# Antisymmetric Aspects of a Perturbed Channel Flow

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**Abstract** This paper aims at studying steady laminar flows of incompressible newtonian fluids in channels at high Reynolds numbers when wall deformations can lead to separation. Thanks to the use of generalized asymptotic expansions, cases are examined for which linearized Euler equations are a good approximation in the core flow. The extraction of the antisymmetric part of the problem leads to a new and promising approach of the flow structure understanding. Comparisons with Navier–Stokes solutions demonstrate the relevance of the proposed approach.

## 1 Introduction

We consider a steady, two-dimensional, incompressible, laminar flow in a channel at high Reynolds numbers. When the walls are parallel the fully developed flow, Poiseuille's flow, constitutes the reference flow. The channel geometry is perturbed by wall deformations, troughs or bumps, which can be sufficiently severe to induce flow separation.

Here, the flow is analyzed by using the Successive Complementary Expansion Method [1], SCEM, in which we seek a Uniformly Valid Approximation, UVA, based on generalized asymptotic expansions.

In the study of high Reynolds number flows, the first idea is to consider Euler equations formally obtained from Navier–Stokes equations when the Reynolds number tends to infinity. Then, an asymptotic analysis can be applied and it is tempting to call for a hierarchical process. The first step is to solve the Euler equations. In the vicinity of singular zones, near the walls or in the wakes, the second step consists in trying to correct the first approximation by a boundary layer analysis.

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However, in many problems involving a strong coupling, this type of hierarchical approach is known not to be possible. Excluding a multi-layer approach of triple deck type [4, 5], which introduces very restrictive hypotheses on the scales, a possibility is to use generalized asymptotic expansions. According to this method, the small parameters of the problem can be included in the functions which form the expansions. This idea is very different because the small parameters are not considered as tending towards zero but are only small. Thanks to the generalized expansions, the effects of the eulerian region on the boundary layer region and the reciprocal effects are considered simultaneously and not hierarchically. Moreover, the construction of a UVA does not require any matching principle, only the boundary conditions of the problem are used.

After the formulation of the problem (Sect. 2), a direct analysis (Sect. 3) with small wall deformations shows that the Navier–Stokes equations reduce to a coupled system consisting of generalized boundary layer equations *uniformly valid in the whole flow* – the so-called field equations – and linearized Euler equations – the so-called core equations. A deeper study is performed by separating geometrically the symmetric and antisymmetric parts (Sect. 4). The analysis of the flow enables us to improve the usual asymptotic hypotheses and to consider original configurations. Comparisons of the evolution of the skin-friction coefficient with Navier–Stokes solutions show the relevance of the proposed approach (Sect. 5).

## 2 Formulation of the Problem

The Navier–Stokes equations are written in nondimensional form in an orthogonal axis-system $(x, y)$

$$\frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{V}}{\partial y} = 0, \tag{1a}$$

$$\mathcal{U}\frac{\partial \mathcal{U}}{\partial x} + \mathcal{V}\frac{\partial \mathcal{U}}{\partial y} = -\frac{\partial \mathcal{P}}{\partial x} + \frac{1}{\mathcal{R}}\left(\frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2}\right), \tag{1b}$$

$$\mathcal{U}\frac{\partial \mathcal{V}}{\partial x} + \mathcal{V}\frac{\partial \mathcal{V}}{\partial y} = -\frac{\partial \mathcal{P}}{\partial y} + \frac{1}{\mathcal{R}}\left(\frac{\partial^2 \mathcal{V}}{\partial x^2} + \frac{\partial^2 \mathcal{V}}{\partial y^2}\right). \tag{1c}$$

with $\mathcal{R}$ denoting the Reynolds number based on the width of the non-perturbed channel and a reference velocity such that the basic plane Poiseuille flow is

$$u_0 = \frac{1}{4} - y^2,\ v_0 = 0,\ p_0 = -\frac{2x}{\mathcal{R}} + p_c. \tag{2}$$

where $p_c$ is an arbitrary constant. The channel is perturbed by indentations of the lower and upper walls

$$y_l = -\frac{1}{2} + F(x, \varepsilon),\ y_u = \frac{1}{2} - G(x, \varepsilon), \tag{3}$$

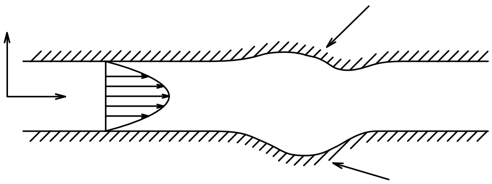

**Fig. 1** Flow in a two-dimensional channel with deformed walls. In this figure, all quantities are dimensionless

where $\varepsilon$ is a parameter (Fig. 1). At high Reynolds number, the reduced equations obtained formally by taking their limit when the Reynolds number goes to infinity are of first order. A singular perturbation problem arises.

## 3 Direct Analysis

To go further, it is usual [4, 5] to consider small wall perturbations leading to assumption $(H1)$

$$(H1) : F = \varepsilon f, \; G = \varepsilon g. \tag{4}$$

A perturbation is said to be significant when flow separation is possible. To translate this, it is required that, in boundary layers of thickness $\varepsilon$, the perturbation of the longitudinal velocity is of the same order as $u_0$, i.e. of order $\mathrm{O}(\varepsilon)$. Thus, according to SCEM, we are seeking a UVA of the form

$$\mathcal{U} = u_0(y) + \varepsilon\hat{u}(x, y, \varepsilon) + \cdots = u(x, y, \varepsilon) + \cdots , \tag{5a}$$

$$\mathcal{V} = \varepsilon\hat{v}(x, y, \varepsilon) + \cdots = v(x, y, \varepsilon) + \cdots , \tag{5b}$$

$$\mathcal{P} = p_0(x) + \varepsilon\hat{p}(x, y, \varepsilon) + \cdots = p(x, y, \varepsilon) + \cdots . \tag{5c}$$

It must be noted that $\hat{u}$, $\hat{v}$, $\hat{p}$ are functions not only of $x$ and $y$ but also of $\varepsilon$. Expansions (5a–5c) are said to be generalized to underline the difference with regular expansions in which $\hat{u}$, $\hat{v}$, $\hat{p}$ would not depend on $\varepsilon$. An asymptotic expansion is not necessarily based on regular expansions and it has been shown that generalized expansions are more powerful for certain boundary layer problems [1].

In the *whole flow field*, Navier–Stokes equations reduce to [1]

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0, \tag{6a}$$

$$u_0 \frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\mathrm{d}u_0}{\mathrm{d}y} + \varepsilon\left(\hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial \hat{u}}{\partial y}\right) = -\frac{\partial \hat{p}_1}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 \hat{u}}{\partial y^2}, \tag{6b}$$

where the index “1” denotes the characteristics of the flow perturbation in the core. As shown in [1], it must be noted that in the streamwise momentum equation, $\frac{\partial \hat{p}}{\partial x}$ is replaced by $\frac{\partial \hat{p}_1}{\partial x}$. Equations (6a–6b) have the same form as the standard boundary layer equations but $\hat{p}_1(x, y)$ is a solution of core flow equations given below. In the core, (6a–6b) reduce to the core flow equations up to negligible terms and therefore (6a–6b) *are valid in the whole field.*

The global interactive boundary layer model described by (6a–6b) and the core flow equations is the best approximation of Navier–Stokes model we can propose but it is not easy to solve. Fortunately, it can be shown that the core flow (Euler) equations can be linearized and the solution of the resulting model is much easier [2]. Thus, the field equations are structurally non-linear whereas the core flow equations are linear. With notations defined by (5a–5c), the field and core flow equations can now be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p_1}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial y^2}, \tag{7b}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{8a}$$

$$u_0\frac{\partial u_1}{\partial x} + v_1\frac{\mathrm{d}u_0}{\mathrm{d}y} = -\frac{\partial}{\partial x}\left(p_1 - p_0\right), \tag{8b}$$

$$u_0\frac{\partial v_1}{\partial y} = -\frac{\partial}{\partial y}\left(p_1 - p_0\right). \tag{8c}$$

In the above equations index “1” refers to quantities satisfying the core flow equations. From (8a–8c), it is found that $v_1$ is solution of Poisson’s equation

$$u_0\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right) = v_1\frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2}, \tag{9}$$

and the $x$-component of the pressure gradient required to solve the generalized boundary layer equations is given by (8b) in which the continuity equation (8a) is taken into account

$$-u_0\frac{\partial v_1}{\partial y} + v_1\frac{\mathrm{d}u_0}{\mathrm{d}y} = -\frac{\partial}{\partial x}\left(p_1 - p_0\right). \tag{10}$$

It can be shown that (9) associated to (10) gives the $y$-momentum equation (8c) if the perturbations vanish at upstream infinity. This establishes the equivalence between (8a–8c) and (9–10).

To sum up, the problem to solve comprises (7a–7b), (9) and (10). At the walls, the boundary conditions are

$$y = y_\ell \text{ and } y = y_u : u = 0,\ v = 0, \tag{11}$$

and the coupling between the core flow equations and the generalized boundary layer equations is expressed by identifying $u, v$ and $u_1, v_1$ in the core

$$(u, v) \to (u_1, v_1). \tag{12}$$

The model presented above belongs to a class of strong coupling method since there is no hierarchy between the boundary layer equations and the core flow equations. The triple deck theory, or more precisely its equivalent for channel flows as developed by Smith [4, 5], belongs also to this class of strong coupling models. In fact, Smith's model is included in the present model since the expansions are regular whereas in the present model the expansions are generalized. It is interesting to note that the first approximation of Smith's model for $v_1$ is symmetric with respect to $y$ and corresponds to a geometrically antisymmetric problem. In the core, Smith's model gives

$$v_1 = -u_0(y)\frac{\mathrm{d}A(x)}{\mathrm{d}x}, \tag{13}$$

where $A$ is defined as the displacement function. It must be noted that (13) is an eigensolution of (10) but not of (9). This remark leads us to try to separate as far as possible the symmetric and antisymmetric problems which leads, as we will see, to a new approach of the asymptotic problem. The issue of asymmetry has been approached earlier by Lagrée et al. [3].

## 4 Influence of Asymmetry

The analysis starts from (1a–1c) in which we introduce the transformation

$$X = x,\ Y = y - \nu H(\mu x, \varepsilon),\ U = \mathcal{U},\ V = \mathcal{V} - \mathcal{U}\nu\frac{\mathrm{d}H}{\mathrm{d}x},\ P = \mathcal{P}, \tag{14}$$

where $\nu$ and $\mu$ are order functions such that $\nu \preceq 1$ and $\mu \prec 1$. We have

$$H = \mathrm{O_S}(1) \text{ and } \frac{\mathrm{d}^n H}{\mathrm{d}x^n} = \mathrm{O}(\mu^n). \tag{15}$$

where $\mathrm{O_S}$ means "is of strict order of" whereas O means "is at most of order of" [1]. With these hypotheses, Navier–Stokes equations become

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{16a}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{\mathcal{R}}\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + \mathrm{O}(\nu\mu), \tag{16b}$$

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{\partial P}{\partial Y} + \frac{1}{\mathcal{R}}\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) + \mathrm{O}(\nu\mu). \tag{16c}$$

If we set

$$E = \frac{F+G}{2},\ H = \frac{F-G}{2}, \tag{17}$$

the problem is geometrically symmetrized. Note that the channel is deformed symmetrically when $H = 0$. The wall conditions then become

$$Y = Y_\ell = -\frac{1}{2} + E \text{ and } Y = Y_u = \frac{1}{2} - E : U = 0,\ V = 0. \tag{18}$$

Moreover, for small $\mu$, the basic flow corresponding to $E = 0$ is

$$U_0 = \frac{1}{4} - Y^2,\ V_0 = 0,\ P_0 = -2\frac{X}{\mathcal{R}} + P_c, \tag{19}$$

where $P_c$ is an arbitrary constant. We introduce assumption $(H2)$

$$(H2) : E = \varepsilon e,\ \nu\mu^2 \preceq \varepsilon. \tag{20}$$

With $(H2)$, the complete system to solve comprises the field equations

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{21a}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P_1}{\partial X} + \frac{1}{\mathcal{R}}\frac{\partial^2 U}{\partial Y^2}, \tag{21b}$$

where $P$ is replaced by $P_1$ and the core flow equations which can be linearized

$$\frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} = 0, \tag{22a}$$

$$U_0\frac{\partial U_1}{\partial X} + V_1\frac{\mathrm{d}U_0}{\partial Y} = -\frac{\partial}{\partial X}\left(P_1 - P_0\right), \tag{22b}$$

$$U_0\frac{\partial V_1}{\partial X} + \nu\frac{\mathrm{d}^2 H}{\mathrm{d}X^2}U_0^2 = -\frac{\partial}{\partial Y}\left(P_1 - P_0\right). \tag{22c}$$

This system is solved with (18) and the coupling condition in the core

$$V \to V_1. \tag{23}$$

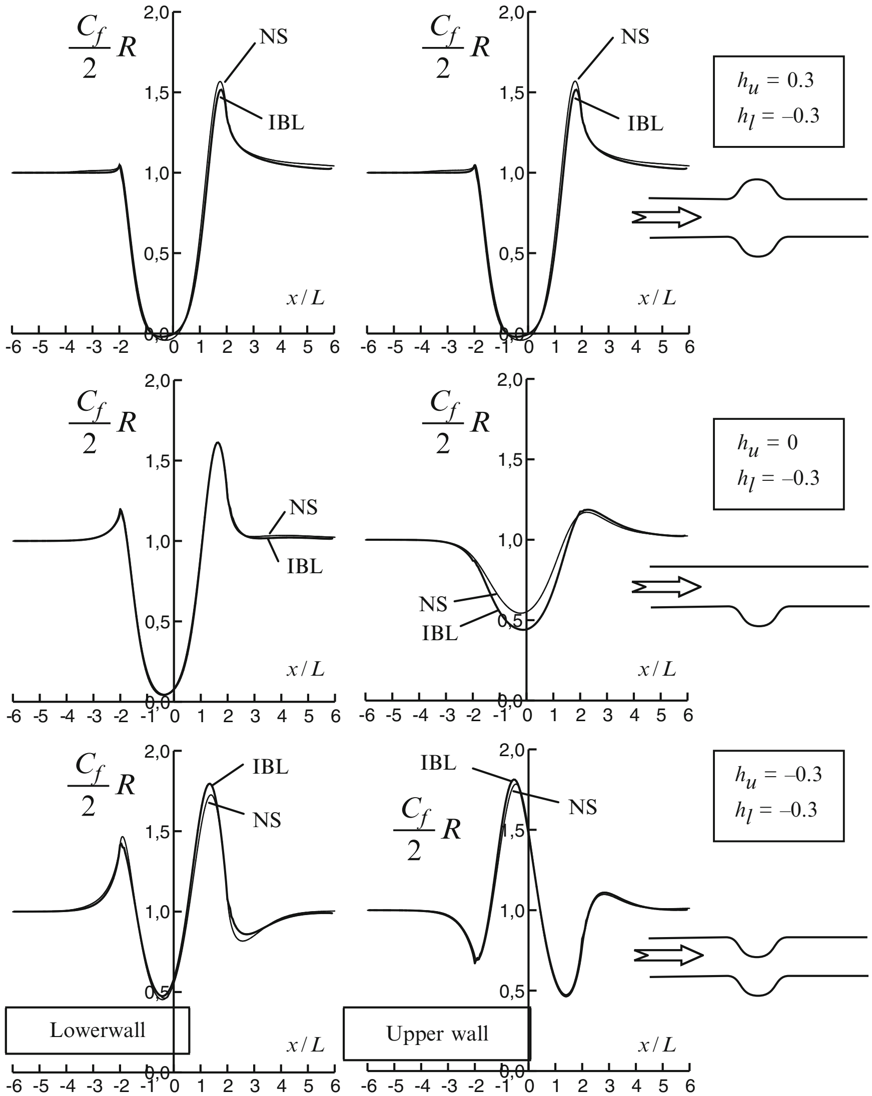


**Fig. 2** Flow produced by a trough in the lower wall and different upper wall deformations. $\mathcal{R} = 1{,}000$. NS-Navier–Stokes results, IBL-Interactive Boundary Layer results

## 5 Results and Conclusions

To assess the validity of the Interactive Boundary Layer method, IBL, it is chosen to examine the evolution of the skin-friction coefficient which is a very sensitive flow feature, $C_f = \frac{2}{\mathcal{R}}\tau_w$ where $\tau_w$ is the reduced wall shear stress. Details on the

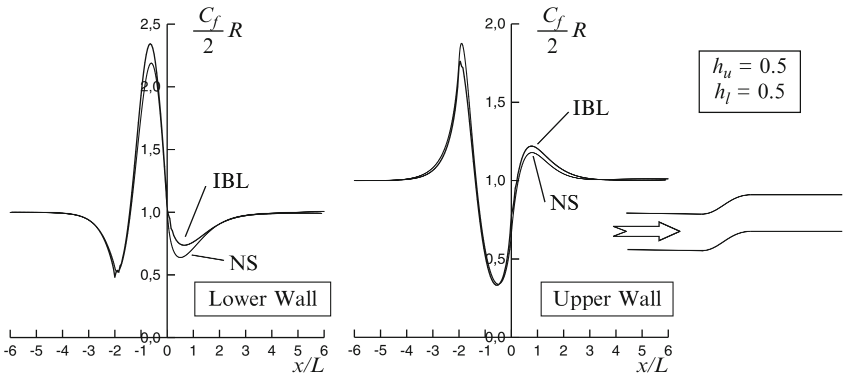


**Fig. 3** Flow produced by a channel bend. $\mathcal{R} = \mathbf{1{,}000}$. NS-Navier–Stokes results; IBL = Interactive Boundary Layer results

numerical procedure can be found elsewhere [2]. The Navier–Stokes equations are solved with the commercial code FLUENT. Different cases are calculated in which the walls are deformed in a domain $x_1 \leq x \leq x_2$

$$F = \frac{h_l}{2}\left(1 + \cos\frac{2\pi x}{L}\right); G = -\frac{h_u}{2}\left(1 + \cos\frac{2\pi x}{L}\right); L = 4. \tag{24}$$

For all cases, the Reynolds number is $\mathcal{R} = 1{,}000$.

At first, comparisons between IBL and Navier–Stokes results are given in Fig. 2. The lower wall is deformed by a trough located in the domain $-2 \leq x \leq 2$ with $h_l = -0.3$. The upper wall is deformed in the same domain $-2 \leq x \leq 2$ but different upper wall shapes have been investigated between the symmetric case ($h_u = 0.3$) and the antisymmetric case ($h_u = -0.3$). Even though the amplitude of the wall deformation is not really small as required by the theory, an excellent agreement with Navier–Stokes results is observed.

The IBL model enables us also to treat original problems. In the case of a bend, when the channel does not recover its initial position at the downstream end, the usual techniques of small perturbations do not work any longer. As an example, the walls are deformed in the domain $-2 \leq x \leq 0$ with $h_l = 0.5$ and $h_u = 0.5$; for $x > 0$, we have $y_l = 0$, $y_u = 1$ so that the channel axis is displaced from $y = 0$ upstream to $y = 0.5$ downstream. In this case again, a good agreement with Navier–Stokes results is observed (Fig. 3). This shows that $\frac{\mathrm{d}H}{\mathrm{d}x}$ which characterizes the influence of the antisymmetric part of the wall deformation plays an important role in the definition of what could be the small parameter of the problem.

Other non usual cases can be treated by this method, for example dilated or constricted channels, ... The IBL calculations are much faster than Navier–Stokes calculations and, in addition, the new asymptotic analysis helps us to understand the flow structure. Moreover, this step is necessary to approach the important problem of separation control.

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