A Patched Mesh Method for Singularly Perturbed Reaction–Diffusion Equations

C. de Falco and E. O'Riordan

Abstract A singularly perturbed elliptic problem of reaction–diffusion type is examined. The solution is decomposed into a sum of a regular component, boundary layer components and corner layer components. Numerical approximations are generated separately for each of these components. These approximations are patched together to form a global approximation to the solution of the continuous problem. An asymptotic error bound in the pointwise maximum norm is established; whose dependence on the values of the singular perturbation parameter is explicitly identified. Numerical results are presented to illustrate the performance of the numerical method.

1 Introduction

Consider the singularly perturbed diffusion reaction problem

$$-\varepsilon \triangle u(\mathbf{x}) + b(\mathbf{x})\, u(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \Omega \subset \mathbb{R}^d,\ u|_{\partial\Omega} = g(\mathbf{x}), \tag{1}$$

with $0 < \varepsilon << 1$ and $b(\mathbf{x}) \geq \beta > 0$ for $\mathbf{x} \in \overline{\Omega}$. The solution displays boundary layers whose width depends on the parameter ε. For $d = 1$ a very simple yet effective strategy to construct parameter uniform numerical methods is the use of piecewise uniform Shishkin meshes [1], i.e. meshes with a refinement region near the boundary whose width is selected a priori to match the length-scale of the layer. In the case of $d = 2$ and when the domain Ω is a rectangle, it is well established [1,3] that the natural extension of this approach to a tensor product of two one dimensional piecewise uniform Shishkin meshes yields a parameter uniform [1] second order (ignoring logarithmic factors) rate of convergence. The extension of this approach to other geometries is non-trivial. Curvilinear tensor product meshes [8] can deal with

C. de Falco (✉)
School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland,
E-mail: carlo.defalco@dcu.ie

A.F. Hegarty et al. (eds.), *BAIL 2008 - Boundary and Interior Layers.*
Lecture Notes in Computational Science and Engineering,
DOI: 10.1007/978-3-642-00605-0,

a limited set of geometries, while creating a single globally conforming unstructured triangulation with a uniform refinement in the layer region can produce inefficient or pathologically deformed meshes when ε is small. Although such inconveniences might be overcome by discretisation methods allowing for non conforming meshes (see, e.g. [9, Chap. 2, Sect. 2.5]) at the interface between the interior and boundary layer region, this would still involve producing a different triangulation for the whole domain Ω for each value of ε. This may require a significant computational effort which, for general domains, may outweigh that required for the solution of the discrete problem itself. To cope with these issues we investigate a method inspired by Chimera Overset Grid Methods [2] and by the method of Patches of Finite Elements of [6]. Note that one cannot expect this general approach to be parameter uniform without some modification that would resolve all layers within the solution. In contrast to the methods in [2, 6], which can be viewed as variants of the Schwartz iterative technique, our approach makes use of an a priori expansion to decompose the solution u of (1) into a sum of a *regular component* v, a set of *boundary layer components* $w_q,\ q = 1, \dots n_w$ and a set of *corner layer components* $z_p,\ p = 1, \dots n_z$. Each component is implicitly defined as the solution of a boundary value problem. In this paper, we consider the case of $\varepsilon \leq CN^{-1}$, where N^d is the dimension of the discrete problem. Hence, quantities of order ε are considered negligible compared to the discretisation errors. In Sect. 3, the pointwise bounds established on the layer components allow us identify subdomains or *patches* $\Omega_q, \Omega_p \subset \Omega,\ q = 1, \dots n_w,\ p = n_w + 1, \dots n_w + n_z$ outside of which a component is negligible. This decomposition also allows one to compute a discrete approximation to u by solving $n_w + n_z + 1$ problems *once* without any further iteration. Furthermore, as the decomposition is performed at the continuous level, this approach does not pose restrictions on the method used to discretise each boundary value problem. For example, in the case of the regular component defined in (4), one could use the results in [10] to analyze the error (in the case of a sufficiently smooth regular component) if one employed a finite element method on an unstructured quasi-uniform mesh instead of the numerical method analyzed in Sects. 4 and 5, which is based on a standard finite difference operator on a tensor product mesh. We finally point out that, although in the sections below we present theoretical results for a problem posed on the simple geometry of the unit square, the encouraging numerical results presented in [4] and Sect. 6.2 indicate the practical viability of the same approach for singularly perturbed problems on more complicated geometries. Throughout the paper $\|\cdot\|$ denotes the global pointwise maximum norm over the domain $\overline{\Omega}$ and C is a constant independent of ε and N.

2 Continuous Problem

Consider the singularly perturbed elliptic problem

$$L_\varepsilon u := -\varepsilon \triangle u + b(x,y)u = f(x,y), (x,y) \in \Omega = (0,1)^2, \tag{2a}$$

$$u = g,\ (x,y) \in \partial\Omega, \tag{2b}$$

$$f, b \in C^{4,\alpha}(\overline{\Omega}),\ g \in C(\partial\Omega),\ b(x,y) \geq \beta > 0,\ (x,y) \in \bar{\Omega}, \tag{2c}$$

where $0 < \varepsilon \leq 1$ is a singular perturbation parameter. We adopt the following notation for the edges and the boundary conditions:

$$\partial\Omega_1 = \{(x,0)|0 \leq x \leq 1\}, \partial\Omega_2 = \{(1,y)|0 \leq y \leq 1\},$$
$$\partial\Omega_3 = \{(x,1)|0 \leq x \leq 1\}, \partial\Omega_4 = \{(0,y)|0 \leq y \leq 1\},$$
$$g(x,y) = g_i(x), (x,y) \in \partial\Omega_i, i = 1,3;\ g(x,y) = g_i(y), (x,y) \in \partial\Omega_i, i = 2,4.$$

Assume further that $g_s \in C^{4,\alpha}([0,1]),\ s = 1,2,3,4$. From Han and Kellogg [7] and Andreev [1] we note the following levels of compatibility conditions: for the corner $(0,0)$,

$$g_1(0) = g_4(0), \tag{3a}$$
$$-\varepsilon g_1''(0) - \varepsilon g_4''(0) + b(0,0)g_1(0) = f(0,0), \tag{3b}$$

and similarly for the other corners. If (3a) is assumed at all four corners then $u \in C^{1,\alpha}(\overline{\Omega})$ and if (3a) and (3b) are assumed at all four corners then $u \in C^{3,\alpha}(\overline{\Omega})$. The reduced solution u_0 is defined via the reduced problem

$$b(x,y)u_0(x,y) = f(x,y),\ (x,y) \in \overline{\Omega}.$$

The regular component v of u is the solution of the elliptic problem

$$L_\varepsilon v = f(x,y), (x,y) \in \Omega,\ v = u_0,\ (x,y) \in \partial\Omega. \tag{4}$$

Note that the regular component can be written as $v = u_0 + \varepsilon R$, where

$$L_\varepsilon R = \triangle u_0, (x,y) \in \Omega,\ R = 0,\ (x,y) \in \partial\Omega.$$

Hence $R \in C^{0,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ and by the maximum principle $\|R\| \leq C$.

Remark 1. Note that at the corner $(0,0)$ the necessary compatibility condition for $u \in C^{3,\alpha}(\overline{\Omega})$ is that $b(0,0)u(0,0) = f(0,0) + \varepsilon(g_4''(0) + g_1''(0))$ which is that

$$u(0,0) - u_0(0,0) = O(\varepsilon). \tag{5}$$

3 Solution Decomposition

The solution is decomposed into a sum of a regular component v, boundary layer components $w_i(x,y), i = 1,2,3,4$ and corner layer components $z_i(x,y), i = 1,2, 3,4$

$$u = v + \sum_{i=1}^{4} w_i - \sum_{i=1}^{4} z_i.$$

Similar but different decompositions are given in [1,3]. Note that v is defined in (4) and the boundary layer function w_1 associated with the edge $y=0$ is defined as the solution of the problem

$$L_\varepsilon w_1 = -\varepsilon(1-y)s_1''(x), (x,y) \in \Omega, \tag{6a}$$
$$w_1(0,y) = q_4(y), \quad w_1(1,y) = q_2(y), \qquad 0 \le y \le 1, \tag{6b}$$
$$w_1(x,0) = s_1(x) := (u-v)(x,0), \quad w_1(x,1) = 0, 0 \le x \le 1, \tag{6c}$$
$$-\varepsilon q_4'' + b(0,y)q_4 = 0,\ y \in (0,1), \quad q_4(0) = s_1(0),\ q_4(1) = 0 \tag{6d}$$
$$-\varepsilon q_2'' + b(1,y)q_2 = 0,\ y \in (0,1), \quad q_2(0) = s_1(1),\ q_2(1) = 0. \tag{6e}$$

Lemma 1. *The solution of (6) satisfies the bounds*

$$|w_1(x,y)| \le Ce^{-y\sqrt{\beta/\varepsilon}} + C\varepsilon(1-y), \tag{7a}$$
$$\left\|\frac{\partial^{i+j} w_1}{\partial x^i \partial y^j}\right\| \le C\varepsilon^{-(i+j)/2}, 1 \le i+j \le 3,$$
$$\left\|\frac{\partial^j w_1}{\partial x^j}\right\| \le C,\ j = 1,2. \tag{7b}$$

Proof. Note that

$$|q_4(y)| \le C|s_1(0)|e^{-y\sqrt{\beta/\varepsilon}}, \quad |q_2(y)| \le C|s_1(1)|e^{-y\sqrt{\beta/\varepsilon}}.$$

Consider the following interpolant of the boundary data

$$h(x,y) =(s_1(x) - s_1(0)(1-x))(1-y) + (q_4(y) - q_4(1)y)(1-x) + (q_2(y) - q_2(0)(1-y))x.$$

Then

$$L_\varepsilon h = -\varepsilon(1-y)s_1''(x) + T(x,y),$$

where $T(x,y) := bh-(1-x)b(0,y)q_4(y)-xb(1,y)q_2(y)$. Note that $T(x,y)=0$ at each of the four corners. Then since $L_\varepsilon(w_1-h) = T(x,y)$, we have sufficiently compatibility (3b) for $w_1 \in C^{3,\alpha}(\overline{\Omega})$ and

$$|(w_1-h)(x,y)| \le Cx(1-x).$$

Using the maximum principle and classical bounds on the derivatives [3] we have that

$$|w_1(x,y)| \le Ce^{-y\sqrt{\beta/\varepsilon}} + C\varepsilon(1-y),\ \left\|\frac{\partial^{i+j} w_1}{\partial x^i \partial y^j}\right\| \le C\varepsilon^{-(i+j)/2},\ i+j \le 3.$$

Also $|T(x,y)| \le Cx(1-x)$, which implies that

$$\left|\frac{\partial w_1}{\partial x}(0,y)\right| \le C,\ \left|\frac{\partial w_1}{\partial x}(1,y)\right| \le C,$$

and using the differential equation (6a), we conclude that

$$\Big|\frac{\partial^2 w_1}{\partial x^2}(0,y)\Big| \le C(1-y)|s_1''(0)|,\ \Big|\frac{\partial^2 w_1}{\partial x^2}(1,y)\Big| \le C(1-y)|s_1''(1)|.$$

$$\begin{aligned} \text{Since,}\qquad L_\varepsilon \frac{\partial w_1}{\partial x} &= -\varepsilon(1-y)s_1^{(3)}(x) - b_x w_1, \\ \text{and}\qquad L_\varepsilon \frac{\partial^2 w_1}{\partial x^2} &= -\varepsilon(1-y)s_1^{(4)}(x) - b_{xx} w_1 - 2b_x \frac{\partial w_1}{\partial x}, \end{aligned}$$

we can use the maximum principle to establish the bounds $\|\frac{\partial^i w_1}{\partial x^i}\| \le C,\ i = 1,2$ on the derivatives orthogonal to the layer. □

Define the corner layer function z_1 associated with the corner $(0,0)$ as follows:

$$L_\varepsilon z_1 = 0, (x,y) \in \Omega, \tag{8a}$$
$$z_1(0,y) = w_1(0,y) = q_4(y),\ z_1(1,y) = 0,\ 0 \le y \le 1, \tag{8b}$$
$$z_1(x,0) = w_4(x,0) = q_1(x),\ z_1(x,1) = 0, 0 \le x \le 1, \tag{8c}$$
$$-\varepsilon q_1'' + b(x,0)q_1 = 0,\ x \in (0,1),\ q_1(0) = s_1(0),\ q_1(1) = 0. \tag{8d}$$

Then $z_1 \in C^{1,\alpha}(\overline{\Omega})$ and we have that

$$|z_1(x,y)| \le Ce^{-x\sqrt{\beta/\varepsilon}}e^{-y\sqrt{\beta/\varepsilon}}. \tag{9a}$$

Analogous bounds hold for the other boundary (corner) layer functions associated with the other three edges (corners).

4 Discrete Algorithm

We employ the standard central finite difference operator

$$L^N U^N := -\varepsilon(\delta_x^2 + \delta_y^2)U^N + bU^N = f,$$

which can also be generated from a standard finite element formulation on a structured tensor product grid with lumping as a quadrature rule. Here δ_x^2 denotes the classical three-point finite difference approximation to u_{xx} on a non-uniform mesh. We initially solve for an approximation $\bar{V}$ to the regular component v on a uniform coarse grid $\bar{\Omega}_u^N = \{(x_i,y_j)|x_i = i/N, y_j = j/N, 0 \le i,j \le N\}$. That is, the mesh function V^N is the solution of

$$L^N V^N = f,\ (x_i,y_j) \in \Omega_u^N;\ V^N = v,\ (x_i,y_j) \in \partial\Omega \cap \bar{\Omega}_u^N.$$

A global approximation to v is a simple interpolant of the form

$$\bar{V} = \sum_{i,j} V^N(x_i, y_j)\phi_i(x)\psi_j(y),$$

where $\phi_i(x)$ and $\psi_j(y)$ are the standard hat functions associated with x_i and y_j respectively. Define the following subdomains: $\Omega_1 = (0,1) \times (0,\sigma)$, $\Omega_2 = (1-\sigma, 1) \times (0,1)$, $\Omega_3 = (0,1) \times (1-\sigma, 1), \Omega_4 = (0,\sigma) \times (0,1)$. On each of these subdomains we define a tensor product of two uniform meshes. That is, $\bar{\Omega}_1^N := \{(x_i, y_j) | x_i = i/N, y_j = j\sigma/N, 0 \le i, j \le N\}$, where the Shishkin transition parameter σ is taken to be

$$\sigma := \min\left\{1, 2\sqrt{\frac{\varepsilon}{\beta}} \ln N\right\}. \tag{10}$$

The nodal values of an approximation $\bar{W}_1$ (defined solely on the layer region $\bar{\Omega}_1$) to the boundary layer function w_1 are computed by solving

$$\begin{aligned} L^N W_1^N = 0, (x_i, y_j) \in \Omega_1^N, \\ W_1^N(0, y_j) = s_1(0)e^{-y_j\sqrt{b(0,0)/\varepsilon}},\ W_1^N(1, y_j) = s_1(1)\sigma^{-1}(\sigma - y_j), 0 \le y_j \le \sigma, \\ W_1^N(x_i, 0) = s_1(x_i),\ W_1^N(x_i, \sigma) = W_1^N(0,\sigma)(1 - x_i) + W_1^N(1,\sigma)x_i, 0 < x_i < 1. \end{aligned}$$

The nodal values of an approximation $\bar{Z}_1$ (defined solely on the corner layer region $\bar{\Omega}_5 \equiv \bar{\Omega}_1 \cap \bar{\Omega}_4$) to the corner layer function z_1 are computed by solving

$$\begin{aligned} L^N Z_1^N = 0, (x_i, y_j) \in \Omega_1^N \cap \Omega_4^N, \\ Z_1^N(0, y_j) = W_1^N(0, y_j),\ Z_1^N(\sigma, y_j) = \sigma^{-1} W_4^N(0,\sigma)(\sigma - y_j),\ 0 \le y_j \le \sigma, \\ Z_1^N(x_i, 0) = W_4^N(x_i, 0),\ Z_1^N(x_i, \sigma) = \sigma^{-1} W_1^N(0,\sigma)(\sigma - x_i),\ 0 < x_i < \sigma. \end{aligned}$$

The approximations to the other six layer functions are defined analogously. The approximation $\bar{U}$ to the solution is patched together using the sum

$$\bar{U} = \bar{V} + \sum_{i=1}^{4} \bar{W}_i - \sum_{i=1}^{4} \bar{Z}_i.$$

5 Error Analysis

Theorem 1. *For the solution of (2a) and the approximation defined in Sect. 4*

$$\|u - \bar{U}\| \le CN^{-1} \ln N + C\sqrt{\varepsilon}.$$

Proof. Note that on the coarse uniform mesh Ω_u^N

$$\begin{aligned}\left|L^N(u_0 - V^N)(x_i, y_j)\right| &= \left|(L^N - L_\varepsilon)u_0(x_i, y_j)\right| + C\varepsilon,\ (x_i, y_j) \in \Omega_u^N \\ &\le CN^{-2}\varepsilon + C\varepsilon \le CN^{-2}\varepsilon + C\varepsilon.\end{aligned}$$

Then

$$\|v - \bar{V}\| \le \|v - u_0\| + \|u_0 - \bar{V}\| \le CN^{-2}\varepsilon + C\varepsilon + CN^{-2}. \tag{11}$$

Within the boundary layer region Ω_1^N, by (6) and the bounds in Lemma 1, we have that

$$\left|L^N(w_1 - W_1^N)(x_i, y_j)\right| = \left|(L^N - L_\varepsilon)w_1(x_i, y_j)\right| + C\varepsilon \le CN^{-1}\ln N + C\varepsilon.$$

Note that, if $\Psi(y) := s_1(0)e^{-y\sqrt{b(0,0)/\varepsilon}}$ then $(\Psi - q_4)(0) = 0$,

$$-\varepsilon(\Psi(y) - q_4(y))'' + b(0, y)(\Psi(y) - q_4(y)) = (b(0, y) - b(0, 0))\Psi(y),$$

and $|(b(0, y) - b(0, 0))|\Psi(y) \le C\sqrt{\varepsilon}$. From this, on the boundary $\partial\Omega_1^N$ we have

$$\begin{aligned}&|(W_1^N - w_1)(0, y_j)| \le C\sqrt{\varepsilon},\ |(W_1^N - w_1)(1, y_j)| \le C\sqrt{\varepsilon}, 0 \le y_j \le \sigma, \\ &(W_1^N - w_1)(x_i, 0) = 0,\ (W_1^N - w_1)(x_i, \sigma)| \le CN^{-2} + C\varepsilon, 0 < x_i < 1.\end{aligned}$$

Then we can conclude that over the entire domain Ω

$$\|w_1 - \bar{W}_1\| \le C(N^{-1}\ln N + \sqrt{\varepsilon}). \tag{12}$$

Within the corner region, we follow closely the approach of Andreev [1]. We first further decompose the corner layer function z_1. Let $z_1 = q_1(x)q_4(y) + z_{00} + \sqrt{\varepsilon}R_2$, where

$$\begin{aligned}|L_\varepsilon R_2| &= |(b(0, 0) - L_\varepsilon)q_1(x)q_4(y)| \le C\sqrt{q_1(x)q_4(y)},\ R_2 = 0, (x, y) \in \partial\Omega \\ L_\varepsilon z_{00} &= b(0, 0)q_1(x)q_4(y), (x, y) \in \Omega, \qquad z_{00} = 0, (x, y) \in \partial\Omega.\end{aligned}$$

Note that $|z_{00}(x, y)| \le Cq_1(x)q_4(y)$. The discrete version of this secondary decomposition is

$$\begin{aligned}Z_1^N &= q_1(x_i)q_4(y_j) + Z_{00}^N + \sqrt{\varepsilon}R_2^N \\ L^N Z_{00}^N &= b(0, 0)q_1(x_i)q_4(y_j) + (L_\varepsilon - L^N)q_1(x_i)q_4(y_j), (x_i, y_j) \in \Omega_1^N \cap \Omega_4^N, \\ Z_{00}^N &= 0, (x_i, y_j) \in \partial(\Omega_1^N \cap \Omega_4^N).\end{aligned}$$

Hence $|R_2^N| \le C$ and on the boundary of the corner patch we have that

$$|R_2^N(x_i, y_j)| \le C(N^{-1}\ln N + \sqrt{\varepsilon}),\ (x_i, y_j) \in \partial(\Omega_1^N \cap \Omega_4^N).$$

It remains to estimate the error in $|z_{0,0} - Z_{0,0}^N|$. Set $\tau := \sum_{(x_i,y_j)\in\Omega_1^N\cap\Omega_4^N} |(L^N - L_\varepsilon)z_{0,0}(x_i, j_j)|$. We decompose z_{00} as in [1, Theorem 2.1], $(\chi_{1,1} = -b(0, 0)$, $\chi_{1,2} = 0)$ and from [1, p. 962], we have that $\tau \le C\ln N$. In the corner layer region, we then bound the nodal error using the discrete stability bound given in [1, Theorem 3.1], as follows

$$
\begin{aligned}
|z_{0,0} - Z^N_{0,0}| &\le CN^{-2}(\ln N)^4 + CN^{-2}\ln N \\
&\qquad \sum_{(x_i,y_j)\in\Omega_1^N\cap\Omega_4^N} |L^N(z_{0,0} - Z^N_{0,0})(x_i,y_j)| \\
&\le CN^{-2}(\ln N)^4 + CN^{-2}\ln N \\
&\qquad \sum_{(x_i,y_j)\in\Omega_1^N\cap\Omega_4^N} |(L_\varepsilon - L^N)q_1(x_i)q_4(y_j)| \\
&\le CN^{-2}(\ln N)^4 + CN^{-2}\ln N \\
&\qquad \sum_{(x_i,y_j)\in\Omega_1^N\cap\Omega_4^N} \frac{h^2}{\varepsilon} e^{-x_{i-1}\sqrt{\beta/\varepsilon}} e^{-y_{j-1}\sqrt{\beta/\varepsilon}} \\
&\le CN^{-2}(\ln N)^4 + CN^{-2}\ln N \Big(\frac{\rho}{1-e^{-\rho}}\Big)^2, \ \rho = h\sqrt{\beta/\varepsilon}, h = \sigma/N \\
&\le CN^{-2}(\ln N)^4.
\end{aligned}
$$

By explicitly differentiating the leading term in the representation given in [1, Theorem 2.1], we can deduce the following bound on the first derivatives:

$$
\Big\| \frac{\partial^{i+j} z_{0,0}}{\partial x^i \partial y^j} \Big\| \le C\varepsilon^{-1/2}, \ i + j = 1.
$$

Use of the interpolation bound in [11, Lemma 4.1] completes the proof. □

Remark 2. It is worth noting that if the additional compatibility conditions (3b) are assumed to hold at all four corners, then $|s_1(0)| \le C\varepsilon$ and $|s_1(1)| \le C\varepsilon$. It follows that is not necessary to patch in the corners (i.e. it is not required to compute Z) in order to derive the following error bound

$$
\|u - \bar{U}\| \le CN^{-1}\ln N + C\varepsilon.
$$

6 Numerical Results

6.1 Test Example 1

We consider a particular example of problem (2a) with the following coefficients:

$$
b(x,y) = 1 + x^2y^2, \ f(x,y) = 1 + 2xy \tag{13}
$$

and boundary data

$$
g_1(x) \equiv g_4(y) \equiv 1, \ g_3 = 1 - x^2, \ g_2 = 1 - y^2. \tag{14}
$$

Table 1 Parameter-uniform global two-mesh differences D^N and rates ρ^N on a patched mesh for test example 1 over the range $R_\varepsilon = [2^{-40}, 2^{-7}]$

N	2^5	2^6	2^7	2^8	2^9
D^N	1.16×10^{-2}	2.92×10^{-3}	7.32×10^{-4}	1.83×10^{-4}	4.57×10^{-5}
ρ^N	1.99	2.00	2.00	2.00	

Note that in this particular example the zero level compatibility conditions (3a) are satisfied at all four corners, but the compatibility condition at the first level (corresponding to (3b)) is not satisfied at the corner $(1, 1)$. Tensor product meshes with N steps in each direction are used both for the boundary and corner patches, while a triangular mesh with N^2 degrees of freedom is used in computing an approximation to the regular component over the entire domain. The convergence behaviour of the numerical method is reported in Table 1 where the global two mesh differences D^N and the approximate uniform rates of convergence ρ^N were computed over a certain range R_ε of values for ε, using

$$D^N := \max_{\varepsilon \in R_\varepsilon} \|\bar{U}^N - \bar{U}^{2N}\|_{\Omega_S^{10N}}, \ \rho^N := \log_2 \frac{D^N}{D^{2N}}.$$

Here Ω_S^{10N} is a tensor product piecewise-uniform Shishkin mesh [3] with $10N$ elements in each coordinate direction. We choose to measure the difference between the two interpolants on this finer mesh Ω_S^{10N}, as the maximum difference between the two interpolants may not occur over the set of mesh points $\Omega^N \cup \Omega^{2N}$. The computed uniform rate of convergence for this example is greater than what is established theoretically in Theorem 1.

6.2 Test Example 2

To assess the applicability of the patched mesh method to a problem posed on a non-rectangular domain, we consider a problem of the form (1) set in a domain $\Omega \equiv \Omega_1 \cup \Omega_2$ with $\Omega_1 \equiv (-1, 1) \times (-1, 0)$ and $\Omega_2 \equiv \left\{(x, y)|x^2 + y^2 < 1\right\}$. For this test example, the coefficients b, f and g are given by

$$\begin{cases} f(x, y) = b(x, y) = 1, \quad (x, y) \in \Omega \\ g(x, y) = \frac{2-\tanh(12y)-\tanh(12)}{2}, \quad (x, y) \in \partial\Omega. \end{cases}$$

For this choice of data no boundary layer occurs near the side $y = -1$. Let $\partial\Omega_L : = \partial\Omega \setminus \{(x, -1), 0 < x < 1\}$ be the remainder of the boundary. The patch for this problem is taken to be $\bar{\Omega}_p : = \{\mathbf{x} \in \bar{\Omega} | \mathrm{dist}(\mathbf{x}, \partial\Omega_L) \leq \sigma\}$, where σ is as given in (10).

The solution to this second test example is shown in Fig. 1b, while Tables 2 and 3 show the performance of the patched mesh method and of a standard finite

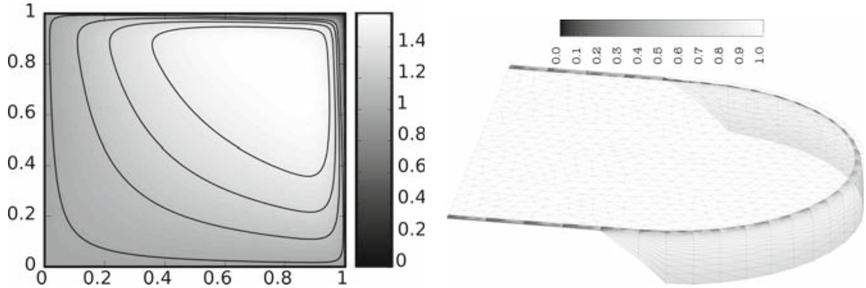

Fig. 1 Computed solutions to the two test examples using a patched mesh method with $N = 64$

Table 2 Parameter-uniform global two-mesh differences D^N and rates ρ^N on a patched mesh for test example 2 over the range $R_\varepsilon = 10^{-4}[2^{-20}, 1]$

N	2^4	2^5	2^6	2^7	2^8
D^N	1.47×10^{-2}	5.61×10^{-3}	1.99×10^{-3}	6.77×10^{-4}	2.22×10^{-4}
ρ^N	0.91	1.26	1.67	2.12	

Table 3 Parameter-uniform global two-mesh differences D^N and rates ρ^N of a standard finite element method on a quasi-uniform mesh for test example 2 over the range $R_\varepsilon = 10^{-4}[2^{-20}, 1]$

N	2^4	2^5	2^6	2^7	2^8
D^N	0.257	0.684	0.58	0.437	0.471
ρ^N	1.58	0.00	−0.04	0.44	

element method on a quasi uniform mesh respectively. The rates in Table 2 suggest that the patched method is parameter uniform for this problem, which contrasts with the apparent lack of uniform convergence displayed in Table 3 for a standard finite element method on a quasi-uniform mesh.

Acknowledgement This research was supported by the Mathematics Applications Consortium for Science and Industry in Ireland (MACSI) under the Science Foundation Ireland (SFI) mathematics initiative.

References

1. V.B. Andreev. On the accuracy of grid approximations to nonsmooth solutions of a singularly perturbed reaction–diffusion equation in a square. *Differential Equations*, 42(7):954–966, 2006.
2. F. Brezzi, J.-L. Lions, and O. Pironneau. Analysis of a chimera method. *Comptes Rendus de l'Academie des Sciences, Series I Mathematics*, 332(7):655–660, 2001.
3. C. Clavero, J.L. Gracia and E. O'Riordan. A parameter robust numerical method for a two-dimensional reaction–diffusion problem. *Mathematics of Computation*, 74:1743–1758, 2005.

4. M. Culpo, C. de Falco, and E. O' Riordan. Patches of finite elements for singularly-perturbed diffusion reaction equations with discontinuous coefficients. In *Proceedings of the ECMI 2008 Conference*, 2008.
5. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, and G.I. Shishkin. *Robust Computational Techniques for Boundary Layers.* Chapman and Hall/CRC, New York/Boca Raton, 2000.
6. R. Glowinski, J. He, A. Lozinski, J. Rappaz, and J. Wagner. Finite element approximation of multi-scale elliptic problems using patches of elements. *Numerische Mathematik*, 101(4):663–687, 2005.
7. H. Han and R.B. Kellogg. Differentiability properties of solutions of the equation $-\varepsilon^2\triangle u + ru = f(x,y)$ in a square. *SIAM Journal of Mathematical Analysis*, 21:394–408, 1990.
8. N. Kopteva. Maximum norm error analysis of a 2D singularly perturbed semilinear reaction–diffusion problem. *Mathematics of Computation*, 76(258):631–646, 2007.
9. A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Numerical Mathematics and Scientific Computation. Clarendon Press, Oxford, 1999.
10. A.H. Schatz and L.B. Wahlbin. On the finite element method for singularly perturbed reaction–diffusion problems in two and one dimensions. *Mathematics of Computation*, 40(161):47–89, 1983.
11. M. Stynes and E. O'Riordan. A uniformly convergent galerkin method on a shishkin mesh for a convection-diffusion problem. *Journal of Mathematical Analysis and Applications*, 214:36–54, 1997.