A Locally Adapting Parameter Design for the Divergence Stabilization of FEM Discretizations of the Navier–Stokes Equations

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Abstract We will first briefly summarize the previous efforts in constructing a parameter design for local projection and grad-div stabilization based on a-priori convergence analysis for the linearized problem given in [LRL08] and [MT07]. Especially for Taylor-Hood type elements this leads to a grad-div stabilization parameter $\mu \sim 1$. While this design works well for some academic testproblems it does not give satisfactory results for others. A review of the convergence estimate suggests an a-posteriori parameter design including local norms of velocity and pressure. Some first numerical results based on this parameter design will be presented.

1 Introduction

Consider the non-dimensional, unsteady, incompressible Navier–Stokes equations:

$$\begin{aligned} \partial_t \mathbf{u} - Re^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \tilde{\mathbf{f}} \qquad & \text{in } \Omega \times (0,T) \\ \nabla \cdot \mathbf{u} = 0 \qquad & \text{in } \Omega \times (0,T) \end{aligned} \tag{1}$$

in the primitive variables velocity $\mathbf{u}$ and pressure p in a bounded, polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$ and with given source term $\tilde{\mathbf{f}}$. The dimensionless Reynolds number is given by $Re = \frac{UL}{\nu}$ with U and L being a characteristic velocity and length, respectively, and ν the kinematic viscosity.

A standard approach for solving (1) is to apply a semi-discretization in time with an implicit A-stable scheme first and then to linearize the problem with a fixed point

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or Newton-type method. The fixed point iteration leads to a series of Oseen-type problems:

$$
\begin{aligned}
-Re^{-1}\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u} + \nabla p = \mathbf{f} \qquad & \text{in } \Omega \\
\nabla\cdot\mathbf{u} = 0 \qquad & \text{in } \Omega\,.
\end{aligned}
$$

We consider σ to be constant and proportional to the inverse of the chosen timestep size and $\mathbf{b} \in \mathbf{H}_{div}(\Omega) \cap \mathbf{L}^{\infty}(\Omega)$ with $\sigma - \frac{1}{2}\nabla\cdot\mathbf{b} \geq \sigma_0 \geq 0$ almost everywhere. For simplicity we impose homogeneous Dirichlet boundary conditions $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$.

The appropriate solution space for the continuous problem is

$$
(\mathbf{u}, p) \in \mathbf{V} \times Q := \left[H_0^1(\Omega)\right]^d \times L_0^2(\Omega).
$$

The weak formulation for the Oseen problem then reads

Find $U = (\mathbf{u}, p) \in \mathbf{V} \times Q$ s.t.

$$
A(U, V) = (\mathbf{f}, \mathbf{v}) \qquad \forall V = (\mathbf{v}, q) \in \mathbf{V} \times Q
$$

with the bilinear form

$$
A(U, V) := Re^{-1}(\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u}, \mathbf{v}) - (\nabla\cdot\mathbf{v}, p) + (\nabla\cdot\mathbf{u}, q),
$$

where $(\,,\,)$ denotes the inner product on $L^2(\Omega)$ or $[L^2(\Omega)]^d$.

As a spatial discretization we consider quadrilateral ($d = 2$) and hexahedral elements ($d = 3$) and require a shape-regular triangulation $\mathcal{T}_h$. Let F_K be the mapping from the reference cell $\hat{K}$ to real cell K and let $\mathbb{Q}_r$ be the space of tensor polynomials, i.e. polynomials of maximum degree r in each coordinate direction. Then we can define the mapped finite element space

$$
Y_{r,h} = \{v \in C(\overline{\Omega}) \mid v|_K \circ F_K \in \mathbb{Q}_r(\hat{K}) \quad \forall K \in \mathcal{T}_h\}\,.
$$

We choose the discrete ansatz spaces $\mathbf{V}_h = \left[Y_{s,h}\right]^d \cap \mathbf{V}$ and $Q_h = Q_{t,h} \cap Q$ for velocity and pressure with polynomial degrees s and t, respectively.

2 The Local Projection Stabilization Framework

The standard Galerkin approximation with finite elements suffers from two problems. On the one hand the case $Re \gg 1$ gives raise to spurious oscillations in the velocity component of the solution due to dominating advection and poor mass conservation; on the other hand, a pressure instability occurs for spaces that do not satisfy the discrete inf-sup condition.

A widespread framework to deal with all these problems is the residual based stabilization. Especially the combination of *Streamline-Upwind/Petrov-Galerkin (SUPG)* and *Pressure-Stabilization/Petrov-Galerkin (PSPG)* is often used, sometimes supplemented with *Grad-Div stabilization*, see [BBJL07] and references therein.

The class of residual based methods has several drawbacks. For example the SUPG and PSPG methods are non-symmetric and introduce additional coupling terms between velocity and pressure. These create some difficulties in the analysis and lead to upper bounds on the stabilization parameters in order to prove the stability of the method.

As a remedy for the drawbacks of the class of residual based methods several symmetric stabilization methods have been proposed. They all have in common that they add a symmetric, positive semi-definite bilinear form S_h to the original weak formulation of the problem.

The stabilized variational formulation is then given by:

$$\text{Find } U_h = (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \text{ s.t.}$$
$$(A + S_h)(U_h, V_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall V_h = (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h.$$

There are several ways to define the penalty term S_h, see [BBJL07]. Here we will focus on the local projection stabilization (LPS) following the framework introduced in [MST07]. The idea of LPS is to penalize only the small scales of the quantities of interest defined by some fluctuation operator.

Let V_H/Q_H be a pair of scalar and discontinuous coarse spaces on a suitable macro triangulation $\mathcal{M}_h$ and let $\pi^{v/q} : L^2(\Omega) \to V_H/Q_H$ be the local L^2-projections into the coarse spaces. Then we can define the fluctuation operators

$$\kappa^{v/q} := id - \pi^{v/q} : L^2(\Omega) \to L^2(\Omega).$$

We will use boldface notation $\boldsymbol{\kappa}^v$ if we apply the operator component-wise. The stabilizing bilinear form S_h can then be defined as

$$\begin{aligned} S_h(U, V) := &\sum_{M \in \mathcal{M}_h} \tau_M \big(\boldsymbol{\kappa}^v((\nabla \cdot \mathbf{b})\mathbf{u}), (\nabla \cdot \mathbf{b})\mathbf{v}\big)_M \\ &+ \sum_{M \in \mathcal{M}_h} \mu_M \big(\kappa^q(\nabla \cdot \mathbf{u}), \nabla \cdot \mathbf{v}\big)_M + \sum_{M \in \mathcal{M}_h} \alpha_M \big(\boldsymbol{\kappa}^v(\nabla p), \nabla q\big)_M. \end{aligned}$$

It contains penalty terms for the fluctuations of the streamline derivative and divergence of the velocity and the pressure gradients, weighted element-wise by user chosen parameters τ_M, μ_M and α_M. Other variants that stabilize fluctuations of the full gradient of the velocity are possible.

3 Parameter Design

Two typically used conforming spatial discretizations are the family of Taylor-Hood elements (TH, $s = t + 1$) and approximations with equal order for velocity and pressure (EO, $s = t$). The coarse spaces are chosen in a so called two-level manner, where $\mathcal{T}_h$ is a suitable global refinement of $\mathcal{M}_h := \mathcal{T}_{2h}$. The full a-priori analysis on stability and error estimates for this method can be found in [LRL08] and [MT07].

Under the assumptions given there one can derive the following estimate for the error between the continuous solution $U = (\mathbf{u}, p)$ and the discrete solution $U_h = (\mathbf{u}_h, p_h)$ in the stabilized energy norm:

$$|||U - U_h|||_{LP}^2 \leq C \sum_{M \in \mathcal{M}_h} \Big(\tau_M h_M^{2s} |(\mathbf{b} \cdot \nabla)\mathbf{u}|_{s,\omega_M}^2 + C_M^u h_M^{2s} |\mathbf{u}|_{s+1,\omega_M}^2 + C_M^p h_M^{2t} |p|_{t+1,\omega_M}^2 \Big) \quad (2)$$

where ω_M denotes a certain neighborhood of the macro element and

$$C_M^u := Re^{-1} + h_M^2(\sigma + \tau_M^{-1} + \alpha_M^{-1}) + \mu_M + \tau_M \|\mathbf{b}\|_{\infty,M}^2,$$
$$C_M^p := \alpha_M + \mu_M^{-1} h_M^2.$$

The energy norm itself is given by:

$$|||(\mathbf{v}, q)|||_{LP}^2 = Re^{-1} |\mathbf{v}|_1^2 + \sigma_0 \|\mathbf{v}\|_0^2 + \delta \|q\|_0^2 + S_h(\mathbf{v}, q; \mathbf{v}, q).$$

In order to get asymptotically optimal rates of convergence, the stabilization parameters must satisfy a certain scaling with respect to h_M given in Table 1. These parameter designs are based on the assumption $|\mathbf{u}|_{k+1,M} \sim |p|_{k,M}$ and obtained by balancing the parameter dependent terms in the a-priori error estimate (2) in order to minimize the upper bound on the error.

For the Taylor-Hood element the divergence parameter μ_M is notably conspicuous because it is of order 1 and might dominate the whole PDE. In [OR04], where the grad-div stabilization for the Stokes problem is analyzed, it is remarked, that the larger the norm of the pressure is compared to the norm of the velocity, the more important the divergence stabilization is. We propose that balancing the μ_M–dependent terms should include the local norms of $\mathbf{u}$ and p because there may be large differences in the scaling of both. Following this approach gives:

Table 1 Selected space combinations with parameter scaling ($Re^{-1} < h_M$)

	V_h	Q_h	V_H	Q_H	τ_M	μ_M	α_M	error
TH	$Y_{k,h}$	$Y_{k-1,h}$	$Y_{k-1,2h}^{\text{disc}}$	$\{0\}$	$\sim h_M$	~ 1	0	$\mathcal{O}\left(h_M^k\right)$
EO	$Y_{k,h}$	$Y_{k,h}$	$Y_{k-1,2h}^{\text{disc}}$	$\{0\}$	$\sim h_M$	$\sim h_M$	$\sim h_M$	$\mathcal{O}\left(h_M^{k+1/2}\right)$

$$\mu_M \, |\mathbf{u}|^2_{k+1,\omega_M} \sim \mu_M^{-1} \, |p|^2_{k,\omega_M} \quad \Longrightarrow \quad \mu_M \sim \frac{|p|_{k,\omega_M}}{|\mathbf{u}|_{k+1,\omega_M}} \, .$$

Since the solution $(\mathbf{u}, p)$ is generally unknown, these norms must be replaced by norms of the discrete solution $(\mathbf{u}_h, p_h)$. This leads to a local and nonlinear parameter design. We should further note, that it may be difficult to recover approximations of the high order derivatives from the discrete solution to evaluate the norms for large k.

4 Numerical Results

As test cases we considered two stationary Navier–Stokes problems with special properties.

Problem 1. On the unit square $\Omega = (0,1)^2$ we define

$$\mathbf{u}(x,y) = \begin{pmatrix} \cos(2x-1)e^{2y-1} \\ \sin(2x-1)e^{2y-1} \end{pmatrix} , \qquad p(x,y) = \frac{e^2 - e^{-2}}{8} - \frac{e^{4y-2}}{2}$$

and right hand side $\mathbf{f} = 0$. Then the Laplacian vanishes, $\Delta\mathbf{u} = 0$. The sole contribution from the velocity field to the PDE is the nonlinear term that cancels out with the pressure gradient.

Problem 2. Again on the unit square $\Omega = (0,1)^2$ we prescribe a fixed velocity profile and a channel-like linear pressure

$$\mathbf{u}(x,y) = \begin{pmatrix} \sin(\pi y) \\ 0 \end{pmatrix} , \qquad p(x,y) = Re^{-1}\pi(x - \tfrac{1}{2})$$

and get a non-vanishing right-hand side. This time the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is zero and the pressure is scaled with the inverse of the Reynolds number. A vector plot of the velocity field for both examples is given in Fig. 1.

Remark. We did not use the quadratic Poiseuille profile for the second example because it is contained in the ansatz spaces for $k \geq 2$.

The following numerical tests were carried out on an unstructured, quasi uniform mesh with $h \approx \frac{1}{32}$ and the Taylor-Hood element with $k = 2$. The nonlinearity was resolved by a damped defect correction iteration and the norm of the residual was reduced below 10^{-12}.

Figure 2 shows how the various errors of the discrete solution depend on the Reynolds number without stabilization. For the first problem we see almost a linear increase of the errors in the velocity with the Reynolds number, while the pressure

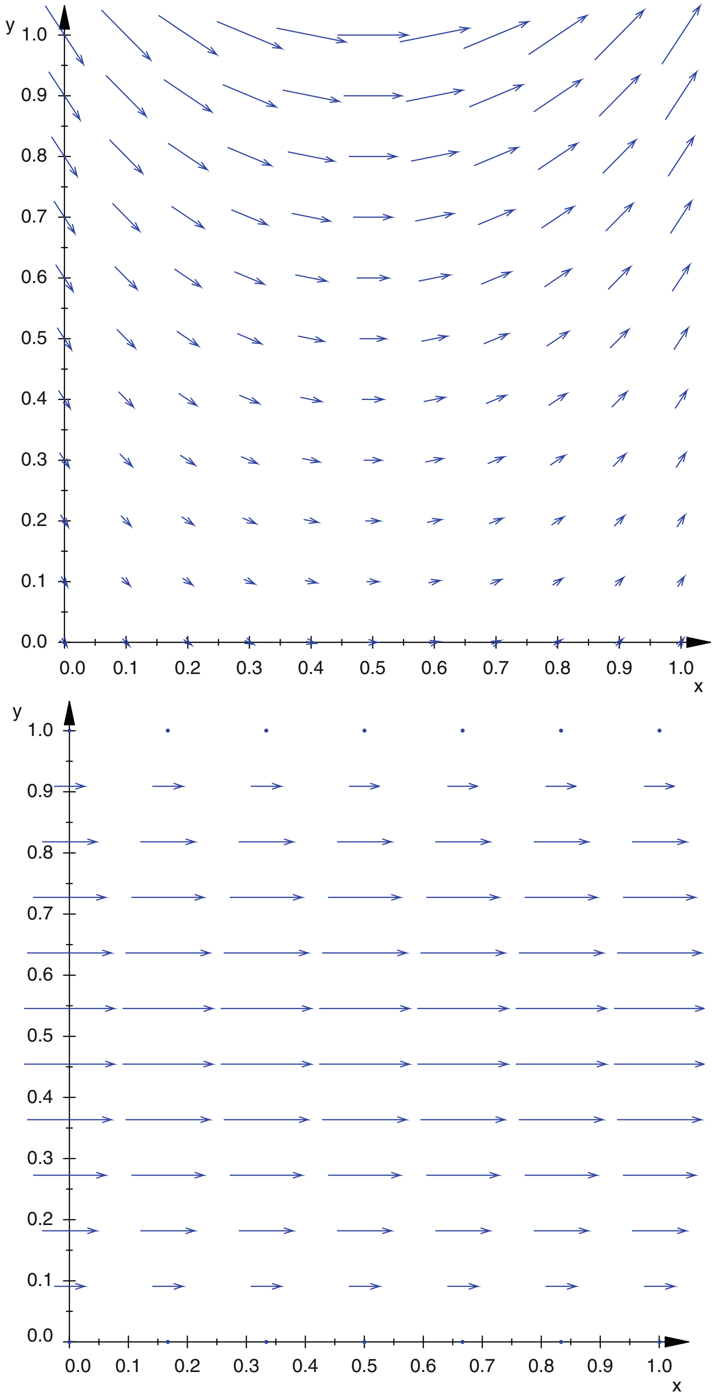

Fig. 1 Vector plot of velocity for problems 1 (*top*) and 2 (*bottom*)

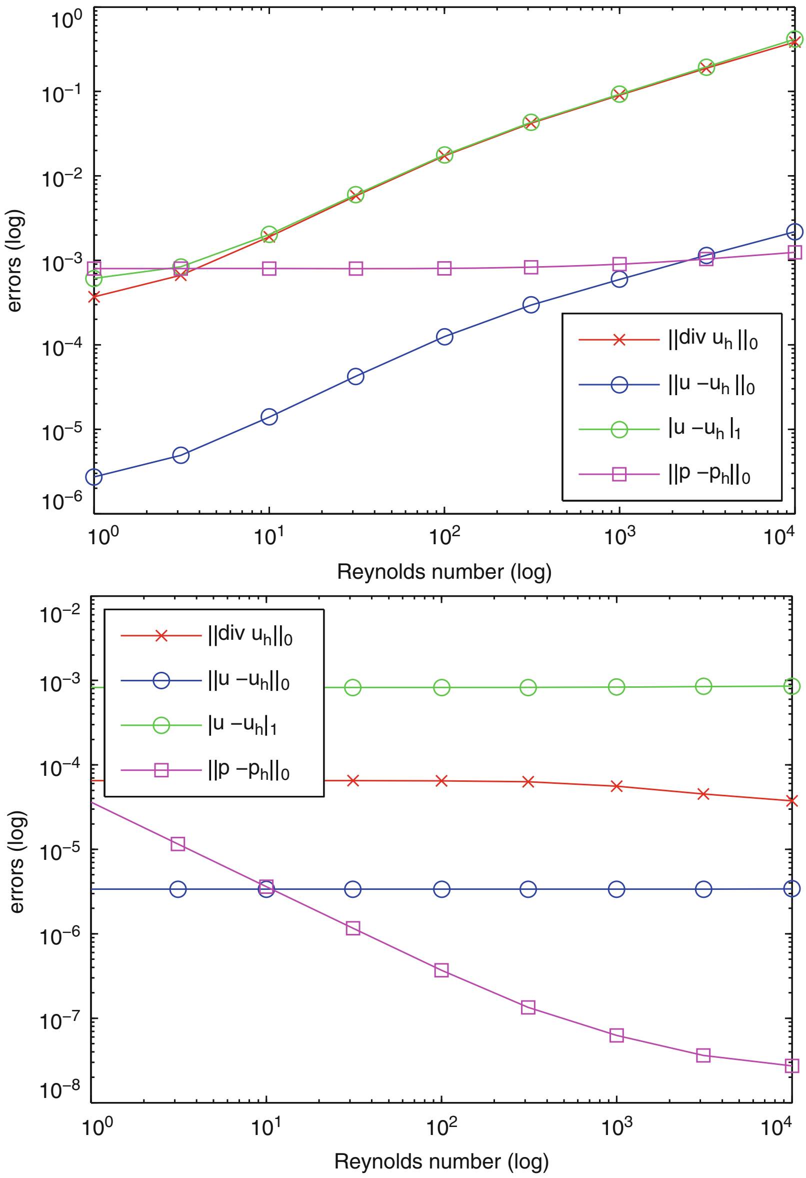

Fig. 2 Errors vs. Reynolds number Re for problems 1 (*top*) and 2 (*bottom*)

error remains constant. The error of the velocity in the H^1-seminorm is dominated by the divergence error. For the second problem we can observe a linear decrease of the pressure error that is caused by the scaling of the pressure with Re^{-1}. The velocity errors are not affected by the Reynolds number and the divergence error is smaller than the H^1-seminorm error.

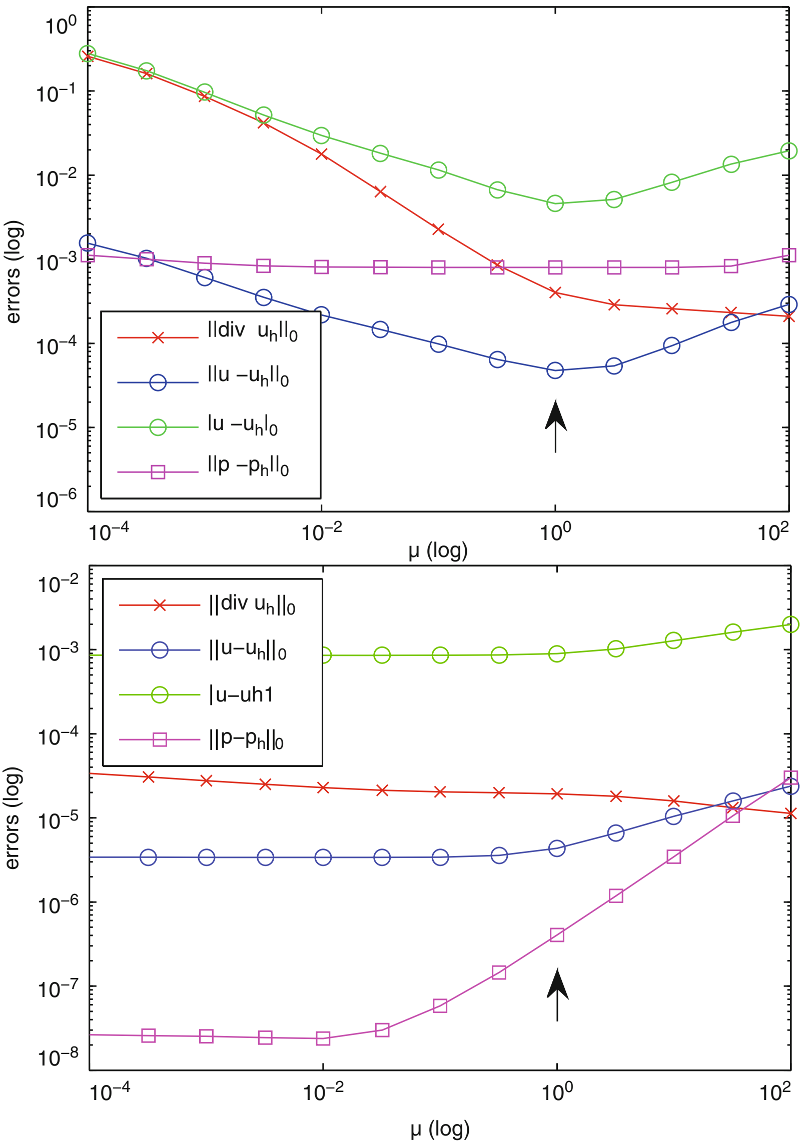

Fig. 3 Stabilization with the old parameter design, problems 1 (*top*) and 2 (*bottom*)

The effect of divergence stabilization on the errors for the original parameter design and both examples with $Re = 10^4$ is shown in Fig. 3. For the first problem the divergence stabilization improves the velocity errors by several orders of magnitude and decouples the divergence error from the H^1-seminorm error. The optimal parameter $\mu_M \approx 1$ reduces the divergence error to the level it had for $Re = 1$. However, the behavior is different for the second problem. At some point

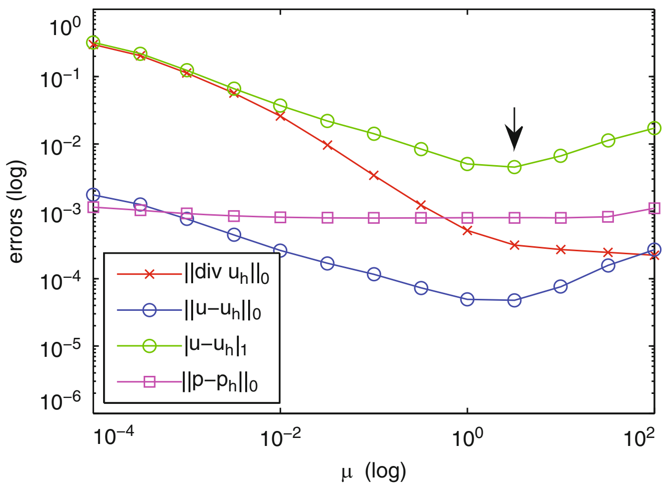

Fig. 4 Stabilization with the new parameter design, Example 1

the pressure error starts to increase linearly with the stabilization parameter. The previously optimal value now increases the pressure error by more than one order of magnitude. Over the whole range of tested parameters only a marginal improvement of the error can be observed. The errors without stabilization are almost optimal.

To get some first results for the new parameter design we used the reference solution and inserted it into the new parameter design. Due to vanishing second derivatives of the pressure for the second problem the parameter design reduces to $\mu_M = 0$ and reproduces what we could see in the previous numerical result: for this problem the divergence stabilization is superfluous. For the first problem the original assumption on the norms is valid and the new parameter design gives results (shown in Fig. 4) comparable to the old parameter design.

More realistic flows, like the flow around a cylinder used in benchmark computations [TS96], show locally varying properties. Close to the cylinder nonlinear effects are stronger, while far behind the cylinder channel like flow can be observed. The proposed parameter is an indicator for the flow type and varies by two orders of magnitude for the flow around the cylinder.

5 Conclusion

Parameter designs for the divergence stabilization did not take into account the local norms of velocity and pressure so far. This leads to parameters far from being optimal for some types of flow (e.g. channel type flow) that actually increase the errors. By a careful look into existing a-priori analysis and error estimates we were able to derive a new parameter design for the divergence stabilization that includes local norms of velocity and pressure in order to minimize the upper bound of the error. The rate of convergence is not affected by the new choice. Unfortunately the new parameter design has several drawbacks that are an obstacle to an efficient implementation.

We should note, that similar observations can be made for the pressure stabilization parameter, because it appears in front of velocity and pressure norms in the error estimate. In practice the effect of badly chosen parameters is less visible there, because the parameter typically is proportional to h_M or h_M^2 for pressure stabilization.

We have not yet implemented the proposed nonlinear parameter design, because we belive that balancing the parameter using the asymptotic a-priori error estimate is still not optimal. What we finally want to do is to determine the load on the divergence constraint, for example by using a Helmholtz-decomposition of the convective and external forcing terms in the momentum equation.

The question whether it is possible to construct a reliable and robust parameter design, that works over a broad range of problems without case by case parameter tuning and can be efficiently implemented, is still open.

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