Nonlinear Singular Kelvin Modes in a Columnar Vortex

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Abstract This paper considers the propagation of helical neutral modes within a cylindrical vortex and the subsequent formation of nonlinear critical layers around the radius where the mean-flow angular velocity and the mode frequency are comparable. Analogy can be done with the stratified critical layers. We formulate a steady-state theory valid when the analogous Richardson number is small at the critical radius. The apparent singularity is removed by retaining nonlinear terms in the critical-layer equations of motion. The result from the interaction is the emergence of multipolar vortices whose poles are located around the critical radius, spiral along the basic vortex axis and are embedded in a distorted mean flow caused by a slow diffusion of the three-dimensional vorticity field from the critical layer.

1 Introduction

The propagation of helical perturbations to a columnar and bounded vortex has been studied first by Lord Kelvin. In cylindrical coordinates (r, θ, z), the problem involves the investigation of infinitesimal perturbations (u_r, u_θ, u_z) superimposed on a flow with azimuthal velocity profile $\overline{V}(r)$. In this paper, we are interested in waves propagating in an unbounded vortex. A model that has often been employed is the discontinuous Rankine vortex, a constant-vorticity cylinder embedded in a zero-vorticity space. The related modes are called Kelvin modes. The motivation stems with the study of the stability of interacting vortices. For instance, we can cite the aircraft trailing vortices: a pair of counter-rotating vortex filaments shed from the wingtips of aircraft. A prevailing instability in such problems is the elliptic instability that involves resonantly interacting Kelvin waves. Tsai and Widnall [11] found that the most unstable perturbations of the Rankine vortex corresponded to a pair of

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Kelvin modes having zero frequency and azimuthal wavenumbers $m = \pm 1$. Real vortices, however, have continuous profiles and it is important to ask what effect the use of a continuous-vorticity profile might have on this instability mechanism. Sipp and Jacquin [9] have recently done so in a linear study and they concluded that the "Widnall instabilities" would not occur because of the presence of a critical layer. The neutral Kelvin modes required for the resonant interaction would be damped in the continuous case. In this paper, we reexamine the question by emphasizing the effect of nonlinearity rather than viscosity in the critical layer.

Due to the similarity between both critical-layer singularities, it is possible to anticipate certain results based on those that have been demonstrated for stratified shear flows in [5] which is a companion paper. For Kelvin modes on vortices, we will extract an equivalent Richardson number and will show that when the latter is small at the critical level, inviscid nonlinear modes exist while they would be damped if viscosity were used to deal with the critical layer.

The reason of the nonlinear neutral mode existence is the absence of any phase change across the critical point. We will show in Sect. 4 by means of an inviscid analysis valid when the vorticity is small at the critical level that the only solution compatible with a nonlinear critical layer has no phase jump. Section 5 yields the same result when the axial wavelength is large. This result was found by Caillol and Grimshaw (2004) in the two-dimensional-motion assumption with the same small-vorticity approximation and for a Bessel function J_1 azimuthal-velocity basic profile [3]. In that particular case, neutral modes have an analytical expression.

2 Outer Flow

We consider small-amplitude helical perturbations to a swirling flow $\overline{V}(r)$ corresponding to a pressure distribution $\overline{p}(r)$, of phase $\xi = kz + m\theta - \omega t$, k and m being respectively the axial and azimuthal wavenumbers, and ω the frequency. Dealing with neutral modes, ξ can be used as an independent variable. The momentum and continuity equations can then be written

$$\frac{Du_r}{Dt} = \frac{u_\theta^2}{r} - \frac{\partial p}{\partial r} + \frac{1}{Re}\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u_r}{\partial r}\Big), \tag{1}$$

$$\frac{Du_\theta}{Dt} = -\frac{u_r\,u_\theta}{r} - \frac{m}{r}\frac{\partial p}{\partial \xi} + \frac{1}{Re}\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u_\theta}{\partial r}\Big) + F_b,$$

$$\frac{Du_z}{Dt} = -k\frac{\partial p}{\partial \xi} + \frac{1}{Re}\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u_z}{\partial r}\Big),$$

where $$\frac{D}{Dt} = \Big(\frac{m}{r}u_\theta + ku_z - \omega\Big)\frac{\partial}{\partial \xi} + u_r\frac{\partial}{\partial r}$$

and $$\frac{\partial (r\,u_r)}{\partial r} + m\frac{\partial u_\theta}{\partial \xi} + k\,r\frac{\partial u_z}{\partial \xi} = 0\,.$$

Our analysis being primarily inviscid, we have retained in the momentum equations only those viscous terms that will be the largest in the critical layer and, consequently, required in the analysis to follow. The basic-vortex small viscous damping is balanced by a body force F_b whose expression will be given later on. The equations have been nondimensionalized by using the angular velocity of the vortex at its center and a vortex characteristic radius.

2.1 The Singular Mode

$O(\varepsilon)$ disturbances are superimposed to a mean flow $\overline{V}$ and $\overline{W}$:

$$u_r = \varepsilon\, U_r\,, \quad u_\theta = \overline{V} + \varepsilon\, U_\theta\,, \quad u_z = \overline{W} + \varepsilon\, U_z\,, \quad p = \overline{p} + \varepsilon\, P_r\,. \tag{2}$$

ε is a small dimensionless parameter. The mean shear flow induces axial and azimuthal mean vorticities $Q_z = D_*[\overline{V}]$ and $Q_\theta = -D[\overline{W}]$ where $D = d/dr$ and $D_* = D + 1/r$. The angular rotation of the vortex is denoted $\Omega = \overline{V}/r$. We study the asymptotic steady régime following the critical layer formation induced by the wave/vortex interaction. Mean axial and azimuthal motions are generated while the critical layer is forming as results from this interaction. To have an analytically tractable problem, $\overline{W}$ will be of smaller amplitude than the basic-vortex azimuthal velocity $\overline{V}_0$. In the same way, $\overline{V}$ contains additional smaller contributions to $\overline{V}_0$. Such a mean flow is produced by viscous diffusion of momentum through the critical layer over a very long time due to small viscosity [4]. Away from the critical layer, the perturbations are taken sinusoidal: $U_r = u \sin\xi$, $U_\theta = v\cos\xi$, $U_z = w\sin\xi$ and $P_r = p_r\cos\xi$. Introducing these into (1), the linearized system can be reduced to the Howard–Gupta equation [7]

$$D[S(r)D_*u] + \Big[\frac{m}{\gamma(r)r^2}\Big(2S(r)Q_z(r) - rD[S(r)Q_z(r)]\Big) + 2k^2S(r)\Omega(r)\frac{Q_z(r)}{\gamma^2(r)} + \frac{k}{r\gamma(r)}\Big(rD[S(r)Q_\theta(r)] - S(r)Q_\theta(r)\Big) + 2mk\frac{\Omega(r)}{r\gamma^2(r)}S(r)Q_\theta(r) - 1\Big]u = 0, \tag{3}$$

where $\gamma = m\Omega + k\overline{W} - \omega$ and $S = r^2/(m^2 + k^2r^2)$. This equation admits a singularity at the critical radius r_c where $\gamma(r_c) = 0$. Following [8], we expand all terms in (3) around r_c to obtain a solution valid locally having the form

$$u(\eta) = A\,u_+(\eta) + B\,u_-(\eta)\,,\;\; u_\pm(\eta) = \eta^{\frac{1}{2}(1\pm\nu)}\,\hat{u}_\pm(\eta), \text{ and } \eta = r - r_c. \tag{4}$$

The functions $\hat{u}_\pm(\eta)$ are regular in 0. We define ν as $\nu = (1 - 4\,J_c)^{1/2}$, and the equivalent local Richardson number as

$$J_c = \frac{2\,k^2\,\Omega_c}{(m\Omega_c' + k\overline{W}_c')^2}\Big(Q_{z,c} + \frac{m}{kr_c}Q_{\theta,c}\Big)\,. \tag{5}$$

3 Critical Layer Analysis

The critical-layer scaling is determined by balancing the perturbation with the swirling flow in a frame moving with the wave angular speed $\omega/m = \Omega_c$. Let us concentrate on the case $J_c < 1/4$ corresponding to (4). The most singular Frobenius solution is characterized by the exponent $\delta = (1-\nu)/2$. Consideration of the system (1) leads to the conclusion that the inner cross-stream coordinate is $R = (r-r_c)/\varepsilon^\beta$ where $\beta = (2-\delta)^{-1}$. The azimuthal velocity V in the new frame is defined by

$$u_\theta - \overline{V}_c \sim \overline{V}_c{}'(r-r_c) + \varepsilon\, v(r) \cos\xi = \varepsilon^\beta [V(R,\xi) + \Omega_c R] .$$

The remaining dependent variables are scaled as

$$u_r = \varepsilon^{2\beta} U(R,\xi), \;\; u_z = \varepsilon^\beta W(R,\xi) \quad \text{and} \quad p - \frac{1}{2}\Omega_c^2 r^2 = \varepsilon^{2\beta} P(R,\xi) .$$

The Reynolds number scales as $1/Re = \lambda\, \varepsilon^{3\beta}$. Substituting these new variables into (1) leads to the critical-layer equations. The latter are highly nonlinear and the solution even at lowest order involves all the harmonics. For that reason, we consider the case of small Richardson number because a simple closed form solution is possible. This number is small for different régimes according to the value of the shear, ratio of the vorticity over the inertial frequency at the critical radius: (1) S_z, $S_\theta \ll 1$, (2) $k r_c$, $S_\theta \ll 1$ where $S_{z/\theta} = Q_{z/\theta,c}/(2\Omega_c)$. (1) may apply to a rapidly rotating vortex or a critical layer occurring far away from the core of the vortex. Numerical solutions of the eigenvalue problem reveal that it is indeed often the case. (2) corresponds to a long axial wavelength mode. Assuming J_c small, the expansions (4) become

$$\phi_a = \eta + \sum_{n=2}^{\infty} a_{0,n}\eta^n, \;\; \phi_b = 1 + \sum_{n=2}^{\infty} b_{0,n}\eta^n + b_0\phi_a(\eta) \ln \eta^* . \tag{6}$$

η^* is a normalized cross-stream coordinate: $\eta^* = \eta/\eta_0$ where η_0 is determined while matching the outer flow with the critical-layer flow. So, u becomes

$$u(\eta) = (\Theta\, r_c\, \phi_a + \phi_b) \sin\xi + b_0 r_c\, \Phi\, \phi_a \cos\xi. \tag{7}$$

The logarithmic term in (6) is expressed by writing $\ln|r-r_c|$ for $r > r_c$ and $\ln|r-r_c| + i\,\Phi$ when $r < r_c$; $\Phi(\lambda)$ is defined as the phase change. On either side of the critical level, Θ takes different values denoted $\Theta^\pm$. u must be continuous at $r = r_c$, so the integration constant in front of ϕ_b is unique and chosen without loss of generality equal to 1.

4 The Small-Vorticity Limit

Mean fields are expanded in the outer flow in this way

$$\overline{V} = \overline{V}_0 + \varepsilon^{\frac{1}{2}}\,\overline{V}_1 + \varepsilon\,\overline{V}_2 + \,\ldots,\; \overline{W} = \varepsilon^{\frac{1}{2}}\overline{W}_1 + \varepsilon\,\overline{W}_2 + \ldots \tag{8}$$

with the related vorticities

$$Q_z = Q_0 + \varepsilon^{\frac{1}{2}}Q_{z,1} + \,\ldots,\; Q_\theta = \varepsilon^{\frac{1}{2}}Q_{\theta,1} + \varepsilon Q_{\theta,2} + \ldots$$

and similar expressions for U_θ, U_z and P_r. We omit the subscript z to the zero-order axial mean vorticity. The first-order Richardson number is then since $Q_{0,c} = 0$

$$J_{1,c} = \frac{k^2 r_c^2}{2m^2\,\Omega_{0,c}}\left(Q_{z,1,c} + \frac{m}{k r_c}Q_{\theta,1,c}\right). \tag{9}$$

The additional mean flow is likely to be distorted, that is the velocity and its successive derivatives may be different at either side of the critical radius. Similar velocity and temperature distortions were shown by [10] to be necessary for the stratified critical layer. Distortions enable one to match the three components of vorticity and the normal velocity on the separatrices bounding open and closed-streamline flows within the critical layer.

4.1 Critical-Layer Flow Outside of the Separatrices

The critical-layer equations are in the small-vorticity limit ($\beta = 1/2$)

$$\frac{\partial P}{\partial R} - 2\,\Omega_c V = \varepsilon^{\frac{1}{2}}\,\frac{V^2}{r_c}$$

$$\frac{m}{r}\frac{\partial P}{\partial \xi} + U\left(2\Omega_c + \frac{\partial V}{\partial R}\right) + \left(\frac{m}{r}V + kW\right)\frac{\partial V}{\partial \xi} + \varepsilon^{\frac{1}{2}}\frac{UV}{r_c} = \lambda\left(\frac{\partial^2 V}{\partial R^2} + \frac{\varepsilon^{\frac{1}{2}}}{r_c}\frac{\partial V}{\partial R}\right) + \frac{F_b}{\varepsilon}$$

$$k\frac{\partial P}{\partial \xi} + U\frac{\partial W}{\partial R} + \left(\frac{m}{r}V + kW\right)\frac{\partial W}{\partial \xi} = \lambda\left(\frac{\partial^2 W}{\partial R^2} + \frac{\varepsilon^{\frac{1}{2}}}{r_c}\frac{\partial W}{\partial R}\right)$$

$$\text{and}\qquad \frac{\partial U}{\partial R} + \frac{m}{r}\frac{\partial V}{\partial \xi} + k\frac{\partial W}{\partial \xi} = 0. \tag{10}$$

The body force is $F_b = -\lambda\varepsilon^{\frac{3}{2}}\,\Delta\overline{V}_0 \simeq -\lambda\varepsilon^{\frac{3}{2}}\,(\overline{V}_0'' + \overline{V}_0'/r_c)$. Writing the outer expansion in inner coordinate determines how the inner variables are to be expanded in the critical layer. For example, for U,

$$U \sim U^{(0)} + \varepsilon^{1/2}\log\varepsilon\, U^{(1)} + \varepsilon^{1/2}U^{(2)} + \cdots,\; U^{(j)} = U_i^{(j)} + \lambda U_v^{(j)} + O(\lambda^2), \tag{11}$$

with similar expansions for V, W and P. Each field at each order j, as $\lambda \to 0$, can be expanded in an inviscid and a viscous parts. Injecting such a decomposition in (10) leads to secularity conditions on the viscous velocity and pressure components. Requiring them to match with the secular terms in the outer flow will fix the arbitrary functions that arise from integrating the governing PDEs [2]. In contrast with the $O(1)$ vorticity case, it is straightforward to find a leading-order solution of the system (10). It consists simply of a radial oscillation superimposed on the mean flow, specifically,

$$\begin{aligned} U^{(0)} &= \sin\xi,\, V^{(0)} = \overline{V}_{1,c} - 2\,\Omega_{0,c} R,\, W^{(0)} = \overline{W}_{1,c} \;\; P^{(0)} \\ &= 2\,\Omega_{0,c} R\,(\overline{V}_{1,c} - \Omega_{0,c}\, R). \end{aligned}$$

The $O(\varepsilon^{1/2}\log\varepsilon)$ solution is also obtained from the outer flow. The non trivial equations are obtained at the order $O(\varepsilon^{\frac{1}{2}}.)$ Eliminating the pressure and replacing $V^{(2)}$ by a streamfunction-like variable $\psi^{(2)}$ lead to two coupled PDEs that can be integrated once with respect to R. Before displaying these, we accomplish a transformation in order to have the streamwise-motion equation written in a standard way [6].

$$\xi = X - \frac{\pi}{2}\,[\,1 + s_i\,],\;\; R = s_i\, R_0 R^*,\;\; \psi^{(2)} = s_i \frac{\Omega_{0,c}}{r_c} R_0^3 \hat{\psi}^{(2)},\;\; U^{(2)} = \frac{R_0}{r_c}\,\hat{U}^{(2)},$$

$$V^{(2)} = \frac{s_i}{2\,m}\,\hat{V}^{(2)},\;\; W^{(2)} = s_i\,\frac{k\,r_c}{2\,m^2}\,\hat{W}^{(2)},\;\; P^{(2)} = 2\,s_i\,\Omega_{0,c}^2 \frac{R_0^3}{r_c}\,\hat{P}^{(2)},$$

$$V^{(2)} = \psi_R^{(2)} + R^2, \quad s = \mathrm{sgn}(R)\,, \quad s_i = \mathrm{sgn}(m\,\Omega_{0,c})\,, \text{ and } R_0 = \left|\frac{r_c}{2\,m\,\Omega_{0,c}}\right|^{1/2}.$$

We then get

$$\sin X\, \hat{\psi}^{(2)}_{R^*\,R^*} + R^*\,\hat{\psi}^{(2)}_{X\,R^*} = \hat{\psi}^{(2)}_X - \overline{\lambda}\left(\hat{\psi}^{(2)}_{R^*\,R^*\,R^*} - \frac{r_c\,Q'_{0,c}}{\Omega_{0,c}}\right) - \frac{r_c F(X)}{R_0\Omega_{0,c}}, \quad (12)$$

$$\text{and} \qquad \sin X\, \hat{W}^{(2)}_{R^*} + R^*\,\hat{W}^{(2)}_X = \hat{\psi}^{(2)}_X - \overline{\lambda}\,\hat{W}^{(2)}_{R^*\,R^*} - \frac{r_c F(X)}{R_0\Omega_{0,c}}, \quad (13)$$

where $\overline{\lambda} = \lambda/R_0$. In the following analysis, we drop the hats with the understanding that it is the new variables with which we are dealing. The radial-momentum equation integrated with respect to R determines $P^{(2)}$. Finally, the continuity equation provides an expression for $U^{(2)}$. $F(X)$ is an arbitrary function arising from the integration. Matching to the outer solution leads to the expression of $F(X)$ and then to

$$Q_{\theta,1,c} = -\frac{kr_c}{m}\,Q_{z,1,c} \quad \text{and} \quad J_{1,c} = 0. \qquad (14)$$

In order to relate the mean-vorticity jumps to the phase change, we integrate (12) and (13) over one wavelength in X and then over R. The obtained relations are valid as $R \to \infty$, so $\psi^{(2)}$ and $W^{(2)}$ are replaced by their asymptotic expansions. Clearly, when there is a phase change $\Phi \neq 0$, the only way is if $Q_{z,1,c}$ and $Q_{\theta,1,c}$ are discontinuous across the critical layer. Finally, the relations of the vorticity jumps to the phase change are given by

$$[Q_{z,1,c}]_-^+ = -\frac{m}{kr_c}[Q_{\theta,1,c}]_-^+ = -s_i\,\frac{1}{2}\,Q'_{0,c}R_0\,\frac{\Phi(\overline{\lambda})}{\overline{\lambda}}\,, \tag{15}$$

where $\Phi(\overline{\lambda})$ must be determined by solving (12) and (13) numerically. When we consider the limit $\overline{\lambda} \ll 1$, as in other critical-layer problems, there are regions of closed flow in the cartesian frame (X, R) and the solutions within such regions must be matched to those outside across separatrices. $\psi^{(2)}$ and $W^{(2)}$ can be then determined by solving (12) and (13). Each streamline can be defined univoquely by the variable: $Z = 1/2R^2 + \cos X$, Fig. 1 for a picture of the current lines projected on the plane $z = Cst$ in the case of the nonlinear neutral Kelvin mode $m = 2$.

4.2 Flow Within the Separatrices

First, the three vorticity components are matched across the separatrices. In Appendix, we have extended the Prandtl–Batchelor theorem and shown that $\psi_{RR}^{(2,\odot)} = const. = Q^{(2,\odot)}$ within a region of closed flow according to (A4). $\odot$ defines the flow within the separatrices. Matching the axial vorticities along the upper and lower separatrices $Z = 1$ give

$$Q^{(2,\odot)} = \frac{1}{2}\,\frac{s_i r_c}{\Omega_{0,c}R_0}\,(Q_{z,1,c}^{+} + Q_{z,1,c}^{-}) \quad \text{and}$$
$$[Q_{z,1,c}]_-^+ = -2\,s_i\,Q'_{0,c}\,R_0\,[K(1) + \sqrt{2}]\,. \tag{16}$$

The second equation in (16) shows that a jump in axial vorticity takes place across the critical layer even in the inviscid limit. Equating now the two expressions for $[Q_{z,1,c}]_-^+$ derived in (15) and in (16), we obtain

$$\frac{\Phi(\overline{\lambda})}{\overline{\lambda}} \sim 4\,[K(1) + \sqrt{2}] \quad \text{as} \quad \overline{\lambda} \to 0\,. \tag{17}$$

This is exactly the result obtained in [6] but with the opposite sign. Numerical evaluation of the integral below defining K yields $K(1) + \sqrt{2} \simeq 1.3788$,

$$K(Z) = \frac{1}{\sqrt{2}}\int_{\infty}^{Z}\Big[\frac{2\pi}{\int_0^{2\pi}(Z_1 - \cos X)^{\frac{1}{2}}\,dX} - \frac{1}{Z_1^{\frac{1}{2}}}\Big]\,dZ_1\,.$$

There are two conditions that should be satisfied along the separatrix, namely, a kinematic condition and, secondly, continuity of pressure. The kinematic condition

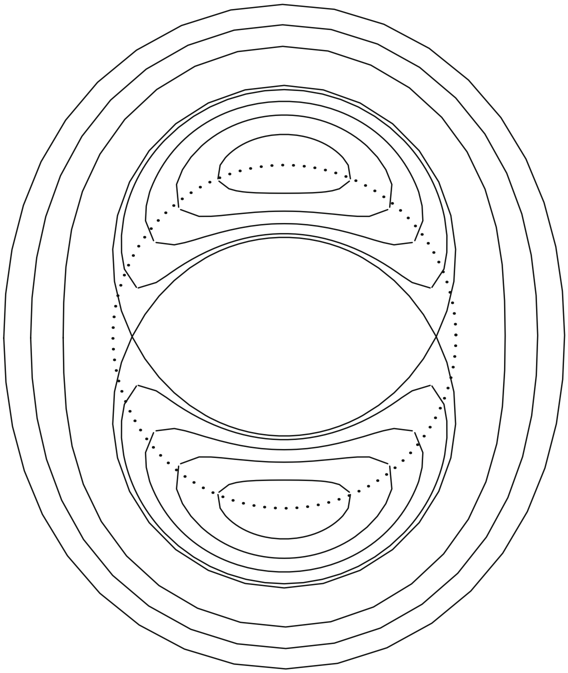

Fig. 1 Nonlinear neutral mode ($\varepsilon = 0.2$) $m = 2$, $k = 2$, $r_c = 1$ and $r_c Q'_{z,0,c} = Q^+_{z,1,c} = \Omega_{0,c}$; view taken at a height $z = Cst$. The *dotted circle* is $r = r_c$.

requires the normal velocity to the separatrix $Z = 1$ to be continuous. At the order $\varepsilon^{\frac{1}{2}}$, the kinematic condition plus the azimuthal-vorticity matching yield jointly with the PB theorem the determination of $W^{(2,\odot)}$. Radial-vorticity matching is equivalent to azimuthal-vorticity matching. Moreover, if we require continuities of $V^{(2,\odot)}$ across $R = 0$ and $V^{(2)}$ across the separatrix, then $\Theta^+ = \Theta^-$; as a result, there is no phase change across the nonlinear critical layer in the inviscid limit.

5 The Long-Wave Nonlinear Critical Layer

In this section, the Richardson number J_c is still taken to be $O(\varepsilon^{1/2})$. However, J_c is small because $k \ll 1$. Specifically, (5) shows that we must scale $kr_c = \kappa\,\varepsilon^{1/4}$. u_z then possesses a new scaling; $u_z = \varepsilon^{1/4}\kappa\, W(R, \xi)$. As in Sect. 4, a simple leading-order solution of the system (10) can readily be found and consists of a radial oscillation superimposed on the mean flow with an additional oscillatory component to the pressure. The axial-vorticity jump and the interior axial vorticity turn out to be the same as in (15) and (16), where R_0 is now defined as

$R_0 = |r_c/[m\,(2\,\Omega_{0,c} - Q_{0,c})]|^{1/2}$ and $J_{1,c}$ as

$$J_{1,c} = \frac{2\,\kappa^2}{m^2}\,\Omega_{0,c}\,\frac{Q_{0,c} + mQ_{\theta,1,c}}{(Q_{0,c} - 2\,\Omega_{0,c})^2}. \tag{18}$$

Matching of the azimuthal vorticity across the separatrix $Z = 1$ leads to

$$m\,Q_{\theta,1,c} = -Q_{0,c}\,, \quad \text{or} \quad J_{1,c} = 0\,, \quad \text{and} \quad [Q_{\theta,1,c}]_-^+ = 0 \quad \text{or} \quad [\overline{W}_1']_-^+ = 0. \tag{19}$$

Matching of the normal velocity across the separatrix determines $W^{(2,\odot)}$. Continuities of $V^{(2,\odot)}$ at $R = 0$ and $Z = 1$ respectively give $\Theta^+ = \Theta^-$ and $[\overline{W}_{1,c}'']_-^+ = 0$. To conclude this section, we say a few words about the mean-flow distortions that are present even in the limit $\lambda \to 0$. Caillol and Grimshaw [4] have used the method of strained coordinates to parametrize the streamlines in the critical layer in order to have the velocity at the cat's eye core and at the stagnation points obeying certain topological conditions. We have done this here as well, but omit the details; these points belong to helices, so all the velocity components are linked.

6 Concluding Remarks

We have analytically investigated the critical-layer like interaction of a neutral Kelvin mode with a swirling shear flow. The linear theory is similar to this of a stratified shear flow. The nonlinear study is made possible due to the small-Richardson-number assumption relevant for instance, to rapidly rotating vortices and yields a classic Kelvin cat's eye pattern within the critical layer. The result of this interaction is an additional and distorted mean flow of higher magnitude than the mode amplitude. Axial and azimuthal mean vorticities may be distorted. The vorticity jump is then proportional to the gradient of the basic axial vorticity. The equivalent Richardson number J_c reveals to be smaller than expected, of order the mode amplitude, which implies that the streamlines within the critical layer need to be even more distorted, in order to describe an $O(\varepsilon^{\frac{1}{2}})\ J_c$ critical-layer flow.

Appendix: Generalized Prandtl–Batchelor Theorem

Batchelor proved that for a steady, inviscid and plane flow the vorticity inside a bounded region is constant. We follow that basic procedure, but the three dimensionality of the present problem naturally adds complications. Our starting point is the momentum equation

$$\partial_t \mathbf{u} + \mathbf{Q} \times \mathbf{u} + \nabla H = \frac{1}{Re}\,\triangle\,\mathbf{u} + \mathbf{F}_b, \tag{1}$$

$H = p/\rho + |\mathbf{u}|^2/2$, $\mathbf{Q}$ is the vorticity and $\mathbf{F}$ is a body force to enable a viscous, parallel flow. We decompose the motion into an inviscid and a viscous components. The development by Batchelor, at this point, involved a curvilinear integration along a streamline, whereas in our case the integration is on a surface $Z = const$ (recall that $Z = R^2/2 + \cos X$). This surface is a cylinder that spirals with respect to the axis of the vortex. The integration will be done either in a plane $z = const$, in which case θ varies over a $2\pi/m$ range or in a plane $\theta = const$, in which case z varies over a $2\pi/k$ range. In that way, we obtain two conditions that are sufficient to determine the flow inside the separatrices. Performing the integration now, we obtain at the leading orders

$$\oint(\mathbf{Q}_i \times \mathbf{u}_v) \cdot d\mathbf{l} + \oint(\mathbf{Q}_v \times \mathbf{u}_i) \cdot d\mathbf{l} + \varepsilon^{3/2} \oint[\nabla \times \mathbf{Q}_i + \triangle \overline{V}_0 \, \mathbf{e}_\theta] \cdot d\mathbf{l} = \mathbf{0}. \tag{2}$$

In the two dimensional case, the first two integrals vanish, but that does not happen here because $\mathbf{u}$ is three dimensional. First, we carry out an integration with respect to θ in a plane $z = const$. The integration is along a "streamline" $Z = Z_0$, say, where Z_0 is constant. At the lowest order, we obtain

$$\oint s[R\, \psi^{(2,\odot)}_{RRZ} - 1 - r_c \frac{Q'_{0,c}}{\Omega_{0,c}}]\, d\theta = 0,$$

where the integration is in the clockwise sense. The body force vanishes because of symmetry which permits us to write the integral as

$$\int_0^{2\pi/m} [\psi^{(2,\odot)+}_{RRZ} + \psi^{(2,\odot)-}_{RRZ}] \sqrt{Z_0 - \cos[m\theta]}\, d\theta = 0\,. \tag{3}$$

Given that $\psi^{(2,\odot)}_{RR}$ depends only on Z, (A3) leads us to conclude that

$$\psi^{(2,\odot)}_{RR} = const. \tag{4}$$

A second condition is determined by integrating in a plane $\theta = const$ with z traversing a $2\pi/k$ path. At leading order, this leads to a condition that helps determine the axial velocity, namely,

$$\oint sR[(RW^{(2,\odot)}_Z)_Z + W^{(2,\odot)}_{v,X}]\, dz = 0. \tag{5}$$

The governing equations for $\psi^{(2,\odot)}$ and $W^{(2,\odot)}$ used in the foregoing development were those appropriate to the small-vorticity case $Q_{z,c} \ll 1$. A similar analysis can be carried out in the long-wavelength problem. Again, we begin with an integration with respect to θ in a plane $z = const$. At the lowest order, we obtain (A3). A second condition will now be determined by integrating in a plane $\theta = const$. The leading-order equation is the following:

$$\oint sR\, W^{(2,\odot)}_{RZ}\, dz = 0. \tag{6}$$

Writing $W^{(2,\odot)} = \kappa Q_{0,c}/m\, R + \Psi^{\odot}_{R}$, further differentiations with respect to R and Z followed by a substitution into (6) leads to

$$\int_0^{2\pi/k} [\,\Psi^{(\odot)+}_{RRZ} + \Psi^{(\odot)-}_{RRZ}\,] \sqrt{Z_0 - \cos[kz]}\, dz = 0\,. \tag{7}$$

That leads to the conclusion that $\Psi^{(\odot)}_{R} = 0$, the reason being that the general solution involves $\sqrt{2\,Z}$ and when substituted into (7), a singularity at $Z = 0$ would result from differentiating with respect to Z.

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