cong F_{\mathcal{C}'}(X, *).$$

Proof. Let $\iota : (X, *) \longrightarrow F_{\mathcal{C}}(X, *)$ be the canonical embedding and

$$\mu : F_{\mathcal{C}}(X, *) \longrightarrow F_{\mathcal{C}}(X, *)/R_{\mathcal{C}'}(F_{\mathcal{C}}(X, *))$$

the natural epimorphism. Then one easily checks (using Lemma 3.4.1) that the pair

$$(F_{\mathcal{C}}(X, *)/R_{\mathcal{C}'}(F_{\mathcal{C}}(X, *)), \mu\iota),$$

where

$$\mu\iota : (X, *) \longrightarrow F_{\mathcal{C}}(X, *)/R_{\mathcal{C}'}(F_{\mathcal{C}}(X, *)),$$

satisfies the universal property of a free pro-$\mathcal{C}'$ group on the pointed space $(X, *)$. □

We say that a variety of finite groups $\mathcal{C}$ is closed under 'extensions with abelian kernel' if whenever

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

is an exact sequence of finite groups such that $A, H \in \mathcal{C}$ and A is abelian, then $G \in \mathcal{C}$.

Lemma 3.4.3 *Let $\mathcal{C}$ be a variety of finite groups and let $\mathcal{C}_e$ be the smallest extension closed variety of finite groups containing $\mathcal{C}$. For a given pointed profinite space $(X, *)$, denote by K_X the kernel of the natural epimorphism*

$$\varphi_X : F_{\mathcal{C}_e}(X, *) \longrightarrow F_{\mathcal{C}}(X, *).$$

Then, K_X is perfect (i.e., $K_X = \overline{[K_X, K_X]}$) for every profinite space X if and only if $\mathcal{C}$ is closed under extensions with abelian kernel.

Proof. Express $(X,*) = \varprojlim (X_i,*)$ as a surjective inverse limit of pointed finite discrete spaces. Then $K_X = \varprojlim K_{X_i}$. Hence one may assume that X is finite and discrete (non pointed).

Suppose that $\mathcal{C}$ is closed under extensions with abelian kernel. Choose a finite discrete space X. We have to show that K_X is perfect. Put $K = K_X$ and $\varphi = \varphi_X$. Then, one has a short exact sequence

$$1 \longrightarrow K/\overline{[K,K]} \longrightarrow F_{\mathcal{C}_e}(X)/\overline{[K,K]} \longrightarrow F_{\mathcal{C}}(X) \longrightarrow 1.$$

From the definition of $\mathcal{C}_e$ and the assumption on $\mathcal{C}$, one sees that $\mathcal{C}$ and $\mathcal{C}_e$ contain the same abelian groups. Hence, $K/\overline{[K,K]}$ is a pro-$\mathcal{C}$ group. Again, from our assumption on $\mathcal{C}$, it follows that $F_{\mathcal{C}_e}(X)/\overline{[K,K]}$ is a pro-$\mathcal{C}$ group. Therefore, there exists a continuous epimorphism

$$\mu : F_{\mathcal{C}}(X) \longrightarrow F_{\mathcal{C}_e}(X)/\overline{[K,K]}.$$

By Proposition 2.5.2, the epimorphism

$$F_{\mathcal{C}}(X) \stackrel{\mu}{\longrightarrow} F_{\mathcal{C}_e}(X)/\overline{[K,K]} \longrightarrow F_{\mathcal{C}_e}(X)/K \stackrel{\cong}{\longrightarrow} F_{\mathcal{C}}(X)$$

is an isomorphim. Thus, $K = \overline{[K,K]}$.

Conversely, suppose that $\mathcal{C}$ is not closed under extensions with abelian kernel. Consider a short exact sequence

$$1 \longrightarrow A \longrightarrow G \stackrel{\alpha}{\longrightarrow} H \longrightarrow 1,$$

where $A, H \in \mathcal{C}$, A is finite abelian and $G \notin \mathcal{C}$. We shall show that K_X is not perfect for a certain finite discrete space X. Choose X to be such that $|X| = d(G)$. Choose a continuous epimorphism $\beta : F_{\mathcal{C}}(X) \longrightarrow H$. By a property of free pro-$\mathcal{C}$ groups that we prove in the next section (see Theorem 3.5.8), one has a continuous epimorphism $\psi : F_{\mathcal{C}_e}(X) \longrightarrow G$ such that $\alpha\psi = \beta\varphi_X$. This implies that $\psi(K_X)$ is contained in A. We claim that K_X is not perfect. To see this, it suffices to show that $\psi(K_X) \neq 1$, since A is abelian. Now, if we had $\psi(K_X) = 1$, then ψ would factor through $F_{\mathcal{C}}(X)$. Thus, G would be in $\mathcal{C}$, a contradiction. □

3.5 Characterization of Free Pro-$\mathcal{C}$ Groups

Definition 3.5.1 *Let G be a profinite group. Let $\mathcal{E}$ be a nonempty class of continuous epimorphisms*

$$\alpha : A \longrightarrow B \tag{4}$$

of profinite groups. Denote by $\mathcal{E}_f$ the subclass of $\mathcal{E}$ consisting of those epimorphisms (4) such that $K = \mathrm{Ker}(\alpha)$ is a finite minimal normal subgroup of A.

(a) *An $\mathcal{E}$-embedding problem for G is a diagram*

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle\varphi} \\ A & \xrightarrow{\alpha} & B \end{array}$$

or, written more explicitly,

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1 \end{array} \tag{5}$$

with exact row, where $\alpha \in \mathcal{E}$ and φ is a continuous epimorphism of profinite groups. We say that the $\mathcal{E}$-embedding problem (5) is 'solvable' or that it 'has a solution' if there exists a continuous epimorphism

$$\bar{\varphi} : G \longrightarrow A$$

such that $\alpha\bar{\varphi} = \varphi$. The above $\mathcal{E}$-embedding problem is said to be 'weakly solvable' or to have a 'weak solution' if there is a continuous homomorphism

$$\bar{\varphi} : G \longrightarrow A$$

such that $\alpha\bar{\varphi} = \varphi$.

(b) *The kernel of the $\mathcal{E}$-embedding problem (5) is the group $K = \mathrm{Ker}(\alpha)$. We say that the $\mathcal{E}$-embedding problem (5) has 'finite minimal normal kernel' if α is in $\mathcal{E}_f$.*

(c) *The nonempty class $\mathcal{E}$ of extensions is 'admissible' if whenever*

$$\alpha : A \longrightarrow B$$

is in $\mathcal{E}$, so are the corresponding epimorphisms

$$A \longrightarrow A/N \quad \textit{and} \quad A/N \longrightarrow B,$$

for any closed normal subgroup N of A contained in $\mathrm{Ker}(\alpha)$.

(d) *An infinite profinite group G is said to have the 'strong lifting property' over a class of epimorphisms $\mathcal{E}$ if every $\mathcal{E}$-embedding problem (5) with $w_0(B) < w_0(G)$ and $w_0(A) \le w_0(G)$ is solvable.*

Remark 3.5.2 The term 'embedding problem' has its origins in Galois theory. Denote by $\bar{F}$ an algebraic separable closure of a given field F. The Galois group $G_{\bar{F}/F}$ of the extension $\bar{F}/F$ is called the *absolute Galois group of F*. Let K/F be a Galois extension of fields and let $\alpha : H' \longrightarrow H$ be a continuous epimorphism of profinite groups. Assume that $H = G_{K/F}$, the Galois group of K/F. Then there is an epimorphism

$$\varphi : G_{\bar{F}/F} \longrightarrow H = G_{K/F}$$

defined by restricting the automorphisms in $G_{\bar{F}/F}$ to K. One question that arises often in Galois theory is the following: does there exist a subfield K' of $\bar{F}$ containing K in such a way that $H' = G_{K'/F}$ and the natural epimorphism $G_{K'/F} \longrightarrow G_{K/F}$ is precisely α? Observe that this question is equivalent to asking whether there is a solution of the following embedding problem:

$$\begin{array}{ccc} & & G_{\bar{F}/F} \\ & & \downarrow{\scriptstyle\varphi} \\ H' & \xrightarrow{\alpha} & H. \end{array}$$

This question is sometimes referred to as the 'inverse problem of Galois theory'.

Let $\mathbf{Q}$, the field of rational numbers. A well-known question in algebraic number theory is whether every finite group appears as a Galois group of a Galois extension of $\mathbf{Q}$. Or, equivalently,

Open Question 3.5.3 *Is every finite group a continuous homomorphic image of the absolute Galois group $G_{\bar{\mathbf{Q}}/\mathbf{Q}}$ of the field $\mathbf{Q}$ of rational numbers?*

For some additional information on this question see Section 3.7.

Let $\mathcal{C}$ be a formation. Observe that if $\mathcal{E}$ is an admissible class, then so is $\mathcal{E}_f$. The class of all continuous epimorphisms of pro-$\mathcal{C}$ groups is an example of admissible class that we shall use frequently.

Lemma 3.5.4 *Let $\mathcal{E}$ be an admissible class of continuous epimorphisms of profinite groups and let G be a profinite group. The following conditions are equivalent.*

(a) *G has the strong lifting property over $\mathcal{E}$;*
(b) *G has the strong lifting property over $\mathcal{E}_f$.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a): Suppose G has the strong lifting property over $\mathcal{E}_f$ and let (5) be a $\mathcal{E}$-embedding problem with $w_0(B) < w_0(G)$ and $w_0(A) \leq w_0(G)$. By Corollary 2.6.5, there exist an ordinal number μ and a chain of closed subgroups of K (see diagram (5))

$$K = K_0 > K_1 > \cdots > K_\lambda > \cdots > K_\mu = 1$$

such that

(i) each K_λ is a normal subgroup of A with $K_\lambda/K_{\lambda+1}$ finite; moreover, $K_{\lambda+1}$ is maximal in K_λ with respect to these properties;
(ii) if λ is a limit ordinal, then $K_\lambda = \bigcap_{\nu<\lambda} K_\nu$; and

(iii) if $w_0(A) = w_0(G)$ (therefore K is an infinite group and $w_0(A/K) < w_0(A)$), then $w_0(A/K_\lambda) < w_0(A)$ whenever $\lambda < \mu$.

We must prove that there exists an epimorphism $\bar{\varphi} : G \longrightarrow A$ such that $\alpha\bar{\varphi} = \varphi$. To do this we show in fact that for each $\lambda \le \mu$ there exists an epimorphism

$$\varphi_\lambda : G \longrightarrow A/K_\lambda$$

such that if $\lambda_1 \le \lambda$ the diagram

$$\begin{array}{ccc} & G & \\ {\scriptstyle\varphi_\lambda}\swarrow & & \searrow{\scriptstyle\varphi_{\lambda_1}} \\ A/K_\lambda & \longrightarrow & A/K_{\lambda_1} \end{array}$$

commutes, where the horizontal mapping is the natural epimorphism. Then we can take $\bar{\varphi} = \varphi_\mu$. To show the existence of φ_λ, we proceed by induction (transfinite, if K is infinite) on λ. Note that $A/K_0 = B$; so, put $\varphi_0 = \varphi$. Let $\lambda \le \mu$ and assume that φ_ν has been defined for all $\nu < \lambda$ so that the above conditions are satisfied. If λ is a limit ordinal, observe that since $K_\lambda = \bigcap_{\nu<\lambda} K_\nu$, then

$$A/K_\lambda = \varprojlim_{\nu<\lambda} A/K_\nu;$$

in this case, define $\varphi_\lambda = \varprojlim{}_{\nu<\lambda}\varphi_\nu$.

If, on the other hand, $\lambda = \sigma + 1$, we define φ_λ to be a solution to the $\mathcal{E}_f$-embedding problem with finite minimal normal kernel

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & {\scriptstyle\varphi_\lambda}\swarrow & \downarrow{\scriptstyle\varphi_\sigma} & & \\ 1 & \longrightarrow & K_\sigma/K_\lambda & \longrightarrow & A/K_\lambda & \longrightarrow & A/K_\sigma & \longrightarrow & 1 \end{array}$$

To see that such a solution exists, we have to verify that $w_0(A/K_\sigma) < w_0(G)$ and $w_0(A/K_\lambda) \le w_0(G)$. If $w_0(A) < w_0(G)$, these inequalities are clear. On the other hand, if $w_0(A) = w_0(G)$, we have

$$w_0(A/K_\lambda) = w_0(A/K_\sigma) < w_0(A) = w_0(G),$$

since K_σ/K_λ is a finite group and since condition (iii) above holds.

It is clear that in either case φ_λ satisfies the required conditions. □

Next we consider equivalent conditions to weak solvability of embedding problems for some special types of admissible classes.

Lemma 3.5.5 *Let $\mathcal{C}$ and $\mathcal{C}'$ be varieties of finite groups. Let $\mathcal{E}$ be the class of all continuous epimorphisms* (4) *of pro-$\mathcal{C}$ groups such that* $\mathrm{Ker}(\alpha)$ *is pro-$\mathcal{C}'$, and let $\bar{\mathcal{E}}$ consist of those epimorphisms* (4) *in $\mathcal{E}$ for which* $\mathrm{Ker}(\alpha)$ *is finite. Let G be a profinite group. The following conditions are equivalent.*

(a) *Every $\mathcal{E}$-embedding problem* (5) *for G has a weak solution*;
(b) *Every $\bar{\mathcal{E}}$-embedding problem* (5) *for G has a weak solution*;
(c) *Every $\bar{\mathcal{E}}_a$-embedding problem* (5) *for G has a weak solution, where $\bar{\mathcal{E}}_a$ consists of those epimorphisms* (4) *in $\bar{\mathcal{E}}$ such that* $\mathrm{Ker}(\alpha)$ *is a finite abelian minimal normal subgroup of A.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

(b) $\Rightarrow$ (a): Consider the embedding problem (5) with $\alpha \in \mathcal{E}$. Define a set $\mathcal{P}$ to consist of all pairs (K', η'), where K' is a closed normal subgroup of A contained in K, and $\eta' : G \longrightarrow A/K'$ is a continuous homomorphism such that the diagram

$$\begin{array}{ccc} & & G \\ & \swarrow^{\eta'} & \downarrow{\scriptstyle\varphi} \\ A/K' & \longrightarrow & B \end{array}$$

commutes. The set $\mathcal{P}$ is nonempty since $(K, \varphi) \in \mathcal{P}$. Define $(K', \eta') \preceq (K'', \eta'')$ if $K' \geq K''$ and

$$\begin{array}{ccc} & & G \\ & \swarrow^{\eta'} & \downarrow{\scriptstyle\eta''} \\ A/K' & \longleftarrow & A/K'' \end{array}$$

commutes. Then $\mathcal{P}$ is an inductive poset. Indeed, if $\{(K'_i, \eta'_i)\}_i$ is a totally ordered subset of $\mathcal{P}$, put

$$K' = \bigcap_i K'_i \quad \text{and} \quad \eta' = \varprojlim_i \eta'_i;$$

then $(K', \eta') \in \mathcal{P}$ and $(K', \eta') \succeq (K'_i, \eta'_i)$ for all i.

Let $(\tilde{K}, \tilde{\eta})$ be a maximal element of $\mathcal{P}$. We shall show that $\tilde{K} = 1$. Suppose $\tilde{K} \neq 1$; then there exists an open normal subgroup L of $\tilde{K}$ which is normal in A, such that $L \neq \tilde{K}$ (if $\tilde{K} \neq 1$, it contains a proper open subgroup $\tilde{K} \cap U$ where U is open in A; then U contains an open normal subgroup V of A; put $L = K \cap V$).

Since $\tilde{K}/L$ is finite, it follows from (b) that there exists a continuous homomorphism

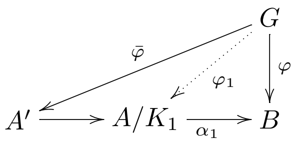
$$\psi : G \longrightarrow A/L$$

such that

$$\begin{array}{ccc} & & G \\ & \swarrow^{\psi} & \downarrow{\scriptstyle\tilde{\eta}} \\ A/L & \longrightarrow & A/\tilde{K} \end{array}$$

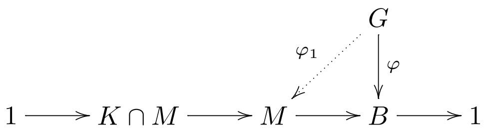
commutes. Hence, $(L, \psi) \in \mathcal{P}$ and $(L, \psi) \succ (\tilde{K}, \tilde{\eta})$, contradicting the maximality of $(\tilde{K}, \tilde{\eta})$. Thus $\tilde{K} = 1$.

(c) $\Rightarrow$ (b): We show in fact something stronger, namely that if (c) holds and we have a diagram (5) with $\alpha \in \bar{\mathcal{E}}$ and K finite, then there exists a continuous homomorphism $\bar{\varphi} : G \longrightarrow A$ making the diagram commutative. We prove this by induction on the order of K. We distinguish two cases depending on whether K is minimal normal in A or not. Suppose first the latter. Then there exists a normal subgroup K_1 of A such that $1 < K_1 < K$.



Let $\alpha_1 : A/K_1 \longrightarrow B$ be the epimorphism induced by α. Then, by induction, there exists a continuous homomorphism $\varphi_1 : G \longrightarrow A/K_1$ such that $\alpha_1\varphi_1 = \varphi$. Let $\beta : A \longrightarrow A/K_1$ be the canonical epimorphism, and set $A' = \beta^{-1}(\varphi_1(G))$. By induction again, there exists a continuous homomorphism $\bar{\varphi} : G \longrightarrow A'$ such that $\beta_{|A'}\bar{\varphi} = \varphi_1$. If we think of $\bar{\varphi}$ as a mapping $G \longrightarrow A$, then $\bar{\varphi}$ is the desired lifting.

Next assume that K is finite minimal normal in A. Consider the Frattini subgroup $\Phi(A)$ of A, and recall that $\Phi(A)$ is pronilpotent (see Corollary 2.8.4). By the minimality of K, either $K \leq \Phi(A)$ or $K \cap \Phi(A) = 1$. Assume first that $K \leq \Phi(A)$. Hence K is nilpotent, since it is finite. Observe that $[K, K] = 1$, for otherwise $[K, K] = K$, contradicting the nilpotency of K. Therefore, K is abelian. Then the existence of $\bar{\varphi}$ follows from (c). Suppose now that $K \cap \Phi(A) = 1$. Then there exists a maximal open subgroup M of A such that $K \not\leq M$. Hence $K \cap M < K$. Thus, by induction, there exists a continuous homomorphism $\varphi_1 : G \longrightarrow M$ making the diagram



commutative. Finally, define $\bar{\varphi} : G \longrightarrow A$ to be the composition

$$G \xrightarrow{\varphi_1} M \hookrightarrow A. \qquad \square$$

Having the strong lifting property over a suitable class of epimorphisms is a powerful property for a profinite group; in the following result it is used as a key tool to determine when two groups are isomorphic.

Proposition 3.5.6 *Let $\mathcal{E}$ be an admissible class of continuous epimorphisms of profinite groups and let G_1 and G_2 be infinite profinite groups with the* *strong lifting property over $\mathcal{E}$ and such that $w_0(G_1) = w_0(G_2) = \mathfrak{m}$. Assume that $N_i \triangleleft_c G_i$ such that $w_0(G_i/N_i) < \mathfrak{m}$ and that the epimorphisms*

$$G_i \longrightarrow G_i/N_i \longrightarrow 1$$

belong to $\mathcal{E}$ $(i = 1, 2)$. Then, any isomorphism $\varphi : G_1/N_1 \longrightarrow G_2/N_2$ lifts to an isomorphism $\bar{\varphi} : G_1 \longrightarrow G_2$ such that the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{\varphi}} & G_2 \\ \downarrow & & \downarrow \\ G_1/N_1 & \xrightarrow[\varphi]{} & G_2/N_2 \end{array}$$

commutes.

Proof. Let μ be the smallest ordinal with cardinality $\mathfrak{m}$. By Corollary 2.6.5, there exists a chain of closed normal subgroups of G_i $(i = 1, 2)$

$$N_i = N_{i,0} \geq N_{i,1} \geq \cdots \geq N_{i,\lambda} \geq \cdots \geq N_{i,\mu} = 1$$

indexed by the ordinals $\lambda \leq \mu$, such that

(i) $N_{i,\lambda}/N_{i,\lambda+1}$ is finite for $\lambda \geq 0$;
(ii) if λ is a limit ordinal, then $N_{i,\lambda} = \bigcap_{\nu<\lambda} N_{i,\nu}$, and
(iii) $w_0(G_i/N_{i,\lambda}) < \mathfrak{m}$, for $\lambda < \mu$.

We shall use transfinite induction to construct chains of closed normal subgroups of G_i $(i = 1, 2)$

$$N_i = N'_{i,0} \geq N'_{i,1} \geq \cdots \geq N'_{i,\lambda} \geq \cdots \geq N'_{i,\mu} = 1$$

satisfying conditions analogous to (i), (ii), (iii), and in addition

(iv) $N'_{i,\lambda} \leq N_{i,\lambda}$ and $w_0(G_i/N'_{i,\lambda}) \leq w_0(G_i/N_{i,\lambda})$, for all λ $(i = 1, 2)$.

Note that conditions (iii) and (iv) imply that $w_0(G_i/N'_{i,\lambda}) < w_0(G_i)$ for all $\lambda < \mu$ $(i = 1, 2)$.

Furthermore, we construct isomorphisms

$$\varphi_\lambda : G_1/N'_{1,\lambda} \longrightarrow G_2/N'_{2,\lambda}$$

for each $\lambda \leq \mu$, in such a way that if $\lambda < \nu \leq \mu$, then the diagram

$$\begin{array}{ccc} G_1/N'_{1,\nu} & \xrightarrow{\varphi_\nu} & G_2/N'_{2,\nu} \\ \downarrow & & \downarrow \\ G_1/N'_{1,\lambda} & \xrightarrow[\varphi_\lambda]{} & G_2/N'_{2,\lambda} \end{array}$$

commutes. Set $N'_{i,0} = N_{i,0} = N_i$ $(i = 1, 2)$, and let $\varphi_0 : G_1/N'_{1,0} \longrightarrow G_2/N'_{2,0}$ be the given isomorphism φ. Let $\rho \leq \mu$ and assume we have constructed chains indexed by $\lambda < \rho$

$$N_i = N'_{i,0} \geq N'_{i,1} \geq \cdots \geq N'_{i,\lambda} \geq \cdots \quad (i = 1, 2)$$

as well as isomorphisms φ_λ $(\lambda < \rho)$, satisfying the above conditions. Next we indicate how to construct $N'_{i,\rho}$ $(i = 1, 2)$ and an isomorphism φ_ρ such that the above conditions still hold. If ρ is a limit ordinal, put

$$N'_{i,\rho} = \bigcap_{\lambda<\rho} N'_{i,\lambda} \quad (i = 1, 2).$$

Observe that

$$G_i/N'_{i,\rho} = \varprojlim_{\lambda<\rho} G_i/N'_{i,\lambda} \quad (i = 1, 2).$$

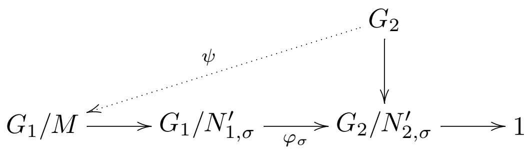
In this case, define

$$\varphi_\rho = \varprojlim_{\lambda<\rho} \varphi_\lambda.$$

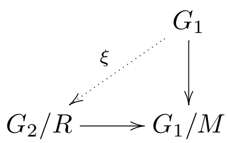
By Theorem 2.6.4, one has that

$$w_0(G_i/N'_{i,\rho}) \leq \sum_{\lambda<\rho} w_0(G_i/N_{i,\lambda}) = w_0(G_i/N_{i,\rho}).$$

If $\rho = \sigma + 1$ for some ordinal σ, we proceed as follows: put $M = N'_{1,\sigma} \cap N_{1,\rho}$ and $P = N'_{2,\sigma} \cap N_{2,\rho}$. Observe that $[N'_{1,\sigma} : M] < \infty$ and $[N'_{2,\sigma} : P] < \infty$. Let the continuous epimorphism $\psi : G_2 \longrightarrow G_1/M$ be a solution to the $\mathcal{E}$-embedding problem for G_2



Set $R = P \cap \mathrm{Ker}(\psi)$. Then ψ induces a natural epimorphism $G_2/R \longrightarrow G_1/M$. Let the continuous epimorphism $\xi : G_1 \longrightarrow G_2/R$ be a solution to the $\mathcal{E}$-embedding problem for G_1



(such a solution exists since $w_0(G_1/M) < w_0(G_2)$). Set $S = \mathrm{Ker}(\xi)$. Therefore ξ induces an isomorphism $\delta : G_1/S \longrightarrow G_2/R$. Set $N'_{1,\rho} = S$, $G'_{2,\rho} = R$, and $\varphi_\rho = \delta$. Then $N'_{1,\rho} \leq N_{1,\rho}$, $N'_{2,\rho} \leq N_{2,\rho}$ and

$$\begin{array}{ccc} G_1/N'_{1\rho} & \xrightarrow{\varphi_\rho} & G_2/N'_{2\rho} \\ \downarrow & & \downarrow \\ G_1/N'_{1,\sigma} & \xrightarrow[\varphi_\sigma]{} & G_2/N'_{2,\sigma} \end{array}$$

commutes. Finally, observe that $w_0(G_1/N'_{1,\rho}) < w_0(G_1)$ and $w_0(G_2/N'_{2,\rho}) < w_0(G_2)$, as desired. □

The following useful special case is obtained by putting $N_i = G_i$ $(i = 1, 2)$.

Corollary 3.5.7 *Let $\mathcal{C}$ be a formation of finite groups. Let G_1 and G_2 be infinite pro-$\mathcal{C}$ groups, with $w_0(G_1) = w_0(G_2)$. Assume that G_1 and G_2 have the strong lifting property over the class of all continuous epimorphisms of pro-$\mathcal{C}$ groups. Then G_1 and G_2 are isomorphic.*

Next we present two results that characterize free pro-$\mathcal{C}$ groups on a set converging to 1 in terms of embedding problems. The first one is about free groups of finite rank. As we shall see in many occasions, the second result is a most useful tool whenever one wants to investigate whether an infinitely generated pro-$\mathcal{C}$ group is free pro-$\mathcal{C}$.

Theorem 3.5.8 *Let $\mathcal{C}$ be a formation of finite groups and let G be a pro-$\mathcal{C}$ group. Assume that $d(G) = m$ is finite. Let $\mathcal{E} = \mathcal{E}_{\mathcal{C}}$ be the class of all epimorphisms of pro-$\mathcal{C}$ groups. Then, the following two conditions are equivalent*

(a) *G is a free pro-$\mathcal{C}$ group of rank m;*
(b) *Every $\mathcal{E}$-embedding problem for G*

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1 \end{array}$$

with $d(B) \leq d(G)$ and $d(A) \leq d(G)$, has a solution.

Proof. (a) ⇒ (b) This implication follows immediately from Proposition 2.5.4.

(b) ⇒ (a) Consider a free pro-$\mathcal{C}$ group F of rank m, and let $\alpha : F \longrightarrow G$ be a continuous epimorphism. By (b) there exists an continuous epimorphism $\varphi : G \longrightarrow F$ such that $\alpha\varphi = \mathrm{id}_G$. Then φ is a monomorphism, and thus an isomorphism. □

Theorem 3.5.9 *Let $\mathcal{C}$ be a formation of finite groups and let G be a pro-$\mathcal{C}$ group. Assume that $d(G) = \mathfrak{m}$ is infinite. Let $\mathcal{E} = \mathcal{E}_{\mathcal{C}}$ be the class of all epimorphisms of pro-$\mathcal{C}$ groups. Then, the following two conditions are equivalent*

(a) *G is a free pro-$\mathcal{C}$ group on a set converging to 1 of rank $\mathfrak{m}$;*

(b) *G has the strong lifting property over $\mathcal{E}$.*

Proof. (a) $\Rightarrow$ (b) Let G be a free pro-$\mathcal{C}$ group of rank $\mathfrak{m}$ on the set X converging to 1. Then $|X| = w_0(G)$ (see Proposition 2.6.2). Consider the $\mathcal{E}$-embedding problem

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \downarrow{\scriptstyle \varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1 \end{array}$$

with $w_0(B) < w_0(G)$ and $w_0(A) \leq w_0(G)$. We must show that there exists a continuous epimorphism $\bar{\varphi} : G \longrightarrow A$ such that $\alpha\bar{\varphi} = \varphi$. According to Lemma 3.5.4, we may assume that K is finite. Put $X_0 = X \cap \mathrm{Ker}(\varphi)$. Let $\mathcal{U}$ be the collection of all open normal subgroups of B. By our assumptions, $|\mathcal{U}| < \mathfrak{m}$. Observe that, since X converges to 1,

$$|X - \mathrm{Ker}(\varphi)| = \left| X - \bigcap_{U \in \mathcal{U}} \varphi^{-1}(U) \right| = \left| \bigcup_{U \in \mathcal{U}} (X - \varphi^{-1}(U)) \right| = |\mathcal{U}|.$$

Therefore, $|X_0| = \mathfrak{m}$. Let Z be a set of generators of K; since Z is finite, we may choose a subset Y of X_0 such that $|Z| = |Y|$. By Proposition 2.2.2, there exists a continuous section $\sigma : B \longrightarrow A$ of α. Think of K as a subgroup of A. Define $\varphi_1 : X \longrightarrow A$ as a map that sends Y to Z bijectively, and such that $\varphi_1 = \sigma\varphi$ on $X - Y$. Since X is a set converging to 1 and φ and σ are continuous, the mapping φ_1 converges to 1. Therefore, φ_1 extends to a continuous homomorphism $\bar{\varphi} : G \longrightarrow A$ with $\alpha\bar{\varphi} = \varphi$. Finally note that $\bar{\varphi}$ is onto since $\varphi_1(X)$ generates A.

(b) $\Rightarrow$ (a) This follows immediately from Corollary 3.5.7. □

Combining the theorem above with Lemma 3.5.4, we get the following characterization of free pro-$\mathcal{C}$ groups of infinite countable rank.

Corollary 3.5.10 *Let $\mathcal{C}$ be a formation of finite groups and let G be a pro-$\mathcal{C}$ group with $w_0(G) = \aleph_0$. Then G is a free pro-$\mathcal{C}$ group on a countably infinite set converging to 1 if and only if every embedding problem of the form*

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \downarrow{\scriptstyle \varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1 \end{array}$$

has a solution whenever A is finite.

The next result provides another characterization of free pro-$\mathcal{C}$ groups from a slightly different point of view.

Proposition 3.5.11 *Let $\mathcal{C}$ be a formation of finite groups and let G be a pro-$\mathcal{C}$ group. Assume that $d(G) = \mathfrak{m}$ is infinite. Then G is a free pro-$\mathcal{C}$ group of rank $\mathfrak{m}$ if and only if the following condition is satisfied:*

(∗) *every embedding problem of pro-$\mathcal{C}$ groups*

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \Big\downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & C & \xrightarrow{\alpha} & D & \longrightarrow & 1, \end{array}$$

with $1 \neq C \in \mathcal{C}$, has $\mathfrak{m}$ different solutions $\psi : G \longrightarrow C$.

Proof. Assume that G is a free pro-$\mathcal{C}$ group on a set X converging to 1 with $|X| = \mathfrak{m}$. Consider an embedding problem for G as above, with C finite. Since D is finite, $U = \mathrm{Ker}(\varphi)$ is open in G. Hence, $X - U$ is finite and $|X \cap U| = \mathfrak{m}$. Since K is finite, there exists an indexing set I of cardinality $\mathfrak{m}$ and a collection $\{X_i\}_{i\in I}$ of distinct subsets of $X \cap U$, each of them of size $|K|$. Let $\sigma : D \longrightarrow C$ be a section of α. For each $i \in I$, define a map $\varphi_i : X \longrightarrow C$ as follows: $\varphi_i = \sigma\varphi$ on $X - U$, φ_i sends X_i to K bijectively (we think of K as a subgroup of C), and $\varphi_i(X \cap U - X_i)) = 1$. Clearly, $\varphi_i(X)$ generates C. Thus φ_i extends to a continuous epimorphism $\psi_i : G \longrightarrow C$ with $\alpha\psi_i = \varphi$. Furthermore, the maps ψ_i $(i \in I)$ are all distinct.

Conversely, assume that condition (∗) holds. Consider an embedding problem

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \Big\downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1, \end{array}$$

where A and B are pro-$\mathcal{C}$ groups and where $w_0(B) < \mathfrak{m}$ and $w_0(A) \leq \mathfrak{m}$. According to Theorem 3.5.9, it suffices to show that such an embedding problem has a solution. By Lemma 3.5.4, we may assume that K is a finite minimal normal subgroup of A. Let $V \triangleleft_o A$ be such that $V \cap K = 1$. Consider the commutative diagram

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \Big\downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle\beta_{|K}} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\ 1 & \longrightarrow & K_V = KV/V & \longrightarrow & A/V & \xrightarrow{\alpha_V} & B/\alpha(V) & \longrightarrow & 1, \end{array}$$

where β and γ are the canonical epimorphisms, α_V is the epimorphism induced by α and $K_V = \mathrm{Ker}(\alpha_V)$. One shows easily that the maps $\alpha, \beta, \alpha_V, \gamma$

form a pullback diagram (see Exercise 2.10.1); moreover, $\beta_{|K}$ is an isomorphism and K_V is minimal normal in A/V.

By assumption, since $A/V \in \mathcal{C}$, there exists an indexing set I with $|I| = \mathfrak{m}$ and distinct continuous epimorphisms $\psi_i : G \longrightarrow A/V$ such that $\alpha_V \psi_i = \gamma\varphi$ ($i \in I$). By definition of pullback, for each $i \in I$, there exists a unique continuous homomorphism $\bar{\varphi}_i : G \longrightarrow A$ such that $\alpha\bar{\varphi}_i = \varphi$ and $\beta\bar{\varphi}_i = \psi_i$. The proof will be finished if we can prove that $\bar{\varphi}_j$ is an epimorphism for some $j \in I$. Observe that for this it suffices to prove the following claim: $\mathrm{Ker}(\varphi) \not\leq \mathrm{Ker}(\psi_j)$, for some $j \in I$. Indeed, if the claim holds, $\psi_j(\mathrm{Ker}(\varphi))$ is a nontrivial normal subgroup of A/V. Hence either $K_V \cap \psi_j(\mathrm{Ker}(\varphi)) = 1$ or $K_V \leq \psi_j(\mathrm{Ker}(\varphi))$, since K_V is minimal normal in A/V. On the other hand, $\alpha_V(\psi_j(\mathrm{Ker}(\varphi))) = (\gamma\varphi)(\mathrm{Ker}(\varphi)) = 1$; so, we deduce that $\psi_j(\mathrm{Ker}(\varphi)) = K_V$. Therefore, $\mathrm{Ker}(\alpha_V\psi_j) = \mathrm{Ker}(\varphi)\mathrm{Ker}(\psi_j)$. Thus, by Lemma 2.10.2, $\bar{\varphi}_j$ is surjective.

It remains to prove the claim. Let $N = \bigcap_{i\in I} \mathrm{Ker}(\psi_i)$. It follows that $w_0(G/N) = \mathfrak{m}$. Indeed, assume that $w_0(G/N) = \mathfrak{n} < \mathfrak{m}$; then G/N is a quotient of a free pro-$\mathcal{C}$ group F of rank $\mathfrak{n}$; so, F would have $\mathfrak{m}$ distinct continuous epimorphisms onto the finite group A, which is plainly impossible, since each such an epimorphism is completely determined by its values on a finite subset of a basis of F. Therefore, $w_0(G/N) = w_0(G) > w_0(B) = w_0(G/\mathrm{Ker}(\varphi))$. This implies that $\mathrm{Ker}(\varphi) \not\leq \mathrm{Ker}(\psi_j)$, for some $j \in I$. □

Next we prove that all free pro-$\mathcal{C}$ groups are in fact free pro-$\mathcal{C}$ groups on some set converging to 1. Nevertheless, it is sometimes more natural and more convenient to describe certain free pro-$\mathcal{C}$ group as being free on a topological space, rather than on a set; this becomes apparent when one studies subgroups of free groups (see Section 8.1).

Proposition 3.5.12 *Let $\mathcal{C}$ be a formation of finite groups and let $F = F_{\mathcal{C}}(X,*)$ be a free pro-$\mathcal{C}$ group on a pointed profinite space $(X,*)$. Then F is a free pro-$\mathcal{C}$ group on a certain set converging to* 1. *Furthermore, let $\mathcal{R}$ be the collection of all open equivalence relations R on X. Then if $\mathcal{R}$ is finite, so is the rank of F, and if $\mathcal{R}$ is infinite*, $\mathrm{rank}(F) = |\mathcal{R}|$.

Proof. If X is finite, there is nothing to prove. So, we assume from now on that $(X,*)$ is an infinite pointed profinite space. Clearly $|\mathcal{R}| = \rho(X)$, where $\rho(X)$ denotes the cardinality of the set of clopen subsets of X. We seek to prove that $F = F_{\mathcal{C}}(X,*)$ is a free pro-$\mathcal{C}$ group on a set of cardinality $\rho(X)$ converging to 1. Let $\mathcal{E} = \mathcal{E}_{\mathcal{C}}$ be the class of all epimorphisms of pro-$\mathcal{C}$ groups and consider an $\mathcal{E}$-embedding problem

$$\begin{array}{ccccccccc} & & & & & & F & & \\ & & & & & & \Big\downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1 \end{array} \tag{6}$$

where $w_0(B) < w_0(F)$ and $w_0(A) \leq w_0(F)$. According to the characterization of free pro-$\mathcal{C}$ groups on a set converging to 1 established in Theorem 3.5.9, we must show that there exists a continuous epimorphism $\bar{\varphi} : F \longrightarrow A$ such that $\alpha\bar{\varphi} = \varphi$. By Lemma 3.5.4, we may assume that the kernel K is finite.

Put $Y = \varphi(X)$, and let $\psi : X \longrightarrow Y$ be the restriction of φ to X. Note that ψ is a mapping of pointed spaces, if we think of 1 as the distinguished point of Y. It follows from Proposition 2.6.2 and our hypotheses that $\rho(Y) < \rho(X)$. In particular, if Y is finite, then $\psi^{-1}(y)$ is infinite for some $y \in Y$.

So in any case we may choose points $y_1, \ldots, y_m \in Y$, and for each $i = 1, \ldots, m$, points $x_{i,0}, \ldots, x_{i,n_i} \in \psi^{-1}(y_i)$, none of them equal to $*$, such that $n_1 + \cdots + n_m = |K| - 1$. Represent the set of elements of K as

$$\{1\} \cup \{k_{i,j} \mid i = 1, \ldots, m; j = 0, \ldots, n_i\}.$$

Choose clopen subsets U and $U_{i,j}$ of X such that $* \in U, x_{i,j} \in U_{i,j}$ $(i = 1, \ldots, m; j = 0, \ldots, n_i)$ and $X = U \cupdot U_{1,0} \cupdot \cdots \cupdot U_{m,n_m}$. Define

$$\delta : X \longrightarrow K$$

as follows: $\delta(x) = 1$ if $x \in U$ or if $x \in U_{i,0}$ $(i = 1, \ldots, m)$, and $\delta(x) = k_{i,j}$ if $x \in U_{i,j}$ $(i = 1, \ldots, m; j = 1, \ldots, n_i)$. Then δ is a continuous mapping. Next, consider a continuous section

$$\sigma : B \longrightarrow A$$

of α such that $\sigma(1) = 1$ (see Proposition 2.2.2), and define

$$\xi : X \longrightarrow A$$

by $\xi(x) = \delta(x)\sigma(\psi(x))$ for $x \in X$. Plainly, ξ is continuous and $\xi(*) = 1$. Therefore there exists a continuous homomorphism

$$\bar{\xi} : F \longrightarrow A$$

extending ξ. Observe that $\alpha(\bar{\xi}(x)) = \psi(x)$ for all $x \in X$. It follows that $\alpha\bar{\xi} = \xi$. We claim that $\bar{\varphi} = \bar{\xi}$ is the desired solution of the $\mathcal{E}$-embedding problem (6). To verify this claim it remains to show that $\bar{\xi}$ is an epimorphism. Note first that

$$\xi(x_{i,j})\xi(x_{i,0})^{-1} = \delta(x_{i,j})\sigma(\psi(x_{i,j}))(\delta(x_{i,0})\sigma(\psi(x_{i,0})))^{-1} = k_{i,j}$$

$(i = 1, \ldots, m; j = 1, \ldots, n_i)$; therefore, $K \leq \bar{\xi}(F)$. On the other hand, $\alpha(\bar{\xi}(F)) = B$, and thus $\bar{\xi}(F) = A$, as required. □

The proof of the theorem above is not constructive, in the sense that it does not exhibit an explicit basis of F converging to 1. The following theorem shows that a construction of such a basis cannot be expected. It answers negatively Open Question 3.5.13 in the first edition of this book.

Theorem 3.5.13 *Let X be a profinite space and let $F = F(X)$ be the free pro-$\mathcal{C}$ group on X. There is no basis S of F converging to 1 that can be obtained from X in a canonical way, or more precisely, there is no such S that is left invariant under the action of the group* $\mathrm{Aut}(X)$ *of homeomorphisms from X to X.*

Proof. We prove this by exhibiting a concrete example of a variety $\mathcal{C}$ and a space X such that no basis S of F converging to 1 is left invariant under the action of the group of automorphisms of F induced by the homeomorphisms in $\mathrm{Aut}(X)$.

Choose $\mathcal{C}$ to be the variety of all finite p-groups, where p is a fixed prime number. Observe that the Frattini quotient $F/\Phi(F)$ of F is a vector space over the field $\mathbf{F}_p$ with p elements and it is also the free pro-$\underline{\mathcal{C}}$ group on the space X, where $\underline{\mathcal{C}}$ is the variety of all finite abelian p-groups of exponent p (the vector spaces of finite dimension over $\mathbf{F}_p$). A basis S of F converging to 1 can be consider to be also a basis of $F/\Phi(F)$ converging to 1; moreover every $\varphi \in \mathrm{Aut}(X)$ induces a continuous automorphism of $F/\Phi(F)$; hence we may replace F by $F/\Phi(F)$.

Consider the Pontryagin dual $\mathrm{Hom}(F/\Phi(F), \mathbf{F}_p)$ of $F/\Phi(F)$. Under this duality, a basis S of $F/\Phi(F)$ converging to 1 is transformed into an ordinary basis of the discrete vector space $\mathrm{Hom}(F/\Phi(F), \mathbf{F}_p) = C(X, \mathbf{F}_p)$ over the field $\mathbf{F}_p$; furthermore, every $\varphi \in \mathrm{Aut}(X)$ is transformed into an automorphism of $C(X, \mathbf{F}_p)$. Therefore, it suffices to prove that, after an appropriate choice of X, there exists no basis of the vector space $C(X, \mathbf{F}_p)$ which is left invariant under the action of $\mathrm{Aut}(X)$. Fix a prime q, and let $X = \mathbf{Z}_q$. The result will follow if we prove the following stronger assertion:

Let $f : X \longrightarrow \mathbf{F}_p$ be a nonconstant continuous function. Then the transforms of f under $\mathrm{Aut}(X)$ *are linearly dependent.*

For simplicity we restrict ourselves to the case $p = 2$ (the argument can be easily extended to any prime p). Consider the decomposition

$$\mathbf{Z}_q = \varprojlim_{n \in \mathbf{N}} \mathbf{Z}/q^n\mathbf{Z}.$$

By Lemma 1.1.16, f factors through $\mathbf{Z}/q^{n_0}\mathbf{Z}$, for some $n_0 \in \mathbf{N}$, i.e., there exists $\tilde{f} : \mathbf{Z}/q^{n_0}\mathbf{Z} \longrightarrow \mathbf{F}_2 = \{0, 1\}$ such that

$$f = \tilde{f}\varphi_{n_0},$$

where $\varphi_{n_0} : \mathbf{Z}_q \longrightarrow \mathbf{Z}/q^{n_0}\mathbf{Z}$ is the projection.

Let a be the number of elements $z \in \mathbf{Z}/q^{n_0}\mathbf{Z}$ such that $\tilde{f}(z) = 0$, and let b be the number of elements $z \in \mathbf{Z}/q^{n_0}\mathbf{Z}$ such that $\tilde{f}(z) = 1$. Note $a + b = q^{n_0}$. Since f is nonconstant, $a, b > 0$. In fact we may assume $a, b > 1$: simply replace $\mathbf{Z}/q^{n_0}\mathbf{Z}$ by $\mathbf{Z}/q^{n_0+1}\mathbf{Z}$ (this has the effect of multiplying a and b by q). Now consider the set T of functions obtained by transforming $\tilde{f}$ by the permutations of $\mathbf{Z}/q^{n_0}\mathbf{Z}$. Then

$$|T| = \frac{q^{n_0}!}{a!b!} = \binom{q^{n_0}}{a}.$$

Since $a > 1$, we get $|T| > q^{n_0}$. Since

$$\dim C(\mathbf{Z}/q^{n_0}\mathbf{Z}, \mathbf{F}_2) = q^{n_0},$$

we deduce that the elements of T are linearly dependent. Finally observe that every permutation of $\mathbf{Z}/q^{n_0}\mathbf{Z}$ is induced by a homeomorphism

$$\mathbf{Z}_q \longrightarrow \mathbf{Z}_q,$$

i.e., an element of $\mathrm{Aut}(X)$. This proves the above assertion and the theorem. □

Exercise 3.5.14 Let $\mathcal{C}$ be a nontrivial formation of finite groups and X a set. Prove

(a) If $X \neq \emptyset$ is finite, $|F_{\mathcal{C}}(X)| = 2^{\aleph_0}$.
(b) Let C be a finite cyclic group in $\mathcal{C}$, and let $G = \prod_X C$ be the direct product of $|X|$ copies of C. Then G can be generated by a set converging to 1 of cardinality $|X|$.
(c) If X is infinite and let F be the free pro-$\mathcal{C}$ group on the set X converging to 1, then $|F| = 2^{|X|}$. (Hint: use Proposition 2.6.2.)
(d) Assume that X is infinite and let $\Phi = \Phi(X)$ be a free abstract group on X. Then the pro-$\mathcal{C}$ completion of Φ is a free pro-$\mathcal{C}$ group of rank $2^{|X|}$. (Hint: see Exercise 3.3.3.)
(e) Let $\mathfrak{m}$ be an infinite cardinal and let p be a fixed prime number. Consider the direct sum $A = \bigoplus_{\mathfrak{m}} \mathbf{Z}/p\mathbf{Z}$ of $\mathfrak{m}$ copies of $\mathbf{Z}/p\mathbf{Z}$. Then $d(\widehat{A}) = 2^{\mathfrak{m}}$.
(f) Let Y be an infinite topological space with the discrete topology. Show that
$$|F_{\mathcal{C}}(Y)| = 2^{2^{|Y|}}.$$

In Proposition 3.3.9 we saw that an inverse limit of free pro-$\mathcal{C}$ groups is a free pro-$\mathcal{C}$ group if the canonical mappings in the inverse system send bases to bases. As we shall exhibit later (see Example 9.1.14), a general inverse limit G of free pro-$\mathcal{C}$ groups need not be free pro-$\mathcal{C}$. However, in the following theorem we show that if, in addition, G has a countable fundamental system of neighborhoods of the identity (i.e., $w_0(G) = \aleph_0$), then G is free pro-$\mathcal{C}$.

Theorem 3.5.15 *Let*

$$G = \varprojlim_{i \in I} F_i$$

be an inverse limit of a surjective inverse system of free pro-$\mathcal{C}$ groups (F_i, φ_{ij}) indexed by a poset I. Assume that G admits a countable set of generators converging to 1 (i.e., G is second countable as a topological space). Then G is a free pro-$\mathcal{C}$ group.

Proof. Suppose first that G is finitely generated. Then the free groups F_i have finite rank bounded by $d(G)$, the minimal number of generators of G. It follows that there exists some $i_o \in I$ such that $\operatorname{rank}(F_i) = \operatorname{rank}(F_{i_o})$ if $i \geq i_o$. Therefore, by the Hopfian property (see Proposition 2.5.2), $\varphi_{ii_o} : F_i \longrightarrow F_{i_o}$ is an isomorphism for each $i \geq i_o$. Thus $G \cong F_{i_o}$ is a free pro-$\mathcal{C}$ group.

Assume next that G admits an infinite countable set of generators converging to 1. Let $\mathcal{E} = \mathcal{E}_{\mathcal{C}}$ be the class of all epimorphisms of pro-$\mathcal{C}$ groups. Then, according to Corollary 3.5.10, it suffices to prove that every $\mathcal{E}$-embedding problem for G of the form

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \Big\downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 1 \end{array}$$

has a solution, whenever A is a finite group.

Denote by

$$\varphi_r : G \longrightarrow F_r$$

the canonical epimorphism. Since B is finite, there exists some $r \in I$ and an epimorphism

$$\psi_r : F_r \longrightarrow B$$

such that $\varphi = \psi_r \varphi_r$ (see Lemma 1.1.16). Since G is not finitely generated, we may choose r in such a way that $\operatorname{rank}(F_r) > |A|$. By Theorem 3.5.8, there exists an epimorphism $\mu : F_r \longrightarrow A$ such that $\alpha\mu = \psi_r$. Therefore, $\mu\varphi_r : G \longrightarrow A$ is the desired solution to the above embedding problem. $\square$

3.6 Open Subgroups of Free Pro-$\mathcal{C}$ Groups

In this section we begin the study of the structure of closed subgroups of free pro-$\mathcal{C}$ groups. Unlike the situation for subgroups of abstract free groups, a closed subgroup of a free pro-$\mathcal{C}$ group is not necessarily a free pro-$\mathcal{C}$ group. For example, $\mathbf{Z}_p$ is a closed subgroup of the free profinite group of $\widehat{\mathbf{Z}}$, but obviously $\mathbf{Z}_p$ is not a free profinite group. Nevertheless, we shall describe several types of closed subgroups of a free pro-$\mathcal{C}$ group, and we shall see that in some cases they are free pro-$\mathcal{C}$. We revisit this topic at other places in this book; in particular, in Chapter 7, where we deal with subgroups of free pro-p groups, and in Chapter 8, where we study normal subgroups of free pro-$\mathcal{C}$ groups.

Before we state the next theorem, we fix notation and recall some results about subgroups of abstract free groups. For the details one can consult Magnus, Karras and Solitar [1966], Lyndon and Schupp [1977], or Serre [1980], for example. Let D be an abstract free group on a set X, and let L be a subgroup of D. Recall that a right transversal T of L in D is a complete system of representatives of the right cosets of L in D, so that $D = \dot\bigcup_{t\in T} Lt$; we shall assume that $1 \in T$. Write $t \in T$ as a reduced word in term of the elements of X, i.e., $t = x_1^{\epsilon_1} \cdots x_r^{\epsilon_r}$ for some $x_1, \dots, x_r \in X$, with $\epsilon_i = \pm 1$ for all $i = 1, \dots, r$, and $\epsilon_i = \epsilon_{i+1}$ if $x_i = x_{i+1}$ $(i = 1, \dots, r-1)$. We refer to the elements $x_1^{\epsilon_1} \cdots x_i^{\epsilon_i}$ $(i = 0, \dots, r)$ as the initial segments of $t = x_1^{\epsilon_1} \cdots x_r^{\epsilon_r}$. We say that the transversal T is a *right Schreier transversal* if whenever t is in T, so is any initial segment of t. Every subgroup L of D admits a right Schreier transversal. A final piece of notation: if $f \in D$, denote by $\tilde f$ the unique element $\tilde f \in T$ such that $L\tilde f = Lf$. Then one has the following theorem due to Nielsen and Schreier.

Theorem 3.6.1 *Let D be an abstract free group on a set X, L a subgroup of D, and let T be a right Schreier transversal of L in D. Then L is a free group on the set*

$$\{tx(\widetilde{tx})^{-1} \mid x \in X, t \in T, tx(\widetilde{tx})^{-1} \neq 1\}.$$

Furthermore, if L has finite index in D, then

$$\mathrm{rank}(L) - 1 = [D : L](\mathrm{rank}(D) - 1).$$

Theorem 3.6.2 *Assume that $\mathcal{C}$ is an extension closed variety of finite groups (respectively, an NE-formation of finite groups). Let F be a free pro-$\mathcal{C}$ group on a set X converging to 1, and let H be an open (respectively, open normal) subgroup of F. Then*

(a) *The set*

$$Z = \{tx(\widetilde{tx})^{-1} \mid x \in X, t \in T, tx(\widetilde{tx})^{-1} \neq 1\},$$

converges to 1, where T is an appropriate right transversal of H in F; moreover, H is a free pro-$\mathcal{C}$ group on the set Z.

(b) *If* $\mathrm{rank}(F)$ *is infinite, then* $\mathrm{rank}(H) = \mathrm{rank}(F)$; *while if* $\mathrm{rank}(F)$ *is finite,then so is* $\mathrm{rank}(H)$, *and*

$$\mathrm{rank}(H) - 1 = [F : H](\mathrm{rank}(F) - 1).$$

Proof. Let D be the abstract subgroup of F generated by X. By Corollary 3.3.14 and Proposition 3.3.15, D is an abstract free group with basis X. Choose a Schreier transversal T of $D \cap H$ in D.

Case 1. $X = \{x_1, \dots, x_n\}$ is finite.

As pointed out above, $D \cap H$ is a free abstract group. By Proposition 3.2.2, $\overline{D \cap H} = H$. By Lemmas 3.1.4, 3.2.4 and 3.2.6, H is the pro-$\mathcal{C}$ completion of $D \cap H$; hence H is a free pro-$\mathcal{C}$ group. Then, by Theorem 3.6.1,

$$\{tx(\widetilde{tx})^{-1} \mid x \in X, t \in T, tx(\widetilde{tx})^{-1} \neq 1\}$$

is a basis of $D \cap H$, and so of H (see Proposition 3.3.6). Therefore, using again Theorem 3.6.1, $\mathrm{rank}(H) - 1 = [F : H](\mathrm{rank}(F) - 1)$, as desired.

Case 2. X is an infinite set.

By Proposition 3.3.12, we may express the free pro-$\mathcal{C}$ group $F = F_{\mathcal{C}}(X)$ on the set X converging to 1 as an inverse limit

$$F = \varprojlim_{j \in J} F_{\mathcal{C}}(X_j),$$

with $[F_{\mathcal{C}}(X_j) : \varphi_j(H)] = [F : H]$, for every $j \in J$, where each X_j is a finite subset of X, and $\varphi_j : F \longrightarrow F_{\mathcal{C}}(X_j)$ denotes the canonical epimorphism. Let D_j be the abstract subgroup of $F_{\mathcal{C}}(X_j)$ generated by X_j ($j \in J$). Therefore, $\varphi_j(T) = \{\varphi_j(t) \mid t \in T\}$ is a Schreier transversal of the subgroup $D_j \cap \varphi_j(H)$ in D_j ($j \in J$). Put $\tilde{X} = X \cup \{1\}$ and $\tilde{X}_j = X_j \cup \{1\}$ ($j \in J$). Then $F_{\mathcal{C}}(X_j) = F_{\mathcal{C}}(\tilde{X}_j, 1)$. By Case 1, $\varphi_j(H)$ is a free pro-$\mathcal{C}$ group on the finite pointed space

$$(Y_i, 1) = (\{x\varphi_j(t)(\widetilde{x\varphi_j(t)})^{-1} \mid x \in \tilde{X}_j, t \in T\}, 1).$$

Observe that $\varphi_{jk}(\tilde{Y}_j, 1) = (\tilde{Y}_k, 1)$ ($j \succeq k$), and that

$$H = \varprojlim_{j \in J} \varphi_j(H).$$

Hence, by Proposition 3.3.9, H is a free pro-$\mathcal{C}$ group on the pointed topological space

$$(Y, 1) = (\varprojlim_{j \in J} Y_j, 1).$$

It remains to prove that Y is the one-point compactification of the set Z in the statement. Clearly Z is a discrete subspace of F since X is discrete and T is finite. Moreover, $Z \cup \{1\}$ is compact (it is the continuous image of the compact space $(X \cup \{1\}) \times T$), in fact, it is the one-point compactification of Z. Since $\varphi_j(Z \cup \{1\}) = \tilde{Y}_j$ ($j \in J$), we infer that $Z \cup \{1\} = Y$ (see Corollary 1.1.8). This proves the theorem. □

Corollary 3.6.3 *Let G be a finitely generated profinite group with* $\mathrm{d}(G) = d$ *and let $U \leq_o G$. Then U is also finitely generated as a profinite group and* $\mathrm{d}(U) \leq 1 + [G : U](d - 1)$.

Proof. Consider a free profinite group F of rank d and an epimorphism

$$\varphi : F \longrightarrow G.$$

Then $\varphi(\varphi^{-1}(U)) = U$. So the result follows from Theorem 3.6.2 applied to the open subgroup $\varphi^{-1}(U)$ of F. □

A subgroup H of a group G is called *subnormal* if there exists a finite chain of subgroups of G

$$H = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 = G.$$

If G is profinite and H is closed, we only refer to H as subnormal if there is a chain as above with every G_i closed.

Corollary 3.6.4

(a) *For* $r, i \in \mathbf{N}$, *define* $\mathrm{T}(r,i) = 1 + i(r-1)$. *If* $r, i, j \in \mathbf{N}$, *then*

$$\mathrm{T}(\mathrm{T}(r,i), j) = \mathrm{T}(r, ij).$$

(b) *Let* $\mathcal{C}$ *be an NE-formation of finite groups. Let* F *be a free pro-*$\mathcal{C}$ *group of finite* rank r, *and let* H *be an open subnormal subgroup of* F. *Then* H *is a free pro-*$\mathcal{C}$ *group of* rank $1 + [F : H](r-1)$.

Proof. Part (a) is a routine calculation. Part (b) follows from the theorem and an easy induction. □

3.7 Notes, Comments and Further Reading

Profinite topologies are used sometimes to express some algebraic facts in a succinct manner. For example, an abstract group G is called LERF or *subgroup separable* if every finitely generated subgroup of G is closed in the profinite topology of G (cf. Scott [1978]). In Hall [1949] Theorem 5.1, it is proved that finitely generated subgroups of abstract free groups are closed in the profinite topology; see also Hall [1950]. For a study of the induced topology on the Fitting subgroup of certain groups, see Pickel [1976] and Kilsch [1986].

Lemma 3.1.5 and Corollary 3.1.6 appear in Ribes and Zalesskii [1994]. Corollary 3.2.8 was proved by Dixon, Formanek, Poland and Ribes [1982]. Theorem 3.2.9 appears in Fried and Jarden [2008]. For polycyclic groups with isomorphic finite quotients see Grunewald, Pickel and Segal [1980].

Free pro-$\mathcal{C}$ groups appear in Iwasawa [1953], where $\mathcal{C}$ is a variety of finite groups, although he does not use the name 'free pro-$\mathcal{C}$'. In the same paper (Theorem 8) Iwasawa proves a precursor of the results of Douady and Harbater mentioned in Example 3.3.8(e): let F be an algebraically closed countable field and let K be the maximal solvable extension of $F(T)$; then the Galois group of the extension $K/F(T)$ is a free prosolvable group of countable rank. The first explicit reference to the universal property of freeness for pro-p groups seems to appear in the first edition of Serre's Cohomologie Galoisienne. The first systematic study of free pro-$\mathcal{C}$ groups over topological spaces was began by Gildenhuys and Lim [1972]. At the time it was known, using cohomological methods, that every free pro-p group on a topological space is free on a set converging to 1 (Tate); see Section 7.6. Proposition 3.5.12, showing that this is also the case for general pro-$\mathcal{C}$ groups, was proved by Mel'nikov [1980]. Proposition 3.3.9 appears in Gildenhuys and Lim [1972]. Proposition 3.3.12 was established in Gildenhuys and Ribes [1973]. A version of Theorem 3.3.16 is shown in Iwasawa [1953].

The embedding problem, as indicated in Remark 3.5.2, seems to have been posed first in Brauer [1932]. The literature about the inverse problem of

Galois theory is very extensive. Open Question 3.5.3 has been partially answered in many special cases. Shafarevich [1954] answered it for finite solvable groups (this paper had a difficulty related to the prime number 2, but Shafarevich indicated how to overcome this difficulty shortly after); see Schmidt and Wingberg [1998] for a simplified proof of Shafarevich's result. The book of Matzat [1987] describes the construction of field extensions corresponding to some finite simple groups. See Pop [1996] for the study of embedding problems over certain fields. For a general survey of results and methods see Serre [1992] and Völklein [1996].

Iwasawa [1953] makes a pioneering use of embedding problems for groups to characterize free pro-$\mathcal{C}$ groups of countable rank (see Corollary 3.5.10). This was generalized by Mel'nikov [1978] (see Theorem 3.5.9).

Proposition 3.5.11 was proved by Chatzidakis in her 1984 thesis and appears in Chatzidakis [1998]; this paper contains several other results on free profinite groups. In Jarden [1995], profinite groups with solvable finite embedding problems (i.e., embedding problems such as (1) of Section 3.5, where A is finite) are studied. Theorem 3.5.13 was proved by J-P. Serre (private communication) to answer negatively Open Question 3.5.13 in the first edition of the present book. Theorem 3.5.15 is due to Mel'nikov [1980]. Theorem 3.6.1 appears in Binz, Neukirch and Wenzel [1971]; see a different proof, independent of the Kurosh theorem for abstract groups, in Appendix D, Theorem D.2.2.

Let F be a free nonabelian pro-p group; Zubkov [1987] proves that F cannot be embedded as a closed subgroup of $\mathrm{GL}_2(R)$, if $p \neq 2$ and R is a commutative profinite ring; Barnea and Larsen [1999] show the same result for $\mathrm{GL}_n(F)$, if F is a local field.

3.7.1 A Problem of Grothendieck on Completions

Assume that $\varphi : G_1 \longrightarrow G_2$ is a homomorphism of finitely generated residually finite abstract groups such that the corresponding homomorphism $\widehat{\varphi} : \widehat{G}_1 \longrightarrow \widehat{G}_2$ of the profinite completion is an isomorphism.

Question: Is φ necessarily an isomorphism?

This question was posed in Grothendieck [1970] for groups G_1 and G_2 which in addition are finitely presented. Finite presentability is a natural condition for the groups Grothendieck was studying, namely fundamental groups of certain complex varieties which are compact and locally simply connected; such fundamental groups are finitely presented.

Here we indicate some results related to this question as well as some references. The motivation of Grothendieck was the study of the functor induced by φ

$$\varphi^* : \mathrm{Rep}_A(G_2) \longrightarrow \mathrm{Rep}_A(G_1),$$

where A is a commutative ring and $\mathrm{Rep}_A(G)$ stands for the category of finitely presented A-modules on which the group G operates. Grothendieck [1970], Theorem 1.2, proved that if $\widehat{\varphi}$ is an isomorphism, then φ^* is an equivalence of categories. In this connection see also Lubotzky [1980].

In Platonov and Tavgen [1986] an example was found that answers negatively the above question. This example is based on a construction by Higman [1951] of an infinite finitely presented group with no nontrivial finite quotients. Let F be a free abstract group on a basis $\{x_1, x_2, x_3, x_4\}$. Let N be the smallest normal subgroup of F containing the elements $x_2x_1x_2^{-1}x_1^{-2}, x_3x_2x_3^{-1}x_2^{-2}, x_4x_3x_4^{-1}x_3^{-2}, x_1x_4x_1^{-1}x_4^{-2}$. The group constructed by Higman is F/N. Denote by Δ the diagonal subgroup of the direct product $F \times F$, and consider the subgroup $G_1 = (N \times \{1\})\Delta$ of $G_2 = F \times F$. Then Platonov and Tavgen show that the inclusion $G_1 \longrightarrow G_2$ induces an isomorphism $\widehat{G}_1 \longrightarrow \widehat{G}_2$.

Further examples with negative answers to the question above have been given in Bass and Lubotzky [2000] and Pyber [2004]. All these examples involve groups which do not appear to be finitely presented. Examples with negative answer to Grothendieck's question, i.e., with groups G_1 and G_2 that are finitely presented, are given in Bridson and Grunewald [2004].

Platonov and Tavgen [1990] contains several results showing that in some interesting cases the above question has a positive answer. For example they prove

Theorem 3.7.1 *The above question has a positive answer if G_2 is a subgroup of $\mathrm{SL}_2(K)$, where K is either the field of real or rational numbers.*

In connection with Theorems 3.2.7 and Corollary 3.2.8, one may ask

Open Question 3.7.2 *What pro-$\mathcal{C}$ groups are pro-$\mathcal{C}$ completions of finitely generated abstract groups?*

For partial answers to this question see Segal [2001], and Kassabov and Nikolov [2006].