9 Satellite Gravity Gradiometry (SGG): From Scalar to Tensorial Solution

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Abstract Satellite gravity gradiometry (SGG) is an ultra-sensitive detection technique of the space gravitational gradient (i.e., the Hesse tensor of the Earth's gravitational potential). In this note, SGG – understood as a spacewise inverse problem of satellite technology – is discussed under three mathematical aspects: First, SGG is considered from potential theoretic point of view as a continuous problem of "harmonic downward continuation." The space-borne gravity gradients are assumed to be known continuously over the "satellite (orbit) surface"; the purpose is to specify sufficient conditions under which uniqueness and existence can be guaranteed. In a spherical context, mathematical results are outlined by decomposition of the Hesse matrix in terms of tensor spherical harmonics. Second, the potential theoretic information leads us to a reformulation of the SGG-problem as an ill-posed pseudodifferential equation. Its solution is dealt within classical regularization methods, based on filtering techniques. Third, a very promising method is worked out for developing an immediate interrelation between the Earth's gravitational potential at the Earth's surface and the known gravitational tensor.

1 Introduction

Due to the nonspherical shape, the irregularities of its interior mass density, and the movement of the lithospheric plates, the external gravitational field of the Earth shows significant variations. The recognition of the structure of the Earth's gravitational potential is of tremendous importance for many questions in geosciences, for example, the analysis of present day tectonic motions, the study of the Earth's interior, models of deformation analysis, the determination of the sea surface topography, and circulations of the oceans, which, of course, have a great influence on the global climate and its change. Therefore, a detailed knowledge of the global gravitational field including the local high-resolution microstructure is essential for various scientific disciplines.

Satellite gravity gradiometry (SGG) is a modern domain of studying the characteristics, the structure, and the variation process of the Earth's graviational field. The principle of satellite gradiometry can be explained roughly by the following model (cf. *Fig. 1*): several test masses in a low orbiting satellite feel, due to their distinct positions and the local changes of the gravitational field, different forces, thus yielding different accelerations. The measurements of the relative accerlerations between two test masses provide information about the second-order partial derivatives of the gravitational potential. More concretely, measured are differences between the

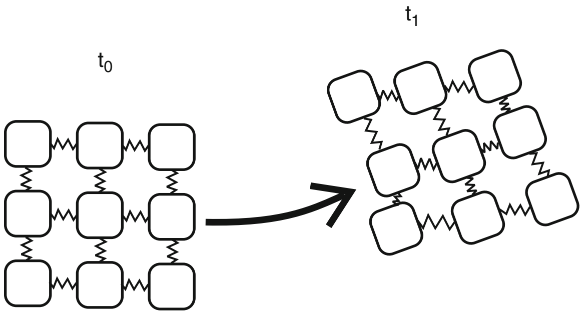


Fig. 1
The principle of a gradiometer

displacements of opposite test masses. This yields information on the differences of the forces. Since the gradiometer itself is small, these differences can be identified with differentials, so that a so-called full gradiometer gives information on the whole tensor consisting out of all second-order partial derivatives of the gravitational potential , i.e., the Hesse matrix. In an ideal case, the full Hesse matrix can be observed by an array of test masses.

On 17 March 2009, the European Space Agency (ESA) began to realize the concept of SGG with the launch of the most sophisticated mission ever to investigate the Earth's gravitational field, viz. GOCE (Gravity and Ocean Circulation Explorer). ESA's 1-ton spacecraft carries a set of six state-of-the-art high-sensitivity accelerometers to measure the components of the gravity field along all three axes (see the contribution of R. Rummel in this issue for more details on the measuring devices of this satellite). GOCE is producing a coverage of the entire Earth with measurements (apart from gaps at the polar regions). For around 20 months GOCE will be gathering gravitational data. In order to make this mission possible, ESA and its partners had to overcome an impressive technical challenge by designing a satellite that is orbiting the Earth close enough (at an altitude of only 250 km) to collect high-accuracy gravitational data while being able to filter out disturbances caused, e.g., by the remaining traces of the atmosphere.

It is not surprising that, during the last decade, the ambitious mission GOCE motivated many scientific activities such that a huge number of written material is available in different fields concerned with special user group activities, mission synergy, calibration as well as validation procedures, geoscientific progress (in fields like gravity field recovery, ocean circulation, hydrology, glacialogy, deformation, climate modeling, etc), data management, and so on. A survey about the recent status is well-demonstrated by the "ESA Living Planet Programme," which also contains a list on GOCE-publications (see also the contribution by the ESA-Frascati Group in this issue, for information from geodetic point of view the reader is referred, e.g., to the notes (Beutler et al. 2003; ESA 1999; ESA 2007; Rummel et al. 1993), too). Mathematically, the literature dealing with the solution procedures of problems related to SGG can be divided essentially into two classes: the timewise approach and the spacewise approach. The former one considers the measured data as a time series, while the second one supposes that the data are given in advance on a (closed) surface.

This chapter is part of the spacewise approach, its goal is a potential theoretically reflected approach to SGG with strong interest in the characterization of SGG-data types and tensorial oriented solution of the occuring (pseudodifferential) SGG-equations by regularization. Particular emphasis is laid on the transition from scalar data types (such as the second-order radial derivative) to full tensor data of the Hesse matrix.

2 SGG in Potential Theoretic Perspective

Gravity as observed on the Earth's surface is the combined effect of the gravitational mass attraction and the centrifugal force due to the Earth's rotation. The force of gravity provides a directional structure to the space above the Earth's surface. It is tangential to the vertical plumb lines and perpendicular to all level surfaces. Any water surface at rest is part of a level surface. As if the Earth were a homogeneous, spherical body gravity turns out to be constant all over the Earth's surface, the well-known quantity 9.8 ms^{-2}. The plumb lines are directed toward the Earth's center of mass, and this implies that all level surfaces are nearly spherical, too. However, the gravity decreases from the poles to the equator by about 0.05 ms^{-2}. This is caused by the flattening of the Earth's figure and the negative effect of the centrifugal force, which is maximal at the equator. Second, high mountains and deep ocean trenches cause the gravity to vary. Third, materials within the Earth's interior are not uniformly distributed. The irregular gravity field shapes as virtual surface the geoid. The level surfaces are ideal reference surfaces, for example, for heights. In more detail, the *gravity acceleration (gravity)* w is the resultant of gravitation v and centrifugal acceleration c, i.e., $w = v + c$. The centrifugal force c arises as a result of the rotation of the Earth about its axis. We assume here a rotation of constant angular velocity ω_0 about the rotational axis x_3, which is further assumed to be fixed with respect to the Earth. The centrifugal acceleration acting on a unit mass is directed outward perpendicularly to the spin axis. If the ε^3-axis of an Earth-fixed coordinate system coincides with the axis of rotation, then we have $c(x) = -\omega_0^2 \varepsilon^3 \wedge \left(\varepsilon^3 \wedge x\right)$. Using the so–called *centrifugal potential* $C(x) = (1/2)\omega_0^2(x_1^2 + x_2^2)$ we can write $c = \nabla C$.

The direction of the gravity w is known as the direction of the *plumb line*, the quantity $|w|$ is called the *gravity intensity* (often just *gravity*). The *gravity potential of the Earth* can be expressed in the form: $W = V + C$. The gravity acceleration w is given by $w = \nabla W = \nabla V + \nabla C$. The surfaces of constant gravity potential $W(x) = \text{const}, x \in \mathbb{R}^3$, are designated as *equipotential (level,* or *geopotential) surfaces of gravity*. The *gravity potential* W of the Earth is the sum of the *gravitational potential* V and the *centrifugal potential* C, i.e., $W = V + C$. In an Earth's fixed coordinate system the centrifugal potential C is explicitly known. Hence, the determination of equipotential surfaces of the potential W is strongly related to the knowledge of the potential V. The gravity vector w given by $w(x) = \nabla_x W(x)$ where the point $x \in \mathbb{R}^3$ is located outside and on a sphere around the origin with Earth's radius R, is normal to the equipotential surface passing through the same point. Thus, equipotential surfaces intuitively express the notion of tangential surfaces, as they are normal to the plumb lines given by the direction of the gravity vector (for more details see, for example, Heiskanen and Moritz (1967), (Freeden and Schreiner 2009) and the contribution by H. Moritz in this issue).

According to the classical Newton's Law of Gravitation (1687), knowing the density distribution ρ of a body, the gravitational potential can be computed everywhere in $\mathbb{R}^3$. More explicitly, the gravitational potential V of the Earth's exterior is given by

$$V(x) = G \int_{\text{Earth}} \frac{\rho(y)}{|x - y|} \, dV(y), \quad x \in \mathbb{R}^3 \backslash \text{Earth}, \tag{1}$$

where G is the gravitational constant ($G = 6.6742 \cdot 10^{-11} \text{m}^3 \text{ kg}^{-1} \text{ s}^{-2}$) and dV is the (Lebesgue-) volume measure. The properties of the gravitational potential (1) in the Earth's exterior are appropriately described by the Laplace equation:

$$\Delta V(x) = 0, \ x \in \mathbb{R}^3 \backslash \text{Earth}. \tag{2}$$

The gravitational potential V as defined by (1) is regular at infinity, i.e.,

$$|V(x)| = O\left(\frac{1}{|x|}\right), |x| \to \infty. \tag{3}$$

For practical purposes, the problem is that in reality the density distribution ρ is very irregular and known only for parts of the upper crust of the Earth. It is actually so that geoscientists would like to know it from measuring the gravitational field. Even if the Earth is supposed to be spherical, the determination of the gravitational potential by integrating Newton's potential is not achievable. This is the reason why, in simplifying spherical nomenclature, we first expand the so-called reciprocal distance in terms of harmonics (related to the Earth's mean radius R) as a series

$$\frac{1}{|x-y|} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{4\pi R}{2n+1} H^R_{-n-1,k}(x) H^R_{n,k}(y), \tag{4}$$

where $H^R_{n,k}$ is an *inner harmonic* of degree n and order k given by

$$H^R_{n,k}(x) = \frac{1}{R}\left(\frac{|x|}{R}\right)^n Y_{n,k}(\xi), \quad x = |x|\xi, \xi \in \Omega, \tag{5}$$

and $H^R_{-n-1,k}$ is an *outer harmonic* of degree n and order k given by

$$H^R_{-n-1,k}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+1} Y_{n,k}(\xi), \quad x = |x|\xi, \xi \in \Omega. \tag{6}$$

Note that the family $\{Y_{n,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ is an $\mathcal{L}^2(\Omega)$-orthonormal system of scalar spherical harmonics (for more details concerning spherical harmonics see, e.g., Müller (1966), Freeden et al. (1998), Freeden and Schreiner 2009). Insertion of the series expansion (4) into the Newton formula for the external gravitational potential yields

$$V(x) = G \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{4\pi R}{2n+1} \int_{\Omega_R^{\text{int}}} \rho(y)\, H^R_{n,k}(y)\, dV(y)\, H^R_{-n-1,k}(x). \tag{7}$$

The expansion coefficients of the series (7) are not computable, since their determination requires the knowledge of the density function ρ in the Earth's interior. In fact, it turns out that there are infinitely many mass distributions, which have the given gravitational potential of the Earth as exterior potential.

Nevertheless, collecting the results from potential theory on the Earth's gravitational field v for the outer space (in spherical approximation) we are confronted with the following (mathematical) characterization: v is an infinitely often differentiable vector field in the exterior of the Earth such that (v1) div $v = \nabla \cdot v = 0$, curl $v = \mathrm{L} \cdot v = 0$ in the Earth's exterior, (v2) v is regular at infinity: $|v(x)| = O\left(1/(|x|^2)\right), |x| \to \infty$. Seen from mathematical point of view, the properties (v1) and (v2) imply that the Earth's gravitational field v in the exterior of the Earth is a gradient field $v = \nabla V$, where the gravitational potential V fulfills the properties: V is an infinitely often differentiable scalar field in the exterior of the Earth such that (V1) V is harmonic in the Earth's exterior, and vice versa. Moreover, the gradient field of the Earth's gravitational field (i.e., the *Jacobi matrix field*) $\mathbf{v} = \nabla \mathbf{v}$, obeys the following properties: $\mathbf{v}$ is an infinitely often differentiable tensor field in the exterior of the Earth such that (**v**1) div $\mathbf{v} = \nabla \cdot \mathbf{v} = 0$, curl $\mathbf{v} = \mathrm{L} \cdot \mathbf{v} = 0$ in the Earth's exterior, (**v**2) $\mathbf{v}$ is regular at infinity: $|\mathbf{v}(\mathbf{x})| = O\left(1/(|x|^3)\right), |x| \to \infty$, and vice versa. Combining our identities we finally see that $\mathbf{v}$ can be represented as the *Hesse tensor of the scalar field* V, i.e., $\mathbf{v} = (\nabla \otimes \nabla)\, \mathbf{V} = \nabla^{(2)} \mathbf{V}$.

The technological SGG-principle of determining the tensor field $\mathbf{v}$ at satellite altitude is illustrated graphically in ➋ *Fig. 2*. The position of a low orbiting satellite is tracked using GPS. Inside the satellite there is a gradiometer. A simplified model of a gradiometer is sketched in ➋ *Fig. 1*. An array of test masses is connected with springs. Once more, the measured quantities are the differences between the displacements of opposite test masses. According to Hooke's law the mechanical configuration provides information on the differences of the forces. They, however, are due to local differences of ∇V. Since the gradiometer itself is small, these differences can be identified with differentials, so that a so-called full gradiometer gives information on the whole tensor consisting out of all second order partial derivatives of V, i.e., the Hesse matrix $\mathbf{v}$ of V.

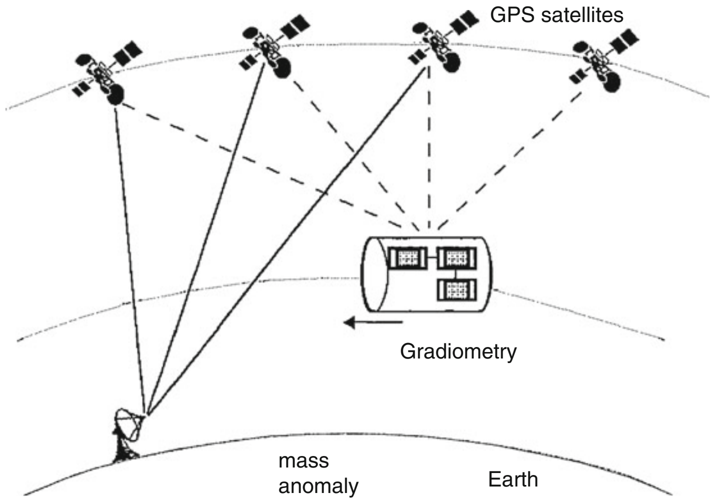


Fig. 2
The principle of satellite gravity gradiometry (from ESA (1999))

From our preparatory remarks it becomes obvious that the potential theoretic situation for the SGG problem can be formulated briefly as follows: Suppose that the satellite data $\mathbf{v} = (\nabla \otimes \nabla)\,\mathbf{V}$ are known continuously over the "orbital surface," the satellite gravity gradiometry problem amounts to the problem of determining V from $\mathbf{v} = (\nabla \otimes \nabla)\,\mathbf{V}$ at the "orbital surface."

Mathematically, SGG is a nonstandard problem of potential theory. The reasons are obvious:

- SGG is ill–posed since the data are not given on the boundary of the domain of interest, i.e., on the Earth's surface, but on a surface in the exterior domain of the Earth, i.e., at a certain height.
- Tensorial SGG-data (or scalar manifestations of them) do not form the standard equipment of potential theory (such as, e.g., Dirichlet or Neumann data). Thus, it is – at first sight – not clear whether these data ensure the uniqueness of the SGG-problem or not.
- SGG-data have its natural limit because of the strong damping of the high-frequency parts of the (spherical harmonic expansion of the) gravitational potential with increasing satellite heights. For a heuristic explanation of this calamity, let us start from the assumption that the gravitational potential outside the spherical Earth's surface Ω_R with the mean radius R is given by the ordinary expansion in terms of outer harmonics (confer the identity (7))

$$V(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \int_{\Omega_R} V(y)\, H^R_{-n-1,k}(y)\, d\omega(y)\, H^R_{-n-1,k}(x) \qquad (8)$$

($d\omega$ is the usual surface measure). Then it is not hard to see that those parts of the gravitational potential belonging to the outer harmonics $H^R_{-n-1,k}$ of order n at height H above the Earth's surface Ω_R are damped by a factor $[R/(R+H)]^{n+1}$. Just a way out of this difficulty is seen in SGG, where, e.g., second-order radial derivatives of the gravitational potential are available at a height of typically about 250 km. The second derivatives cause (roughly speaking) an amplification of the potential coefficients by a factor of order n^2. This compensates the damping effect due to the satellite's height if n is not too large. Nevertheless, in spite of the amplification, the SGG-problem still remains (exponentially) ill-posed. Altogether, the graviational potential decreases exponentially with increasing height, and therefore the process of transforming, the data down to the Earth surface (often called "downward continuation") is unstable.

The non-canonical (SGG)-situation of uniqueness within the potential theoretic framework can be demonstrated already by a simple example in spherical context: Suppose that one scalar component of the Hesse tensor is prescribed for all points x at the sphere $\Omega_{R+H} = \{x \in \mathbb{R}^3 \mid |x| = R + H\}$. Is the gravitational potential V unique on the sphere $\Omega_R = \{x \in \mathbb{R}^3 \mid |x| = R\}$? The answer is not positive, in general. To see this, we construct a counterexample: If $b \in \mathbb{R}^3$ with $|b| = 1$ is given, the second-order directional derivative of V at the point x is $b^T \nabla \otimes \nabla V(x) b$. Given a potential V, we construct a vector field b on Ω_{R+H}, such that the second-order directional derivative $b^T \nabla \otimes \nabla V b$. is zero: Assume that V is a solution of (2) and (3). For each $x \in \Omega_{R+H}$, we know that the Hesse tensor $\nabla \otimes \nabla V(x)$ is symmetric. Thus, there exists an orthogonal matrix $A(x)$ so that $A(x)^T(\nabla\otimes\nabla V(x))A(x) = \mathrm{diag}(\lambda_1(x), \lambda_2(x), \lambda_3(x))$, where $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ are the eigenvalues of $\nabla \otimes \nabla V(x)$. From the harmonicity of V it is clear that $0 = \Delta V(x) = \lambda_1(x) + \lambda_2(x) + \lambda_3(x)$. Let $\mu_0 = 3^{-1/2}(1,1,1)^T$. We define the vector field $\mu : \Omega_{R+H} \to \mathbb{R}^3$ by $\mu(x) = A(x)\mu_0$, $x \in \Omega_{R+H}$. Then we obtain

$$\begin{aligned} \mu^T(x)(\nabla \otimes \nabla V(x))\mu(x) &= \mu_0^T A(x)^T (\nabla \otimes \nabla V(x)) A(x) \mu_0 \qquad (9) \\ &= \frac{1}{3}(1\,1\,1) \begin{pmatrix} \lambda_1(x) & 0 & 0 \\ 0 & \lambda_2(x) & 0 \\ 0 & 0 & \lambda_3(x) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{3}(\lambda_1(x) + \lambda_2(x) + \lambda_3(x)) \\ &= 0. \qquad (10) \end{aligned}$$

Hence, we have constructed a vector field μ such that the second-order directional derivative of V in the direction of $\mu(x)$ is zero for every point $x \in \Omega_{R+H}$. It can be easily seen that, for a given V, there exist many vector fields showing the same properties for uniqueness as the vector field μ. Observing these arguments we are led to the conclusion that the function V is undetectable from the directional derivatives corresponding to μ (see also Schreiner 1994a,b).

It is, however, good news that we are not lost here: As a matter of fact, there do exist conditions under which only one quantity of the Hesse tensor yields a unique solution (at least up to low order harmonics). In order to formulate these results, a certain decomposition of the Hesse tensor is necessary, which strongly depends on the separation of the Laplace operator in terms of polar coordinates. In order to follow this path, we start to reformulate the SGG-problem more easily in spherical context. For that purpose we start with some basic facts specifically formulated on the unit sphere $\Omega = \{x \in \mathbb{R}^3 \mid |x| = 1\}$: As is well-known, any $x \in \mathbb{R}^3, x \neq 0$, can be decomposed uniquely in the form $x = r\xi$, where the directional part is an element of the unit sphere: $\xi \in \Omega$. Let $\{Y_{n,m}\} : \Omega \to \mathbb{R}^3$, $n = 0,1,\ldots$, $m = 1,\ldots,2n+1$, be an orthonormal set of spherical harmonics. As is well–known (see, e.g., Freeden and Schreiner 2009), the system is complete in $\mathcal{L}^2(\Omega)$, hence, each function $F \in \mathcal{L}^2(\Omega)$ can be represented by the spherical harmonic expansion

$$F(\xi) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} F^{\wedge}(n,m) Y_{n,m}(\xi), \quad \xi \in \Omega, \qquad (11)$$

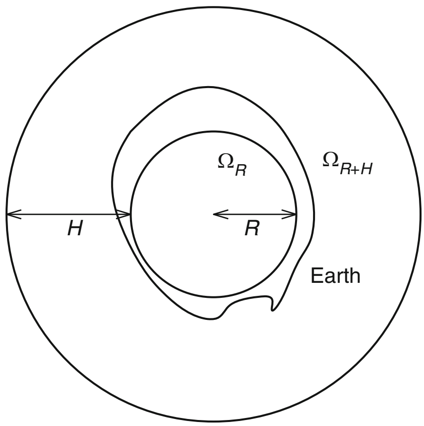
with "Fourier coefficients" given by

$$F^{\wedge}(n,m) = (F, Y_{n,m})_{\mathcal{L}^2(\Omega)} = \int_{\Omega} F(\xi) Y_{n,m}(\xi)\, d\omega(\xi). \tag{12}$$

Furthermore, the (outer) harmonics $H_{-n-1,m} : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$ related to the unit sphere Ω are denoted by $H_{-n-1,m}(x) = H^1_{-n-1,m}(x)$, where $H^1_{-n-1,m}(x) = (1/|x|^{n+1}) Y_{n,m}(x/|x|)$. Clearly, they are harmonic functions and their restrictions coincide on Ω with the corresponding spherical harmonics. Any function $F \in \mathcal{L}^2(\Omega)$ can, thus, be identified with a harmonic potential via the expansion (11), in particular, this holds true for the Earth's external gravitational potential. This motivates the following mathematical model situation of the SGG-problem to be considered next:

(i) *Isomorphism*: Consider the sphere $\Omega_R \subset \mathbb{R}^3$ around the origin with radius $R > 0$. Ω_R^{int} is the inner space of Ω_R, and Ω_R^{ext} is the outer space. By virtue of the isomorphism $\Omega \ni \xi \mapsto R\xi \in \Omega_R$ we assume functions $F : \Omega_R \to \mathbb{R}$ to be defined on Ω. It is clear that the function spaces defined on Ω admit their natural generalizations as spaces of functions defined on Ω_R. Obviously, an $\mathcal{L}^2(\Omega)$–orthonormal system of spherical harmonics forms an orthogonal system on Ω_R (with respect to $(\cdot,\cdot)_{\mathcal{L}^2(\Omega_R)}$). Moreover, with the relationship $\xi \leftrightarrow R\xi$, the differential operators on Ω_R can be related to operators on the unit sphere Ω. In more detail, the *surface gradient* $\nabla^{*;R}$, the *surface curl gradient* $\mathrm{L}^{*;\mathrm{R}}$ and the *Beltrami operator* $\Delta^{*;R}$ on Ω_R, respectively, admit the representation $\nabla^{*;R} = (1/R)\nabla^{*;1} = (1/R)\nabla^*$, $\mathrm{L}^{*;\mathrm{R}} = (1/\mathrm{R})\mathrm{L}^{*;1} = (1/\mathrm{R})\mathrm{L}^*$, $\Delta^{*;R} = (1/R^2)\Delta^{*;1} = (1/R^2)\Delta^*$, where ∇^*, L^*, Δ^* are the surface gradient, surface curl gradient, and the Beltrami operator of the unit sphere Ω, respectively. For Y_n being a spherical harmonic of degree n we have $\Delta^{*;R} Y_n = -(1/R^2)\, n(n+1) Y_n = -(1/R^2)\, \Delta^* Y_n$.

(ii) *Runge Property*: Instead of looking for a harmonic function outside and on the (real) Earth, we search for a harmonic function outside the unit sphere Ω (assuming the units are chosen in such a way that the sphere Ω with radius 1 is inside of the Earth and at the same time not too "far away" from the Earth's boundary). The justification of this simplification (see ➋ *Fig. 3*) is based on the Runge approach (see, e.g., Freeden 1980a; Freeden and Michel 2004 as well as the remarks in ➋ Chap. 6 of this handbook): To any harmonic function V



◘ Fig. 3
The role of the "Runge sphere" within the spherically reflected SGG-problem

outside of the (real) Earth and any given $\varepsilon > 0$, there exists a harmonic function U outside of the unit sphere inside the (real) Earth such that the absolute error $|V(x) - U(x)| < \varepsilon$ holds true for all points x outside and on the (real) Earth's surface.

3 Decomposition of Tensor Fields by Means of Tensor Spherical Harmonics

Let us recapitulate that any point $\xi \in \Omega$ may be represented by polar coordinates in a standard way

$$\xi = t\varepsilon^3 + \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2), \quad -1 \le t \le 1, \quad 0 \le \varphi < 2\pi, \quad t = \cos\vartheta, \tag{13}$$

($\vartheta \in [0, \pi]$: (co-)latitude, φ: longitude, t: polar distance). Consequently, any element $\xi \in \Omega$ may be represented using its coordinates (φ, t) in accordance with (13).

For the representation of vector and tensor fields on the unit sphere Ω, we are led to use a local triad of orthonormal unit vectors in the directions r, φ, and t as shown by ➋ *Fig. 4* (for more details the reader is referred to Freeden and Schreiner (2009) and the references therein).

As is well–known, the second-order tensor fields on the unit sphere, i.e., $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, can be separated into their tangential and normal parts as follows:

$$\mathbf{p}_{*,\mathrm{nor}}\mathbf{f} = (\mathbf{f}\xi) \otimes \xi, \tag{14}$$

$$\mathbf{p}_{\mathrm{nor},*}\mathbf{f} = \xi \otimes (\xi^{\mathrm{T}}\mathbf{f}), \tag{15}$$

$$\mathbf{p}_{*,\tan}\mathbf{f} = \mathbf{f} - \mathbf{p}_{*,\mathrm{nor}}\mathbf{f} = f - (\mathbf{f}\xi) \otimes \xi, \tag{16}$$

$$\mathbf{p}_{\tan,*}\mathbf{f} = \mathbf{f} - \mathbf{p}_{\mathrm{nor},*}\mathbf{f} = \mathbf{f} - \xi \otimes (\xi^{\mathrm{T}}\mathbf{f}), \tag{17}$$

$$\begin{aligned}\mathbf{p}_{\mathrm{nor},\tan}\mathbf{f} &= \mathbf{p}_{\mathrm{nor},*}(\mathbf{p}_{*,\tan}\mathbf{f}) = \mathbf{p}_{*,\tan}(\mathbf{p}_{\mathrm{nor},*}\mathbf{f}) \\ &= \xi \otimes (\xi^{\mathrm{T}}\mathbf{f}) - (\xi^T\mathbf{f}\xi)\xi \otimes \xi.\end{aligned} \tag{18}$$

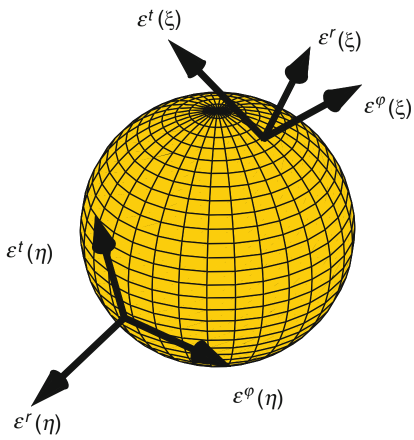


Fig. 4
Local triads $\varepsilon^r, \varepsilon^\varphi, \varepsilon^t$ with respect to two different points ξ and η on the unit sphere

The operators $\mathbf{p}_{\mathrm{nor,nor}}$, $\mathbf{p}_{\mathrm{tan,nor}}$, and $\mathbf{p}_{\mathrm{tan,tan}}$ are defined analogously. A vector field $\mathbf{f} : \Omega \to \mathbb{R} \otimes \mathbb{R}$ is called normal if $\mathbf{f} = \mathbf{p}_{\mathrm{nor,nor}}\mathbf{f}$ and tangential if $\mathbf{f} = \mathbf{p}_{\mathrm{tan,tan}}\mathbf{f}$. It is called left normal if $\mathbf{f} = \mathbf{p}_{\mathrm{nor},*}\mathbf{f}$, left normal/right tangential if $\mathbf{f} = \mathbf{p}_{\mathrm{nor,tan}}\mathbf{f}$, and so on.

The constant tensor fields $\mathbf{i}_{\mathrm{tan}}$ and $\mathbf{j}_{\mathrm{tan}}$ can be defined using the local triads by

$$\mathbf{i}_{\mathrm{tan}} = \varepsilon^{\varphi} \otimes \varepsilon^{\varphi} + \varepsilon^{t} \otimes \varepsilon^{t}, \quad \mathbf{j}_{\mathrm{tan}} = \xi \wedge \mathbf{i}_{\mathrm{tan}} = \varepsilon^{t} \otimes \varepsilon^{\varphi} - \varepsilon^{\varphi} \otimes \varepsilon^{t}. \tag{19}$$

Spherical tensor fields can be discussed in an elegant manner by the use of certain differential processes. Let u be a continuously differentiable vector field on Ω, i.e., $u \in \mathrm{c}^{(1)}(\Omega)$, given in its coordinate form by

$$u(\xi) = \sum_{i=1}^{3} U_i(\xi)\varepsilon^{i}, \quad \xi \in \Omega, \quad U_i \in \mathcal{C}^{(1)}(\Omega). \tag{20}$$

Then we define the operators $\nabla^* \otimes$ and $\mathrm{L}^* \otimes$ by

$$\nabla_{\xi}^* \otimes u(\xi) = \sum_{i=1}^{3} \left(\nabla_{\xi}^* U_i(\xi)\right) \otimes \varepsilon^{i}, \quad \xi \in \Omega, \tag{21}$$

$$\mathrm{L}_{\xi}^* \otimes u(\xi) = \sum_{i=1}^{3} \left(\mathrm{L}_{\xi}^* U_i(\xi)\right) \otimes \varepsilon^{i}, \quad \xi \in \Omega. \tag{22}$$

Clearly, $\nabla^* \otimes u$ and $\mathrm{L}^* \otimes u$ are left tangential. But it is an important fact, that even if u is tangential, the tensor fields $\nabla^* \otimes u$ and $\mathrm{L}^* \otimes u$ are generally not tangential. It is obvious, that the product rule is valid. To be specific, let $F \in \mathrm{C}^{(1)}(\Omega)$ and $u \in \mathrm{c}^{(1)}(\Omega)$ (once more, note that the notation $u \in \mathrm{c}^{(1)}(\Omega)$ means that the vector field u is a continuously differentiable on Ω), then

$$\nabla_{\xi}^* \otimes (F(\xi)u(\xi)) = \nabla_{\xi}^* F(\xi) \otimes u(\xi) + F(\xi)\nabla_{\xi}^* \otimes u(\xi), \quad \xi \in \Omega. \tag{23}$$

In view of the above equations and definitions we accordingly introduce operators $\mathbf{o}^{(i,k)} : \mathrm{C}^{(2)}(\Omega) \to \mathbf{c}^{(0)}(\Omega)$ (note that $\mathbf{c}^{(0)}(\Omega)$ is the class of continuous second-order tensor fields on the unit sphere Ω) by

$$\mathbf{o}_{\xi}^{(1,1)} F(\xi) = \xi \otimes \xi F(\xi), \tag{24}$$

$$\mathbf{o}_{\xi}^{(1,2)} F(\xi) = \xi \otimes \nabla_{\xi}^* F(\xi), \tag{25}$$

$$\mathbf{o}_{\xi}^{(1,3)} F(\xi) = \xi \otimes \mathrm{L}_{\xi}^* F(\xi), \tag{26}$$

$$\mathbf{o}_{\xi}^{(2,1)} F(\xi) = \nabla_{\xi}^* F(\xi) \otimes \xi, \tag{27}$$

$$\mathbf{o}_{\xi}^{(3,1)} F(\xi) = \mathrm{L}_{\xi}^* F(\xi) \otimes \xi, \tag{28}$$

$$\mathbf{o}_{\xi}^{(2,2)} F(\xi) = \mathbf{i}_{\mathrm{tan}}(\xi) F(\xi), \tag{29}$$

$$\mathbf{o}_{\xi}^{(2,3)} F(\xi) = \left(\nabla_{\xi}^* \otimes \nabla_{\xi}^* - \mathrm{L}_{\xi}^* \otimes L_{\xi}^*\right) F(\xi) + 2\nabla_{\xi}^* F(\xi) \otimes \xi, \tag{30}$$

$$\mathbf{o}_{\xi}^{(3,2)} F(\xi) = \left(\nabla_{\xi}^* \otimes \mathrm{L}_{\xi}^* + \mathrm{L}_{\xi}^* \otimes \nabla_{\xi}^*\right) F(\xi) + 2\mathrm{L}_{\xi}^* F(\xi) \otimes \xi, \tag{31}$$

$$\mathbf{o}_{\xi}^{(3,3)} F(\xi) = \mathbf{j}_{\mathrm{tan}}(\xi) F(\xi), \tag{32}$$

$\xi \in \Omega$.

After our preparations involving spherical second-order tensor fields it is not difficult to prove the following lemma.

Lemma 3.1
Let $F : \Omega \to \mathbb{R}$ be sufficiently smooth. Then the following statements are valid:

1. $\mathbf{o}^{(1,1)}F$ *is a normal tensor field.*
2. $\mathbf{o}^{(1,2)}F$ *and* $\mathbf{o}^{(1,3)}F$ *are left normal/right tangential.*
3. $\mathbf{o}^{(2,1)}F$ *and* $\mathbf{o}^{(3,1)}F$ *are left tangential/right normal.*
4. $\mathbf{o}^{(2,2)}F$, $\mathbf{o}^{(2,3)}F$, $\mathbf{o}^{(3,2)}F$ *and* $\mathbf{o}^{(3,3)}F$ *are tangential.*
5. $\mathbf{o}^{(1,1)}F$, $\mathbf{o}^{(2,2)}F$, $\mathbf{o}^{(2,3)}F$ *and* $\mathbf{o}^{(3,2)}F$ *are symmetric.*
6. $\mathbf{o}^{(3,3)}F$ *is skew-symmetric.*
7. $\left(\mathbf{o}^{(1,2)}F\right)^T = \mathbf{o}^{(2,1)}F$ *and* $\left(\mathbf{o}^{(1,3)}F\right)^T = \mathbf{o}^{(3,1)}F$.
8. *For* $\xi \in \Omega$

$$\text{trace } \mathbf{o}_\xi^{(i,k)}F(\xi) = \begin{cases} F(\xi) & \textit{for} \quad (i,k) = (1,1) \\ 2F(\xi) & \textit{for} \quad (i,k) = (2,2) \\ 0 & \textit{for} \quad (i,k) \neq (1,1),\ (2,2) \end{cases} .$$

The tangent representation theorem (cf. Backus 1966, 1967) asserts that if $\mathbf{p}_{\tan,\tan}\mathbf{f}$ is the tangential part of a tensor field $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$, as defined above, then there exist unique scalar fields $F_{2,2}, F_{3,3}, F_{2,3}, F_{3,2}$ such that

$$\int_\Omega F_{2,2}(\xi)\, d\omega(\xi) = \int_\Omega F_{3,3}(\xi)\, d\omega(\xi) = 0, \tag{33}$$

$$\int_\Omega F_{3,2}(\xi)(\varepsilon^i \cdot \xi)\, d\omega(\xi) = \int_\Omega F_{2,3}(\xi)(\varepsilon^i \cdot \xi)\, d\omega(\xi) = 0, \quad i = 1,2,3, \tag{34}$$

and

$$\mathbf{p}_{\tan,\tan}\mathbf{f} = \mathbf{o}^{(2,2)}F_{2,2} + \mathbf{o}^{(2,3)}F_{2,3} + \mathbf{o}^{(3,2)}F_{3,2} + \mathbf{o}^{(3,3)}F_{3,3}. \tag{35}$$

Furthermore, the following orthogonality relations may be formulated: Let $F, G : \Omega \to \mathbb{R}$ be sufficiently smooth. Then for all $\xi \in \Omega$, we have $\mathbf{o}_\xi^{(i,k)}F(\xi) \cdot \mathbf{o}_\xi^{(i',k')}F(\xi) = 0$ whenever $(i,k) \neq (i',k')$. The adjoint operators $O^{(i,k)}$ satisfying

$$\int_\Omega \mathbf{o}^{(i,k)}F(\xi) \cdot \mathbf{f}(\xi)\, d\omega(\xi) = \int_\Omega F(\xi)\, O^{(i,k)}\mathbf{f}(\xi)\, d\omega(\xi), \tag{36}$$

for all sufficiently smooth functions $F : \Omega \to \mathbb{R}$ and tensor fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ can be deduced by elementary calculations. In more detail, for $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$, we find (cf. Freeden and Schreiner 2009)

$$O_\xi^{(1,1)}\mathbf{f}(\xi) = \xi^T \mathbf{f}(\xi)\xi, \tag{37}$$

$$O_\xi^{(1,2)}\mathbf{f}(\xi) = -\nabla_\xi^* \cdot p_{\tan}\left(\xi^T \mathbf{f}(\xi)\right), \tag{38}$$

$$O_\xi^{(1,3)}\mathbf{f}(\xi) = -\mathrm{L}_\xi^* \cdot p_{\tan}\left(\xi^T \mathbf{f}(\xi)\right), \tag{39}$$

$$O_\xi^{(2,1)}\mathbf{f}(\xi) = -\nabla_\xi^* \cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \tag{40}$$

$$O_\xi^{(3,1)}\mathbf{f}(\xi) = -\mathrm{L}_\xi^* \cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \tag{41}$$

$$O_\xi^{(2,2)}\mathbf{f}(\xi) = \mathbf{i}_{\tan}(\xi)\cdot\mathbf{f}(\xi), \tag{42}$$

$$\begin{aligned} O_\xi^{(2,3)}\mathbf{f}(\xi) = {} & \nabla_\xi^* \cdot p_{\tan}\left(\nabla_\xi^* \cdot \mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) - \mathrm{L}_\xi^* \cdot p_{\tan}\left(\mathrm{L}_\xi^* \cdot \mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) \\ & - 2\nabla_\xi^* \cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \end{aligned} \tag{43}$$

$$\begin{aligned} O_\xi^{(3,2)}\mathbf{f}(\xi) = {} & \mathrm{L}_\xi^* \cdot p_{\tan}\left(\nabla_\xi^* \cdot \mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) + \nabla_\xi^* \cdot p_{\tan}\left(\mathrm{L}_\xi^* \cdot \mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) \\ & - 2\mathrm{L}_\xi^* \cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \end{aligned} \tag{44}$$

$$O_\xi^{(3,3)}\mathbf{f}(\xi) = \mathbf{j}_{\tan}(\xi)\cdot\mathbf{f}(\xi), \tag{45}$$

$\xi \in \Omega$. Provided that $F : \Omega \to \mathbb{R}$ is sufficiently smooth we see that

$$O_\xi^{(i',k')}\mathbf{o}_\xi^{(i,k)}F(\xi) = 0 \ \text{ if } (i,k) \neq (i',k'), \tag{46}$$

whereas

$$O_\xi^{(i,k)}\mathbf{o}_\xi^{(i,k)}F(\xi) = \begin{cases} F(\xi) & \text{if } (i,k) = (1,1) \\ -\Delta^* F(\xi) & \text{if } (i,k) \in \{(1,2),(1,3) \\ & \qquad\qquad (2,1),(3,1)\} \\ 2F(\xi) & \text{if } (i,k) \in \{(2,2),(3,3)\} \\ 2\Delta^*(\Delta^*+2)F(\xi) & \text{if } (i,k) \in \{(2,3),(3,2)\}. \end{cases} \tag{47}$$

Using this set of operators we can find explicit formulas for the functions $F_{i,k}$ in the tensor decomposition theorem.

Theorem 3.2

Helmholtz decomposition theorem: Let $\mathbf{f}$ be of class $\mathbf{c}^{(2)}(\Omega)$. Then there exist uniquely defined functions $F_{i,k} \in \mathrm{C}^{(2)}(\Omega)$, $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$ with $(F_{i,k}, Y_0)_{\mathcal{L}^2(\Omega)} = 0$ for all spherical harmonic Y_0 of degree 0, if $(i,k) \in \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$ and $(F_{i,k}, Y_1)_{\mathcal{L}^2(\Omega)} = 0$ for all spherical harmonics Y_1 of degree 1 if $(i,k) \in \{(2,3),(3,2)\}$, in such a way that

$$\mathbf{f} = \sum_{i,k=1}^{3} \mathbf{o}^{(i,k)}F_{i,k}, \tag{48}$$

where the functions $\xi \mapsto F_{i,k}(\xi)$, $\xi \in \Omega$, are explicitly given by

$$F_{1,1}(\xi) = O_\xi^{(1,1)}\mathbf{f}(\xi), \tag{49}$$

$$F_{2,2}(\xi) = \frac{1}{2}O_\xi^{(2,2)}\mathbf{f}(\xi), \tag{50}$$

$$F_{3,3}(\xi) = \frac{1}{2}O_\xi^{(3,3)}\mathbf{f}(\xi), \tag{51}$$

$$F_{1,2}(\xi) = -\int_{\Omega} G(\Delta^*;\xi,\eta) O_\eta^{(1,2)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{52}$$

$$F_{1,3}(\xi) = -\int_{\Omega} G(\Delta^*;\xi,\eta) O_\eta^{(1,3)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{53}$$

$$F_{2,1}(\xi) = -\int_{\Omega} G(\Delta^*;\xi,\eta) O_\eta^{(2,1)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{54}$$

$$F_{3,1}(\xi) = -\int_{\Omega} G(\Delta^*;\xi,\eta) O_\eta^{(3,1)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{55}$$

$$F_{2,3}(\xi) = \int_{\Omega} G(\Delta^*(\Delta^*+2);\xi,\eta) O_\eta^{(2,3)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{56}$$

$$F_{3,2}(\xi) = \int_{\Omega} G(\Delta^*(\Delta^*+2);\xi,\eta) O_\eta^{(3,2)} \mathbf{f}(\eta)\, d\omega(\eta). \tag{57}$$

The functions $G(\Delta^*;\cdot,\cdot)$ and $G(\Delta^*(\Delta^*+2);\cdot,\cdot)$ are the Green functions to the Beltrami operator Δ^* and its iteration $\Delta^*(\Delta^*+2)$, respectively. For more details concerning the Green functions we refer to Freeden (1980b) and Freeden and Schreiner (2009).

The decomposition (Theorem 3.2) will be of crucial importance to verify uniqueness results for the satellite gravity gradiometry problem in spherical context.

4 Solution as Pseudodifferential Equation

Suppose that the function $H : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is twice continuously differentiable. We want to show how the Hesse matrix restricted to the unit sphere Ω, i.e.,

$$\mathbf{h}(\xi) = \nabla_x \otimes \nabla_x H(x)|_{|x|=1}, \quad \xi \in \Omega, \tag{58}$$

can be decomposed according to the rules of Theorem 3.2. In order to evaluate

$$\nabla_x \otimes \nabla_x H(x) = \left(\xi \frac{\partial}{\partial r} + \frac{1}{r}\nabla_\xi^*\right) \otimes \left(\xi \frac{\partial}{\partial r} + \frac{1}{r}\nabla_\xi^*\right) H(r\xi), \tag{59}$$

we first see that

$$\xi \frac{\partial}{\partial r} \otimes \xi \frac{\partial}{\partial r} H(r\xi) = \xi \otimes \xi \left(\frac{\partial}{\partial r}\right)^2 H(r\xi), \tag{60}$$

$$\xi \frac{\partial}{\partial r} \otimes \frac{1}{r}\nabla_\xi^* H(r\xi) = -\frac{1}{r^2}\xi \otimes \nabla_\xi^* H(r\xi) + \frac{1}{r}\xi \otimes \nabla_\xi^* \frac{\partial}{\partial r} H(r\xi), \tag{61}$$

$$\frac{1}{r}\nabla_\xi^* \otimes \xi \frac{\partial}{\partial r} H(r\xi) = \frac{1}{r}\mathbf{i}_{\tan}(\xi) \frac{\partial}{\partial r} H(r\xi) + \frac{1}{r}\nabla_\xi^* \left(\frac{\partial}{\partial r} H(r\xi)\right) \otimes \xi, \tag{62}$$

$$\frac{1}{r}\nabla_\xi^* \otimes \frac{1}{r}\nabla_\xi^* H(r\xi) = \frac{1}{r^2}\nabla_\xi^* \otimes \nabla_\xi^* H(r\xi). \tag{63}$$

Summing up these terms we find (cf. Freeden and Schreiner (2009))

$$\begin{aligned}\nabla_x \otimes \nabla_x H(x)|_{|x|=1} &= \xi \otimes \xi \left(\frac{\partial}{\partial r}\right)^2 H(r\xi)|_{r=1} + \xi \otimes \nabla_\xi^* \left(\frac{\partial}{\partial r} H(r\xi)|_{r=1} - H(\xi)\right) \\ &\quad + \left(\nabla^* \frac{\partial}{\partial r} H(r\xi)|_{r=1}\right) \otimes \xi + \nabla_\xi^* \otimes \nabla_\xi^* H(\xi) + \mathbf{i}_{\tan}(\xi) \frac{\partial}{\partial r} H(r\xi)|_{r=1}. \end{aligned} \tag{64}$$

Using the identities (60)–(63) and the definition of the $\mathbf{o}^{(i,k)}$-operators we are able to write

$$\nabla_x \otimes \nabla_x H(x)|_{|x|=1} = \mathbf{o}_\xi^{(1,1)} \left(\left(\frac{\partial}{\partial r} \right)^2 H(r\xi)|_{r=1} \right) + \mathbf{o}_\xi^{(1,2)} \left(\frac{\partial}{\partial r} H(r\xi)|_{r=1} - H(\xi) \right)$$
$$+ \mathbf{o}_\xi^{(2,1)} \left(\frac{\partial}{\partial r} H(r\xi)|_{r=1} - H(\xi) \right) + \mathbf{o}_\xi^{(2,2)} \left(\frac{1}{2} \Delta_\xi^* H(\xi) + \frac{\partial}{\partial r} H(r\xi)|_{r=1} \right)$$
$$+ \mathbf{o}_\xi^{(2,3)} \frac{1}{2} H(\xi). \tag{65}$$

In particular, if we consider an outer harmonic $H_{-n-1,m} : x \mapsto H_{-n-1,m}(x)$ with $H_{-n-1,m}(r\xi) = r^{-(n+1)} Y_{n,m}(\xi)$, $r > 0$, $\xi \in \Omega$, we obtain the following decomposition of the Hesse matrix on the sphere Ω_{R+H}, i.e., for $x \in \mathbb{R}^3$ with $|x| = R + H$:

$$\nabla \otimes \nabla H_{-n-1,m}((R+H)\,\xi) = (n+1)(n+2) \frac{1}{(R+H)^{n+3}} \mathbf{o}_\xi^{(1,1)} Y_{n,m}(\xi)$$
$$- (n+2) \frac{1}{(R+H)^{n+3}} \left(\mathbf{o}_\xi^{(1,2)} Y_{n,m}(\xi) + \mathbf{o}_\xi^{(2,1)} Y_{n,m}(\xi) \right)$$
$$- \frac{(n+1)(n+2)}{2} \frac{1}{(R+H)^{n+3}} \mathbf{o}_\xi^{(2,2)} Y_{n,m}(\xi)$$
$$+ \frac{1}{2} \frac{1}{(R+H)^{n+3}} \mathbf{o}_\xi^{(2,3)} Y_{n,m}(\xi). \tag{66}$$

Keeping in mind, that any solution of the SGG-problem can be expressed as a series of outer harmonics and using the completeness of the spherical harmonics in the space of square-integrable functions on the unit sphere, it follows that the SGG problem is uniquely solvable (up to some low order spherical harmonics) by the $O^{(1,1)}$, $O^{(1,2)}$, $O^{(2,1)}$, $O^{(2,2)}$, and $O^{(2,3)}$ components. To be more specific, we are able to formulate the following theorem:

Theorem 4.1
Let V satisfy the following condition $V \in Pot(\mathcal{C}^{(0)}(\Omega))$, i.e.,

$$V \in \mathcal{C}^{(0)}(\overline{\Omega^{ext}}) \cap \mathcal{C}^{(2)}(\Omega^{ext}), \tag{67}$$

$$\Delta V(x) = 0, \quad x \in \Omega^{ext}, \tag{68}$$

$$|V(x)| = O\left(\frac{1}{|x|} \right), |x| \to \infty, \text{ uniformly for all directions.} \tag{69}$$

Then the following statements are valid:

1. $O^{(i,k)} \nabla \otimes \nabla V((R+H)\xi) = 0$ *if* $(i,k) \in \{(1,3),(3,1),(3,2),(3,3)\}$.
2. $O^{(i,k)} \nabla \otimes \nabla V((R+H)\xi) = 0$ *for* $(i,k) \in \{(1,1),(2,2)\}$ *if and only if* $V = 0$.
3. $O^{(i,k)} \nabla \otimes \nabla V((R+H)\xi) = 0$ *for* $(i,k) \in \{(1,2),(2,1)\}$ *if and only if* $V|_\Omega$ *is constant.*
4. $O^{(2,3)} \nabla \otimes \nabla V((R+H)\xi) = 0$ *if and only if* $V|_\Omega$ *is linear combination of spherical harmonics of degree 0 and 1.*

This theorem gives detailed information, which tensor components of the Hesse tensor ensure the uniqueness of the SGG-problem (see also the considerations due to Schreiner (1994a), Freeden et al. (2002)). Anyway, for a potential V of class $Pot(\mathcal{C}^{(0)}(\Omega))$ with vanishing spherical harmonic moments of degree 0 and 1 such as the Earth's disturbing potential (see, e.g., Heiskanen and Moritz (1967) for its definition) uniqueness is assured in all cases (listed in Theorem 4.1).

Since we now know at least in the spherical setting, which conditions guarantee the uniqueness of an SGG-solution we can turn to the question of how to find a solution and what we mean with a solution, since we have to take into account the ill–posedness. To this end, we are interested here in analyzing the problem step by step. We start with the reformulation of the SGG-problem as pseudo differential equation on the sphere, give a short overview on regularization, and show how this ingredients can be composed together to regularize the SGG-data.

In doing so we find great help by discussing how classical boundary value problems in gravitational field of the Earth as well as modern satellite problems may be transferred into pseudodifferential equations, thereby always assuming the spherically oriented geometry. Indeed, it is helpful to treat the classical Dirichlet and Neumann boundary value problem as well as significant satellite problems such as satellite-to-satellite tracking (SST) and SGG.

4.1 SGG as Pseudodifferential Equation

Let $\Sigma \subset \mathbb{R}^3$ be a regular surface, i.e., we assume the following properties: (i) Σ divides the Euclidean space $\mathbb{R}^3$ into the bounded region Σ^{int} (inner space) and the unbounded region Σ^{ext} (outer space) so that $\Sigma^{\text{ext}} = \mathbb{R}^3 \backslash \overline{\Sigma^{\text{int}}}$, $\Sigma = \overline{\Sigma^{\text{int}}} \cap \overline{\Sigma^{\text{ext}}}$ with $\varnothing = \Sigma^{\text{int}} \cap \Sigma^{\text{ext}}$, (ii) Σ^{int} contains the origin, (iii) Σ is a closed and compact surface free of double points, (iv) Σ is locally of class $\mathcal{C}^{(2)}$ (see Freeden and Michel (2004) for more details concerning regular surfaces).

From our preparatory considerations (in particular, from the Introduction) it can be deduced that a gravitational potential of interest may be understood to be a member of the class $V \in Pot(\mathcal{C}^{(0)}(\Sigma))$, i.e.,

$$V \in \mathcal{C}^{(2)}(\overline{\Sigma^{ext}}) \cap \mathcal{C}^{(2)}(\Sigma^{ext}), \tag{70}$$

$$\Delta V(x) = 0, \quad x \in \Sigma^{ext}, \tag{71}$$

$$|V(x)| = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \text{uniformly for all directions.} \tag{72}$$

Assume that $\Omega_R = \{x \in \mathbb{R}^3 \mid |x| = R\}$ is a (Runge) sphere with radius R around the origin, i.e., a sphere that lies entirely inside Σ, i.e. $\Omega_R \subset \Sigma^{\text{int}}$. On the class $\mathcal{L}^2(\Omega_R)$ we impose the inner product $(\cdot,\cdot)_{\mathcal{L}^2(\Omega_R)}$. Then we know that the functions $\frac{1}{R} Y_{n,m}\left(\frac{\cdot}{R}\right)$ form an orthonormal set of functions on Ω_R, i.e., given $F \in \mathcal{L}^2(\Omega_R)$, its Fourier expansion reads

$$F(x) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{R^2} \left(F, Y_{n,m}\left(\frac{\cdot}{R}\right)\right)_{\mathcal{L}^2(\Omega_R)} Y_{n,m}\left(\frac{x}{R}\right), \quad x \in \Omega_R. \tag{73}$$

Instead of considering potentials that are harmonic outside Σ and continuous on Σ, we now consider potentials that are harmonic outside Ω_R and that are of class $\mathcal{L}^2(\Omega_R)$. In accordance with our notation we define

$$\text{Pot}(\mathcal{L}^2(\Omega_R)) = \left\{x \mapsto \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{R^2} \left(F, Y_{n,m}\left(\frac{\cdot}{R}\right)\right)_{\mathcal{L}^2(\Omega_R)} \frac{R^{n+1}}{|x|^{n+1}} Y_{n,m}\left(\frac{x}{|x|}\right) \mid F \in \mathcal{L}^2(\Omega_R)\right\}. \tag{74}$$

Clearly, $\mathrm{Pot}(\mathcal{L}^2(\Omega_R))$ is a "subset" of $\mathrm{Pot}(\mathcal{C}^{(0)}(\Sigma))$ in the sense that if $V \in \mathrm{Pot}(\mathcal{L}^2(\Omega_R))$, then $V|_{\overline{\Sigma^{\mathrm{ext}}}} \in \mathrm{Pot}(\mathcal{C}^{(0)}(\Sigma))$. The "difference" of these two spaces is not "too large": Indeed, we know from the Runge approximation theorem (cf. Freeden 1980a), that for every $\varepsilon > 0$ and every $V \in \mathrm{Pot}(\mathcal{C}^{(0)}(\Sigma))$ there exists a $\hat{V} \in \mathrm{Pot}(\mathcal{L}^2(\Omega_R))$ such that $\sup_{x\in\Sigma^{\mathrm{ext}}} |V(x) - \hat{V}(x)| < \varepsilon$. Thus, in all geosciences, it is common (but not strictly consistent with the Runge argumentation) to identify Ω_R with the surface of the Earth and to assume that the restriction $V|_{\Omega_R}$ is of class $\mathcal{L}^2(\Omega_R)$. Clearly, we have a canonical isomorphism between $\mathcal{L}^2(\Omega_R)$ and $\mathrm{Pot}(\mathcal{L}^2(\Omega_R))$, which is defined via the trace operator, i.e., the restriction to Ω_R and its harmonic continuation, respectively.

4.2 Upward/Downward Continuation

Let Ω_{R+H} be the sphere with radius $R + H$. The system $\frac{1}{R+H} Y_{n,m}\left(\frac{\cdot}{R+H}\right)$ is then orthonormal in $\mathcal{L}^2(\Omega_{R+H})$. (We assume H to be the height of a satellite above the Earth's surface). Let $F \in \mathrm{Pot}(\mathcal{L}^2(\Omega_R))$ be represented in the form

$$x \mapsto \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \frac{1}{R^2}\left(F, Y_{n,m}\left(\frac{\cdot}{R}\right)\right)_{\mathcal{L}^2(\Omega_R)} \frac{R^{n+1}}{|x|^{n+1}} Y_{n,m}\left(\frac{x}{|x|}\right). \tag{75}$$

Then the restriction of F on Ω_{R+H} reads

$$F|_{\Omega_{R+H}} : x \mapsto \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \frac{1}{R^2}\left(F, Y_{n,m}\left(\frac{\cdot}{R}\right)\right)_{\mathcal{L}^2(\Omega_R)} \frac{R^{n+1}}{(R+H)^{n+1}} Y_{n,m}\left(\frac{x}{R+H}\right). \tag{76}$$

Hence, any element $\frac{1}{R} Y_{n,m}\left(\frac{\cdot}{R}\right)$ of the orthonormal system in $\mathcal{L}^2(\Omega_R)$ is mapped to a function $R^n/(R+H)^n\ 1/R+H\ Y_{n,m}(\cdot/R+H)$. The operation defined in such away is called *upward continuation*. It is representable by the pseudodifferential operator (for more details on pseudodifferential operators the reader should consult Svensson (1983), Schneider (1997), Freeden et al. (1998), and Freeden 1999) as well as ➋ Chap. 27 of this handbook

$$\Lambda_{\mathrm{up}}^{R,H} : \mathcal{L}^2(\Omega_R) \longrightarrow \mathcal{L}^2(\Omega_{R+H})$$

with associated symbol

$$\left(\Lambda_{\mathrm{up}}^{R,H}\right)^{\wedge}(n) = \frac{R^n}{(R+H)^n}. \tag{77}$$

In other words, we have

$$\Lambda_{\mathrm{up}}^{R,H}\left(\frac{1}{R} Y_{n,m}\left(\frac{\cdot}{R}\right)\right) = \left(\Lambda_{\mathrm{up}}^{R,H}\right)^{\wedge}(n) \frac{1}{R+H} Y_{n,m}\left(\frac{\cdot}{R+H}\right). \tag{78}$$

The image of $\Lambda_{\mathrm{up}}^{R,H}$ is given by Picard's criterion (cf. Theorem 4.4):

$$\Lambda_{\mathrm{up}}^{R,H}(\mathcal{L}^2(\Omega_R)) = \left\{ F \in \mathcal{L}^2(\Omega_{R+H}) \middle| \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \left(\frac{(R+H)^n}{R^n}\right)^2 \times \left(F, \frac{1}{R+H} Y_{n,m}\left(\frac{\cdot}{R+H}\right)\right)^2_{\mathcal{L}^2(\Omega_{R+H})} < \infty \right\}. \tag{79}$$

The inverse of $\Lambda_{\text{up}}^{R,H}$ is called the *downward continuation* operator, $\Lambda_{\text{down}}^{R,H} = \left(\Lambda_{\text{up}}^{R,H}\right)^{-1}$. It brings down the gravitational potential at height $R+H$ to the height R:

$$\Lambda_{\text{down}}^{R,H} : \Lambda_{\text{up}}^{R,H}(\mathcal{L}^2(\Omega_{R+H})) \longrightarrow \mathcal{L}^2(\Omega_R)$$

with

$$\Lambda_{\text{down}}^{R,H}\left(\frac{1}{R+H} Y_{n,m}\left(\frac{\cdot}{R+H}\right)\right) = \frac{(R+H)^n}{R^n}\frac{1}{R} Y_{n,m}\left(\frac{\cdot}{R}\right) \tag{80}$$

such that the symbol of $\Lambda_{\text{down}}^{R,H}$ is

$$\left(\Lambda_{\text{down}}^{R,H}\right)^{\wedge}(n) = \frac{(R+H)^n}{R^n}. \tag{81}$$

It is obvious that the upward continuation is well-posed, whereas the downward continuation generates an ill-posed problem.

4.3 Operator of the First-Order Radial Derivative

Let $F \in \text{Pot}(\mathcal{L}^2(\Omega_R))$ have the representation (75). If we restrict F to a sphere Ω_γ with radius γ, we have

$$\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{1}{R^2}\left(F, Y_{n,m}\left(\frac{\cdot}{R}\right)\right)_{\mathcal{L}^2(\Omega_R)} \frac{R^{n+1}}{\gamma^{n+1}} Y_{n,m}\left(\frac{x}{\gamma}\right), \quad x \in \Omega_\gamma. \tag{82}$$

Accordingly, the restriction of $\frac{\partial}{\partial r}F$ to Ω_γ amounts to

$$\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{1}{R^2}\left(F, Y_{n,m}\left(\frac{\cdot}{R}\right)\right)_{\mathcal{L}^2(\Omega_R)} \frac{-(n+1)}{\gamma}\frac{R^{n+1}}{\gamma^{n+1}} Y_{n,m}\left(\frac{x}{\gamma}\right). \tag{83}$$

Thus, the process of forming the first radial derivative at height γ constitutes the pseudodifferential operator $\Lambda_{\text{FND}}^{\gamma}$ (FND stands for first-order normal derivative) with the symbol

$$\left(\Lambda_{\text{FND}}^{\gamma}\right)^{\wedge}(n) = -\frac{n+1}{\gamma}. \tag{84}$$

4.4 Pseudodifferential Operator for SST

The principle of SST is sketched in ➲ *Fig. 5* (note that two variants of SST are discussed in satellite techniques, the so-called high–low and the low–low method. We only explain here the high–low variant, for which the GFZ–satellite CHAMP (CHAllenging Minisatellite Payload) launched in 2000 is a prototype.

The motion of a satellite in a low orbit such as CHAMP, GOCE (typical heights are in the range 200–500 km) is tracked with a GPS receiver. So the relative motion between the satellite and the GPS–satellites (the latter have a height of approximately 2,0000 km) can be measured. Assuming that the motion of the GPS–satellites is known (in fact, their orbit is very stable because of the large height), one can calculate the acceleration of the low orbiting satellite. Since the acceleration and the force acting on the satellite are proportional by Newton's law, one gets information about the gradient field $\nabla V(p)$ at the satellite's position p. Assuming that

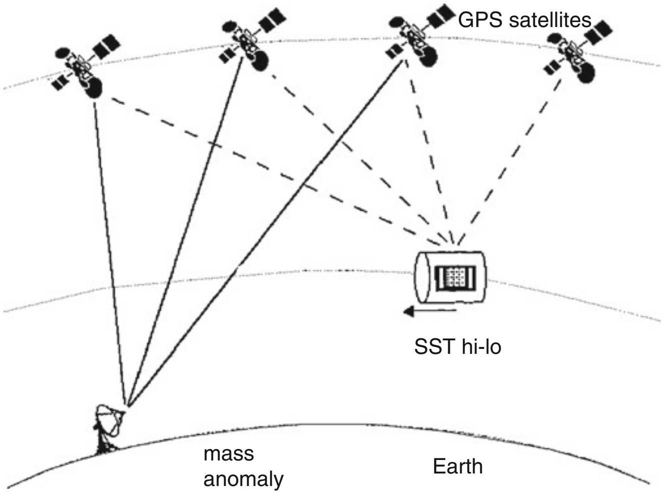


Fig. 5
The principle of satellite–to satellite tracking (from ESA (1999))

the height variations of the satellite are small, we obtain data information of ∇V at height H, that is on the sphere Ω_{R+H}. For simplicity, it is useful to consider only the radial component from these vectorial data, which is the first radial derivative.

Thus, given $F \in \mathrm{Pot}(\mathcal{L}^2(\Omega_R))$, we get the SST–data by a process of upward continuation and then taking the first radial derivative. Mathematically, SST amounts to introduce the operator

$$\Lambda^{R,H}_{\mathrm{SST}} : \mathcal{L}^2(\Omega_R) \longrightarrow \mathcal{L}^2(\Omega_{R+H})$$

via

$$\Lambda^{R,H}_{\mathrm{SST}} = -\Lambda^{R+H}_{\mathrm{FND}} \Lambda^{R,H}_{\mathrm{up}} \tag{85}$$

(we use the minus sign here, to avoid the minus in the symbol), and get

$$\left(\Lambda^{R,H}_{\mathrm{SST}}\right)^{\wedge}(n) = \frac{R^n}{(R+H)^2} \frac{n+1}{R+H}. \tag{86}$$

It is easily seen that the Picard criterion (see, e.g., Engl et al. (1997)) reads for this operator

$$\Lambda^{R,H}_{\mathrm{SST}}(\mathcal{L}^2(\Omega_R)) = \left\{ F \in \mathcal{L}^2(\Omega_{R+H}) \middle| \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \left(\frac{(R+H)^n}{R^n} \frac{R+H}{n+1} \right)^2 \right.$$
$$\left. \times \left(F, \frac{1}{R+H} Y_{n,m} \left(\frac{\cdot}{R+H} \right) \right)^2_{\mathcal{L}^2(\Omega_{R+H})} < \infty \right\}. \tag{87}$$

4.5 Pseudodifferential Operator of the Second-Order Radial Derivative

Analogous considerations applied to the operator $\frac{\partial^2}{\partial r^2}$ on F in (75) at height γ yields

$$\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{1}{R^2}\left(F, Y_{n,m}\left(\frac{\cdot}{R}\right)\right)_{\mathcal{L}^2(\Omega_R)}\frac{(n+1)(n+2)}{\gamma^2}\frac{R^{n+1}}{\gamma^{n+1}}Y_{n,m}\left(\frac{x}{\gamma}\right), \quad x\in\Omega_\gamma. \tag{88}$$

Thus, the second-order radial derivative at height γ is represented by the pseudodifferential operator $\Lambda^{\gamma}_{\mathrm{SND}}$ with the symbol

$$(\Lambda^{\gamma}_{\mathrm{SND}})^{\wedge}(n) = \frac{(n+1)(n+2)}{\gamma^2}. \tag{89}$$

4.6 Pseudodifferential Operator for Satellite Gravity Gradiometry

If we restrict ourselves for the moment to the second-order radial derivative $\frac{\partial^2}{\partial r^2}V$, and assume that the height of the satellite is H, we are led to the pseudodifferential operator describing satellite gravity gradiometry by

$$\Lambda^{R,H}_{\mathrm{SGG}} = \Lambda^{R+H}_{\mathrm{SND}}\Lambda^{R,H}_{\mathrm{up}}$$

so that

$$\left(\Lambda^{R,H}_{\mathrm{SGG}}\right)^{\wedge}(n) = \frac{R^n}{(R+H)^n}\frac{(n+1)(n+2)}{(R+H)^2}. \tag{90}$$

In consequence,

$$\Lambda^{R,H}_{\mathrm{SGG}} : \mathcal{L}^2(\Omega_R) \longrightarrow \mathcal{L}^2(\Omega_{R+H})$$

with

$$\Lambda^{R,H}_{\mathrm{SGG}}(\mathcal{L}^2(\Omega_R)) = \left\{F\in\mathcal{L}^2(\Omega_{R+H})\,\middle|\,\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(\frac{(R+H)^n}{R^n}\frac{(R+H)^2}{(n+1)(n+2)}\right)^2 \times\left(F,\frac{1}{R+H}Y_{n,m}\left(\frac{\cdot}{R+H}\right)\right)^2_{\mathcal{L}^2(\Omega_{R+H})}<\infty\right\}. \tag{91}$$

4.7 Survey on Pseudodifferential Operators Relevant in Satellite Technology

Until now, our purpose was to develop a class of pseudodifferential operators, which describe, in particular, important operations for actual and future satellite missions. In what follows, we are interested in a brief mathematical survey about our investigations. In order to keep the forthcoming notations as simple as possible, we use the fact that all spheres around the origin are isomorphic. Thus, we consider the resulting pseudodifferential operators on the unit sphere and ignore the different heights in the domain of definition of the functions, but not in the symbol of the operators. Hence, we can use the results of the last chapters directly for the regularization of the satellite problems. If one wants to incorporate the different heights, one has only to observe the factors R and $R+H$, respectively.

Operator	Description	Symbol	Order
$\Lambda^{R,H}_{\mathrm{up}}$	Upward continuation operator	$\frac{R^n}{(R+H)^n}$	$-\infty$
$\Lambda^{R,H}_{\mathrm{down}}$	Downward continuation operator	$\frac{(R+H)^n}{R^n}$	∞
$\Lambda^{R}_{\mathrm{FND}}$	First-order radial derivative at the Earth surface	$-\frac{(n+1)}{R}$	1
$\Lambda^{R}_{\mathrm{SND}}$	Second-order radial derivative at the Earth surface	$\frac{(n+1)(n+2)}{R^2}$	2
$\Lambda^{R,H}_{\mathrm{SST}}$	Pseudodifferential Operator for satellite–to–satellite tracking	$\frac{R^n}{(R+H)^n}\frac{n+1}{R+H}$	$-\infty$
$\Lambda^{R,H}_{\mathrm{SGG}}$	Pseudodifferential operator for satellite gravity gradiometry	$\frac{R^n}{(R+H)^n}\frac{(n+1)(n+2)}{(R+H)^2}$	$-\infty$

All pseudodifferential operators are then defined on $\mathcal{L}^2(\Omega)$ or on suitable Sobolev spaces (see Freeden et al. 1998; Freeden 1999). The table above gives a summary of all the aforementioned operators.

In order to show how these operators work, we give some graphical examples. We start from the disturbance potential of the NASA, GSFC, and NIMA Earth's Gravity Model EGM96 (cf. Lemoine et al. 1998). In ➋ *Figs. 6–8* we graphically show the potential at the height of the Earth surface, at the height 250 km and further more the second-order radial derivative at height 250 km.

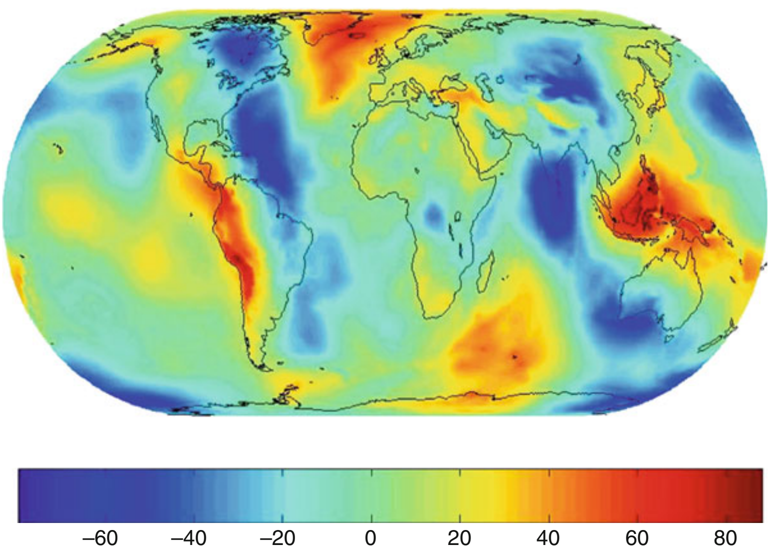


Fig. 6
The disturbance potential from EGM96 at the Earth's surface, at height 250 km, in m^2/s^2

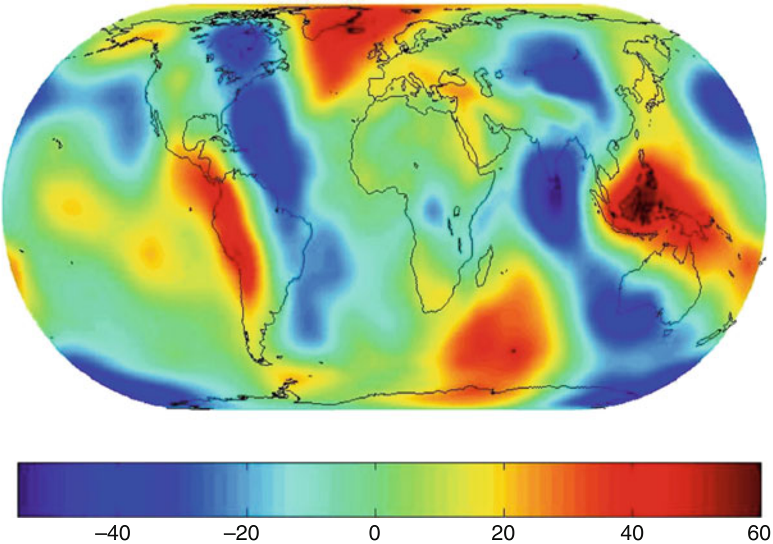


Fig. 7
The disturbance potential from EGM96 at height 250 km in m^2/s^2

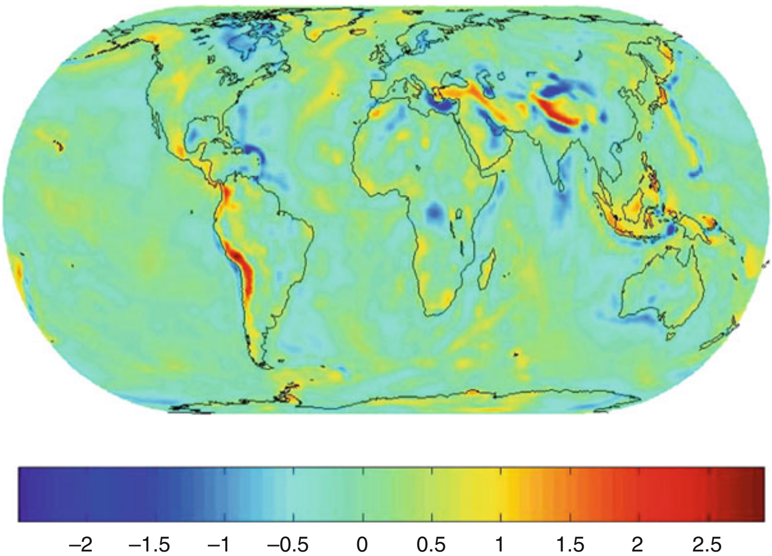


Fig. 8
The second-order radial derivative of the disturbance potential from EGM96 at height 250 km in $10^{-10}/s^2$

4.8 Classical Boundary Value Problems and Satellite Problems

The Neumann problem of potential theory for the outer space of the sphere Ω (based on $\mathcal{L}^2(\Omega)$ boundary data) reads as follows: Find $V \in \text{Pot}(\mathcal{L}^2(\Omega))$ such that $\frac{\partial}{\partial r} V|_\Omega = G$. Since the trace of V is assumed to be a member of the class $\mathcal{L}^2(\Omega)$, the appropriate space for G is the Sobolev space $\mathcal{H}^{-1}(\Omega)$. Using pseudodifferential operators as described earlier, this problem reads in an $\mathcal{L}^2(\Omega)$-context as follows: Given $G \in \mathcal{L}^2(\Omega)$, find $F \in \mathcal{L}^2(\Omega)$ such that

$$\Lambda^R_{\text{FND}} F = G \tag{92}$$

with $\left(\Lambda^R_{\text{FND}}\right)^\wedge(n) = -\frac{n+1}{R}$, $n = 0, 1, \ldots$. Similar considerations show that the Dirichlet problem transfers to the trivial form $Id\, F = G$, where Id is the identity operator with $Id^\wedge(n) = 1$, $n = 0, 1, \ldots$.

Evidently, the classical problems of potential theory expressed in pseudodifferential form are well–posed in the sense that the inverse operators $\left(\Lambda^R_{\text{FND}}\right)^{-1}$ and Id^{-1} are bounded in $\mathcal{L}^2(\Omega)$. In contrary, the problems coming from SST and SGG are ill–posed, as we will see in a moment. To be more concrete, SST intends to obtain information of V at the Earth's surface (radius R) from measurements of the first radial derivative at the satellite's height H. Thus, we obtain the problem: Given $G \in \mathcal{L}^2(\Omega)$, find $F \in \mathcal{L}^2(\Omega)$ so that

$$\Lambda^{R,H}_{\text{SST}} F = G \tag{93}$$

with

$$\left(\Lambda^{R,H}_{\text{SST}}\right)^\wedge(n) = \frac{R^n}{(R+H)^n}\,\frac{n+1}{R+H}. \tag{94}$$

Similarly, SGG is formulated as pseudodifferential equation as follows: Given $G \in \mathcal{L}^2(\Omega)$, find $F \in \mathcal{L}^2(\Omega)$ so that

$$\Lambda^{R,H}_{\text{SGG}} F = G \tag{95}$$

with

$$\left(\Lambda^{R,H}_{\text{SGG}}\right)^\wedge(n) = \frac{R^n}{(R+H)^n}\,\frac{(n+1)(n+2)}{(R+H)^2}. \tag{96}$$

For more detailed studies in a potential theoretic framework, the reader may wish to consult Freeden et al. (2002). The inverses of these operators possess a symbol which is exponentially increasing as $n \to \infty$. Thus, the inverse operators are unbounded, or in the jargon of regularization, these two problems are exponentially ill–posed. By a naive application of the inverse operator on the right–hand side, one cannot expect to obtain a useful solution. Thus, so–called regularization strategies have to be applied. Therefore, the basic aspects on regularization should be presented next.

4.9 A Short Introduction to the Regularization of Ill-Posed Problems

For the convenience of the reader, we present here a brief course of basic facts on regularization in a Hilbert space setting, which is useful to understand the solution strategies in the framework of pseudodifferential equations. The explanations are based on the monographs of Engl et al.

(1996) and Kirsch (1996), where much more additional material can be found even for more general reference spaces, too. .

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces with inner products $(\cdot,\cdot)_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{K}}$, respectively. Let

$$\Lambda : \mathcal{H} \longrightarrow \mathcal{K} \tag{97}$$

be a linear bounded operator. Given $y \in \mathcal{K}$, we are looking for a solution of

$$\Lambda x = y. \tag{98}$$

In accordance to Hadamard (1923), we call such a problem *well–posed*, if the following properties are valid:

- For all admissible data, a solution exists.
- For all admissible data, the solution is unique.
- The solution depends continuously on the data.

In our setting, these requirements can be translated into

- Λ is injective, i.e. $\mathcal{R}(\Lambda) = \mathcal{K}$
- Λ is surjective, i.e. $\mathcal{N}(\Lambda) = \{0\}$
- Λ^{-1} is bounded and continuous

If one of the three conditions is not fulfilled, the problem (98) is called *ill-posed*. It will turn out that the satellite problems we are concerned with are ill–posed, the largest problem being the unboundedness of the inverse operator Λ^{-1}.

Let us discuss the consequences of the violations of the above requirements for the well–posedness of (98). The lack of injectivity of Λ is perhaps the easiest problem. The space $\mathcal{H}$ can be replaced by the orthogonal complement $\mathcal{N}(\Lambda)^{\perp}$, and the restriction of the operator Λ to $\mathcal{N}(\Lambda)^{\perp}$ yields to an injective problem.

From practical point of view, one is very often confronted with the problem that $\mathcal{R}(\Lambda) \neq \mathcal{K}$, since the right-hand side is given by measurements and is, therefore, disturbed by errors. We assume now that $y \in \mathcal{R}(\Lambda)$ but only a perturbed right-hand side y^{δ} is known. We suppose

$$\|y - y^{\delta}\|_{\mathcal{K}} < \delta. \tag{99}$$

Our aim is to solve

$$\Lambda x^{\delta} = y^{\delta}. \tag{100}$$

Since y^{δ} might not be in $\mathcal{R}(\Lambda)$, the solution of this equation might not exist, and we have to generalize what is meant by a solution. x^{δ} is called *least–squares solution* of (100), if

$$\|\Lambda x^{\delta} - y^{\delta}\|_{\mathcal{K}} = \inf\{\|\Lambda z - y^{\delta}\|_{\mathcal{K}} \,|z \in \mathcal{H}\}. \tag{101}$$

The solution of (101) might not be unique, and therefore one looks for the solution of (101) with minimal norm. x^{δ} is called *best approximate solution* of $\Lambda x^{\delta} = y^{\delta}$, if x^{δ} is a least–squares solution and

$$\|x^{\delta}\|_{\mathcal{H}} = \inf\{\|z\|_{\mathcal{H}} \,|z \text{ is a least–squares solution of } \Lambda z = y^{\delta}\} \tag{102}$$

holds.

The notion of a best–approximate solution is closely related to the Moore–Penrose (generalized) inverse, of Λ (see Nashed 1976). We let

$$\tilde{\Lambda} : \mathcal{N}(\Lambda)^{\perp} \longrightarrow \mathcal{R}(\Lambda)$$

with

$$\tilde{\Lambda} = \Lambda|_{\mathcal{N}(\Lambda)^\perp} \tag{103}$$

and define the *Moore–Penrose (generalized) inverse* Λ^+ to be the unique linear extension of $\tilde{\Lambda}^{-1}$ to

$$\mathcal{D}(\Lambda^+) := \mathcal{R}(\Lambda) + \mathcal{R}(\Lambda)^\perp \tag{104}$$

with

$$\mathcal{N}(\Lambda^+) = \mathcal{R}(\Lambda)^\perp. \tag{105}$$

A standard result is provided by

Theorem 4.2
If $y \in \mathcal{D}(\Lambda^+)$, then $\Lambda x = y$ has a unique best–approximate solution which is given by

$$x^+ = \Lambda^+ y.$$

Note that the best–approximate solution is defined for all perturbed data $y^\delta \in \mathcal{K}$, whereas the last theorem requires that the right–hand side is an element of $\mathcal{D}(\Lambda^+)$.

A serious problem for ill–posed problems occurs when Λ^{-1} or Λ^+ are not continuous. This means that small errors in the data or even small numerical noise can cause large errors in the solution. In fact, in most cases the application of an unbounded Λ^{-1} or Λ^+ does not make any sense. The usual strategy to overcome this difficulty is to substitute the unbounded inverse operator

$$\Lambda^{-1} : \mathcal{R}(\Lambda) \longrightarrow \mathcal{H}$$

by a suitable bounded approximation

$$R : \mathcal{K} \longrightarrow \mathcal{H}.$$

The operator R is not chosen to be fixed, but dependent on a *regularization parameter* α. According to Kirsch (1996) we are led to introduce the following definition:

Definition 4.3
A regularization strategy is a family of linear bounded operators

$$R_\alpha : \mathcal{K} \longrightarrow \mathcal{H}, \quad \alpha > 0,$$

so that

$$\lim_{\alpha \to 0} R_\alpha \Lambda x = x \quad \textit{for all } x \in \mathcal{H},$$

i.e. the operators $\mathcal{R}_\alpha \Lambda$ converge pointwise to the identity.

From the theory of inverse problems (see, e.g., Kirsch 1996) it is also clear that if $\Lambda : \mathcal{H} \to \mathcal{K}$ is compact and $\mathcal{H}$ has infinite dimension (as it is the case for the application we have in mind), then the operators $\mathcal{R}_\alpha$ are not uniformly bounded, i.e., there exists a sequence (α_j) with $\lim_{j\to\infty} \alpha_j = 0$ and

$$\|R_{\alpha_j}\|_{\mathcal{L}(\mathcal{K},\mathcal{H})} \to \infty \quad \text{for } j \to \infty. \tag{106}$$

Note that the convergence of $\mathcal{R}_\alpha \Lambda x$ in Definition 4.3 is based on $y = \Lambda x$, i.e., on unperturbed data. In practice, the right-hand side is affected by errors and then no convergence is achieved. Instead, one is (or has to be) satisfied with an approximate solution based on a certain choice of the regularization parameter.

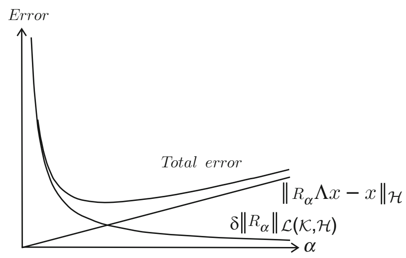


Fig. 9
Typical behavior of the total error in a regularization process

Let us discuss the error of the solution. For this purpose, we let $y \in \mathcal{R}(\Lambda)$ be the (unknown) exact right–hand side and $y^\delta \in \mathcal{K}$ be the measured data with

$$\|y - y^\delta\|_{\mathcal{K}} < \delta. \tag{107}$$

For a fixed $\alpha > 0$, we let

$$x^{\alpha,\delta} = R_\alpha y^\delta, \tag{108}$$

and look at $x^{\alpha,\delta}$ as an approximation of the solution x of $\Lambda x = y$. Then the error can be split as follows:

$$\begin{aligned} \|x^{\alpha,\delta} - x\|_{\mathcal{H}} &= \|R_\alpha y^\delta - x\|_{\mathcal{H}} \\ &\le \|R_\alpha y^\delta - R_\alpha y\|_{\mathcal{H}} + \|R_\alpha y - x\|_{\mathcal{H}} \\ &\le \|R_\alpha\|_{\mathcal{L}(\mathcal{K},\mathcal{H})} \, \|y^\delta - y\|_{\mathcal{K}} + \|R_\alpha y - x\|_{\mathcal{H}}, \end{aligned} \tag{109}$$

such that

$$\|x^{\alpha,\delta} - x\|_{\mathcal{H}} \le \delta \|R_\alpha\|_{\mathcal{L}(\mathcal{K},\mathcal{H})} + \|R_\alpha \Lambda x - x\|_{\mathcal{H}}. \tag{110}$$

We see that the error between the exact and the approximate solution consists of two parts: The first term is the product of the bound for the error in the data and the norm of the regularization parameter R_α. This term will usually tend to infinity for $\alpha \to 0$ if the inverse Λ^{-1} is unbounded and Λ is compact (cf. (106)). The second term denotes the approximation error $\|(R_\alpha - \Lambda^{-1})y\|_{\mathcal{H}}$ for the exact right–hand side $y = \Lambda x$. This error tends to zero as $\alpha \to 0$ by the definition of a regularization strategy. Thus, both parts of the error show a diametrically oriented behavior. A typical picture of the errors in dependence on the regularization parameter α is sketched in *Fig. 9*. Thus, a strategy is needed to choose α dependent on δ in order to keep the error as small as possible, i.e., we would like to minimize

$$\delta \|R_\alpha\|_{\mathcal{L}(\mathcal{K},\mathcal{H})} + \|R_\alpha \Lambda x - x\|_{\mathcal{H}}. \tag{111}$$

In principle, we distinguish two classes of parameter choice rules: If $\alpha = \alpha(\delta)$ does not depend on δ, we call $\alpha = \alpha(\delta)$ an a priori parameter choice rule. Otherwise α depends also on y^δ and we call $\alpha = \alpha(\delta, y^\delta)$ an a posteriori parameter choice rule. It is usual to say a parameter choice rule is convergent, if for $\delta \to 0$ the rule is such that

$$\lim_{\delta \to 0} \sup\{\|R_{\alpha(\delta,y^\delta)} y^\delta - T^+ y\|_{\mathcal{H}} \,|y^\delta \in \mathcal{K},\ \|y^\delta - y\|_{\mathcal{K}} \le \delta\} = 0 \tag{112}$$

and

$$\lim_{\delta \to 0} \sup\{\alpha(\delta, y^\delta) \,|y^\delta \in \mathcal{K}, \|y - y^\delta\|_{\mathcal{K}} \le \delta\} = 0. \tag{113}$$

We stop here the discussion of parameter choice rules. For more material the interested reader is referred to, e.g., Engl et al. (1996) and Kirsch (1996).

The remaining part of this section is devoted to the case that Λ is compact, since then we gain benefits from the spectral representations of the operators. If $\Lambda : \mathcal{H} \to \mathcal{K}$ is compact, a singular system $(\sigma_n; v_n, u_n)$ is defined as follows: $\{\sigma_n^2\}_{n\in\mathbb{N}}$ are the nonzero eigenvalues of the self–adjoint operator $\Lambda^*\Lambda$ (Λ^* is the adjoint operator of Λ), written down in decreasing order with multiplicity. The family $\{v_n\}_{n\in\mathbb{N}}$ constitutes a corresponding complete orthonormal system of eigenvectors of $\Lambda^*\Lambda$. We let $\sigma_n > 0$ and define the family $\{u_n\}_{n\in\mathbb{N}}$ via $u_n = \Lambda v_n / \|\Lambda v_n\|_{\mathcal{K}}$. The sequence $\{u_n\}_{n\in\mathbb{N}}$ forms a complete orthonormal system of eigenvectors of $\Lambda\Lambda^*$, and the following formulas are valid:

$$\Lambda v_n = \sigma_n u_n, \tag{114}$$

$$\Lambda^* u_n = \sigma_n v_n, \tag{115}$$

$$\Lambda x = \sum_{n=1}^{\infty} \sigma_n (x, v_n)_{\mathcal{H}} \; u_n, \;\; x \in \mathcal{H}, \tag{116}$$

$$\Lambda^* y = \sum_{n=1}^{\infty} \sigma_n (y, u_n)_{\mathcal{K}} \; v_n, \;\; y \in \mathcal{K}. \tag{117}$$

The convergence of the infinite series is understood with respect to the Hilbert space norms under consideration. The identities (116) and (117) are called the *singular value expansions* of the corresponding operators. If there are infinitely many singular values, they accumulate (only) at 0, i.e., $\lim_{n\to\infty} \sigma_n = 0$.

Theorem 4.4
Let $(\sigma_n; v_n, u_n)$ be a singular system for the compact linear operator Λ, $y \in \mathcal{K}$. Then we have

$$y \in \mathcal{D}(\Lambda^+) \text{ if and only if } \sum_{n=1}^{\infty} \frac{|(y, u_n)_{\mathcal{K}}|^2}{\sigma_n^2} < \infty, \tag{118}$$

and for $y \in \mathcal{D}(\Lambda^+)$ it holds

$$\Lambda^+ y = \sum_{n=1}^{\infty} \frac{(y, u_n)_{\mathcal{K}}}{\sigma_n} v_n. \tag{119}$$

The condition (118) is the *Picard criterion*. It says that a best–approximate solution of $\Lambda x = y$ exists only if the Fourier coefficients of y decay fast enough relative to the singular values.

The representation (119) of the best–approximate solution motivates a method for the construction of regularization operators, namely by damping the factors $1/\sigma_n$ in such a way that the series converges for all $y \in \mathcal{K}$. We are looking for filters

$$q : (0, \infty) \times (0, \|\Lambda\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}) \longrightarrow \mathbb{R} \tag{120}$$

such that

$$R_\alpha y := \sum_{n=1}^{\infty} \frac{q(\alpha, \sigma_n)}{\sigma_n} (y, u_n)_{\mathcal{K}} \; v_n, \quad y \in \mathcal{K},$$

is a regularization strategy. The following statement is known from Kirsch (1996).

Theorem 4.5
Let $\Lambda : \mathcal{H} \to \mathcal{K}$ be compact with singular system $(\sigma_n; v_n, u_n)$. Assume that q from (120) has the following properties:

(i) $|q(\alpha,\sigma)| \leq 1$ *for all* $\alpha > 0$ *and* $0 < \sigma \leq \|\Lambda\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}$.
(ii) *For every* $\alpha > 0$ *there exists a* $c(\alpha)$ *so that* $|q(\alpha,\sigma)| \leq c(\alpha)\sigma$ *for all* $0 < \sigma \leq \|\Lambda\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}$.
(iii) $\lim_{\alpha\to 0} q(\alpha,\sigma) = 1$ *for every* $0 \leq \sigma \leq \|\Lambda\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}$.

Then the operator $R_\alpha : \mathcal{K} \to \mathcal{H}$, $\alpha > 0$, *defined by*

$$R_\alpha y := \sum_{n=1}^{\infty} \frac{q(\alpha,\sigma_n)}{\sigma_n} (y, u_n)_{\mathcal{K}}\, v_n, \qquad y \in \mathcal{K}, \tag{121}$$

is a regularization strategy with $\|R_\alpha\|_{\mathcal{L}(\mathcal{K},\mathcal{H})} \leq c(\alpha)$.

The function q is called a *regularizing filter* for Λ. Two important examples should be mentioned:

$$q(\alpha,\sigma) = \frac{\sigma^2}{\alpha + \sigma^2} \tag{122}$$

defines the *Tikhonov regularization*, whereas

$$q(\alpha,\sigma) = \begin{cases} 1, & \sigma^2 \geq \alpha, \\ 0, & \sigma^2 < \alpha, \end{cases} \tag{123}$$

leads to the regularization by *truncated singular value decomposition*.

4.10 Regularization of the Exponentially Ill-Posed SGG-Problem

We are now in position to have a closer look at the role of the regularization techniques particularly working for the SGG-problem.

In (95), the SGG-problem is formulated as pseudodifferential equation: Given $G \in \mathcal{L}^2(\Omega)$, find $F \in \mathcal{L}^2(\Omega)$ so that $\Lambda^{R,H}_{\mathrm{SGG}} F = G$ with

$$\left(\Lambda^{R,H}_{\mathrm{SGG}}\right)^{\wedge}(n) = \frac{R^n}{(R+H)^n} \frac{(n+1)(n+2)}{(R+H)^2}. \tag{124}$$

Switching now to a finite dimensional space (which is then the realization of the regularization by a singular value truncation), we are interested in a solution of the representation

$$F_N = \sum_{n=1}^{N} F^{\wedge}(n,m) Y_{n,m}. \tag{125}$$

Using a decomposition of G of the form

$$G = \sum_{n=1}^{\infty} G^{\wedge}(n,m) Y_{n,m}, \tag{126}$$

we end up with the spectral equations

$$\left(\Lambda^{R,H}_{\mathrm{SGG}}\right)^{\wedge}(n) F^{\wedge}(n,m) = G^{\wedge}(n,m), \quad n = 1,\ldots,m, \quad m = 1,\ldots,2n+1. \tag{127}$$

In other words, in connection with (125) and (126), we find the relations

$$F^{\wedge}(n,m) = \frac{G^{\wedge}(n,m)}{\left(\Lambda^{R,H}_{\mathrm{SGG}}\right)^{\wedge}(n)}, \quad n = 1,\ldots,m, \quad m = 1,\ldots,2n+1. \tag{128}$$

For the realization of this solution we have to find the coefficients $G^{\wedge}(n,m)$. Of course, we are confronted with the usual problems of integration, aliasing, and so on.

The identity (128) also opens the perspective for SGG-applications by bandlimited regularization wavelets in Earth's gravitational field determination. For more details we refer to Schneider (1997), Freeden et al. (1997), Freeden and Schneider (1998), Glockner (2002), and Hesse (2003). The book written by Freeden (1999) contains non-bandlimited versions of (harmonic) regularization wavelets. Multiscale regularization by use of spherical up-functions is the content of the papers by Schreiner (2004) and Freeden and Schreiner (2004).

5 Future Directions

The regularization schemes described above are based on the decomposition of the Hesse tensor at satellite's height into scalar ingredients due to geometrical properties (normal, tangential, mixed) as well as to analytical properties originated by differentiation processes involving physically defined quantities (such as divergence, curl, etc). SGG-regularization, however, is more suitable and effective if it is based on algorithms involving the full Hesse tensor such as from the GOCE mission (for more insight into the tensorial decomposition of GOCE-data the reader is referred to the contribution of R. Rummel in this issue, in addition, see Rummel and van Gelderen (1992) and Rummel (1997).

Our context initiates another approach to tensor spherical harmonics. Based on cartesian operators (see Freeden and Schreiner (2009)), the construction principle starts from operators $\tilde{\mathbf{o}}_n^{(i,k)}$, $i,k \in \{1,2,3\}$ given by

$$\tilde{\mathbf{o}}_n^{(1,1)} F(x) = \left((2n+3)x - |x|^2 \nabla_x\right) \otimes \left((2n+1)x - |x|^2 \nabla_x\right) F(x), \tag{129}$$

$$\tilde{\mathbf{o}}_n^{(1,2)} F(x) = \left((2n-1)x - |x|^2 \nabla_x\right) \otimes \nabla_x F(x), \tag{130}$$

$$\tilde{\mathbf{o}}_n^{(1,3)} F(x) = \left((2n+1)x - |x|^2 \nabla_x\right) \otimes (x \wedge \nabla_x) F(x), \tag{131}$$

$$\tilde{\mathbf{o}}_n^{(2,1)} F(x) = \nabla_x \otimes \left((2n+1)x - |x|^2 \nabla_x\right) F(x), \tag{132}$$

$$\tilde{\mathbf{o}}_n^{(2,2)} F(x) = \nabla_x \otimes \nabla_x F(x), \tag{133}$$

$$\tilde{\mathbf{o}}_n^{(2,3)} F(x) = \nabla_x \otimes (x \wedge \nabla_x) F(x), \tag{134}$$

$$\tilde{\mathbf{o}}_n^{(3,1)} F(x) = (x \wedge \nabla_x) \otimes \left((2n+1)x - |x|^2 \nabla_x\right) F(x), \tag{135}$$

$$\tilde{\mathbf{o}}_n^{(3,2)} F(x) = (x \wedge \nabla_x) \otimes \nabla_x F(x), \tag{136}$$

$$\tilde{\mathbf{o}}_n^{(3,3)} F(x) = (x \wedge \nabla_x) \otimes (x \wedge \nabla_x) F(x) \tag{137}$$

for $x \in \mathbb{R}^3$ and sufficiently smooth function $F : \mathbb{R}^3 \to \mathbb{R}$.

Elementary calculations in cartesian coordinates lead us in a straightforward way to the following result.

Lemma 5.1
Let $H_n, n \in \mathbb{N}_0$, be a homogeneous harmonic polynomial of degree n. Then, $\tilde{\mathbf{o}}_n^{(i,k)} H_n$ is a homogeneous harmonic tensor polynomial of degree $\deg^{(i,k)}(n)$, where

$$\deg^{(i,k)}(n) = \begin{cases} n-2 & \textit{for} \quad (i,k) = (2,2) \\ n-1 & \textit{for} \quad (i,k) \in \{(2,3),(3,2)\} \\ n & \textit{for} \quad (i,k) \in \{(1,2),(2,1),(3,3)\} \\ n+1 & \textit{for} \quad (i,k) \in \{(1,3),(3,1)\} \\ n+2 & \textit{for} \quad (i,k) = (1,1) \end{cases}. \tag{138}$$

($\deg^{(i,k)}(n) < 0$ *means that* $\tilde{\mathbf{o}}_n^{(i,k)} H_n = 0$).

Applying the operator $\tilde{\mathbf{o}}_n^{(1,1)}$ to the inner harmonic $x \mapsto |x|^n Y_n(x/|x|)$, we are able to deduce the following relation after some easy calculations

$$\begin{aligned} \tilde{\mathbf{o}}_n^{(1,1)} r^n Y_n(\xi)|_{r=1} &= (n+2)(n+1)\mathbf{o}^{(1,1)} Y_n(\xi) - (n+2)\mathbf{o}^{(1,2)} Y_n(\xi) - (n+2)\mathbf{o}^{(2,1)} Y_n(\xi) \\ &\quad - \frac{1}{2}(n+2)(n+1)\mathbf{o}^{(2,2)} Y_n(\xi) + \frac{1}{2}\mathbf{o}^{(2,3)} Y_n(\xi). \end{aligned} \tag{139}$$

(compare with the identity (66)).

Assuming that $\{Y_{n,m}\}_{n=0,\ldots,m=1,\ldots,2n+1}$ is an orthonormal set of scalar spherical harmonics as before, we are led to introduce the following tensor spherical harmonics

$$\tilde{\mathbf{y}}_{n,m}^{(i,k)} = \left(\tilde{\mu}_n^{(i,k)}\right)^{-1/2} \tilde{\mathbf{o}}^{(i,k)} Y_{n,m}, \tag{140}$$

$n = \tilde{0}_{ik}, \ldots, m = 1, \ldots, 2n+1$, where

$$\tilde{0}_{ik} = \begin{cases} 0, & (i,k) \in \{(1,1),(2,1),(3,1)\} \\ 1, & (i,k) \in \{(1,2),(1,3),(2,3),(3,3)\} \\ 2, & (i,k) \in \{(2,2),(3,2)\} \end{cases}. \tag{141}$$

and

$$\tilde{\mu}_n^{(1,1)} = (n+2)(n+1)(2n-3)(2n-1), \tag{142}$$

$$\tilde{\mu}_n^{(1,2)} = 3n^4, \tag{143}$$

$$\tilde{\mu}_n^{(2,1)} = (n+2)(n+1)(2n-3)(2n-1), \tag{144}$$

$$\tilde{\mu}_n^{(2,2)} = n(n-1)(2n+1)(2n-1), \tag{145}$$

$$\tilde{\mu}_n^{(3,3)} = n^2(n-1)(2n+1), \tag{146}$$

$$\tilde{\mu}_n^{(1,3)} = n(n+1)^2(2n+1), \tag{147}$$

$$\tilde{\mu}_n^{(2,3)} = n^2(n+2)(n+1), \tag{148}$$

$$\tilde{\mu}_n^{(3,1)} = n^2(n+1)(2n+1), \tag{149}$$

$$\tilde{\mu}_n^{(3,2)} = n(n+1)^2(2n+1). \tag{150}$$

They are suitable for the solution of tensorial problems due to the following result involving the spaces $\mathbf{l}^2(\Omega)$ and $\mathbf{c}(\Omega)$ of square-integrable and continuous tensor fields on Ω, respectively.

Theorem 5.2
Let $\{Y_{n,m}\}_{n=0,1,\ldots,\ m=1,\ldots,2n+1}$ be an $\mathrm{L}^2(\Omega)$-orthonormal set of scalar spherical harmonics. Then, the set

$$\left\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\right\}_{i,k=1,2,3,\ n=\tilde{0}_{ik},\ldots,\ m=1,\ldots,2n+1}, \tag{151}$$

as defined by (140) forms an $\mathbf{l}^2(\Omega)$-orthonormal set of tensor spherical harmonics which is closed in $\mathbf{c}(\Omega)$ with respect to $\|\cdot\|_{\mathbf{c}(\Omega)}$ and complete in $\mathbf{l}^2(\Omega)$ with respect to $(\cdot,\cdot)_{\mathbf{l}^2(\Omega)}$.

Finally we introduce the tensor outer harmonics of degree n, order m, and kind (i,k) by (see Freeden and Schreiner (2009))

$$\mathbf{h}_{-n-1,m}^{(1,1);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+3}\tilde{\mathbf{y}}_{n,m}^{(1,1)}\left(\frac{x}{|x|}\right), \tag{152}$$

$$\mathbf{h}_{-n-1,m}^{(1,2);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+1}\tilde{\mathbf{y}}_{n,m}^{(1,2)}\left(\frac{x}{|x|}\right), \tag{153}$$

$$\mathbf{h}_{-n-1,m}^{(2,1);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+1}\tilde{\mathbf{y}}_{n,m}^{(2,1)}\left(\frac{x}{|x|}\right), \tag{154}$$

$$\mathbf{h}_{-n-1,m}^{(2,2);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n-1}\tilde{\mathbf{y}}_{n,m}^{(2,2)}\left(\frac{x}{|x|}\right), \tag{155}$$

$$\mathbf{h}_{-n-1,m}^{(3,3);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+1}\tilde{\mathbf{y}}_{n,m}^{(3,3)}\left(\frac{x}{|x|}\right), \tag{156}$$

$$\mathbf{h}_{-n-1,m}^{(1,3);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+2}\tilde{\mathbf{y}}_{n,m}^{(1,3)}\left(\frac{x}{|x|}\right), \tag{157}$$

$$\mathbf{h}_{-n-1,m}^{(2,3);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n}\tilde{\mathbf{y}}_{un,m}^{(2,3)}\left(\frac{x}{|x|}\right), \tag{158}$$

$$\mathbf{h}_{-n-1,m}^{(3,1);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+2}\tilde{\mathbf{y}}_{n,m}^{(3,1)}\left(\frac{x}{|x|}\right), \tag{159}$$

$$\mathbf{h}_{-n-1,m}^{(3,2);R}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n}\tilde{\mathbf{y}}_{n,m}^{(3,2)}\left(\frac{x}{|x|}\right), \tag{160}$$

$x \in \mathbb{R}^3\backslash\{0\}$.

These definitions (in particular the one of kind (1,1)) offer an easy way to represent the gravitational potential V in the exterior of the sphere with radius R in terms of the gravitational tensor $\nabla \otimes \nabla V$ at the satellite's height H. We start with the observation that

$$\nabla \times \nabla \left(\frac{R}{|x|} \right)^{n+1} \frac{1}{R} Y_{n,m} \left(\frac{x}{|x|} \right) = \sqrt{(n+2)(n+1)(2n-3)(2n-1)}\, \mathbf{h}^{(1,1);R}_{-n-1,m}. \tag{161}$$

Using the orthonormal basis $\{1/R\ Y_{n,m}\}$ of the space of square-integrable functions on Ω_R and $\{1/(R+H)\ \tilde{\mathbf{y}}^{(i,k)}_{n,m}\}$ of the space of square-integrable tensor fields on Ω_{R+H}, the relation (161) can be rewritten as

$$\nabla \times \nabla \left(\frac{R}{|x|} \right)^{n+1} \frac{1}{R} Y_{n,m} \left(\frac{x}{|x|} \right) = \sqrt{(n+2)(n+1)(2n-3)(2n-1)} \frac{R+H}{R} \left(\frac{R}{|x|} \right)^{n+3} \times \frac{1}{R+H} \tilde{\mathbf{y}}^{(1,1)}_{n,m}. \tag{162}$$

In other words, the transformation of the potential at height R to the Hesse tensor at height $R+H$ can be expressed by a pseudodifferential operator $\tilde{\boldsymbol{\lambda}}^{R,H}_{SGG}$ with the tensorial symbol

$$\tilde{\boldsymbol{\lambda}}^{R,H\wedge}_{SGG}(n) = \begin{pmatrix} \tilde{\Lambda}^{(1,1);R,H\wedge}_{SGG}(n) & \tilde{\Lambda}^{(1,2);R,H\wedge}_{SGG}(n) & \tilde{\Lambda}^{(1,3);R,H\wedge}_{SGG}(n) \\ \tilde{\Lambda}^{(2,1);R,H\wedge}_{SGG}(n) & \tilde{\Lambda}^{(2,2);R,H\wedge}_{SGG}(n) & \tilde{\Lambda}^{(2,3);R,H\wedge}_{SGG}(n) \\ \tilde{\Lambda}^{(3,1);R,H\wedge}_{SGG}(n) & \tilde{\Lambda}^{(3,2);R,H\wedge}_{SGG}(n) & \tilde{\Lambda}^{(3,3);R,H\wedge}_{SGG}(n) \end{pmatrix},$$

where

$$\tilde{\Lambda}^{(1,1);R,H\wedge}_{SGG}(n) = \sqrt{(n+2)(n+1)(2n-3)(2n-1)} \left(\frac{R}{R+H} \right)^{n+2}$$

and

$$\tilde{\Lambda}^{(i,k);R,H\wedge}_{SGG}(n) = 0, \quad (i,k) \neq (1,1).$$

Hence, the foreward direction of the SGG problem is described by the pseudodifferential operator $\tilde{\boldsymbol{\lambda}}^{R,H}_{SGG}$, so that the SGG problem leads to the pseudodifferential equation

$$\tilde{\boldsymbol{\lambda}}^{R,H}_{SGG} V = \mathbf{h}. \tag{163}$$

In order to formulate this equation more concretely, we show how the potential V is related to its Hesse tensor at height H:

$$V(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \left(\nabla \otimes \nabla V; \mathbf{h}^{(1,1);R+H}_{-n-1,k} \right)_{\mathrm{l}^2(\Omega_{R+H})} \left(\tilde{\mu}^{(1,1)}_n \right)^{-1/2} \left(\frac{R+H}{R} \right)^{n+2} \frac{1}{|x|^{n+1}} Y_{n,m} \left(\frac{x}{|x|} \right). \tag{164}$$

Obviously, the last formula may serve as point of departure for (regularization) solution techniques to determine V at the Earth's surface from the full Hesse tensor $\mathbf{v} = \nabla \otimes \nabla V$ at the satellite altitude. Furthermore, as described in Freeden and Schreiner (2009), it is not difficult to define tensor zonal kernels in accordance with this expansion. In particular, they allow multiscale regularization (solution) schemes based on wavelet methods.

6 Conclusion

Although an impressive rate of the Earth's gravitational potential can be detected globally at the orbit of a satellite (like GOCE), the computational drawback of satellite techniques in geoscientific research is the fact that measurements must be performed at a certain altitude. Consequently, a "downward continuation" process must be applied to handle the potential at the Earth's surface, hence, a loss of information for the signal is unavoidable. Indeed, "downward continuation" causes severe problems, since the amount of amplification for the potential is not known suitably (as an a priori amount) and even small errors in the measurements may produce huge errors in the potential at the Earth's surface.

However, it is of great advantage that satellite data are globally available, at least in principle. Nevertheless, from a mathematical point of view, we are not confronted with a boundary value of potential theory. Satellite techniques such as SST and/or SGG require the solution of an inverse problem to produce gravitational information at the Earth's surface, where it is needed actually. SST/SGG can be formulated adequately as (Fredholm) pseudodifferential equation of the first kind, which is exponentially ill-posed, and this fact makes indispensable the development of suitable mathematical methods with strong relation to the nature and structure of the data.

In this respect it should be mentioned that each approximation's theoretic method has its own aim and character. Even more, it is the essence of any numerical realization that it becomes optimal only with respect to certain specified features. For example, Fourier expansion methods with polynomial trial functions (spherical harmonics) offer the canonical "trend-approximation" of low-frequency phenomena (for global modeling), they offer an excellent control and comparison of spectral properties of the signal, since any spherical harmonic relates to one frequency. This is of tremendous advantage for relating data types under spectral aspects. But it is at the price that the polynomials are globally supported such that local modeling results into serious problems of economy and efficiency. Bandlimited kernels can be used for the transition from long-wavelength to short-wavelength phenomena (global to local modeling) in the signal. Because of their excellent localization properties in the space domain, the non-bandlimited kernels can be used for the modeling of short-wavelength phenomena. Local modeling is effective and economic. But the information obtained by kernel approximations is clustered in frequency bands so that spectral investigations are laborious and time consuming. In other words, for numerical work to be done, we have to make an a priori choice. We have to reflect the different stages of space/frequency localization so that the modeling process can be adapted to the localization requirements necessary and sufficient for our geophysical or geodetic interpretation.

In conclusion, an algorithm establishing an approximate solution for the inverse SGG-problem has to reflect the intention of the applicant. Different techniques for regularization are at the disposal of the numerical analyst for global as well as local purposes. Each effort does give certain progress in the particular field of pre–defined interest. If a broad field of optimality should be covered, only a combined approach is the strategic instrument to make an essential step forward. Thus, for computational aspects of determining the Earth's gravitational potential, at least a twofold combination is demanded, viz. combining globally available satellite data (including the SGG-contribution) with local airborne and/or terrestrial data and combining tools and means of constructive approximation such as polynomials, splines, wavelets, etc. Altogether, in numerical modeling of the Earth's gravitational potential, there is no best universal method, there exist only optimized procedures with respect to certain features and the option and the feasibility for their suitable combination.

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