Distributed and Boundary Control of Singularly Perturbed Advection–Diffusion–Reaction Problems

G. Lube and B. Tews

Abstract We consider the numerical analysis of quadratic optimal control problems with distributed and Robin boundary control governed by an elliptic problem. The Galerkin discretization is stabilized via the local projection approach which leads to a symmetric discrete optimality system. In the singularly perturbed case, the Robin control at parts of the boundary can be seen as regularized Dirichlet control.

1 Introduction

Let $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ be a bounded polyhedral domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_R \cup \Gamma_D}$, $\Gamma_D \cap \Gamma_R = \emptyset$ and outer normal unit vector $\mathbf{n}$. We address some aspects of the numerical analysis of the quadratic optimal control problem

$$\begin{aligned} \text{Minimize } \; J(u, q_\Omega, q_\Gamma) := & \frac{1}{2}\lambda_\Omega \|u - \tilde{u}_\Omega\|^2_{L^2(\Omega)} + \frac{1}{2}\lambda_\Gamma \|u - \tilde{u}_\Gamma\|^2_{L^2(\Gamma_R)} \\ & + \frac{1}{2}\alpha_\Omega \|q_\Omega\|^2_{L^2(\Omega)} + \frac{1}{2}\alpha_\Gamma \|q_\Gamma\|^2_{L^2(\Gamma_R)} \end{aligned} \tag{1}$$

where $(u, q_\Omega, q_\Gamma) \in V \times Q_\Omega \times Q_\Gamma := \{v \in H^1(\Omega) : \, u|_{\Gamma_D} = 0\} \times L^2(\Omega) \times L^2(\Gamma_R)$ solves the mixed boundary value problem of advection-diffusion-reaction type

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \sigma u &= f + q_\Omega \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_D, \qquad \varepsilon\nabla u\cdot\mathbf{n} + \beta u &= g + q_\Gamma \quad \text{on } \Gamma_R. \end{aligned} \tag{2}$$

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We assume that $\varepsilon > 0$ and $\sigma \geq 0$ are constants and that the advective field $\mathbf{b}$ is divergence-free. In (1), the desired states are $\tilde{u}_\Omega$ and $\tilde{u}_\Gamma$. The constants $\lambda_\Omega, \lambda_\Gamma \geq 0$ with $\lambda_\Omega^2 + \lambda_\Gamma^2 > 0$ describe the weights of the distributed and boundary control in (1) whereas $\alpha_\Omega, \alpha_\Gamma \geq 0$ with $\alpha_\Omega^2 + \alpha_\Gamma^2 > 0$ serve as regularisation parameters. The state equation (2) describes the dependence of the state u on the control (q_Ω, q_Γ).

Problem (1)–(2) with $\Gamma_R = \emptyset$ has been considered in [3, 10] for the singularly perturbed case $0 < \epsilon \ll 1$, see also the references therein. Here one goal is to consider problem (1)–(2) simultaneously for distributed and (Robin) boundary control. Notably, for $0 < \epsilon \ll 1$, the Robin control can be seen as regularized Dirichlet control.

The Galerkin discretization is stabilized as in [3] via the local projection approach (LPS for short below) which leads to a symmetric optimality system. This implies that discretization and optimization commute as opposed to residual-based stabilization techniques. Another aim of the present paper is a more general LPS approach, including a two-level variant (as in [3]) and a one-level variant introduced in [9]. Let us emphasize two aspects of the analysis: (1) The regularity of the solution of problem (2) is taken into account by using Sobolev–Slobodeckij spaces and adapting the analysis of the LPS method. (2) The analysis is performed for shape regular meshes (as opposed to quasi-uniform meshes in [3]) which allows for (isotropic) mesh refinement at corners or edges of the domain and in boundary layers.

An outline of the paper is as follows: In Sect. 2, we address the solvability of problem (1)–(2). Then, in Sect. 3, we consider the finite element (FE) discretization of the optimality system whereas Sect. 4 presents its convergence properties. In Sects. 5 and 6, we address a numerical experiment and the interpretation of Robin control as regularized Dirichlet control. For full proofs we refer to [8].

Standard notations for Lebesgue and Sobolev spaces are used, e.g., the L^2-inner product and the L^2-norm in $G \subseteq \Omega$ are denoted by $(\cdot,\cdot)_G$ and $\|\cdot\|_{0,G}$.

2 Continuous Optimal Control Problem

Here we consider the optimality system for the continuous optimal control problem (1)–(2). To this goal, we first consider the solvability of the state equation (2) with $\tilde{f} := f + q_\Omega$ and $\tilde{g} := g + q_\Gamma$. The variational form of problem (2) reads:

$$\text{Find } u \in V \text{ such that } a(u,v) = f(v) \ \forall v \in V, \tag{3}$$
$$a(u,v) := \varepsilon(\nabla u, \nabla v)_\Omega + (\mathbf{b}\cdot\nabla u + \sigma u, v)_\Omega + (\beta u, v)_{\Gamma_R},$$
$$f(v) := (\tilde{f}, v)_\Omega + (\tilde{g}, v)_{\Gamma_R}.$$

Lemma 1. *There exists a unique solution $u \in H^1(\Omega)$ of problem (3) under the assumptions:*

i) $\mathbf{b} \in [L^\infty(\Omega)]^d$, $\quad \tilde{f} \in L^2(\Omega)$, $\quad \tilde{g} \in L^2(\Gamma_R)$, $\quad \beta \in L^\infty(\Gamma_R)$,
ii) $\varepsilon > 0$, $\sigma \geq 0$ *and* $\nabla\cdot\mathbf{b} = 0$ $\quad$ *a.e. in* Ω,

iii) $\tilde{\beta} := \beta + \frac{1}{2}(\mathbf{b}\cdot\mathbf{n}) \ge \beta_0 \ge 0, \quad \beta \ge 0$ *a.e. on* Γ_R,
iv) There holds: $(iv)_1 \quad \mu_{d-1}(\Gamma_D) > 0,$ *and/or* $(iv)_2 \quad \sigma > 0$ *or* $\beta_0 > 0$.

Moreover, the optimal control problem (1)–(2) has a unique solution $(\overline{u}, \overline{q}_\Omega, \overline{q}_\Gamma)$.

The proof can be found in [8], Lemma 2.1. Please note that the assumption $\beta \ge 0$ is not needed for this result, but it will be used later on in the analysis in Sect. 4.

In general, the solution of (3) is not in $W^{2,2}(\Omega)$. Let $\mathcal{S}$ be the set of points (for $d = 2$) or edges (for $d = 3$) which subdivide the polyhedral boundary $\partial\Omega$ into smooth disjoint connected components. The weighted Sobolev space $V_\delta^{k,2}(\Omega)$ denotes the closure of $C^\infty(\Omega)$ w.r.t.

$$\|v\|_{V_\delta^{k,2}(\Omega)} = \Big(\sum_{|\alpha|\le k} \int_\Omega r^{2(\delta-k+|\alpha|)} |D^\alpha u|^2 \, dx \Big)^{\frac{1}{2}}$$

where $r = r(x) = \text{dist}(x, \mathcal{S})$, $\delta \in \mathbb{R}$, and $k \in \mathbb{N}$. The parameter δ is defined via eigenvalues of eigenvalue problems (in local coordinate systems at parts of the set $\mathcal{S}$) associated with problem (3). As it is not the goal here to give sufficient conditions for the solution of problem (3) to belong to $V_\delta^{k,2}(\Omega)$, we refer to [6]. Moreover, we do not intend to consider graded FE meshes in the neighborhood of the set $\mathcal{S}$ although the forthcoming numerical analysis allows such kind of refinement. For such approach to optimal control problems, see [1].

Here we consider on a subdomain $G \subseteq \Omega$ the Sobolev–Slobodeckij spaces

$$W^{k+\lambda,2}(G) := \Big\{ v \in W^{k,2}(G) : \ \|u\|_{k+\lambda,2,G} < \infty \Big\}, \qquad k \in \mathbb{N}_0, \ \lambda \in [0,1)$$
$$\|u\|_{k+\lambda,2,G} := \Big(\|u\|_{k,2,G}^2 + \sum_{|\alpha|=k} \int_G \int_G \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{d+2\lambda}} dx\, dy \Big)^{\frac{1}{2}}.$$

The spaces $W^{k+\lambda,2}(\Gamma_R)$ are defined in a similar way.

Remark 1. The embeddings $V_\delta^{2,2}(\Omega) \subset W^{\frac{d}{2}+\kappa,2}(\Omega) \subset C(\overline{\Omega})$ are valid for $\delta < 2 - \frac{d}{2} + \kappa$ with $\kappa > 0$, cf. [6]. In particular, for the case $\partial\Omega = \Gamma_D$ in polyhedral domains, the conditions $\delta \le \frac{1}{2} + \kappa, \kappa > 0$ are sufficient.

As problem (3) is uniquely solvable, we define the affine linear solution operator $S : L^2(\Omega) \times L^2(\Gamma_R) \to V, \ u = S(q_\Omega + f, q_\Gamma + g)$. Due to the linearity of (2) we can split S in its linear and affine linear part. Inserting $u = S(q_\Omega + f, q_\Gamma + g) = S(q_\Omega, q_\Gamma) + S(f, g)$ in (1), we obtain (with trace operator γ) and the definitions $u_\Omega := \tilde{u}_\Omega - S(f,g)$ and $u_\Gamma := \tilde{u}_\Gamma - \gamma \circ S(f,g)$ the reduced cost functional:

$$\begin{aligned} j(q_\Omega, q_\Gamma) = J\,(q_\Omega, q_\Gamma, S(q_\Omega, q_\Gamma)) &= \frac{1}{2}\lambda_\Omega \|S(q_\Omega, q_\Gamma) - u_\Omega\|_{0,\Omega}^2 \\ &+ \frac{1}{2}\lambda_\Gamma \|\gamma \circ S(q_\Omega, q_\Gamma) - u_\Gamma\|_{0,\Gamma_R}^2 + \frac{1}{2}\alpha_\Omega \|q_\Omega\|_{0,\Omega}^2 \\ &+ \frac{1}{2}\alpha_\Gamma \|q_\Gamma\|_{0,\Gamma_R}^2. \end{aligned} \tag{4}$$

Now the reduced optimization problem reads

$$\text{Minimize} \quad j(q_\Omega, q_\Gamma), \quad (q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma. \tag{5}$$

The reduced cost functional j is continuously differentiable. In order to formulate the optimality conditions for problem (5), we define the associated adjoint state $\overline{p} \in V$ to $(\overline{q}_\Omega, \overline{q}_\Gamma)$ as the solution of

$$\text{Find } p \in V: \quad a_{adj}(p, v) = \lambda_\Omega(\overline{u} - u_\Omega, v)_\Omega + \lambda_\Gamma(\overline{u} - u_\Gamma, v)_{\Gamma_R} \quad \forall v \in V, \tag{6}$$
$$a_{adj}(p, v) := \epsilon(\nabla p, \nabla v)_\Omega - (\mathbf{b} \cdot \nabla p, v)_\Omega + \sigma(p, v)_\Omega + ((\beta + \mathbf{b} \cdot \mathbf{n})p, v)_{\Gamma_R}.$$

The necessary (and sufficient) optimality conditions read

$$D_{q_\Omega} j(\overline{q}_\Omega, \overline{q}_\Gamma) \cdot (k_\Omega - \overline{q}_\Omega) = (\alpha_\Omega \overline{q}_\Omega + \overline{p}, k_\Omega - \overline{q}_\Omega)_\Omega = 0, \quad \forall k_\Omega \in Q_\Omega, \tag{7}$$
$$D_{q_\Gamma} j(\overline{q}_\Omega, \overline{q}_\Gamma) \cdot (k_\Gamma - \overline{q}_\Gamma) = (\alpha_\Gamma \overline{q}_\Gamma + \gamma \circ \overline{p}, k_\Gamma - \overline{q}_\Gamma)_{\Gamma_R} = 0, \quad \forall k_\Gamma \in Q_\Gamma, \tag{8}$$

leading to

$$\alpha_\Omega \overline{q}_\Omega + \overline{p} = 0, \text{ in } \Omega \qquad \alpha_\Gamma \overline{q}_\Gamma + \gamma \circ \overline{p} = 0 \text{ on } \Gamma_R. \tag{9}$$

The optimality system (KKT-system) for problem (1)–(2) is formed by (9) together with the state problem (3) and the adjoint state problem (6). The second order derivatives of $j(q_\Omega, q_\Gamma)$ do not depend on (q_Ω, q_Γ) and are positive definite.

As already said, the solution of (1)–(2) is in general not arbitrarily smooth.

Assumption 1: The optimal solution $(\overline{u}, \overline{p}, \overline{q}_\Omega, \overline{q}_\Gamma)$ of the optimal control problem (1)–(2) belongs to $[W^{1+\lambda,2}(\Omega)]^3 \times W^{\frac{1}{2}+\lambda,2}(\Gamma_R)$ with $1 + \lambda > \frac{d}{2}$.

Assume that $\alpha_\Omega, \alpha_\Gamma > 0$. Then Assumption 1 is valid if the solution u of (3) belongs to $W^{1+\lambda,2}(\Omega)$, $1 + \lambda > d/2$, eventually for sufficiently smooth data $\tilde{f}, \tilde{g}, \beta$. For sufficient conditions, see Remark 1. Then the same statement is valid for the solution p of (6) for sufficiently smooth data u_Ω, u_Γ. Moreover, the regularity of $\overline{q}_\Omega$ and $\overline{q}_\Gamma$ follows via (9). Finally, we remark that Assumption 1 allows later on Lagrangian interpolation of the solution.

3 Stabilized Discrete Optimality System

Here we introduce the discretized optimal control problem to (1)–(2). A more general approach to the discretization as in [3] is applied by considering shape-regular FE meshes and a more flexible stabilization concept.

Consider a family of shape-regular, admissible decompositions $\mathcal{T}_h$ of Ω into d-dimensional simplices, quadrilaterals ($d = 2$) or hexahedra ($d = 3$). Let h_T be the diameter of a cell $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$. Assume that, for each $T \in \mathcal{T}_h$, there exists an affine mapping $F_T : \hat{T} \to T$ which maps the reference element $\hat{T}$ onto T. This quite restrictive assumption for quadrilaterals/ hexahedra can be weakened to asymptotically affine linear mappings [2]. Let e_h denote the set of element faces (for $d = 3$) or element edges (for $d = 2$) induced by $\mathcal{T}_h$ on $\partial\Omega$. Moreover, we assume that the Robin part Γ_R of the boundary is exactly triangulated by e_h.

Set $P_{\mathcal{T}_h} = \{v_h \in L^2(\Omega) : v_h \circ F_T \in \mathbb{P}_1(\hat{T}), T \in \mathcal{T}_h\}$ within $\mathbb{P}_1(\hat{T})$, the space of complete linear polynomials on $\hat{T}$, and $R_{\mathcal{T}_h} = \{v_h \in L^2(\Omega) : v_h \circ F_T \in \mathbb{Q}_1(\hat{T}), T \in \mathcal{T}_h\}$ within $\mathbb{Q}_1(\hat{T})$, the space of all polynomials on $\hat{T}$ with maximal first degree in each coordinate direction. The state space V is approximated by a FE space $V_h \supset P_{\mathcal{T}_h} \cap V$ or $V_h \supset R_{\mathcal{T}_h} \cap V$. Similarly, let $Q_{h,\Omega} \subset H^1(\Omega)$ be a FE space for the control variable and $Q_{h,\Gamma} = Q_{h,\Omega}|_{\Gamma_R}$ its restriction to Γ_R.

The basic Galerkin discretization of the state problem (3) reads:

$$\text{find } u_h \in V_h \text{ such that} \quad a(u_h, v_h) = f(v_h), \qquad \forall v_h \in V_h. \tag{10}$$

The solution u_h of (10) may suffer from spurious oscillations. As a remedy, we consider the local projection stabilization (LPS) approach which results in a symmetric discrete optimality system. LPS methods split the discrete function spaces into small and large scales and add stabilization terms of diffusion-type acting only on the small scales. There are basically a two- and a one-level variant (indicated by $\mathcal{M}_h = \mathcal{T}_{2h}$ and $\mathcal{M}_h = \mathcal{T}_h$, respectively).

The *two-level variant* starts from the given space $V_h = P_{\mathcal{T}_h} \cap V$ or $V_h = R_{\mathcal{T}_h} \cap V$ for simplicial or hexahedral elements. The large scales are determined by means of a coarse, non-overlapping and shape-regular mesh $\mathcal{M}_h = \{M_i\}_{i\in I}$ which is constructed by coarsening $\mathcal{T}_h$ s.t. each $M \in \mathcal{M}_h$ with diameter h_M is the union of neighboring cells $T \in \mathcal{T}_h$. (A more practical approach is to start from the coarse grid $\mathcal{M}_h$ and to construct $\mathcal{T}_h$ by an appropriate refinement, see [4], Sect. 4.) Moreover, we assume:

$$\exists\, C > 0 : \quad h_M \leq C h_T, \quad \forall T \in \mathcal{T}_h,\ M \in \mathcal{M}_h \text{ with } T \subset M. \tag{11}$$

We introduce a discontinuous FE space $D_h \subset L^2(\Omega)$ of piecewise constant functions on $\mathcal{M}_h$ and its restriction $D_h(M) := \{v_h|_M \,;\, v_h \in D_h\}$ to $M \in \mathcal{M}_h$. The next ingredient is the local L^2-projection $\pi_M : L^2(M) \to D_h(M)$ which defines the global projection $\pi_h : L^2(\Omega) \to D_h$ by $(\pi_h v)|_M := \pi_M(v|_M)$ for all $M \in \mathcal{M}_h$. The fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ is defined by $\kappa_h := id - \pi_h$.

The *one-level variant* starts from the given discontinuous FE space D_h of piecewise constant functions on $\mathcal{M}_h = \mathcal{T}_h$ and uses an appropriate FE space V_h on $\mathcal{T}_h$. For simplicial elements, define

$$P_1^{bub}(\hat{T}) = P_1(\hat{T}) + \hat{b} \cdot P_0(\hat{T}), \quad \hat{b}(\hat{x}) := (d+1)^{d+1}\hat{\lambda}_1(\hat{x}) \cdot \ldots \cdot \hat{\lambda}_{d+1}(\hat{x})$$

with the barycentric coordinates $\hat{\lambda}_1, \ldots, \hat{\lambda}_{d+1}$. The enriched space is defined as

$$V_h = \{v \in H^1(\Omega) \cap V : \ v|_T \circ F_T \in P_1^{bub}(\hat{T})\ \forall T \in \mathcal{T}_h\}.$$

A similar construction is given in Sect. 4 of [9] for hexahedral elements. Then the same framework as in the two-level approach can be used by setting $\mathcal{M}_h = \mathcal{T}_h$.

For both variants, the stabilized discrete formulation reads: find $u_h \in V_h$ such that

$$a_{lps}(u_h, v_h) := a(u_h, v_h) + s_h(u_h, v_h) = f(v_h), \qquad \forall v_h \in V_h, \tag{12}$$

$$s_h(u_h, v_h) := \sum_{M \in \mathcal{M}_h} \tau_M (\kappa_h (\mathbf{b} \cdot \nabla u_h), \kappa_h (\mathbf{b} \cdot \nabla v_h))_M. \tag{13}$$

The stabilization s_h with parameters $\tau_M \geq 0$ acts solely on the small scales. Another variant uses $\tilde{s}_h(u_h, v_h) = \sum_M \tilde{\tau}_M (\tilde{\kappa}_h(\nabla u_h), \tilde{\kappa}_h(\nabla v_h))_M$ instead of $s_h(\cdot,\cdot)$. Here $\tilde{\kappa}_h$ denotes a vector-valued version of the fluctuation operator κ_h.

For a discussion of "pro's and con's" of the two variants, we refer to [4].

The discretized control problem associated with (1)–(2) reads as follows:

$$\min\ J(u_h, q_{h,\Omega}, q_{h,\Gamma}), (u_h, q_{h,\Omega}, q_{h,\Gamma}) \in V_h \times Q_{h,\Omega} \times Q_{h,\Gamma}, \tag{14}$$

$$a_{lps}(u_h, v_h) = (f + q_{h,\Omega}, v_h)_\Omega + (g + q_{h,\Gamma}, v_h)_{\Gamma_R}, \forall v_h \in V_h. \tag{15}$$

Problem (14)–(15) has a unique solution $(\overline{u}_h, \overline{q}_{h,\Omega}, \overline{q}_{h,\Gamma})$ which allows us to define the discrete solution operator $S_h : Q_\Omega \times Q_\Gamma \to V_h$ by

$$a_{lps}(S_h(q_{h,\Omega}, q_{h,\Gamma}), v_h) = (f + q_{h,\Omega}, v_h)_\Omega + (g + q_{h,\Gamma}, v_h)_{\Gamma_R} \quad \forall v_h \in V_h$$

and the discrete reduced cost functional as $j_h(q_{h,\Omega}, q_{h,\Gamma}) = J(S_h(q_{h,\Omega}, q_{h,\Gamma}), q_{h,\Omega}, q_{h,\Gamma})$. The necessary (and here also sufficient) optimality conditions read

$$\alpha_\Omega \overline{q}_{h,\Omega} + \overline{p}_h \;=\; 0, \qquad \alpha_\Gamma \overline{q}_{h,\Gamma} + \gamma \circ \overline{p}_h \;=\; 0.$$

Here the discrete adjoint state $p_h \in V_h$ solves the discrete adjoint state problem

$$a_{lps}(v_h, p_h) = \lambda_\Omega(\overline{u}_h - u_\Omega, v_h)_\Omega + \lambda_\Gamma(\overline{u}_h - u_\Gamma, v_h)_{\Gamma_R}. \tag{16}$$

where $\overline{u}_h = S_h(q_\Omega, q_\Gamma)$ is the discrete state according to (15).

Remark 2. The symmetry of the LPS term implies that the operations "optimize" and "discretize" commute, see [3].

4 A-Priori Error Analysis

Here we provide the error analysis for the optimal control problem (1)–(2). It turns out that additional assumptions for the LPS method are required.

Assumption 2: The fluctuation operator $\kappa_h = id - \pi_h$ has the property:

$$\exists\, C_\kappa > 0 \;:\; \|\kappa_h q\|_{0,M} \leq C_\kappa h_M^s |q|_{s,M}, \quad \forall q \in W^{s,2}(M),\ s \in [0,1],\ \forall M \in \mathcal{M}_h. \tag{17}$$

Remark 3. The original version of (17) in [9] only considers $s \in \{0, 1\}$.

Following [9], we construct an interpolation $j_O : V \to V_h$ such that the error $v - I_h v$ is L^2-orthogonal to D_h for all $v \in V$. The following assumption is valid for the discrete spaces discussed in the previous section and allows us to conserve standard approximation properties.

Assumption 3: There exists a constant $\beta_S > 0$ such that, for any $M \in \mathcal{M}_h$,

$$\inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M}\|q_h\|_{0,M}} \geq \beta_S > 0. \tag{18}$$

where $Y_h(M) := \{v_h|_M \ : \ v_h \in V_h, v_h = 0 \text{ on } \Omega \setminus M\}$.

Condition (18) implies that D_h must not be too rich. On the other hand, D_h must be rich enough to fulfil (17) .

The following result extends the proof in [9] to $\lambda \in \{0, 1\}$, see [8], Lemma 4.1.

Lemma 2. *Under Assumption 3 there exists an operator* $j_O : V \to V_h$ *such that*

$$(v - j_O v, q_h)_\Omega = 0, \ \ \forall q_h \in D_h, \forall v \in V, \tag{19}$$

and for all $M \in \mathcal{M}_h$*, for all* $E \in \mathrm{e}_h$*, and for* $v \in V \cap W^{1+\lambda,2}(\Omega)$ *with* $1 + \lambda > \frac{d}{2}$

$$\|v - j_O v\|_{0,M} + h_M |v - j_O v|_{1,M} + h_M^{\frac{1}{2}} \|v - j_O v\|_{0,E} \lesssim h_M^{1+\lambda} \|v\|_{1+\lambda,2,\omega(M)}. \tag{20}$$

The next goal is to derive error estimates for the state problems (15) and (16). First, the stability of the scheme will be given in the mesh-dependent norm

$$|||v||| := \left(\varepsilon |v|^2_{1,\Omega} + \sigma \|v\|^2_{0,\Omega} + \|\tilde{\beta}^{\frac{1}{2}} v\|^2_{0,\Gamma_R} + s_h(v,v)\right)^{\frac{1}{2}}, \qquad \forall v \in V.$$

Lemma 3. *The LPS schemes (15) and (16) have unique solutions.*

Proof. We consider, e.g., problem (15) with $v_h = u_h$. The application of the Cauchy–Schwarz inequality and the definition of the triple norm yields the a priori estimate

$$|||u_h||| \leq C_\Omega \|f + q_{h,\Omega}\|_{0,\Omega} + C_\Gamma \|g + q_{h,\Gamma}\|_{0,\Gamma_R}$$

with $C_\Omega := \min\{\sigma^{-\frac{1}{2}}; C_P \epsilon^{-\frac{1}{2}}\}$, $C_\Gamma := \min\{\beta_0^{-\frac{1}{2}}; C_P \epsilon^{-\frac{1}{2}}\}$ and Poincare constant C_P.

The following a priori estimates are based on the standard technique of combining stability and consistency results based on the previous auxiliary results. Here, and in the following Lemma, we fix some controls $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$ which will be later on, for the main theorem, chosen as the Lagrangian interpolants of the optimal controls $(\overline{q}_\Omega, \overline{q}_\Gamma)$.

Lemma 4. *For* $(q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma$*, let* $u = S(q_\Omega, q_\Gamma) \in V$ *be the solution of (2). For some* $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$*, let* $w_h = S_h(p_\Omega, p_\Gamma) \in V_h$ *be the solution of*

$$a_{lps}(w_h, v_h) = (f + p_\Omega, v_h)_\Omega + (g + p_\Gamma, v_h)_{\Gamma_R} \qquad \forall v_h \in V_h \tag{21}$$

with

$$\tau_M \sim h_M / \|\mathbf{b}\|_{[L^\infty(M)]^d}. \tag{22}$$

Then, under the assumptions of Lemma 1, there holds the a-priori error estimate

$$\begin{aligned} |||u - w_h||| \le{}& C_\Omega \|q_\Omega - p_\Omega\|_{0,\Omega} + C_\Gamma \|q_\Gamma - p_\Gamma\|_{0,\Gamma_R} \\ &+ C\Big(\sum_{M \in \mathcal{M}_h} h_M^{2\lambda+1} \Big\{ \frac{|\mathbf{b}\cdot\nabla u|^2_{\lambda,2,M}}{\|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M \|u\|^2_{1+\lambda,2,M} \Big\} \Big)^{\frac{1}{2}} \end{aligned} \tag{23}$$

with constants C_M and C_Γ as in the proof of Lemma 3 and

$$C_M := \varepsilon h_M^{-1} + \sigma h_M + \|\mathbf{b}\|_{[L^\infty(M)]^d} + \|\beta\|_{L^\infty(\partial M \cap \Gamma_R)} + \|\mathbf{b}\cdot\mathbf{n}\|_{L^\infty(\partial M \cap \Gamma_R)}.$$

For a full proof of Lemma 4, see [8], Lemma 4.3. Similarly, we obtain an a-priori error estimate for the adjoint problem (16) where $|||u - w_h|||$ in (23) can be further estimated via Lemma 4. A full proof of Lemma 5 is given in [8], Lemma 4.4.

Lemma 5. *For $(q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma$, let $p \in V$ be the solution of the adjoint state problem (6) and for some $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$, let $y_h \in V_h$ be the adjoint discrete solution. Then, there holds the a-priori error estimate*

$$\begin{aligned} |||p - y_h||| \le{}& (C_\Omega^2 \lambda_\Omega + C_\Gamma^2 \lambda_\Gamma) |||u - w_h||| \\ &+ C\Big(\sum_M h_M^{2\lambda+1} \Big\{ \frac{|\mathbf{b}\cdot\nabla p|^2_{\lambda,2,M}}{\|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M \|p\|^2_{1+\lambda,2,M} \Big\} \Big)^{\frac{1}{2}} \end{aligned}$$

with τ_M as in (22) and constants C_M, C_Ω and C_Γ as in the previous Lemma.

We can now give the main result for the optimal control problem. For a full proof of Theorem 1, we refer to [8], Theorem 4.5.

Theorem 1. *Let the assumptions of Lemma 1 and Assumption 2 be valid. Moreover, let $(\overline{u}, \overline{q}_\Omega, \overline{q}_\Gamma)$ be the solution of the optimal control problem (1)–(2) and $(\overline{u}_h, \overline{q}_{h,\Omega}, \overline{q}_{h,\Gamma})$ the solution of the discretized problem (14)–(15). Finally, let α_Ω, $\alpha_\Gamma > 0$. Then there exists a constant $C > 0$ depending on $\lambda_\Omega, \lambda_\Gamma, \alpha_\Omega, \alpha_\Gamma, C_\Omega, C_\Gamma$ such that the following error estimate holds:*

$$\begin{aligned} &\|\overline{q}_\Omega - \overline{q}_{h,\Omega}\|_{0,\Omega} + \|\overline{q}_\Gamma - \overline{q}_{h,\Gamma}\|_{0,\Gamma_R} \\ \le{}& C\Big\{ \Big(\sum_{M \in \mathcal{M}_h} h_M^{1+2\lambda} |\overline{q}_\Omega|^2_{1+\lambda,2,M} \Big)^{\frac{1}{2}} + \Big(\sum_{E \in \mathrm{e}_h \cap \Gamma_R} h_E^{1+2\lambda} |\overline{q}_\Gamma|^2_{1+\lambda,2,E} \Big)^{\frac{1}{2}} \\ &+ \Big(\sum_M h_M^{1+2\lambda} \Big(\frac{|\mathbf{b}\cdot\nabla\overline{u}|^2_{\lambda,2,M}}{\|\mathbf{b}\|_{[L^\infty(M)]^d}} + \frac{|\mathbf{b}\cdot\nabla\overline{p}|^2_{\lambda,2,M}}{\|\mathbf{b}\|_{[L^\infty(M)]^d}} \\ &+ C_M \big(\|\overline{u}\|^2_{1+\lambda,2,M} + \|\overline{p}\|^2_{1+\lambda,2,M} \big) \Big) \Big)^{\frac{1}{2}} \Big\} \end{aligned}$$

with τ_M as in (22), $h_E = diam(E)$, $E \in \mathrm{e}_h$ and C_M, C_Ω, C_Γ as in Lemma 4.

Remark 4. In the limit case $\lambda = 1$, we obtain the optimal convergence rate $\mathcal{O}(h_M^{\frac{3}{2}})$.

5 Numerical Experiment

Consider the following numerical example:

$$\min J(q_\Omega, u) := \frac{1}{2}\|u - \tilde{u}_\Omega\|^2_{L^2(\Omega)} + \frac{1}{2}\alpha_\Omega \|q_\Omega\|^2_{L^2(\Omega)},$$
$$-\epsilon\Delta u + (\mathbf{b}\cdot\nabla)u + \sigma u = f + q_\Omega \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

with $q_\Omega \in L^2(\Omega)$, $\epsilon = 10^{-5}$, $\mathbf{b} = (-1, -2)^t$, $\sigma = 1$, $f = 1$, $\tilde{u}_\Omega = 1$ and $\alpha_\Omega = 0.1$. The numerical solution in [3] (for box-constraints of control) with the two-level LPS method and $\varepsilon = 10^{-3}$ gave strong oscillations in the boundary layer regions.

Table 1 gives the convergence history and the numerical convergence rate of the cost functional J. Figure 1 shows the discrete control and state on the coarse grid for the two-level approach with Q_1-elements and $h = \frac{1}{128}$. Spurious oscillations in the boundary layer regions are significantly reduced as compared to the results in [3].

There is an ongoing scientific discussion on the strength of the LPS-method vs. classical residual-based stabilization techniques (like the streamline diffusion method). In [5] it is shown for the one-level LPS method that the LPS-norm gives additional control of the streamline derivative, i.e., on $(\sum_M \delta_M \|\mathbf{b}\cdot\nabla(\cdot)\|^2_{0,M})^{\frac{1}{2}}$ with $\delta_M \sim \min(h_M/\|\mathbf{b}\|_{0,\infty,M}; h_M^2/\varepsilon)$. A further reduction of remaining spurious oscillations in boundary layers is possible with adaptive mesh refinement based on a posteriori error estimators. For the streamline diffusion method applied to optimization problems for advection-diffusion problems, we refer to [10].

Table 1 h-convergence of the cost functional

$h = 2^{-l}$	$J(\overline{q}_h, \overline{u}_h)$	$J(\overline{q}_h, \overline{u}_h) - J(\overline{q}_{2h}, \overline{u}_{2h})$	num. conv. rate
2	3.082E−01	–	–
3	2.767E−01	3.152E−02	–
4	2.639E−01	1.277E−02	1.303
5	2.602E−01	3.748E−03	1.769
6	2.592E−01	9.138E−04	2.036
7	2.591E−01	1.743E−04	2.390

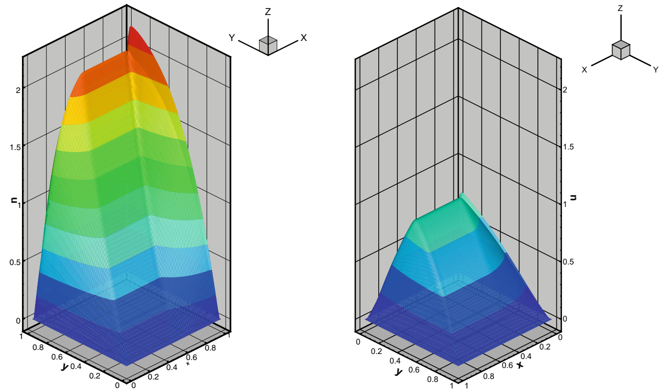

Fig. 1 Optimal discrete control and state for Example 2 with $\varepsilon = 10^{-5}$ and $\tau = 0.1\ h$

6 Further Application: Regularized Dirichlet Control

In applications, a Dirichlet boundary control $u = q$ is desirable. A review of some variants is given in [7]. One possibility is to approximate the Dirichlet control by a Robin control

$$\hat{\varepsilon}\nabla u \cdot \mathbf{n} + \beta(u - q) = 0, \quad \beta = \mathcal{O}(1) \tag{24}$$

for $\hat{\varepsilon} \to +0$, but the choice of $\hat{\varepsilon}$ is delicate. For the singularly perturbed problem (2) with $\hat{\varepsilon} = \varepsilon$, one can interpret the Robin control as regularized Dirichlet control.

Define the subsets Γ_-, Γ_0 and Γ_+ of the boundary $\partial\Omega$, depending on the sign of $(\mathbf{b} \cdot \mathbf{n})(x)$. The solution u of problem (2) has boundary layers at the outflow part Γ_+ with gradient $|\varepsilon\nabla u \cdot \mathbf{n}| \sim 1$ and at characteristic boundaries Γ_0 with (at most) $|\varepsilon\nabla u \cdot \mathbf{n}| \sim \sqrt{\varepsilon}$. At the inflow part Γ_-, one has only $|\varepsilon\nabla u \cdot \mathbf{n}| \sim \varepsilon$. This motivates us to exclude a Dirichlet control at the outflow boundary Γ_+. On $\Gamma_- \cup \Gamma_0$, the Robin regularization (24) with $\hat{\varepsilon} = \varepsilon$ and $\beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n} \geq \beta_0 > 0$ is a good approximation of the Dirichlet control $u = q$.

A typical situation is the flow in a domain of channel type $\Omega = (0, L)\times(-\frac{H}{2}, \frac{H}{2})$ with the flow field $\mathbf{b}(x) = ((\frac{H}{2} - |x_2|)^\kappa, 0)^T$ with $\kappa \geq 0$. The solution u of (2) can be seen as a temperature field or as the density of some chemical reactant. Let us describe potential applications of Dirichlet control: A Dirichlet condition $u = q$ is given at $\Sigma \subset \Gamma_- = \{0\}\times(-\frac{H}{2}, \frac{H}{2})$ whereas a Robin condition $\varepsilon\frac{\partial u}{\partial x_1} + \beta(u - g) = 0$ with $\beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n} \geq \beta_0 > 0$ is prescribed on $\Gamma_- \setminus \Sigma$. A Neumann condition $\varepsilon\frac{\partial u}{\partial x_1} = 0$ might be prescribed on $\Gamma_+ = \{1\} \times (-\frac{H}{2}, \frac{H}{2})$. An "insulation" condition $\varepsilon\frac{\partial u}{\partial x_2} = 0$ is given at the channel walls $\Gamma_0 = (0, L) \times \{-\frac{H}{2}, \frac{H}{2}\}$. Similarly, one can assume a Dirichlet condition $u = q$ at $\Sigma \subset \Gamma_0$ of the channel walls. Finally, replacing the Dirichlet control on $\Sigma \subset \Gamma_- \cup \Gamma_0$ by Robin boundary control leads to the problem considered within this report. An analytical justification of this approach and numerical results will be given elsewhere.

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