Singularly Perturbed Reaction–Diffusion Problem with a Boundary Turning Point

C. de Falco and E. O'Riordan

Abstract Parameter uniform numerical methods for singularly perturbed reaction diffusion problems have been examined extensively in the literature. By using layer adapted meshes of Bakhvalov or Shishkin type, it is now well established that one can achieve second order (or almost second order in the case of the simpler Shishkin meshes) parameter uniform convergence globally in the pointwise maximum norm. Note that, in proving such results, it is often assumed that the coefficient of the reactive term is strictly positive throughout the domain. In this paper, we examine a problem where the reaction coefficient is zero on parts of the boundary.

1 Introduction

Parameter-uniform [6] numerical methods for singularly perturbed reaction diffusion problems of the form

$$-\varepsilon\triangle u + bu = f, \mathbf{x} \in \Omega,\ u = g,\ \mathbf{x} \in \partial\Omega, \tag{1}$$

have been examined extensively [1–3, 10]. By using layer adapted meshes of Bakhvalov [2] or Shishkin [10] type, it is well established that one obtains second order (or almost second order in the case of the simpler Shishkin meshes) uniform convergence globally in the pointwise maximum norm. Note that, normally one assumes that

$$b(\mathbf{x}) \geq \beta > 0,\ \forall \mathbf{x} \in \overline{\Omega}. \tag{2}$$

In this paper, we examine a problem where b is zero on parts of the boundary $\partial\Omega$ and also depends on ε. We are interested in the necessary modifications required when using a piecewise-uniform Shishkin mesh and in the subsequent error analysis.

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A.F. Hegarty et al. (eds.), *BAIL 2008 - Boundary and Interior Layers.*
Lecture Notes in Computational Science and Engineering,
DOI: 10.1007/978-3-642-00605-0,

As for a possible application of the results presented here, in [5] a method was presented for computing the differential capacitance of Metal Oxide Semiconductor (MOS) structure and the advantage of using a layer-adapted mesh was shown to be non negligible for values of the coefficients within a physically reasonable range. The simple example presented in the numerical experiments of Sect. 4 demonstrates that similar benefits are to be expected if the model of [5] is extended to take into account quantization effects [4, 7].

Notation: Throughout this paper C (sometimes subscripted) denotes a generic constant that is independent of ε and N. We also use the following notation

$$|f|_k := \max_{x\in(0,1)} |\frac{d^k f}{dx^k}(x)| \quad \text{and} \quad \|f\| := \max_{x\in[0,1]} |f(x)|.$$

2 Continuous Problem

Consider the following two point boundary value problem

$$Lu := -\varepsilon u'' + b(x;\varepsilon)u = f(x),\ x \in \Omega = (0,1), \tag{3a}$$

$$b(x;\varepsilon) \geq 0,\ x \in \bar{\Omega},\ u(0) = u_0,\ u(1) = u_1, \tag{3b}$$

where f, b are sufficiently smooth and the coefficient b satisfies the following

$$b(0;\varepsilon) = 0;\ |b|_k \leq C\varepsilon^{-k/2}, k \leq 2, \tag{3c}$$

$$(1-\gamma)b(x;\varepsilon) + \sqrt{\varepsilon}\sqrt{\gamma}(\sqrt{b})'(x;\varepsilon) \geq m > 0,\ 0 < \gamma < 1, \tag{3d}$$

$$|b(x;\varepsilon) - b(x;0)| \leq Me^{-x\sqrt{\frac{\theta}{\varepsilon}}},\ \theta \geq \gamma\|b\|, \tag{3e}$$

$$b(x;0) := \lim_{\varepsilon\to 0} b(x;\varepsilon),\ \forall x \in \bar{\Omega}. \tag{3f}$$

For any specific b, we will need to identify the parameters m, M, θ, γ. We note in passing that we do not assume that $f(0) = 0$. We adopt the following notation for the following particular ordering of the two limits

$$b(0;0) := \lim_{x\to 0}\big(\lim_{\varepsilon\to 0} b(x;\varepsilon)\big).$$

From the above assumptions on b, the function $y := \sqrt{b}$ satisfies the following singularly perturbed nonlinear Riccati equation

$$\sqrt{\gamma}\sqrt{\varepsilon}y' + (1-\gamma)y^2 = g^2, x > 0,\ y(0) = 0.$$

We construct a lower solution $\underline{y}$ for y of the form $\underline{y} = C_1(1-e^{-x\sqrt{\frac{\gamma}{\varepsilon}}}),\ C_1 \leq \|g\|$, where $C_1 > 0$ is such that

$$\sqrt{\gamma}\sqrt{\varepsilon}\underline{y}' + (1-\gamma)\underline{y}^2 = C_1\gamma(e^{-t} + K_1(1-e^{-t})^2) \le m,$$
$$\text{and} \qquad t := x\sqrt{\frac{\gamma}{\varepsilon}},\ \gamma K_1 := C_1(1-\gamma).$$

For the function $h(t) = e^{-t} + K_1(1-e^{-t})^2,\ t \ge 0$, we note that

$$\min\left\{K_1, 1 - \frac{1}{4K_1}\right\} \le h(t) \le \max\{1, K_1\}.$$

Hence,

$$\sqrt{\gamma}\sqrt{\varepsilon}\underline{y}' + (1-\gamma)\underline{y}^2 \le C_1 \max\{\gamma, C_1(1-\gamma)\}.$$

Thus, the choice of

$$C_1 := \min\left\{\frac{m}{\gamma}, \sqrt{\frac{m}{1-\gamma}}\right\} \tag{4}$$

suffices for $\underline{y}$ to be a lower solution. It follows that, from (3c) and (3d) that

$$\sqrt{b(x;\varepsilon)} \ge C_1(1 - e^{-x\sqrt{\frac{\gamma}{\varepsilon}}}),\ x \in [0,1], \tag{5a}$$
$$b(x;\varepsilon) \ge \beta := C_1^2(1-e^{-1})^2 > 0,\ x \ge \sqrt{\frac{\varepsilon}{\gamma}}. \tag{5b}$$

Note that (5a) implies that $b(x;0) > 0,\ x \in [0,1]$.

The standard maximum principle [9] is still valid for the linear differential operator L. That is, if $w \in C^0(\bar{\Omega}) \cap C^2(\Omega), w(0) \ge 0, w(1) \ge 0$, and for all $x \in \Omega,\ Lw \ge 0$, then $w \ge 0$ for all $x \in \bar{\Omega}$.

Lemma 1. *For all* $k,\ 0 \le k \le 4,\ |u|_k \le C\left(1 + \varepsilon^{-k/2}\right)$, *where* u *is the solution of problem (3).*

Proof. Consider the following barrier function

$$\phi(x) = A^* + B^*(1 - e^{-\sqrt{\frac{\eta}{\varepsilon}}x}) \ge A^*,$$

where A^*, B^* and η are positive constants specified below. Note that, outside the layer region, if $\beta A^* \ge \|f\|$, then

$$L\phi(x) \ge \beta\phi(x) \ge \|f\|,\ x \ge \sqrt{\frac{\varepsilon}{\gamma}}.$$

In the layer region, where $x \le \sqrt{\frac{\varepsilon}{\gamma}}$, we have that

$$L\phi(x) = b\phi + B^*\eta e^{-x\sqrt{\frac{\eta}{\varepsilon}}} \ge B^*(\eta - b)e^{-x\sqrt{\frac{\eta}{\varepsilon}}} \ge B^*(\eta - \|b\|)e^{-\sqrt{\frac{\eta}{\gamma}}} \ge \|f\|,$$

if $\eta \ge \|b\|$ and $B^*(\eta - \|b\|) \ge \|f\|e^{\sqrt{\frac{\eta}{\gamma}}}$. We can choose $\eta = \gamma + \|b\|$, then by an appropriate choice of A^* and B^* we deduce the stability bound

$$\|u\| \le \max\{\frac{\|f\|}{\beta}, |u_0|, |u_1|\} + \frac{\|f\|}{\gamma} e^{\sqrt{1+\frac{\|b\|}{\gamma}}}(1 - e^{-x\sqrt{\frac{\gamma+\|b\|}{\varepsilon}}}).$$

Recall that (3c) allows the derivatives of the coefficient b to depend adversely on ε. Hence, as we derive parameter explicit bounds on the derivatives of u below, we need to identify how the error constants in these bounds depend on b and its derivatives explicitly. To bound the first derivative of u we use an argument from [2]. Let $x \in \Omega$ and construct a neighbourhood $N_x = (a, a + \sqrt{\varepsilon})$ such that $x \in N_x$ and $N_x \subset \Omega$. Then, by the mean value theorem, for some $y \in \bar{N}_x$,

$$\frac{u(a + \sqrt{\varepsilon}) - u(a)}{\sqrt{\varepsilon}} = u'(y).$$

It follows that $|u'(y)| \le 2\varepsilon^{-1/2}\|u\|_{N_x} \le 2\|u\|\varepsilon^{-1/2}$. Now

$$u'(x) = u'(y) + \int_y^x u''(\xi)d\xi = u'(y) + \varepsilon^{-1}\int_y^x (f - bu)(\xi)\, d\xi$$

and so

$$\|u'\| \le \big(\|f\| + (2 + \|b\|)\|u\|\big)\varepsilon^{-1/2}.$$

The bounds on the higher derivatives are obtained using the differential equation (3a) and (3c). Note that

$$\begin{aligned}
\varepsilon^{1/2}|u|_1 &\le \|f\| + (2 + \|b\|)\|u\|; \qquad \varepsilon|u|_2 \le \|f\| + \|b\|\|u\|;\\
\varepsilon^{3/2}|u|_3 &\le \|b\|\|f\| + \sqrt{\varepsilon}|f|_1 + (\|b\|(2 + \|b\|) + \sqrt{\varepsilon}|b|_1)\|u\|;\\
\varepsilon^2|u|_4 &\le \varepsilon\big(|f|_2 + \|b\||u|_2 + 2|b|_1|u|_1 + \|u\||b|_2\big)\\
&\le \varepsilon|f|_2 + \|b\|\varepsilon|u|_2 + 2\sqrt{\varepsilon}|b|_1\sqrt{\varepsilon}|u|_1 + \|u\|\varepsilon|b|_2.
\end{aligned}$$

□

Define the associated operator

$$L_1\omega(x) := (-\varepsilon\omega'' + b(x;0)\omega),\ x \in \Omega,\ b(x;0) \ge \beta > 0.$$

Decompose the solution into three components of the form

$$u(x) = v(x) + w_L(x) + w_R(x),$$

where

$$\begin{aligned}
L_1 v &= f, x \in \Omega, & v(0) &= \frac{f(0)}{b(0;0)}, \quad v(1) = \frac{f(1)}{b(1;0)},\\
Lw_L &= (L_1 - L)v, x \in \Omega, & w_L(0) &= u(0) - v(0),\ w_L(1) = 0,\\
Lw_R &= 0, x \in \Omega, & w_R(0) &= 0,\ w_R(1) = u(1) - v(1).
\end{aligned}$$

The boundary conditions for the regular component v have been selected [8] so that

$$|v|_k \le C\frac{\|f\|}{\beta}(1+\varepsilon^{1-k/2}), \quad k \le 4. \tag{6}$$

Lemma 2. *The layer components w_L, w_R in the solution of problem (3) satisfy the following bounds*

$$|w_L(x)| \le Ce^{-\int_0^x \sqrt{\frac{\gamma b(t;\varepsilon)}{\varepsilon}}\,dt},\ |w_R(x)| \le Ce^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)}, \tag{7a}$$

$$|w_L|_k \le C\varepsilon^{-k/2},\ |w_R|_k \le C\varepsilon^{-k/2}, 1 \le k \le 4. \tag{7b}$$

Proof. To obtain the pointwise bound on $w_L(x)$, use the barrier function

$$\Psi_L(x) = K_2 e^{-\int_0^x \sqrt{\frac{\gamma b(t;\varepsilon)}{\varepsilon}}\,dt}, \quad K_2 = \max\{\frac{M}{m}\|v\|, |u(0)-v(0)|\},$$

which satisfies

$$\begin{aligned} L\Psi_L &= \big(b(x;\varepsilon)(1-\gamma) + \sqrt{\gamma\varepsilon}(\sqrt{b(x;\varepsilon)})'\big)\Psi_L \ge m\Psi_L \\ &\ge M\|v\|e^{-x\sqrt{\frac{\theta}{\varepsilon}}} \ge |(L-L_1)v|. \end{aligned}$$

To obtain the pointwise bound on $w_R(x)$ use the barrier function

$$\Psi_R(x) = K_3 e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)} + K_4 e^{-\sqrt{\frac{\beta}{\varepsilon}}}(1 - e^{-\sqrt{\frac{1+\|b\|}{\varepsilon}}x}).$$

Outside the left layer, we have that

$$L\Psi_R(x) \ge K_3(b-\beta)e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)} \ge 0,\ x \ge \sqrt{\frac{\varepsilon}{\gamma}},$$

and within the left layer, for $x \le \sqrt{\frac{\varepsilon}{\gamma}}$,

$$\begin{aligned} L\Psi_R(x) &\ge K_4 e^{-\sqrt{\frac{\beta}{\varepsilon}}}(1+\|b\|-b)e^{-\sqrt{\frac{1+\|b\|}{\varepsilon}}x} - K_3\beta e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)} \\ &\ge K_4 e^{-\sqrt{\frac{\beta}{\varepsilon}}} e^{-\sqrt{\frac{1+\|b\|}{\gamma}}} - K_3\beta e^{-\sqrt{\frac{\beta}{\varepsilon}}} e^{\sqrt{\frac{\beta}{\gamma}}} \ge 0, \end{aligned}$$

when we choose K_4 such that $K_4 \ge K_3\beta e^{\sqrt{\frac{1+\|b\|}{\gamma}}} e^{\sqrt{\frac{\beta}{\gamma}}}$ and $K_3 \ge |u(1)-v(1)|$. The bounds on the derivatives are derived as in the proof of Lemma 1. □

3 Discrete Problem

On the domain Ω a piecewise-uniform Shishkin mesh [1] of N mesh intervals is constructed in the usual way. The domain $\overline{\Omega}$ is subdivided into the three subintervals $[0,\sigma_1],\ [\sigma_1, 1-\sigma_2]$ and $[1-\sigma_2, 1]$. On $[0,\sigma_1]$ and $[1-\sigma_2, 1]$ a uniform mesh with

$\frac{N}{4}$ mesh-intervals is placed, while $[\sigma_1, 1-\sigma_2]$ has a uniform mesh with $\frac{N}{2}$ mesh-intervals. The interior mesh points are denoted by Ω_ε^N and $h_i := x_i - x_{i-1}$ is the mesh step. Let

$$\sigma_1 := \min\left\{\frac{1}{4}, \tau\right\}, \quad \sigma_2 := \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\right\}. \tag{8a}$$

Our transition point τ in our Shishkin mesh will be chosen such that

$$\int_0^\tau \sqrt{\gamma b(t;\varepsilon)}\,dt \ge 2\sqrt{\varepsilon}\ln N.$$

For example, based on the lower bound on $\sqrt{b}$ in (5) we take,

$$\tau \ge \frac{\sqrt{\varepsilon}}{\sqrt{\gamma}}(1 + \frac{2}{C_1}\ln N),\ C_1 = \min\left\{\sqrt{\frac{m}{1-\gamma}}, \frac{m}{\gamma}\right\}. \tag{8b}$$

The discrete problem is: Find U such that

$$L^N U(x_i) := -\varepsilon\delta^2 U(x_i) + b(x_i;\varepsilon)U(x_i) = f(x_i),\ x_i \in \Omega_\varepsilon^N, \tag{9a}$$

$$U(0) = u(0),\ U_\varepsilon(1) = u_\varepsilon(1), \tag{9b}$$

$$\text{where} \quad \delta^2 Z(x_i) := \left(\frac{Z(x_{i+1}) - Z(x_i)}{h_{i+1}} - \frac{Z(x_i) - Z(x_{i-1})}{h_i}\right)\frac{2}{h_i + h_{i+1}}.$$

The finite difference operator L^N satisfies a discrete comparison principle. That is for any mesh function Z, if $L^N Z(x_i) \ge 0$ for all $x_i \in \Omega_\varepsilon^N$, $Z(0) \ge 0$, $Z(1) \ge 0$ then $Z(x_i) \ge 0$ for all $x_i \in \overline{\Omega}_\varepsilon^N$.

Lemma 3. *For any mesh function Z then*

$$\|Z\| \le C(\|L^N Z\| + |Z(0)| + |Z(1)|).$$

Proof. Consider the following barrier function

$$\Phi(x_i) = B_1^*(1 - W(x_i)) + \max\{\frac{\|L^N Z\|}{\beta}, |u_0|, |u_1|\}, \quad W(x_i) := \Pi_{j=1}^i (1 + h_j\sqrt{\frac{\zeta}{\varepsilon}})^{-1},$$

where B_1^*, ζ are specified below. Note that $W(x_i) \ge e^{-x_i\sqrt{\frac{\zeta}{\varepsilon}}}$ and

$$L^N\Phi(x_i) = b\Phi(x_i) + B_1^*\frac{2\zeta h_{i+1}}{h_i + h_{i+1}}W(x_{i+1}).$$

We note that outside the layer, $L^N\Phi(x_i) \ge \|L^N Z\|,\ x_i \ge \sqrt{\varepsilon\gamma^{-1}}$. In the layer region, where $x_i < \sqrt{\varepsilon\gamma^{-1}}$, we have that $h_{i+1} = h_i =: h$ and for sufficiently large N

$$h\sqrt{\frac{\zeta}{\varepsilon}} \le \frac{8\sqrt{\zeta}}{C_1\sqrt{\gamma}}N^{-1}\ln N + \frac{4\sqrt{\zeta}}{\sqrt{\gamma}}N^{-1} \le \ln 2.$$

Hence, we have that

$$\begin{aligned} L^N\Phi(x_i) &\ge b\Phi + B_1^*\frac{\zeta}{2}W(x_{i+1}) \ge B_1^*(\zeta W(x_{i+1}) - \|b\|W(x_i)) \\ &\ge B_1^*(\zeta(1+\ln 2)^{-1} - \|b\|)e^{-x_i\sqrt{\frac{\zeta}{\varepsilon}}} \\ &\ge B_1^*(\zeta(1+\ln 2)^{-1} - \|b\|)e^{-\sqrt{\frac{\zeta}{\gamma}}} \ge \|L^N Z\|, \end{aligned}$$

if we choose $\zeta = (1+\ln 2)\|b\| + \gamma$ and $\gamma B_1^* = (1+\ln 2)\|L^N Z\|e^{\sqrt{\frac{\zeta}{\gamma}}}$. □

The discrete solution is decomposed analogously to the continuous solution. That is

$$U(x_i) = V(x_i) + W_L(x_i) + W_R(x_i),$$

where $V(0) = v(0), V(1) = v(1),\ W_L(0) = w_L(0), W_L(1) = 0,\ W_R(0) = 0,$ $W_R(1) = w_R(1)$ and

$$L_1^N V = f,\ L^N W_L = (L_1^N - L^N)V,\ L^N W_R = 0, x_i \in \Omega_\varepsilon^N.$$

From the bounds on the derivatives of the components and Lemma 3, we can follow the argument in [8] to deduce that

$$\|\bar{U} - u\| \le C(N^{-1}\ln N)^2, \tag{10}$$

where $\bar{U}$ is the piecewise linear interpolant of the discrete solution U of the discrete problem (9) and u is the solution of the continuous problem (3).

4 Numerical Experiments

As mentioned in the introduction, a physical problem whose numerical approximation requires relaxing the hypothesis (2) is that of computing the capacitance of an MOS structure where energy quantization in the inversion layer is to be taken into account. By choosing to model such quantization effects following the approach of [7] and performing the scaling and linearization procedure presented in [5], this problem leads to an equation of the form

$$-\varepsilon u'' + e^{A(x)}(1 - e^{-x^2/\lambda^2})u = f(x),$$

where $A(x)$ is the scaled electric potential, λ is the scaled electron wavelength and the semiconductor insulator interface is placed at $x = 0$. Rescaling we get

$$-\varepsilon e^{-A}u'' + (1 - e^{-x^2/\lambda^2})u = e^{-A}f(x),\ b(x;\lambda) = 1 - e^{-t}, t := x^2/\lambda^2. \tag{11a}$$

We set A to be constant and below we will see that it is necessary that

$$\lambda^2 = C\varepsilon. \tag{11b}$$

Let us check that finite values $m^*, M^*, \theta^*, \gamma^*$ exist for the parameters m, M, θ, γ so that the constraints (3c)–(3e) on the coefficient of the zero order term are satisfied, which are required by the theory in the preceding sections. Introduce the additional parameter α_0 defined by $e^A\lambda^2 =: \alpha_0\varepsilon$. Observe that

$$b(0;\varepsilon) = 0;\ |b|_k \le C(\lambda^2)^{-k/2}, k \le 2,\ \|b\| \le 1,$$

$$(1-\gamma)b + \sqrt{\gamma\varepsilon e^{-A}}(\sqrt{b})' = (1-\gamma)(1-e^{-t}) + \sqrt{\frac{\gamma\varepsilon e^{-A}}{\lambda^2}}\sqrt{t}e^{-t}(1-e^{-t})^{-1/2}$$

$$= \sqrt{\frac{\gamma}{\alpha_0}}\big(K(1-e^{-t}) + \sqrt{t}e^{-t}(1-e^{-t})^{-1/2}\big),\ K = (1-\gamma)\sqrt{\frac{\alpha_0}{\gamma}}.$$

Note that $K(1-e^{-t}) + \sqrt{t}e^{-t}(1-e^{-t})^{-1/2} \ge \min\{1, K\}$. We then have that

$$(1-\gamma)b + \sqrt{\gamma\varepsilon e^{-A}}(\sqrt{b})' \ge \min\{1-\gamma, \frac{\sqrt{\gamma}}{\sqrt{\alpha_0}}\} =: m^*.$$

For all $\alpha_0 > 0$, we can choose $0 < \gamma^* < 1$ so that

$$1-\gamma^* = \frac{\sqrt{\gamma^*}}{\sqrt{\alpha_0}}.$$

Hence,

$$\sqrt{\gamma^*} := \frac{2\sqrt{\alpha_0}}{\sqrt{1+4\alpha_0}+1} \quad \text{and} \quad m^* := 1-\gamma^*. \tag{12}$$

Let us examine condition (3e)

$$b(x;\varepsilon) - b(x;0) = e^{-t} \le Me^{-\sqrt{\theta\alpha_0 t}},\ \theta \ge \gamma.$$

Then, we choose M^* and θ^* so that, $\theta^* := \gamma^*$ and $M^* := e^{0.25\gamma^*\alpha_0}$. Under these constraints, we take the transition point in (8) to be

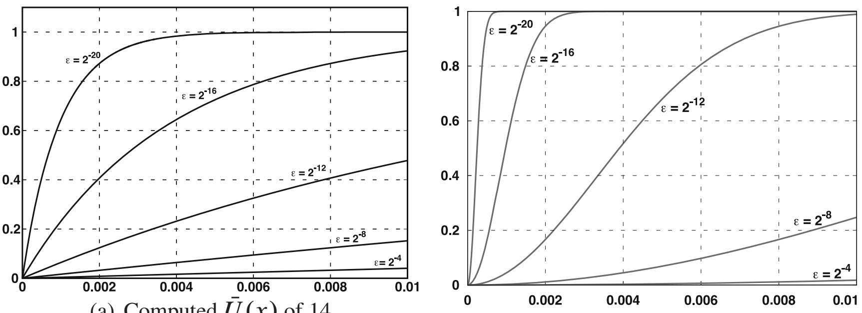
$$\tau = \sqrt{\frac{e^A\,\lambda^2}{\alpha_0\gamma^*}}\left[1 + \frac{2\,\ln(N)}{\min\left\{1,\ \dfrac{1-\gamma^*}{\gamma^*}\right\}}\right], \tag{13}$$

where

$$e^A\lambda^2 =: \alpha_0\varepsilon \quad \text{and} \quad \sqrt{\gamma^*} = \frac{2\sqrt{\alpha_0}}{\sqrt{1+4\alpha_0}+1}.$$

Table 1 The ε-uniform nodal differences D^N, nodal rates p^n, global differences $\bar{D}^N$ and global rates $\bar{p}^N$ for the numerical approximations to the solution of problem (14)

	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1{,}024$	$N = 2{,}048$	$N = 4{,}096$	$N = 8{,}192$	$N = 16{,}384$
D^N	0.01714	0.01196	0.004853	0.001692	0.0008845	0.00036	0.0001223	3.636e-05	1.049e-05	3.024e-06
p^N	0.6178	0.5424	1.287	1.529	0.9384	1.3	1.558	1.75	1.794	1.794
$\bar{D}^N$	0.1871	0.08985	0.04164	0.01445	0.004413	0.00135	0.0004053	0.0001205	3.49e-05	1.006e-05
$\bar{p}^N$	0.8217	1.108	1.097	1.536	1.716	1.714	1.736	1.75	1.788	1.795



(a) Computed $\bar{U}(x)$ of 14

(b) Coefficient $b(x;\epsilon)$ of 14

Fig. 1 The computed approximations $\bar{U}(x)$ of (14), using $N = 4{,}096$, and the coefficient $b(x;\varepsilon)$ for several different values of ε over the interval $[0, 0.01]$

Note that if λ^2 is not bounded above by $C\varepsilon$ then $M^* \to \infty$ as $\varepsilon \to 0$ and so the above error bounds are not uniformly bounded. Hence we require that $\lambda^2 = \mathcal{O}(\varepsilon)$. Below we present numerical results for the particular problem

$$-\varepsilon u'' + (1 - e^{-x^2/\lambda^2})u = 1, \ \lambda^2 = 0.09\varepsilon, \tag{14a}$$

$$u(0) = 0, \ u(1) = (1 - e^{-1/\lambda^2})^{-1}. \tag{14b}$$

The boundary condition at $x = 1$ means that there will be no layer in the vicinity of $x = 1$ and so it suffices to have $\sigma_2 = 0.25$, $\sigma_1 = \min\{0.5, \tau\}$ and to place $N/2$ mesh points in the intervals $[0, \sigma_1]$, $[\sigma_1, 1]$, in this case. In Fig. 1, plots of the layer and of the coefficient $b(x;\varepsilon)$ are displayed for several values of ε.

In Table 1, both the global and the nodal orders of convergence are estimated over the parameter range $\varepsilon = 2^0, 2^{-1}, \ldots, 2^{-20}$, using the double mesh principle [6]. The computed ε-uniform orders of convergence displayed are in line with the theoretical ε-uniform global convergence rate stated in the error bound (10).

Acknowledgement This research was supported by the Mathematics Applications Consortium for Science and Industry in Ireland (MACSI) under the Science Foundation Ireland (SFI) mathematics initiative.

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