Multi-Objective Optimisation Problems: A Symbolic Algorithm for Performance Measurement of Evolutionary Computing Techniques

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Abstract. In this paper, a symbolic algorithm for solving constrained multi-objective optimisation problems is proposed. It is used to get the Pareto optimal solutions as functions of KKT multipliers $\vec{\lambda}$ for multi-objective problems with continuous, differentiable, and convex/pseudo-convex functions. The algorithm is able to detect the relationship between the decision variables that form the exact curve/hyper-surface of the Pareto front. This algorithm enables to formulate an analytical form for the true Pareto front which is necessary in absolute performance measurement of evolutionary computing techniques. Here the proposed technique is tested on some test problems which have been chosen from a number of significant past studies. The results show that the proposed symbolic algorithm is robust to find the analytical formula of the exact Pareto front.

Keywords: Multi-objective optimisation, Evolutionary algorithms, Symbolic algorithm, Pareto front.

1 Introduction

Due to the importance of multi-objective optimisation problems (MOOP) for scientists and engineering designers several mathematical approaches and evolutionary algorithms (EA) have been proposed. In mathematics, the Karush-Kuhn-Tucker conditions (also known as the Kuhn-Tucker or the KKT conditions) are necessary for a solution in MOOP to be optimal. Many valuable theoretical results have been gained and have drawn much attention over the past several years since Kuhn and Tucker published their paper [1] in 1950.

Optimisation algorithms such as evolutionary or particle swarm algorithms are heuristic techniques that have been recently used to deal with multi-objective optimisation problems [2]. They have adequately demonstrated their usefulness in finding a well-converged and a well-distributed set of near Pareto-optimal solutions [3] and [4]. Because of the extensive studies and the available source codes both commercially and freely of these algorithms, they have been popularly applied in various

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problem-solving tasks and have received great attention [5]. However recent studies [6] have shown that multi-objective optimisation with fitness assignment based on Pareto-domination leads to long processing times for large population sizes. This has motivated a considerable amount of research and a wide variety of approaches have been suggested in the last few years [6]. Deb et al. have suggested a verification procedure based on KKT conditions to build confidence about the near-optimality of solutions obtained using an evolutionary optimisation procedure [11].

The aim of this paper is to present a proposed symbolic algorithm which is able to solve constrained multi-objective optimisation analytically. This new algorithm can be used to get an analytical form of the curve/hyper-surface of the Pareto front for a certain class of multi-objective problems. This class involves the set of continuous, differentiable, and convex/pseudo-convex objective functions. Moreover, for this class of functions the algorithm guides to the relationship between the decision variables which describes the Pareto front surface exactly. It is not clear from the mathematical description of the multi-objective optimisation problem (1) what would be the analytical relationship between the decision variables for the solutions to be on the true Pareto front. There is no doubt that such relationship between the decision variables need careful analysis so that one can guarantee that the solutions provided by them are Pareto optimal solutions. The observations emanated from such relationship would be very important for a designer. With such observations, the designer may be able to switch from one optimal solution to another by simple changes in the design, achieving different trade-off requirements of the objectives. This information is not only important for operational purposes; it could also provide vital insight into the problem at hand and may guide evolutionary computing techniques to finding stopping criteria and reduce the time consumption for converging to the true Pareto front. Both the analytical formula of the exact Pareto front and the relationship between the decision variables responsible for constructing this analytical formula are not provided by the state-of-the-art evolutionary algorithm, NSGA-II. The central part of the algorithm is the Karush-Kuhn-Tucker (KKT) theorem [1] which can handle high dimensionality. With this symbolic algorithm one can apply several metrics which need an analytical formula for the exact Pareto front to be known so that one can measure the performance of Evolutionary algorithms.

The layout of the paper is as follows: in section 2 some basic concepts required throughout the paper are presented. A description of the proposed symbolic algorithm is given in section 3. In section 4 some test problems to be solved using the proposed algorithm are described and the results which obtained in the experiments that performed using the algorithm are presented and discussed as well. Section 5 shows NSGA-II performance using the generational distance metric and the analytical formula of the exact Pareto front provided by the symbolic algorithm. Finally, in section 6 some conclusions are presented.

2 Preliminaries

This section highlights some definitions and notations that will be used throughout the paper. The n-dimensional Euclidean space is denoted by R^n . The constrained multi-objective optimisation problem to be handled here in this paper takes the following form [12]:

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x), ..., f_m(x))^T \\ \text{s.t.} \quad & g(x) = (g_1(x), g_2(x), ..., g_p(x))^T \leq 0 \\ & x \in X \subseteq R^n, x^{(L)} \leq x_k \leq x^{(U)}, k = \{1,2,...,n\} \end{aligned} \quad (1)$$

where, $f_M : X \to R, M = \{1,2,...,m\}$ and $g_j : X \to R, j = \{1,2,..., p\}$. In this formulation, $f_i(\vec{x})$ denotes the i^{th} objective function, $g_j(\vec{x})$ denotes inequality type of constraints. The ultimate goal is simultaneous minimisation or maximisation of all given objective functions. When the objective functions conflict each other there may be a set of many alternative solutions. This family of possible solutions cannot improve all the objective functions concurrently. This is called Pareto optimality [12] and the definition is given below. Note that any maximisation objective function can be converted into a minimisation objective by changing its sign.

Definition 2.1. A point $\hat{x} \in X$ is said to be a Pareto optimal solution (or non-inferior or efficient) to the problem (1) if and only if there is no $x \in X$ such that $f(x) \leq f(\hat{x})$.

Definition 2.2. A point $\hat{x} \in X$ is said to be a weak Pareto optimal solution to the problem (1) if and only if there is no $x \in X$ such that $f(x) < f(\hat{x})$.

Roughly speaking, a point $\hat{x} \in X$ is Pareto optimal to problem (1) if and only if one can improve (in the sense of minimisation) the value of one of the objective functions only at the cost of making at least one of the remaining objective function(s) worse; it is weak Pareto optimal if and only if one can not improve all of the objective functions simultaneously.

Definition 2.3. For any $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T \in R^n$, one can define the following [13]:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1,2,...,n$;
(ii) $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1,2,...,n$;
(iii) $x < y$ if and only if $x_i < y_i$ for all $i = 1,2,...,n$

Definition 2.4. A subset $X \subseteq R^n$ is said to be a convex set if for any two points $x, y \in X$ the segment $\alpha\, x + (1-\alpha)\, y \in X$ and $\alpha \in [0,1]$.

Definition 2.5. A function $f : X \subseteq R^n \to R$ is convex if for all $x, y \in X$ is valid that $f(\alpha\, x + (1-\alpha)\, y) \leq \alpha\, f(x) + (1-\alpha)\, f(y)$ for all $\alpha \in [0,1]$.

Definition 2.6. A function $f : X \subseteq R^n \to R$ is differentiable at $\hat{x} \in X$ if $f(\hat{x} + d) - f(\hat{x}) = \nabla f(\hat{x})^T d + \|d\| \varepsilon(\hat{x}, d)$, where $\nabla f(\hat{x})$ is the gradient of f at $\hat{x}$ and $\varepsilon(\hat{x}, d) \to 0$ as $\|d\| \to 0$.

Definition 2.7. Let the function $f : X \subseteq R^n \rightarrow R$ be differentiable at every $x \in X$. Then it is pseudo-convex function if for all $x, y \in X$ such that $\nabla f(x)^T\ (y - x) \geq 0$, we have $f(y) \geq f(x)$.

Theorem 2.1. Suppose $f(\vec{x})$ has continuous second-order partial derivatives at $\vec{x} \in C$ on some open convex set C in R^n. If the Hessian H of $f(\vec{x})$ is positive semi-definite ($H \geq 0$) on C, then $f(\vec{x})$ is convex on C (for a proof one can see [7]).

Theorem 2.2. Let the objective and the constraint functions of problem (1) be convex and continuously differentiable at a decision vector $\hat{x} \in X$. A sufficient condition for $\hat{x}$ to be Pareto optimal is that there exist multipliers $0 < \lambda \in R^m$ and $0 \leq \mu \in R^p$ such that

$$\left.\begin{array}{l} \sum_{i=1}^{m} \lambda_i\ \nabla f_i(\hat{x}) + \sum_{j=1}^{p} \mu_j\ \nabla g_j(\hat{x}) = 0 \\ \mu_j\ g_j(\hat{x}) = 0,\ j = 1,2,\ldots, p \end{array}\right\} \quad (2)$$

Proof. See [8].

3 The Algorithm

The following steps of the algorithm are directly motivated by the KKT theorem. They have been automated and coded in Mathematica® Symbolic Toolbox and run step by step [10]. Later the algorithm is applied to some test problems to illustrate its performance. After that the same problems are solved using the state-of-the-art stochastic algorithm NSGA-II to validate the symbolic algorithm. Below are the steps of the symbolic algorithm:

Step 1. Define the objective functions f_i, $i = 1,2,\ldots, M$ to be minimised.

Step 2. Calculate the Hessian Matrix $H(f_i)$ for each function separately.

If $H \geq 0$ then go to step 4, otherwise go to step 3.

Step 3. Check if the condition $f(\alpha x + (1-\alpha)\ y) \leq \max\left[f(x), f(y)\right]$ or $\nabla f(x)^T (y - x) \leq 0$ for an arbitrary y in the feasible space is satisfied. If yes go to step 4, otherwise terminate.

Step 4. Solve the system $\sum_{i=1}^{m} \lambda_i\ \nabla f_i(\hat{x}) + \sum_{j=1}^{p} \mu_j\ \nabla g_j(\hat{x}) = 0$ to get $x = x(\lambda, \mu)$.

Step 5. Use the system $\mu_j\ g_j(\hat{x}) = 0,\ j = 1,2,\ldots, p$ to get $\mu_j, j = 1,2,\ldots, p$ as a function of $\lambda_i, i = 1,2,\ldots, m$ and substitute in step 4.

Step 6. Substitute the result from step 5 in step 4 to obtain $x = x(\lambda)$.

Step 7. Construct the analytical formula between $f_i, i = 1,2,\ldots, m$ using $x = x(\lambda)$.

Step 8. End.

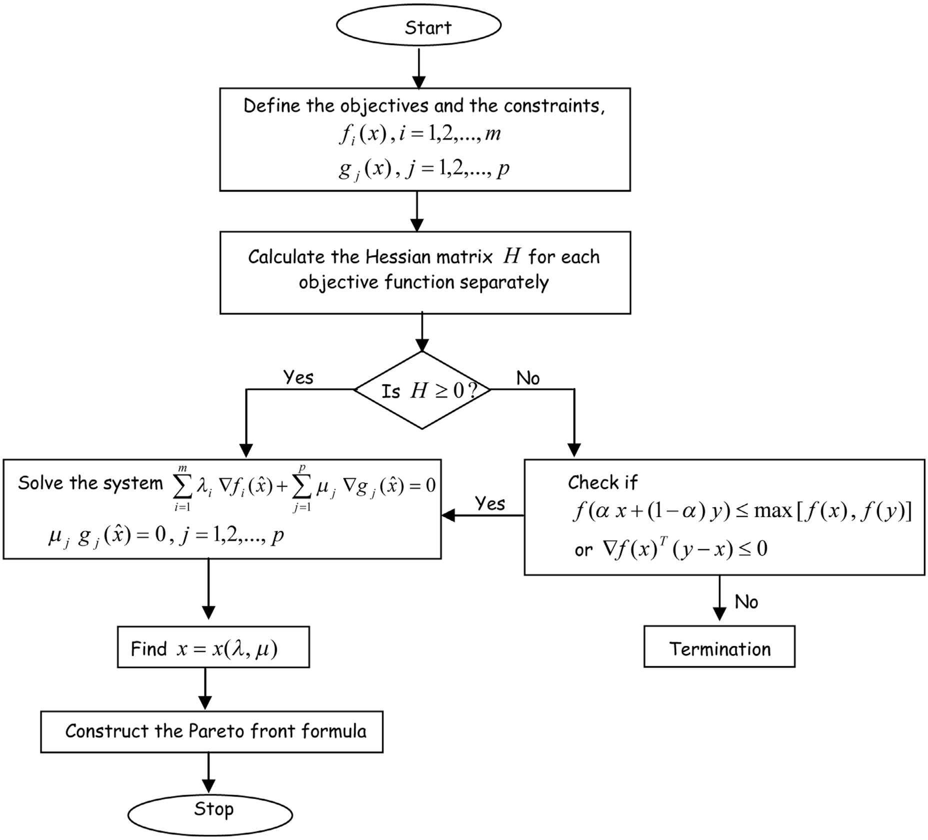

Fig. 1. A flow chart of the proposed symbolic algorithm

4 Test Problems and Results

For the validity of the new symbolic algorithm, some test problems that have been solved using the stat-of-the-art stochastic algorithms were selected from the literature to be solved by the symbolic algorithm. A complete description of these problems is shown below first and after that come a complete discussion for the results provided for each test problem separately:

Problem Formulation 4.1: (Fonseca and Fleming [3])
This problem is a typical multi-objective evolutionary algorithm (MOEA) benchmark problem. It consists of two objective functions and n decision variables as follows:

$$\min \begin{cases} f_1 = 1 - \exp\left(-\sum_{\ell=1}^{n} (x_\ell - \frac{1}{\sqrt{n}})^2\right) \\ f_2 = 1 - \exp\left(-\sum_{\ell=1}^{n} (x_\ell + \frac{1}{\sqrt{n}})^2\right) \end{cases}. \quad (3)$$

Restrictions: $-4 \leq x_\ell \leq 4, \quad \ell = 1,2,...,n$

Problem Formulation 4.2: (Deb [4])
This problem is a two-variable problem. It consists of two objective functions which have the following form:

$$
\begin{aligned}
&\min \quad F = (f_1(x), f_2(x)),\ \textit{where} \\
&f_1(x) = f(x) \\
&f_2(x) = g(y) \cdot h(f, y) \\
&\textit{and} \\
&f(x) = x \\
&g(y) = 1 + 10\, y \\
&h(f, y) = 1 - \left(\frac{f}{g}\right)^{\alpha} - \left(\frac{f}{g}\right) \cdot \sin(2\pi \cdot q \cdot f)
\end{aligned} \tag{4}
$$

where q defines the number of lags in the interval $[0,1]$ and $\alpha = 2$ is a typical choice.
Restrictions: $0 \le x, y \le 1$

Problem Formulation 4.3: (Viennet [3])
This problem is a two-variable problem. It consists of three objective functions that have the following form:

$$
\min \begin{cases} f_1(x, y) = x^2 + (y-1)^2, \\ f_2(x, y) = x^2 + (y+1)^2 + 1, \\ f_3(x, y) = (x-1)^2 + y^2 + 2 \end{cases} \tag{5}
$$

Restrictions: $-2 \le x, y \le 2$

Problem Formulation 4.4: (Constrained problem [4])
This problem is a two-variable problem. It consists of two objective functions and two inequality constraints. It has the following form:

$$
\begin{cases} f_1(x) = x_1, \\ f_2(x) = \dfrac{1 + x_2}{x_1}, \\ g_1(x) \equiv x_2 + 9x_1 \ge 6, \\ g_2(x) \equiv -x_2 + 9x_1 \ge 1 \end{cases} \tag{6}
$$

Restrictions: $0.1 \le x_1 \le 1$ and $0 \le x_2 \le 5$

After executing the symbolic algorithm, the following results and observations are obtained:

Problem Analysis 4.1: For this problem at $n = 2$ the analytical formula of the exact Pareto front is:

$$f_2 = 1 - \exp\left(-\left[2 - \sqrt{-\ell n\,(1 - f_1)}\right]^2\right) \text{ and } 0 \le f_1 < 1 - \exp\left(-2\left[4 + \frac{1}{\sqrt{2}}\right]^2\right) \quad (7)$$

Although both the objective functions are convex functions, the exact Pareto front is non-convex curve as can be seen in Figure 2 (black bold curve). This curve is constructed by the linearity relationship between the decision variables, $x_1 = x_2$ in the interval $[-1.4389, 0.707]$. As can be shown from Figure 3, not all the linearity relationships between the decision variables are used to construct the exact Pareto front. This interesting observation will help the designer to switch from one optimal solution to another. In addition, this linear relationship can be written as functions of KKT multipliers as follows:

$$\left\{ x_1 = \frac{\sqrt{2}\,\lambda_1 - \sqrt{2}\,\lambda_2}{2\,(\lambda_1 + \lambda_2)},\quad x_2 = \frac{\sqrt{2}\,\lambda_1 - \sqrt{2}\,\lambda_2}{2\,(\lambda_1 + \lambda_2)} \right\}$$

This problem has been solved using – a state of the art evolutionary technique – NSGA-II [5] with population size 100 and 100 generations using standard parameters. The result is plotted in Figure 2 (Red squares). It is shown that the robust of NSGA-II in finding uniform solutions on the exact curve of the Pareto front. Here raises the robust of the symbolic algorithm in providing a connected curve of the Pareto front. Furthermore, the symbolic algorithm is guided to the relationship between the decision variables responsible for constructing that curve. In addition, this problem has been solved using the symbolic algorithm at $n = 3$ and it yielded the following:

$$f_2 = 1 - \exp\left(-\left[2 - \sqrt{-\ell n\,(1 - f_1)}\right]^2\right) \text{ and } 0 \le f_1 < 1 - \exp\left(-3\left[4 + \frac{1}{\sqrt{3}}\right]^2\right) \quad (8)$$

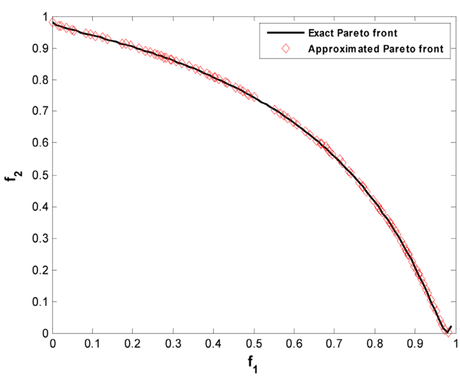

Fig. 2. Objective space of problem 4.1

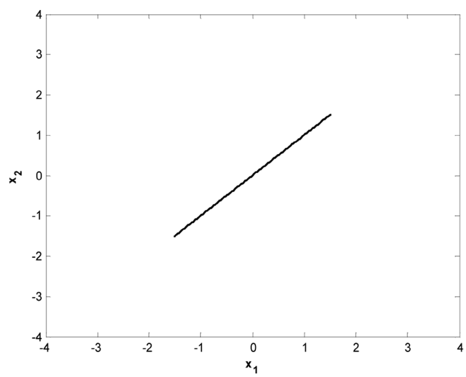

Fig. 3. Decision space of problem 4.1

This means that the dimension has no impact on the shape of the Pareto front; only the constraint imposed on f_1 changed. The algorithm shows also that the Pareto optimal solutions for this problem satisfy at $x_1 = x_2 = x_3$. Again the linearity between the decision variables is the responsible for constructing the analytical formula of the exact Pareto front.

Problem Analysis 4.2: For this problem at $q = 12$ the symbolic algorithm yielded the following results:

Case 1. $x = y = 1 \Rightarrow \vec{f} = (1, 10.9091)$ **,** This is local Pareto front point.

Case 2. $x = y = 0 \Rightarrow \vec{f} = (1, 1)$, This is local Pareto front point.

Case 3. $y = 0$ and x satisfies the equation, $2x + \sin(24\pi x) + 24\pi x \cos(24\pi x) = \dfrac{\lambda_1}{\lambda_2}$.

In this case the analytical formula that involves the exact Pareto front takes the form:

$$f_2 = 1 - f_1^2 - f_1 \sin(24\pi f_1) \text{ and } 0 < f_1 < 1 \tag{9}$$

This formula is plotted in Figure 4 (Black bold curve). As can be seen from Figure 4 it is a disconnected Pareto front. Cases 1 and 2 are dominated by points on this curve. This problem has been solved using NSGA-II with population size 100 and 100 generations using standard parameters. The result is plotted in Figure 4 (Red circles) and same observations found like the previous problem.

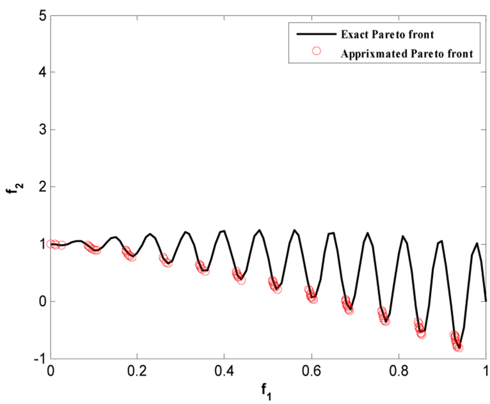

Fig. 4. Exact and approximated Pareto front to problem 4.2

Problem Analysis 4.3: For this problem the equation of the hyper-surface involved the exact Pareto front is:

$$f_3 = \left(\frac{1}{4} \sqrt{10 f_2 + 2 f_1 (3 + f_2) - f_1^2 - f_2^2 - 25} - 1 \right)^2 + \frac{1}{16} (f_1 - f_2 + 1)^2 + 2 \tag{10}$$

$$\text{and } 0 \le f_1 \le 13,\ 0 \le f_2 \le 14$$

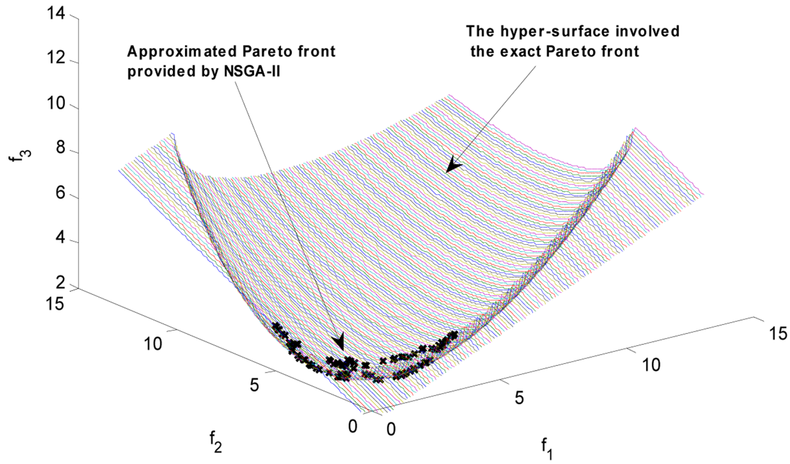

Fig. 5. Exact and approximated Pareto front

The eq. 10 is plotted in Figure 5 (the coloured hyper-surface). This equation is constructed by the following relationship between the decision variable:

$$y = \frac{\lambda_1 - \lambda_2}{\lambda_3} x$$

This problem has been solved using NSGA-II with population size 100 and 100 generations using standard parameters. The result is plotted in Figure 5 (Black crosses) and same observations found like the two previous problems.

Problem Analysis 4.4: For this problem step 4 of the symbolic algorithm yielded:

$$x_1 = \frac{\lambda_2}{\mu_1 - \mu_2 + \mu_5 - \mu_6} \quad , \quad x_2 = \frac{\lambda_2[\lambda_1 - 9(\mu_1 + \mu_2) - \mu_3 + \mu_4]}{(\mu_1 - \mu_2 + \mu_5 - \mu_6)^2} - 1 \qquad (11)$$

with 19 cases for $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and μ_6 have been obtained by step 5. Eight cases are accepted as they make Eq. 11 within its range. The other cases are rejected as they make the decision variables out of their ranges and give complex values for x_1 and x_2. In addition, these rejected cases make the constraints unsatisfied. The accepted cases are:

Case 1. $\mu_1 = \frac{\lambda_1}{18}$, $\mu_2 = \frac{\lambda_1}{18} - \frac{18\lambda_2}{7}$, $\mu_3 = \mu_4 = \mu_5 = \mu_6 = 0.$

This case yields the point $(0.3889, 2.5)$ on the border of the feasible decision space (Bold line, Figure 6). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(0.3889, 8.9997)$ on the region A (Bold curve, Figure 7).

Case 2

$$\mu_1 = 0.111111\lambda_1 - 0.25\lambda_2\,,\ \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0\,,\ \mu_5 = 0.111111\lambda_1 + 1.75\lambda_2\,.$$

This case yields the point $(0.8, 0.44)$ in the feasible decision space (Bold point, Figure 6). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(0.8, 1.8)$ (Bold point, Figure 7). This point is better than points on region C and is dominated by points from regions A and B.

Case 3. $\mu_1 = \sqrt{\dfrac{\lambda_1\lambda_2}{7}}$, $\mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0$.

This case yields: $x_1 = \sqrt{\dfrac{7\,\lambda_2}{\lambda_1}}$, $x_2 = 6 - 9\,x_1$ and $0.3889 \le x_1 \le 0.6667$. This relationship between x_1 and x_2 represents the bold line A in the decision space (Figure 6). All the points satisfying this line A are used to construct the formula, $f_2 = \dfrac{7}{f_1} - 9$, $0.3889 \le f_1 \le 0.6667$. This formula is the first part of the exact Pareto front (Bold curve A, Figure 7)

Case 4. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0, \mu_5 = \sqrt{\lambda_1\lambda_2}$.

This case yields: $x_1 = \sqrt{\dfrac{\lambda_2}{\lambda_1}}$, $x_2 = 0$ and $0.6667 \le x_1 \le 1$. This relationships for x_1 and x_2 represent the bold line B in the decision space (Figure 6). All the points satisfying this line B are used to construct the formula, $f_2 = \dfrac{1}{f_1}$, $0.6667 \le f_1 \le 1$. This formula is the second part of the exact Pareto front (Bold curve B, Figure 7)

Case 5. $\mu_1 = 0$, $\mu_2 = 0.111111\lambda_1 - 1.5\lambda_2$, $\mu_3 = \mu_4 = \mu_5 = 0$, $\mu_6 = -0.111111\lambda_1$.
This case yields the point $(0.6667{,}5)$ in the feasible decision space (Bold point on the bold line C, Figure 6). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(0.6667, 8.9996)$ (Bold point on the curve C, Figure 7). This point is dominated by points on the curve A.

Case 6. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$, $\mu_6 = -\sqrt{\dfrac{\lambda_1\lambda_2}{6}}$.

This case yields: $x_1 = \sqrt{\dfrac{6\,\lambda_2}{\lambda_1}}$, $x_2 = 5$ and $0.6667 \le x_1 \le 1$. This relationships for x_1 and x_2 represent the bold line C in the decision space (Figure 6). All the points satisfying this line C are used to construct the formula, $f_2 = \dfrac{6}{f_1}$, $0.6667 \le f_1 \le 1$. This formula creates the part C (Red curve in Figure 7). It is a local Pareto front and is dominated by both curves A and B.

Case 7. $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\lambda_1 + \lambda_2$, $\mu_5 = \lambda_2$, $\mu_6 = 0$.
This case yields the point $(1,0)$ on the border of the feasible decision space (Bold line B, Figure 6). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(1,1)$ on the border of the region B (Bold curve, Figure 7).

Case 8. $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\lambda_1 + 6\lambda_2$, $\mu_5 = 0$, $\mu_6 = -\lambda_2$.
This case yields the point $(1,5)$ on the border of the feasible decision space (Bold line C, Figure 6). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(1,6)$ on the border of the region C (Red bold curve, Figure 7). This point is dominated by the point $(1,1)$.

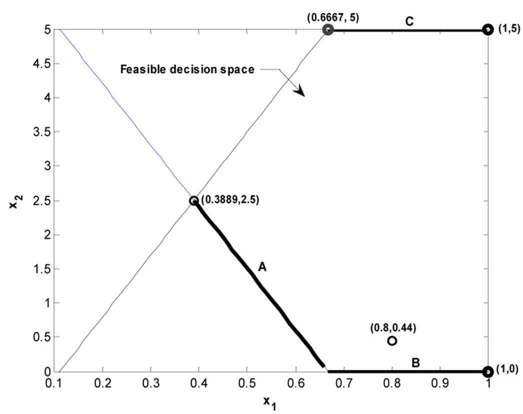

Fig. 6. Feasible search region for problem 1 in the decision variable space

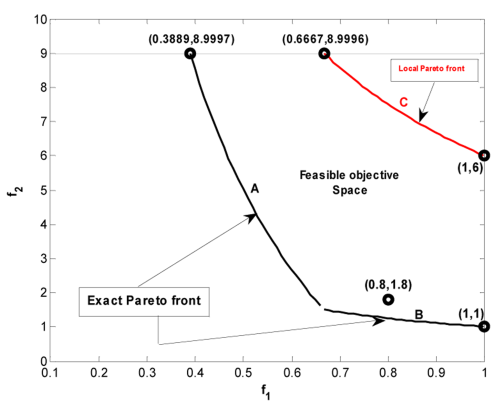

Fig. 7. The exact and local Pareto front for problem 1 using the proposed symbolic algorithm

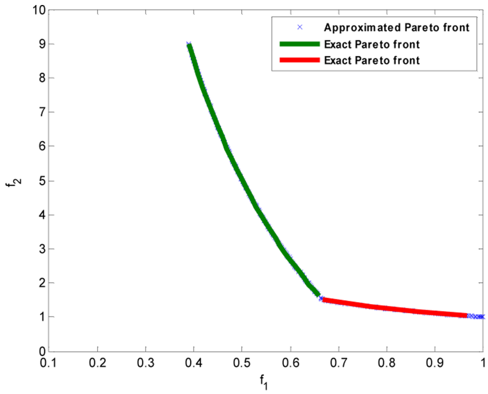

Fig. 8. The approximated Pareto front for problem 1 using NSGA-II algorithm

This problem has been solved using NSGA-II with population size 500 and 500 generations using standard parameters. The result is plotted in Figure 8 and same observations found like the three previous problems.

5 NSGA-II Absolute Performance Measurement

The term performance is always involved when comparing different optimisation techniques experimentally. In the case of multi-objective optimisation, the definition of quality is substantially complex because the optimisation goal itself consists of multiple objectives [9]:

- The distance of the resulting non-dominated set to the Pareto front should be minimised.
- A good (in most cases) uniform distribution of the solutions found is desirable. The assessment of this criterion might be based on a certain distance metric.
- The extent of the obtained non-dominated front should be maximized, i.e., for each objective, a wide range of values should be covered by the non-dominated solutions.

In the literature, some attempts can be found to formalize the above definition (or parts of it) by means of quantitative metrics [2]. Within this paper the generational distance (GD) metric is used. This metric is the average distance from the obtained Pareto front (PF_{known}) to the true Pareto front (PF_{true}) and is defined as follows [2]:

$$GD = \frac{\left(\sum_{i=1}^{n} d_i^{p}\right)^{\frac{1}{p}}}{n} \qquad (12)$$

where n is the number of vectors in PF_{known}, $p = 2$ and d_i is the Euclidean distance (in objective space) between each vector and the nearest vector of PF_{true}. The result $GD = 0$ indicates $PF_{known} = PF_{true}$; any other value indicates PF_{known} deviates from PF_{true}.

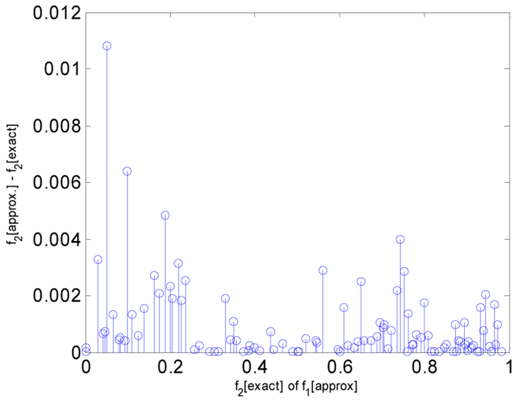

Fig. 9. The residual plot between the approximated Pareto front and the exact Pareto front for problem 4.1

The performance of NSGA-II is shown in Figure 9. This figure illustrates the residuals between the approximated Pareto front and the true Pareto front. As expected for a stochastic technique NSGA-II is able to find some points but not all the points on the true Pareto front in the final generation. There are also some minor deviations from the true Pareto front as shown in the left part of Figure 9.

The NSGA-II with 100 generations and 100 individuals in each generation has been executed 10 times on Fonseca and Fleming problem. By choosing 100 values on the true Pareto front provided by the paper's algorithm near to the 100 individuals obtained by NSGA-II, the generational distance metric has been calculated in each experiment separately using eq. 12.

The experiments show that with a minimum *GD* = 0.000829839 the NSGA-II can approximate the Pareto front in some runs quite well. However, in other runs the approximated Pareto front obtained by NSGA-II is not perfect since maximum *GD* = 0.001065864. The mean and standard deviation of all the 10 experiments are μ= 0.982912*E*-3 and σ = 7.23078*E*-5, respectively. The small standard deviation shows that the GD values after 100 generations of the NSGA-II are already quite close to the mean. However, a GD different from zero indicated an ongoing approximation process.

The proposed KKT-based algorithm providing a closed formula for the Pareto front curve allows for a very precise statistical analysis of the performance of stochastic multi-objective optimisation techniques such as NSGA-II using an absolute performance measure such as the GD.

6 Conclusions

A symbolic algorithm for multi-objective optimisation problems was proposed. It has been applied on some test problems. Exact solutions for these problems have been found by this algorithm. The analytical form of the exact Pareto front has been formulated using the algorithm for these problems as well. Furthermore, a linear relationship between the decision variables has been formulated as a function of KKT multipliers.

This relationship itself is responsible for constructing the true Pareto front. As has been mentioned within this paper, this relationship has a significant contribution in innovation. It guides the designer to switch from one optimal solution to other. Furthermore, it helps to measure the performance of evolutionary algorithms. In addition, it might be used to form the stopping criteria for evolutionary algorithms. The generational distance metric was used to evaluate the performance of the NSGA-II algorithm using the analytical formula of the exact Pareto front found by the proposed algorithm.

References

1. Kuhn, H.W., Tucker, A.W.: Nonlinear Programming. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–491 (1950)
2. Collette, Y., Siarry, P.: Multiobjective Optimisation: Principles and Case Studies. Springer, Berlin (2003)
3. Coello, C.A.C., Van Veldhuizen, D.A., Lamont, G.B.: Evolutionary Algorithms for Solving Multi-Objective Problems. Kluwer Academic Publishers, New York (2002)
4. Deb, K.: Multi-Objective Optimisation using Evolutionary Algorithms. Wiley, Chichester (2001)
5. Deb, K., Sundar, J.: Reference Point Based Multi-Objective Optimisation Using Evolutionary Algorithms. In: Proceedings of the 8th annual conference on Genetic and evolutionary computation, Seattle, Washington, USA, pp. 635–642 (2006)
6. Jensen, M.K.: Reducing the Run-Time Complexity of Multi-objective EAs: The NSGA-II and Other Algorithms. IEEE Transactions on Evolutionary Computation 7(5), 503–515 (2003)
7. Arora, J.S.: Introduction to Optimum Design. Elsevier, Academic Press, UK (2004)
8. Miettinen, K.: Nonlinear Multiobjective Optimisation. Kluwer Academic Publishers, Boston (1999)
9. Zitzler, E., Deb, K., Thiele, L.: Comparison of Multi-objective Evolutionary Algorithms: Empirical Results. Evolutionary Computation 8(2), 173–195 (2002)
10. Askar, S.S., Tiwari, A., Mehnen, J., Ramsden, J.: Solving Real-Life Multi-objective Optimisation Problems: A Mathematical Approach. In: Cranfield Multi-Strand Conference: Creating Wealth Through Research and Innovation (CMC 2008), UK (accepted, 2008)
11. Deb, K., Tewari, R., Dixit, M., Dutta, J.: Finding Trade-off Solutions Close to KKT Points Using Evolutionary Multi-objective Optimisation. In: IEEE Congress on Evolutionary Computation (CEC 2007), pp. 2109–2116 (2007)
12. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of Multi-Objective Optimisation. Academic Press Inc., London (1985)
13. Ehrgott, M.: Multicriteria Optimisation: Lecture notes in Economics and Mathematical Systems. Springer, Germany (2000)