Recent Results on Local Projection Stabilization for Convection-Diffusion and Flow Problems

Lutz Tobiska

Abstract A survey of stabilization methods based on local projection is given. The class of steady problems considered covers scalar convection-diffusion equations, the Stokes problem and the linearized Navier–Stokes equations.

1 Introduction

It is well known that standard finite element discretizations applied to convection-diffusion or incompressible flow problems show spurious oscillations in the case of higher Reynolds numbers, owing to dominating convection. A first proposal to handle this instability for low-order finite element discretization has been the use of upwind finite elements [1]. Another idea, suitable also for higher-order finite elements, is the streamline upwind Petrov-Galerkin (SUPG) stabilization proposed in [2] and analyzed for a scalar convection-diffusion equation in [3]. The method is based on adding weighted residuals to the standard Galerkin method to enhance stability without losing consistency. The same idea is useful in circumventing the Babuška–Brezzi condition which restricts the set of possible finite element spaces that approximate velocity and pressure for incompressible flows. Such a pressure-stabilized Petrov–Galerkin (PSPG) method has been studied for low equal-order interpolations of the Stokes problem in [4]. A detailed error analysis of these SUPG/PSPG-type stabilizations applied to the incompressible Navier–Stokes equations, including both the case of inf–sup stable and equal-order interpolations, can be found in [5]. Recently, local projection stabilization (LPS) [6–8] methods have become quite popular, in particular because of their commutative properties in optimization problems [9] and stabilization properties similar to those of the SUPG

L. Tobiska
Institute for Analysis and Computational Mathematics, Otto von Guericke University Magdeburg, PF 4120, D-39016 Magdeburg, Germany, E-mail: tobiska@ovgu.de

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method [10]. In the following we give an overview of recent developments for this class of stabilizations applied to various problems.

2 Convection-Diffusion Problem

2.1 Standard Galerkin and SUPG

We start with the convection-diffusion equation

$$-\varepsilon\Delta u + b\cdot\nabla u + \sigma u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$. For simplicity we assume $\nabla\cdot b = 0$ and $\sigma > 0$ which guarantees a unique weak solution $u \in H_0^1(\Omega)$. Note that in the interesting case $0 < \varepsilon \ll 1$, the solution exhibits boundary and interior layers whose positions depend on the convection field b. Let $V_h \subset H_0^1(\Omega)$ be a finite element space with mesh size h. Then the discrete problem for the standard Galerkin approach is:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$a(u_h, v_h) := \varepsilon(\nabla u_h, \nabla v_h) + (b\cdot\nabla u_h + \sigma u_h, v_h) = (f, v_h)$$

where $(\cdot,\cdot)$ denotes the inner product in L^2 and its vector-valued analogues. Stability and convergence for piecewise polynomials of degree $r \geq 1$ follow from the coercivity of the bilinear form $a(\cdot,\cdot)$ and the Lemma of Cea:

$$\begin{aligned} a(v,v) \geq ||v||_{1,\varepsilon}^2 &:= \varepsilon|v|_1^2 + \sigma\|v\|_0^2 && \forall v \in V, \\ ||u - u_h||_{1,\varepsilon} &\leq C\, h^r |u|_{r+1}, && u \in H_0^1(\Omega)\cap H^{r+1}(\Omega). \end{aligned}$$

Nevertheless it is well-known that spurious oscillations appear if $\varepsilon \ll h$. This observation shows that the norm $\|\cdot\|_{1,\varepsilon}$ is too weak to suppress global oscillations.

The SUPG [2, 3] modifies the Galerkin method by adding weighted residuals of the strong form of the differential equations, resulting in:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\begin{aligned} &\varepsilon(\nabla u_h, \nabla v_h) + (b\cdot\nabla u_h + \sigma u_h, v_h) \\ &\quad + \sum_{K\in\mathcal{T}_h} \tau_K(-\varepsilon\Delta u_h + b\cdot\nabla u_h + \sigma u_h - f, b\cdot\nabla v_h)_K = (f, v_h) \end{aligned}$$

where $\mathcal{T}_h$ denotes a decomposition of Ω into cells $K \in \mathcal{T}_h$, $(\cdot,\cdot)_K$ is the inner product in $L^2(K)$, and τ_K is a user-chosen stabilization parameter. For $\tau_K \sim h_K$, stability follows again from coercivity of the associated bilinear form, but now with respect to the stronger norm

$$|||v|||_{SUPG} := \left(||v||_{1,\varepsilon}^2 + \sum_{K\in\mathcal{T}_h} \tau_K \|b\cdot\nabla v\|_{0,K}^2 \right)^{1/2},$$

which suppresses global oscillations. A clever estimation of the convection term uses integration by parts and the stability with respect to $|||\cdot|||_{SUPG}$:

$$\begin{aligned}
\left|(b\cdot\nabla(u-i_hu), v_h)\right| &\le |(u-i_hu, b\cdot\nabla v_h)| + |(u-i_hu, v_h\nabla\cdot b)| \\
&\le \sum_{K\in\mathcal{T}_h} \tau_K^{-1/2}\|u-i_hu\|_{0,K}\,\tau_K^{1/2}\|b\cdot\nabla v_h\|_{0,K} + Ch^{r+1}|u|_{r+1}\,\|v_h\|_0 \\
&\le C\left[\left(\sum_{K\in\mathcal{T}_h} \tau_K^{-1}h_K^{2r+2}|u|_{r+1,K}^2\right)^{1/2} + h^{r+1}|u|_{r+1}\right]|||v_h|||_{SUPG}
\end{aligned}$$

resulting in the improved error estimate

$$|||u-u_h|||_{SUPG} \le C\,(\varepsilon^{1/2}+h^{1/2})\,h^r\,|u|_{r+1}$$

for P_r finite elements. Note that in boundary layers we usually have $|u|_{r+1} \sim \varepsilon^{-(r+1/2)}$, which means that the above error estimate becomes useless. Nevertheless local error estimates have been derived that support theoretically the good approximation properties away from layers observed in numerical computations; see, e.g., [11].

Thus the SUPG is a consistent method with improved stability and convergence properties compared to the standard Galerkin approach. However, consistency is obtained at the cost of computing several additional terms to assemble the coefficient matrix of the discrete system.

2.2 Local Projection Stabilization (LPS)

A detailed study of the stability and convergence analysis of the SUPG shows that in the discrete problem only the term

$$\sum_{K\in\mathcal{T}_h} \tau_K(b\cdot\nabla u_h, b\cdot\nabla v_h)_K$$

is responsible for improved stability properties. However, skipping all other terms in the SUPG leads to an inconsistent method for which the consistency error scales with τ_K. A remedy is to add a term that controls only the fluctuations of the derivatives in the streamline direction $b\cdot\nabla u_h$. Let $\mathcal{M}_h$ denote a decomposition of Ω into 'macro' cells $M \in \mathcal{M}_h$ of diameter h_M with $h_K \sim h_M$ for $\overline{K}\cap\overline{M} \neq \emptyset$, D_h a discontinuous projection space associated with the decomposition $\mathcal{M}_h$, $\pi_h : L^2(\Omega) \to D_h$ the L^2 projection, and $\kappa_h := id - \pi_h$ the fluctuation operator. Then our modified discrete problem is:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\varepsilon(\nabla u_h, \nabla v_h) + (b \cdot \nabla u_h + \sigma u_h, v_h) + \sum_{M \in \mathcal{M}_h} \tau_M (\kappa_h(b \cdot \nabla u_h), \kappa_h(b \cdot \nabla v_h))_M = (f, v_h).$$

The modified bilinear form associated with the left-hand side is coercive with respect to the mesh-dependent norm

$$|||v|||_{LPS} := \left(||v||^2_{1,\varepsilon} + \sum_{M \in \mathcal{M}_h} \tau_M \|\kappa_h(b \cdot \nabla v)\|^2_{0,M} \right)^{1/2}.$$

Now the consistency error depends on τ_M and the projection space D_h. If the discontinuous space of piecewise polynomials of degree at most $r-1$ is selected, which we write as $D_h = P^{\mathrm{disc}}_{r-1}(\mathcal{M}_h)$, then for $\tau_M \sim h_M$ we get

$$\begin{aligned} &\left| \sum_{M \in \mathcal{M}_h} \tau_M \, (\kappa_h(b \cdot \nabla u), \kappa_h(b \cdot \nabla v_h))_M \right| \\ &\quad \le \sum_{M \in \mathcal{M}_h} \tau_M^{1/2} h_M^r |b \cdot \nabla u|_{r,M} \; \tau_M^{1/2} \|\kappa_h(b \cdot \nabla v_h)\|_{0,M} \\ &\quad \le C \left(\sum_{M \in \mathcal{M}_h} h_M^{2r+1} |b \cdot \nabla u|^2_{r,M} \right)^{1/2} |||v_h|||_{LPS}. \end{aligned}$$

Using the L^2 stability of the fluctuation operator we see that

$$|||v_h|||_{LPS} \le C \, |||v_h|||_{SUPG} \quad \forall v_h \in V_h$$

which means that the SUPG is at least as stable as the LPS. Having in mind only the coercivity of the bilinear forms with respect to $||| \cdot |||_{SUPG}$ and $||| \cdot |||_{LPS}$, respectively, one might think that the LPS is less stable compared to the SUPG. But in [10] an inf–sup condition for the LPS bilinear form in a stronger norm (which turns out to be equivalent to the SUPG norm) has been shown, i.e., the stability properties of LPS and SUPG are in fact comparable.

2.3 Basics in the Error Analysis of LPS

We assume that $Y_h \approx H^1(\Omega)$ is a finite element space associated with a decomposition of Ω into cells $K \in \mathcal{T}_h$ and $V_h = Y_h \cap H^1_0(\Omega)$ denotes the approximation space. Let the discontinuous projection space $D_h = \oplus_M D_h(M)$ live on a decomposition into macro cells $M \in \mathcal{M}_h$, where the case $\mathcal{T}_h = \mathcal{M}_h$ is assumed to be included. We will see that the key idea of the LPS lies in the existence of a special interpolant $j_h : H^2(\Omega) \to Y_h$ that displays the usual interpolation properties and satisfies in addition the orthogonality property

$$(w - j_h w, q_h) = 0 \qquad \forall w \in H^2(\Omega),\ \forall q_h \in D_h.$$

This orthogonality enables an estimation of the critical part of the convection term after integrating by parts for $\tau_M \sim h_M$ as follows:

$$\begin{aligned} |(u - j_h u, b \cdot \nabla v_h)| &= |(u - j_h u), \kappa_h(b \cdot \nabla v_h))| \\ &\leq \sum_{M \in \mathcal{M}_h} \tau_M^{-1/2} \|u - j_h u\|_{0,M}\ \tau_M^{1/2} \|\kappa_h(b \cdot \nabla v_h)\|_{0,M} \\ &\leq C \left(\sum_{M \in \mathcal{M}_h} h_M^{2r+1} |u|^2_{r+1,M} \right)^{1/2} |||v_h|||_{LPS}. \end{aligned}$$

Dealing with all other terms in the usual way, we end up with the error estimate

$$|||u - u_h|||_{LPS} \leq C\,(\varepsilon^{1/2} + h^{1/2})\, h^r\, |u|_{r+1} \tag{2}$$

for $\tau_M \sim h_M$ [6,7,12,13]. Now the question arises: under which conditions does an interpolation j_h with additional orthogonality properties exist? Examples have been given for the transport equation ($\varepsilon = 0$) in [12] and the Oseen equation in [6], where the two-level variant has been studied in which the decomposition into cells is generated from a macro mesh by certain refinement rules. We indicate this by writing $\mathcal{T}_h = \mathcal{M}_{h/2}$. In the general case we have

Theorem 1 ([7]). *Let the local inf–sup condition*

$$\inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M}\ \|q_h\|_{0,M}} \geq \beta_1 > 0, \qquad \forall M \in \mathcal{M}_h \tag{3}$$

with $Y_h(M) := \{w_h|_M : w_h \in Y_h,\ w_h = 0 \text{ on } \Omega\backslash M\}$ be satisfied. Then there is an interpolation $j_h : H^2(\Omega) \to Y_h$ with the usual interpolation error estimates and the additional orthogonality property

$$(w - j_h w, q_h) = 0, \quad \forall q_h \in D_h,\ \forall w \in H^2(\Omega).$$

In order to fulfil all assumptions of the convergence analysis, two different requirements for the pair (V_h, D_h) of approximation and projection space have to be reconciled:

- D_h has to be rich enough to guarantee a certain order of consistency
- D_h should be small enough w.r.t. V_h to guarantee $j_h u - u \perp D_h$

Two main approaches have been considered in the literature:

$$\text{one-level } (V_h^+, D_h) \quad \Leftrightarrow \quad \text{two-level } (V_h, D_{2h}).$$

In the one-level approach, a standard finite element space is chosen as the projection space D_h to guarantee the consistency order. Then, the approximation space $V_h = Y_h \cap H_0^1(\Omega)$ is (if necessary) enriched to V_h^+ such that the assumptions of Theorem 1 are fulfilled. In the two-level approach, a standard finite element space is chosen as the approximation space V_h and the projection space D_h is thinned out to a space D_{2h} on the next coarser mesh level.

In the following we give explicit examples satisfying all assumptions needed for the above error estimation, see [7] for details. Let b_K and $\tilde{b}_K$ denote the (mapped) bubble functions of lowest polynomial degree that vanish on the boundary ∂K of a simplex and hexahedron respectively. We introduce the enriched approximation spaces on triangles and quadrilaterals respectively:

$$P_r^+ := P_r + \bigoplus_{K\in\mathcal{T}_h} b_K \cdot P_{r-1}(K)$$

$$Q_r^+ := Q_r + \bigoplus_{K\in\mathcal{T}_h} \text{span}\,(\tilde{b}_K \cdot x_i^{r-1},\ i = 1,\dots,d).$$

An overview of different variants is given in Table 1 and illustrated in the two-dimensional case $d = 2$ for $r = 1$ and $r = 2$ in Figs. 1–4.

One disadvantage of the one-level approach is the increasing number of degrees of freedom owing to the enrichments in particular in the case of simplices. However, this can be overcome by static condensation. In the two-level approach the stencil of the stabilizing term increases due to the larger support of $\kappa_h(b \cdot \nabla\varphi_i)$ compared with that of $b \cdot \nabla\varphi_i$ (for each basis function φ_i in V_h). This might not fit into the data structure of an available code.

So far we have only considered the case of boundary conditions of Dirichlet type. Mixed boundary conditions lead often to a limited regularity of the solution of a convection-diffusion problem. In [13], it is shown how the error analysis of the

Table 1 Possible space pairs in the LPS

one-level		two-level	
V_h^+	D_h	V_h	D_{2h}
P_r^+	P_{r-1}^{disc}	P_r	P_{r-1}^{disc}
Q_r^+	P_{r-1}^{disc}	Q_r	Q_{r-1}^{disc}

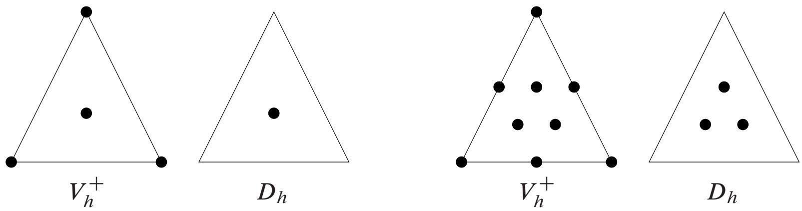

Fig. 1 Approximation and projection spaces on triangles (one-level approach)

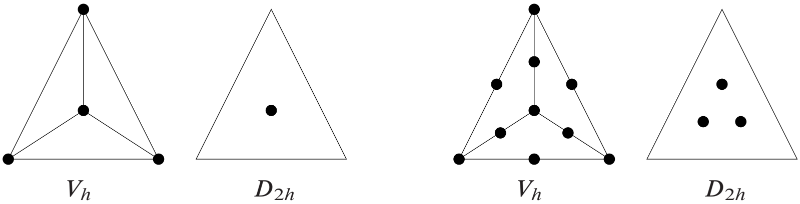

Fig. 2 Approximation and projection spaces on triangles (two-level approach)

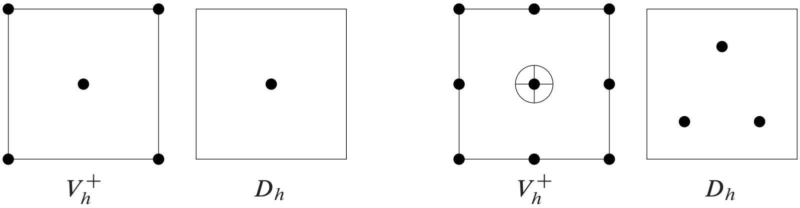

Fig. 3 Approximation and projection spaces on quadrilaterals (one-level approach)

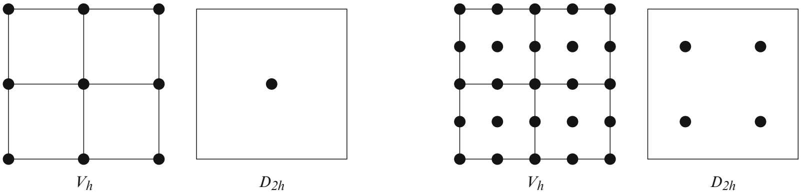

Fig. 4 Approximation and projection spaces on quadrilaterals (two-level approach)

one-level LPS can be extended to the case of boundary conditions of mixed Dirichlet and Neumann type.

2.4 Relationship to Other Stabilization Methods

The LPS is akin to but not exactly equal to the subgrid scale stabilization introduced by Guermond [14], who considered gradients of fluctuations instead of fluctuations of gradients. Thus the stabilizing term has the form

$$\sum_{K\in\mathcal{K}_h} \tau_K(\nabla(id - P_H)u_h, \nabla(id - P_H)v_h)_K$$

where $P_H : v_h \to V_H$ is a projection onto the (resolvable) coarse scales. This can be also interpreted as adding artificial viscosity only for the fine scales of the finite element space V_h. For certain scale separations of $V_h = V_H \oplus V_H^{\perp}$ both methods give spectrally equivalent stabilization terms (simplices) or even coincide (lowest order case on simplices). However, in the general case the stabilizing terms are not spectrally equivalent. For more details see [7].

We mention that a special variant of the LPS has been already introduced by Layton [15] as a mixed method combined with scale separation of the finite element space V_h. The projection space has been chosen as $D_H = \nabla V_H$, where V_H denotes the approximation space on a coarser mesh level. The analysis given in [15] does not use orthogonality of the interpolation and leads to the suboptimal convergence rate of $4/3$ instead of $3/2$ for the scaling $\tau \sim h^{4/3}$, $H \sim h^{2/3}$. In order to gain the full $3/2$-power of h, the orthogonality of the interpolation has been used in [12] for solving the transport equation ($\varepsilon = 0$) discretized by the two-level (Q_1, Q_0)-LPS.

There is also a close relation to the stabilization method using orthogonal subscales (OSS) proposed by Codina in [16,17]. In the OSS the projection π_h is chosen as the L^2 projection into the finite element ansatz space without forcing boundary conditions, i.e., $D_h = Y_h$. Since this projection is no longer local, the stencil of the stabilizing term increases as in the two-level LPS approach or one has to solve a global system in $V_h \times Y_h$ to approximate u and $b \cdot \nabla u$. For details we refer to [16].

2.5 Choice of the Stabilization Parameter

A general strategy to select appropriate stabilization parameters τ_K is to equilibrate different terms in the *a priori* error estimates. In this way, the asymptotic behaviour of τ_K with respect to the meshsize and the polynomial degree of the finite element spaces can be fixed. For convection-diffusion equations in one space dimension, it is known that in the constant coefficient case with $c = 0$ and piecewise linear elements the stabilization parameter in the SUPG method can be chosen in such a way that the discrete solution becomes nodally exact.

It has been shown in [18] that in the one-dimensional, constant coefficient case (with $c = 0$), the one-level version of the $(P_r^+, P_{r-1}^{\text{disc}})$-LPS is equal to the P_{r+1}-differentiated residual method (DRM). Note that in 1D one has $P_r^+ = P_{r+1}$. Moreover, a successive elimination of the higher modes in the P_{r+1}-DRM by static condensation leads to the P_r-DRM, where the P_1-DRM coincides with the SUPG. These observations allow the derivation of explicit formulas for the stabilization parameter in the LPS and DRM such that the P_1 part of the corresponding discrete solutions is nodally exact. For more details, see [18]. The convergence properties of the DRM on arbitrary and on layer-adapted meshes are investigated in [19]. Finally, we mention that the DRM is also closely related to the variational multiscale method (VMS) studied in [20].

2.6 LPS on Layer Adapted Meshes

It has been mentioned already in Sect. 2.1 that $|u|_{r+1} \sim \varepsilon^{-(r+1/2)}$ in boundary layers and error estimates like (2) lose their value as $\varepsilon \to 0$. For the model problem

(1) in the unit square $\Omega = (0,1)^2$, two different types of layers can appear: for $b = (b_1, b_2)$ with $b_1, b_2 > 0$ we observe only exponential layers along the outflow part ($x = 1$ or $y = 1$) of the domain whereas for $b = (b_1, 0)$ with $b_1 > 0$ an exponential layer along $x = 1$ and two characteristic layers along $y = 0$ and $y = 1$ are present. The idea is to use special layer-adapted (so-called S-type) meshes and suitable enriched approximation spaces. Consider the following enrichment of the usual Q_r space of continuous, piecewise (mapped) polynomials of degree r in each variable

$$Q_r^{6+} := Q_r + \bigoplus_{K \in \mathcal{T}_h} Q_K^*(K),$$

$$Q_K^*(\widehat{K}) := \text{span}\left\{(1-\hat{x}_1^2)(1-\hat{x}_2^2)\hat{x}_i^{p-1},\ (1 \pm \hat{x}_{i+1})(1-\hat{x}_i^2)L_{r-1}(\hat{x}_i)\right\},$$

where $i \in \{0,1\}$ modulo 2 and L_{r-1} denotes the Legendre polynomial of order $r-1$. The projection space is set to be $D_h = P_{r-1}^{\text{disc}}$. Note that $P_{r+1} \subset Q_r^{6+}$. The number of subintervals in each coordinate direction of the tensor product mesh on Ω will be denoted by N.

Theorem 2 ([21,22]). *Let $b_1, b_2 > 0$, $(Y_h, D_h) = (Q_r^{6+}, P_{r-1}^{disc})$, and the stabilization parameter be given by $\tau_K \sim N^{-2}$ on the coarse mesh and $\tau_K = 0$ on the fine mesh. Then, there is an interpolant u^I such that*

$$|||u^I - u^N|||_{LPS} \le C\,(N^{-1}\log N)^{r+1},\ \|u - u^N\|_{1,\varepsilon} \le C\,(N^{-1}\log N)^{r+1}$$

on a Shishkin mesh. For the characteristic layers case ($b_1 > 0$, $b_2 = 0$) an appropriate choice of the stabilization parameter in the characteristic layer region leads to the same estimate. Moreover, for $r = 1$ we have the supercloseness result for the spaces $(V_h, D_h) = (Q_1, P_0^{disc})$

$$|||u^I - u^N|||_{LPS} \le C\,(N^{-1}\log N)^2,\ \|u - u^N\|_{1,\varepsilon} \le C\,N^{-1}\log N.$$

Apart from the lowest-order case, we have to handle a considerable set of additional degrees of freedom because of the large enrichment of Q_r. Next, we consider a moderate enrichment of Q_r such that $P_{r+1} \not\subset Q_r^+$ and give a supercloseness result.

Theorem 3 ([23]). *Let the approximation space Y_h be enriched only on the coarse mesh part so that $Y_h = Q_r^+$ on the coarse and $Y_h = Q_r$ on the fine mesh part. Then on Shishkin and Bakhvalov–Shishkin type meshes the interpolant u^I is superclose, i.e.,*

$$|||u^I - u^N|||_{LPS} \le C\,N^{-(r+1/2)}$$

whereas for the solution u one has only

$$\|u - u^N\|_{1,\varepsilon} \le \begin{cases} C\,(N^{-1}\log N)^{-r} & \textit{for a Shishkin mesh,} \\ C\,N^{-r} & \textit{for a Bakhvalov-Shishkin mesh.} \end{cases}$$

Note that the enrichments in Theorem 3 consists of only two additional degrees of freedom per coarse mesh cell. Thus, compared to Theorem 2, a considerable reduction in the number of degrees of freedom has been achieved.

3 Stokes Problem

3.1 Standard Galerkin and PSPG

Now we consider the Stokes Problem

$$-\Delta u + \nabla p = f \quad \text{in } \Omega,\ \operatorname{div} u = 0 \quad \text{in } \Omega,\ u = 0 \quad \text{on } \Gamma \tag{4}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$. There is a unique weak solution $(u, p) \in H_0^1(\Omega)^d \times L_0(\Omega)$. Let $V_h \subset H_0^1(\Omega)^d$ and $Q_h \subset L_0^2(\Omega)$ be finite element spaces with a mesh size h, approximating velocity and pressure respectively. Then the discrete problem of the standard Galerkin approach is:

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$

$$(\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) + (q_h, \operatorname{div} u_h) = (f, v_h).$$

It is well-known [24] that the Babuška–Brezzi condition

$$\exists \beta_0 > 0,\ \forall h: \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(q_h, \operatorname{div} v_h)}{\|q_h\|_0\, |v_h|_1} \geq \beta_0 \tag{5}$$

guarantees stability and convergence of a unique solution $(u_h, p_h) \in V_h \times Q_h$ of the Galerkin method. The condition (5) restricts the possible choices of approximation spaces V_h and Q_h; in particular, equal-order interpolations for velocity and pressure are excluded. One way to circumvent the inf–sup condition is to add weighted residuals of the strong form of the differential equation resulting in the stabilized formulation

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$

$$A_{PSPG}((u_h, p_h); ((v_h, q_h)) = (f, v_h) + \sum_{K \in \mathcal{T}_h} \alpha_K (f, \nabla q_h)_K \tag{6}$$

with the discrete bilinear form A_h given by

$$\begin{aligned} A_{PSPG}((u, p); (v, q)) := \ & (\nabla u, \nabla v) - (p, \operatorname{div} v) + (q, \operatorname{div} u) \\ & + \sum_{K \in \mathcal{T}_h} \alpha_K (-\Delta u + \nabla p, \nabla q)_K. \end{aligned} \tag{7}$$

For continuous pressure approximations $Q_h \subset H^1(\Omega)$, the form A_{PSPG} is coercive on the product space $V_h \times Q_h$ with respect to the norm

$$|||(v,q)|||_{PSPG} := \left(|v|_1^2 + \sum_{K \in \mathcal{T}_h} \alpha_K |q|_{1,K}^2 \right)^{1/2}$$

provided that the stabilization parameter has been chosen as $\alpha_K = \alpha_0 h_K^2$ where the positive constant α_0 satisfies an certain upper bound. This residual-based stabilization technique proposed and analyzed in [4] is also known as the pressure stabilized Petrov–Galerkin (PSPG) approach [25]. Over the years it has been extended and combined with the SUPG for solving the (linearized) Navier–Stokes equations.

3.2 Local Projection Stabilization

Beside the residual-based approach, projection-based stabilization techniques have also been developed for the Stokes problem. A method based on the projection of the pressure gradient onto a continuous finite element space has been proposed in [26]. Although the method is consistent (in a certain sense) it is expensive due to the nonlocality of the projection. Becker and Braack proposed in [27] to project the pressure gradient onto a discontinuous finite element space living on a coarser mesh. This method is not consistent, but it is cheaper owing to the locality of the projection. Nevertheless, as a two-level approach the stabilizing term leads to an larger stencil which might not fit into the data structure of an available code.

A revision of the residual-based PSPG approach shows that the improved stability properties rely on adding the term

$$\sum_{K \in \mathcal{T}_h} \alpha_K (\nabla p, \nabla q)_K \quad \text{instead of} \quad \sum_{K \in \mathcal{T}_h} \alpha_K (-\Delta u + \nabla p - f, \nabla q)_K$$

to the Galerkin method. The other terms are only needed to preserve consistency. Now, replacing in the first term the pressure gradients by the fluctuations, we obtain the LPS for equal-order interpolations [28].

Let the approximation spaces for velocity and pressure be generated by a scalar finite element space $Y_h \approx H^1(\Omega)$, such that $V_h = (Y_h \cap H_0^1(\Omega))^d$ and $Q_h = Y_h \cap L_0^2(\Omega)$. We will consider for simplicity only the one-level approach, thus the discontinuous projection space D_h lives on the same decomposition $\mathcal{M}_h = \mathcal{T}_h$ as the approximation space Y_h. As above we introduce the fluctuation operator $\kappa_h := \mathrm{id} - \pi_h$ with the L^2 projection $\pi_h : L^2(\Omega) \to D_h$. Now the stabilized discrete problem reads:

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$

$$A_h\big((u_h, p_h); (v_h, q_h)\big) := (\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) + (q_h, \operatorname{div} u_h) \qquad (8)$$
$$+ \sum_{K \in \mathcal{T}_h} \alpha_K \big(\kappa_h \nabla p_h, \kappa_h \nabla q_h\big)_K = (f, v_h).$$

As in the Galerkin method the bilinear form is not coercive on the product space $V_h \times Q_h$; indeed we have only

$$A_h((v_h, q_h); (v_h, q_h)) = |v_h|_1^2 + \sum_{K \in T_h} \alpha_K \|\kappa_h(\nabla q_h)\|_{0,K}^2$$

and the right-hand side vanishes for all $(v_h, q_h) = (0, q_h)$ with $\nabla q_h \in D_h$. Therefore, it is essential that an inf–sup condition can be proven in the mesh-dependent norm

$$|||(v, q)||| := \left(|v|_1^2 + \|q\|_0^2 + \sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h(\nabla q)\|_{0,K}^2 \right)^{1/2}.$$

Lemma 1 ([28]). *Let (Y_h, D_h) satisfy the local inf–sup condition in Theorem 1 and let $h_K^2/\alpha_K \le C$. Then there is a positive constant $\beta > 0$ independent of h such that*

$$\inf_{(v_h, q_h) \in V_h \times Q_h} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h\big((v_h, q_h); (w_h, r_h)\big)}{|||(v_h, q_h)|||\, |||(w_h, r_h)|||} \ge \beta.$$

Let us briefly discuss the different properties of PSPG and LPS. In the PSPG method A_{PSPG} is coercive on the product space $V_h \times Q_h$ for a restricted range of the stabilization parameter, more precisely $\alpha_K = \alpha_0 h_K^2$ with an upper bound for α_0 depending on the polynomial degree used in the definition of Y_h. In contrast to that the bilinear form A_h of the LPS satisfies an inf–sup condition on the product space $V_h \times Q_h$ for $\alpha_K = \alpha_0 h_K^2$ and any $\alpha_0 \in \mathbb{R}^+$. The theoretically larger range for α_0 can be also seen in computations [28].

3.3 Error Estimates

As in the case of a scalar convection-diffusion equation one has to balance two requirements: D_h has to be rich enough to guarantee a certain order of consistency and D_h has to be sparse enough to allow the existence of an interpolant $j_h : H^1(\Omega)^d \to Y_h^d$ (needed to prove Lemma 1) such that the interpolation error is perpendicular to D_h^d. The larger domain of definition ($H^1(\Omega)$ instead of $H^2(\Omega)$) is not a problem since interpolants of Scott–Zhang type can be used [29].

We briefly discuss the essential points in the error analysis. For the consistency error we get from the L^2 stability of κ_h the estimate ($\alpha_K \sim h_K^2$)

$$|A_h((u-u_h),(p-p_h);(w_h,r_h))| = \left|\sum_{K\in\mathcal{T}_h} \alpha_K(\kappa_h(\nabla p),\kappa_h(\nabla q_h)\right|$$
$$\leq C\left(\sum_{K\in\mathcal{T}_h} h_K^{2r}|\nabla p|_{r-1}^2\right)^{1/2} |||(w_h,r_h)|||$$

provided D_h comprises piecewise polynomials of degree $r-2$. Using Lemma 1 and the estimation of the consistency error it remains to estimate the approximation error. The most difficult part of it is the estimate

$$|(r_h, \text{div}\,(u-j_hu))| = |(\nabla r_h, u-j_hu)| = |(\kappa_h(\nabla r_h), u-j_hu)|$$
$$\leq C\,h^r|u|_{r+1}\left(\sum_{K\in\mathcal{T}_h}\alpha_K\|\kappa_h\nabla r_h\|_{0,K}^2\right)^{1/2}$$

in which we used the orthogonality property of the interpolant. Putting all pieces together we get the main theorem for the Stokes problem.

Theorem 4 ([28]). *Let the solution of* (4) *be smooth enough such that* $(u,p)\in \left(V\cap H^{r+1}(\Omega)^d\right)\times\left(Q\cap H^r(\Omega)\right)$ *and* $P_{r-2}^{disc}\subset D_h$. *Then, under the assumptions of Lemma 1 and* $\alpha_K\sim\alpha_0 h_K^2$, *there exists a positive constant C independent of h such that*

$$|||(u-u_h,p-p_h)|||\leq C\,h^r(\|u\|_{r+1}+\|p\|_r).$$

Moreover, if the Stokes problem is $H^2(\Omega)^d\times H^1(\Omega)$ *regular, there exists a positive constant C independent of h such that*

$$\|u-u_h\|_0\leq C\,h^{r+1}(\|u\|_{r+1}+\|p\|_r).$$

Note that in contrast to the PSPG approach for the LPS scheme considered we did not require higher regularity of the pressure when using equal-order interpolations.

3.4 Examples

In the following we list approximation and projection spaces from [28] satisfying all assumptions needed for the error estimate in Theorem 4. It turns out that some known stabilization methods in the literature can be recovered as special cases of the one-level LPS.

3.4.1 Simplicial Elements, First-Order Methods

Let the solution and projection spaces be given by $(V_h,Q_h) = (P_1^d,P_1)$ and $D_h = \{0\}$, respectively. Then the fluctuation operator becomes the identity and we get the method proposed by Brezzi and Pitkäranta in [30]. Now, let as above

b_k denote the (mapped) bubble function that belongs to P_{d+1} and vanishes at the boundary ∂K. We enrich the space of continuous, piecewise linear function by adding the bubble functions on each cell, i.e.,

$$P_1^+ = P_1 + \bigoplus_{K \in \mathcal{T}_h} \operatorname{span} b_K.$$

If we enrich only the velocity space so that $(V_h, Q_h, D_h) = ((P_1^+)^d, P_1, \{0\})$, no stabilization is needed since the pair $(V_h, Q_h) = ((P_1^+)^d, P_1)$, the so called 'Mini'-element [31], satisfies the inf–sup condition (5). However, enriching the spaces for approximating velocity and pressure, we get an equal-order interpolation and the LPS becomes necessary. A possible choice with optimal first-order convergence is $(V_h, Q_h, D_h) = ((P_1^+)^d, P_1^+, P_0)$ [28].

3.4.2 Simplicial Elements, Higher-Order Methods

Unlike Subsect. 2.3 we consider the (less) enriched approximation space

$$\widetilde{P}_r^+ := P_r + \bigoplus_{K \in \mathcal{T}_h} b_K \cdot P_{r-2}(K)$$

which fits the projection space $D_h = P_{r-2}^{\mathrm{disc}}$ and the choice $\alpha_K \sim \alpha_0 h_K^2$. Then an LPS method with optimal convergence order $r \geq 2$ is generated by $(V_h, Q_h, D_h) = ((\widetilde{P}_r^+)^d, \widetilde{P}_r^+, P_{r-2}^{\mathrm{disc}})$ [28].

3.4.3 Hexahedral Elements, First-Order Methods

We consider first the case where the approximation and projection spaces are given by $(V_h, Q_h) = (Q_1^d, Q_1)$ and $D_h = \{0\}$, respectively. The fluctuation operator is the identity and we end up again with the stabilization proposed by Brezzi and Pitkäranta in [30]. Now, by enriching only the velocity space, we can derive pairs of finite elements (V_h, Q_h) satisfying the inf–sup condition (5) such that no stabilization is needed. Although similar to the case of triangular elements, where two additional degrees of freedom per cell have been added, in the quadrilateral case ($d = 2$) we have to add at least three additional degrees of freedom in the conforming and non-conforming case [32, 33]. Further examples of enrichments of the velocity space leading to inf–sup stable element pair (V_h^+, Q_1) have been studied in [34,35]. Enriching both the velocity and the pressure space, we get an equal-order interpolation and the LPS becomes needed. Let us enrich the space of continuous, piecewise multi-linear functions by adding the bubble functions on each cell, i.e.,

$$Q_1^+ = Q_1 + \bigoplus_{K \in \mathcal{T}_h} \operatorname{span} \tilde{b}_K.$$

Now a possible choice of spaces with a first-order convergence property is $(V_h, Q_h, D_h) = ((Q_1^+)^d, Q_1^+, Q_0)$ [28].

3.4.4 Hexahedral Elements, Higher-Order Methods

It turns out that for hexahedral elements and $r \geq 2$ the standard spaces Q_r are already rich enough that the pair $(Y_h, D_h) = (Q_r, Q_{r-2}^{\text{disc}})$ satisfies the local inf–sup condition of Theorem 1. Thus an LPS method with optimal convergence order $r \geq 2$ is generated by $(V_h, Q_h, D_h) = (Q_r^d, Q_r, Q_{r-2}^{\text{disc}})$ [28]. Note that the 'smallest' projection space that guarantees the consistency order $r \geq 2$ is the mapped or unmapped space P_{r-2}^{disc}. Since in both cases the inclusion $P_{r-2}^{disc} \subset Q_{r-2}^{disc}$ holds true, the local inf–sup condition of Theorem 1 is still satisfied and we obtain the optimal convergence order also for the choice $(V_h, Q_h, D_h) = (Q_r^d, Q_r, P_{r-2}^{\text{disc}})$. For details and other pairs of finite element spaces we refer to [28].

3.5 Elimination of Enrichments

It has been shown by Bank and Welfert in [36] that the bubble part of the velocity components for the Mini element discretization of the Stokes problem, i.e., $(V_h, Q_h) = ((P_1^+)^d, P_1)$, can be locally eliminated and lead to a formulation equivalent to the stabilized method proposed by Hughes, Franca, and Balestra in [4]. Furthermore, in [37] special enrichments of both the velocity space $V_h = P_1^d$ and the pressure space $Q_h = P_1$ have been introduced and shown to lead by static condensation to a Galerkin least squares stabilized formulation of the Stokes problem. For this, on each cell of the triangulation the velocity components are enriched by two bubble functions and the pressure by a function that does not vanish at the cell boundaries.

Of course, the additional degrees of freedom introduced by the enrichments of the $(V_h, Q_h, D_h) = ((P_1^+)^d, P_1^+, P_0)$-LPS can also be eliminated locally by static condensation. The resulting scheme corresponds to the stabilized method of Hughes, Franca, and Balestra [4] with an additional grad/div stabilization which can be written as

Find $(u_L, p_L) \in V_L \times Q_L = P_1^d \times P_1$ such that for all $(v_L, q_L) \in V_L \times Q_L$

$$(\nabla u_L, \nabla v_L) - (p_L, \operatorname{div} v_L) + \sum_{K \in \mathcal{T}_h} \gamma_K (\operatorname{div} u_L, \operatorname{div} v_L)_K = (f, v_L),$$

$$(q_L, \operatorname{div} u_L) + \sum_{K \in \mathcal{T}_h} (-\Delta u_L + \nabla p_L, \tau_K \nabla q_L)_K = \sum_{K \in \mathcal{T}_h} (f, \tau_K \nabla q_L)_K.$$

Here the parameters γ_K and τ_K behave like [28]

$$\gamma_K \sim \frac{h_K^2}{\alpha_K} \sim 1, \qquad \tau_K(x) \sim h_K^2 b_K(x)$$

which is in agreement with the suggested choice in the literature. The result in [28, 37] demonstrates that pressure bubbles play a role in explaining the addition of the least-squares form of the continuity equation in stabilized methods for the Stokes problem.

4 Oseen Problem

4.1 Standard Galerkin and LPS

We consider finally the Oseen problem

$$-\varepsilon\Delta u + (b\cdot\nabla)u + \sigma u + \nabla p = f \text{ in } \Omega,\ \nabla\cdot u = 0 \quad \text{in } \Omega,\ u = 0 \quad \text{on } \Gamma,$$

which can be understood as a testbed for developing stable and accurate approximations of the incompressible Navier–Stokes equations. The reason for that is that this simpler problem (a unique solution exists for all $\varepsilon > 0$) already includes the two sources of instabilities: the instability due to dominant convection ($\varepsilon \ll 1$) and the instability caused by pairs of finite elements that are not inf–sup stable. The weak formulation of the Oseen problem reads

Find $(u, p) \in V \times Q$ such that for all $(v, q) \in V \times Q$

$$\begin{aligned} A\big((u, p); (v, q)\big) :&= \varepsilon(\nabla u, \nabla v) + \big((b\cdot\nabla)u, v\big) + \sigma(u, v) \\ &\quad - (p, \operatorname{div} v) + (q, \operatorname{div} u) = (f, v) \end{aligned}$$

where $V := H_0^1(\Omega)^d$, $Q := L_0^2(\Omega)$, $\varepsilon > 0$, $\sigma \geq 0$, $b \in W^{1,\infty}(\Omega)$, $\operatorname{div} b = 0$ have been assumed. Now, let us consider the case of equal order interpolation in which the velocity and the pressure space are generated by the same scalar finite element space $Y_h \approx H^1(\Omega)$, namely $V_h := Y_h^d \cap V$ and $Q_h := Y_h \cap Q$ [6,7,38]. Then the stabilized discrete problem is:

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$(A + S)\big((u_h, p_h); (v_h, q_h)\big) = (f, v_h) \qquad \forall (v_h, q_h) \in V_h \times Q_h$$

where the stabilization term is given by

$$\begin{aligned} S\big((u_h, p_h); (v_h, q_h)\big) := \sum_{M\in\mathcal{M}_h} \Big[&\tau_M\big(\kappa_h((b\cdot\nabla)u_h), \kappa_h((b\cdot\nabla)v_h)\big)_M \\ &+ \mu_M\big(\kappa_h(\operatorname{div} u_h), \kappa_h(\operatorname{div} v_h)\big)_M + \alpha_M\big(\kappa_h(\nabla p_h), \kappa_h(\nabla q_h)\big)_M\Big] \end{aligned}$$

with user-chosen parameters τ_M, μ_M, and α_M. Here $\mathcal{M}_h$ denotes a decomposition of Ω into macro cells needed to define the projection spaces D_h while the approximation spaces live on a decomposition $\mathcal{T}_h$ not necessary equal to $\mathcal{M}_h$. Furthermore, $\kappa_h = \mathrm{id} - \pi_h$ is the fluctuation operator and π_h the (vector-valued) L^2 projection

Table 2 Possible (mapped) spaces in the LPS for the Oseen problem

one-level			two-level		
V_h^+	Q_h^+	D_h	V_h	Q_h	D_{2h}
$(P_r^+)^d$	P_r^+	P_{r-1}^{disc}	$(P_r)^d$	P_r	P_{r-1}^{disc}
$(Q_r^+)^d$	Q_r^+	P_{r-1}^{disc}	$(Q_r)^d$	Q_r	Q_{r-1}^{disc}

into the discontinuous projection space D_h. An interesting option is an additional projection space for controlling the fluctuations of the divergence since that term is fully consistent [38]. Under reasonable assumptions the bilinear form $A + S$ satisfies an inf–sup condition on the spaces Y_h and D_h with respect to the mesh-dependent norm

$$|||(v,q)|||_{OSE} := \Big(\varepsilon|v|_1^2 + \sigma\|v\|_0^2 + \rho\|q\|_0^2 + S\big((v,q);(v,q)\big)\Big)^{1/2}$$

with $\rho > 0$ [7]. Moreover, the following error estimate holds true.

Theorem 5 ([7]). *Let $\alpha_M, \mu_M, \tau_M \sim h_M$, b piecewise smooth, $P_{r-1}^{disc} \subset D_h$ and (Y_h, D_h) satisfies the local inf–sup condition* (3)*. Then there is a positive constant C independent of h such that*

$$|||(u-u_h, p-p_h)|||_{OSE} \le C(\varepsilon^{1/2} + h^{1/2})\, h^r \big(\|u\|_{r+1} + \|p\|_{r+1}\big).$$

We show in Table 2 examples of spaces that satisfy all assumptions which guarantee the stated error estimate. Note that in the two-level approach we divide a macro simplex M into $d+1$ simplices K by connecting the barycenter with the vertices. A macro hexahedron is subdivided into 2^d hexahedrons in the usual way. For more details we refer to [7].

4.2 LPS for Inf–Sup Stable Elements

The local projection stabilization has been also applied to inf–sup stable discretizations of the Oseen equation in [8, 39]. An interesting point is that for inf–sup stable finite element pairs one does not need an $H^1(\Omega)$ stable interpolation operator with additional orthogonality properties to prove stability of the discrete problem, unlike the case of equal-order interpolation. Consequently, one has much more flexibility in choosing the approximation and projection spaces [39]. We replace the stabilizing term above by

$$\begin{aligned}S((u_h,p_h),(v_h,q_h)) := \sum_{K\in\mathcal{T}_h} \Big(&\tau_K(\kappa_h^1(b\cdot\nabla u_h),\kappa_h^1(b\cdot\nabla v_h))_K \\ &+\mu_K(\kappa_h^2(\operatorname{div} u_h),\kappa_h^2(\operatorname{div} v_h))_K\Big) \\ &+\sum_{E\in\mathcal{E}_h}\gamma_E\,\langle[p_h]_E,[q_h]_E\rangle_E\end{aligned}$$

in order to handle both continuous ($\gamma_E = 0$) and discontinuous ($\gamma_E > 0$) pressure spaces Q_h. Note that a pressure $p \in H^1(\Omega)$ does not cause any consistency error and that we have introduced two projection spaces resulting in two fluctuation operators. Most of the known inf–sup stable elements approximate the velocity components by elements of order r and the pressure by elements of order $r-1$, which yields error estimates of order r, cf. [8], which in the convection-dominated case ($\varepsilon < h$) is half an order less than the LPS with equal-order interpolation. However, the same convergence order can be achieved in the one-level case by standard finite element spaces without any enrichments [39]; a possible variant is

$$(V_h, Q_h, D_h^1, D_h^2) = ((Q_r)^d, P_{r-1}^{\text{disc}}, (Q_{r-2}^{\text{disc}})^d, P_{t-1}^{\text{disc}})$$

with the parameter choice $\tau_K \sim h_K$, $\mu_K \sim 1$, and $\gamma_E \sim h_E$. Furthermore, there are inf–sup stable elements approximating both the velocity components and the pressure by elements of order r, which yield error estimates of order $r + 1/2$ in the convection-dominated case. For details see [11, 39].

4.3 LPS as an hp-Method

The a priori error analysis and the parameter design of the LPS have been extended to study the dependence of the error not only on the mesh size but also on the polynomial degree [8, 38]. As an example we give a result for the two-level variant of equal order interpolation, i.e., we assume $(Y_h, D_{2h}) = (P_r, P_{r-1})$ with $r \geq 1$.

Theorem 6 ([38]). *Let $\nu \leq h_M/r^2$ and let the stabilization parameters be chosen as $\tau_M \sim h_M/r^2$, $\mu_M \sim h_M/r^2$, and $\alpha_M \sim h_M/r^2$. Then there is a constant $C = C(\beta_1) > 0$ independent of h such that for $l \leq r$*

$$|||(u-u_h, p-p_h)|||_{OSE} \leq C(\beta_1)\,\frac{h^{l+1/2}}{r^l}(\|u\|_{l+1} + \|p\|_{l+1}).$$

Compared with the interpolation error, this estimate is optimal with respect to h but in general not with respect to r.

4.4 LPS on Anisotropic Meshes

In Sect. 2.6 we discussed the convergence properties of the LPS on layer-adapted meshes. Unfortunately, no precise information is known regarding how the derivatives of the solution of the Oseen problem behave in different parts of the domain. Thus, an important ingredient for the construction of layer-adapted meshes is missing. Nevertheless, highly anisotropic meshes are often used to resolve layers. In [40] an extension of the LPS has been proposed which uses different scalings for the fluctuations of the derivatives in x and y direction. For the two-level approach with equal-order interpolation, i.e., $(V_h, Q_h, D_{2h}) = ((Q_1)^d, Q_1, (Q_0)^d)$, optimal anisotropic error estimates have been established.

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