# A System of Singularly Perturbed Semilinear Equations

**J.L. Gracia, F.J. Lisbona, M. Madaune-Tort, and E. O'Riordan**

**Abstract** In this paper systems of singularly perturbed semilinear reaction-diffusion equations are examined. A numerical method is constructed for these systems which involves an appropriate layer–adapted piecewise-uniform mesh. The numerical approximations generated from this method are shown to be uniformly convergent with respect to the singular perturbation parameters.

## 1 Introduction

In this paper we consider semilinear systems of the form

$$\mathbf{Tu} := -E\mathbf{u}'' + \mathbf{b}(x,\mathbf{u}) = \mathbf{0},\ x \in \Omega = (0,1),\ \mathbf{u}(0) = \mathbf{a},\ \mathbf{u}(1) = \mathbf{b}, \quad (1a)$$

$$\mathbf{b}(x,\mathbf{u}) = (b_1(x,\mathbf{u}),\ldots,b_m(x,\mathbf{u}))^T \in C^4(\bar{\Omega}\times\mathbb{R}^m), \quad (1b)$$

and $\forall (x,\mathbf{y}) \in \bar{\Omega}\times\mathbb{R}^m$ we assume that the nonlinear terms satisfy

$$\frac{\partial b_i}{\partial u_j}(x,\mathbf{y}) \le 0, \forall i \ne j, \text{ and } \sum_{j=1}^{m} \frac{\partial b_i}{\partial u_j}(x,\mathbf{y}) > \beta^2 > 0,\ \beta > 0,\ \forall i = 1,\ldots,m, \quad (1c)$$

where $E = \operatorname{diag}\{\varepsilon_1^2,\ldots,\varepsilon_m^2\}$ is a diagonal matrix, $0 < \varepsilon_1 \le \ldots \le \varepsilon_m \le 1$ and $\mathbf{u} = (u_1,\ldots,u_m)^T$.

In [1, 3], information about the layer structure for linear singularly perturbed reaction–diffusion systems was obtained via linear decompositions of the solution into regular and singular components. Here we show that these techniques are applicable to a semilinear system. The preprint [2] is available to the reader to supplement this paper with some additional details.

J.L. Gracia (✉)
Departamento de Matemática Aplicada, Universidad de Zaragoza, Spain,
E-mail: jlgracia@unizar.es

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For any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$, we write $\mathbf{v} \leq \mathbf{w}$ if $v_i \leq w_i$, $\forall i$ and $|\mathbf{v}| := (|v_1|, |v_2|, \ldots, |v_m|)^T$ ; $\|f\|_\infty := \max_x |f(x)|$ and $\|\mathbf{f}\|_\infty := \max_i \|f_i\|_\infty$; $\mathbf{C} := C(1, 1, \ldots, 1)^T$ is a constant vector and $C$ denotes a generic positive constant independent of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ and the discretization parameter.

## 2 Singularly Perturbed Semilinear Systems

Conditions (1b), (1c) and the implicit function theorem ensure that there exists a unique solution $\mathbf{u} \in (C^4(\bar{\Omega}))^m$ to (1a), and that the corresponding reduced problem $\mathbf{b}(x, \mathbf{r}) = \mathbf{0}$, $x \in \bar{\Omega}$, also has a unique solution in $\mathbf{r} \in (C^4(\bar{\Omega}))^m$. Note that the conditions (1c) on the Jacobian matrix $J$ where

$$J(x, \mathbf{y}) := \left(\frac{\partial b_i}{\partial u_j}\right)(x, \mathbf{y}),$$

are the natural extension of the linear case [6] for the coupling matrix. These conditions guarantee that $J$ is an M–matrix for all $(x, \mathbf{y}) \in \bar{\Omega} \times \mathbb{R}^m$.

To deduce the asymptotic behaviour of the solution, we consider the following decomposition $\mathbf{u} = \mathbf{v} + \mathbf{w} + \mathbf{w}_R$, where the regular component $\mathbf{v}$ is the solution of the problem

$$-\mathbf{E}\mathbf{v}'' + \mathbf{b}(x, \mathbf{v}) = \mathbf{0}, \; x \in \Omega, \; \mathbf{v}(0) = \mathbf{r}(0), \; \mathbf{v}(1) = \mathbf{r}(1), \tag{2}$$

and the singular components $\mathbf{w}, \mathbf{w}_R$ are the solutions of

$$\begin{aligned} &-\mathbf{E}\mathbf{w}'' + (\mathbf{b}(x, \mathbf{v} + \mathbf{w}) - \mathbf{b}(x, \mathbf{v})) = \mathbf{0}, \; x \in \Omega, \\ &\quad \mathbf{w}(0) = (\mathbf{u} - \mathbf{v})(0), \; \mathbf{w}(1) = \mathbf{0}, \end{aligned} \tag{3}$$

$$\begin{aligned} &-\mathbf{E}\mathbf{w}_R'' + (\mathbf{b}(x, \mathbf{v} + \mathbf{w} + \mathbf{w}_R) - \mathbf{b}(x, \mathbf{v} + \mathbf{w})) = \mathbf{0}, \; x \in \Omega, \\ &\quad \mathbf{w}_R(0) = \mathbf{0}, \; \mathbf{w}_R(1) = (\mathbf{u} - \mathbf{v})(1). \end{aligned} \tag{4}$$

Note (1c) guarantees existence and uniqueness of $\mathbf{v}, \mathbf{w}, \mathbf{w}_R$ and it will also be used below to establish existence and uniqueness for several further decompositions of these components. Below we state bounds on the derivatives of the left layer component $\mathbf{w}$. The corresponding bounds on the right layer component $\mathbf{w}_R$ are obtained by simply replacing $x$ with $1 - x$.

**Lemma 1.** *The regular component* $\mathbf{v}$ *satisfies*

$$\left\|\frac{d^k \mathbf{v}}{dx^k}\right\|_\infty \leq C, \; k = 0, 1, 2, \; \left\|\frac{d^k v_i}{dx^k}\right\|_\infty \leq C\varepsilon_i^{2-k}, \; k = 3, 4, \; i = 1, \ldots, m. \tag{5}$$

*Proof.* Consider the secondary decomposition of $\mathbf{v} = \sum_{i=1}^{m} \mathbf{q}^{[i]}$, where

$$-E_m \frac{d^2\mathbf{q}^{[m]}}{dx^2} + \mathbf{b}(x, \mathbf{q}^{[m]}) = \mathbf{0},\ (\mathbf{q}^{[m]})_m(0) = r_m(0),\ (\mathbf{q}^{[m]})_m(1) = r_m(1), \quad (6)$$

$$-E_j \frac{d^2\mathbf{q}^{[j]}}{dx^2} + \mathbf{b}(x, \sum_{i=j}^{m} \mathbf{q}^{[i]}) - \mathbf{b}(x, \sum_{i=j+1}^{m} \mathbf{q}^{[i]}) = \varepsilon_j^2 \sum_{i=j+1}^{m} (\mathbf{q}^{[i]})_j \mathbf{e_j},\ x \in \Omega,$$
$$(\mathbf{q}^{[j]})_i(0) = (\mathbf{q}^{[j]})_i(1) = 0,\ j \le i \le m,\ 1 \le j < m, \quad (7)$$

with the matrix $E_i$ is the zero matrix except that on the main diagonal $(E_i)_{jj} = \varepsilon_j^2,\ j \ge i$, (note that in this notation $E_1 = E$) and $\mathbf{e_i}$ is the $i$th vector of the canonical basis. Conditions (1c) imply that $\mathbf{q}^{[m]}(0) = \mathbf{r}(0),\ \mathbf{q}^{[m]}(1) = \mathbf{r}(1)$, and $\mathbf{q}^{[j]}(0) = \mathbf{q}^{[j]}(1) = \mathbf{0}$, for $1 \le j < m$.

To obtain estimates for the component $\mathbf{q}^{[m]}$, we introduce the function $\mathbf{z} = \mathbf{q}^{[m]} - \mathbf{r}$, which is the solution of the problem

$$-E_m \mathbf{z}'' + \int_{s=0}^{1} J(x, \mathbf{r} + s\mathbf{z}) ds\, \mathbf{z} = E_m \mathbf{r}'', \quad \mathbf{z}(0) = \mathbf{z}(1) = \mathbf{0}.$$

The conditions (1c) ensure that a maximum principle holds for this system. Thus $\|\mathbf{z}\|_\infty \le C\varepsilon_m^2$ and $\|z_m''\|_\infty \le C$ and follows that $\|z_m'\|_\infty \le C$. We conclude that

$$\left\| \frac{d^k (\mathbf{q}^{[m]})_m}{dx^k} \right\|_\infty \le C,\ k = 0, 1, 2, \quad \text{and} \quad \left\| \mathbf{q}^{[m]} \right\|_\infty \le C.$$

In addition, from the nonlinear system $b_1\left(x, \mathbf{q}^{[m]}\right) = \cdots = b_{m-1}\left(x, \mathbf{q}^{[m]}\right) = 0$, we have that

$$\left\| \frac{d^k (\mathbf{q}^{[m]})_i}{dx^k} \right\|_\infty \le C,\ k = 1, 2,\ 1 \le i < m.$$

Differentiating the $m$th equation of (6) twice and using the above bound we conclude that $\|d^4(\mathbf{q}^{[m]})_m/dx^4\|_\infty \le C\varepsilon_m^{-2}$. Hence $\|d^3(\mathbf{q}^{[m]})_m/dx^3\|_\infty \le C\varepsilon_m^{-1}$ and, using the first $m-1$ equations of (6), we have that

$$\left\| \frac{d^k (\mathbf{q}^{[m]})_i}{dx^k} \right\|_\infty \le C\varepsilon_m^{2-k},\ k = 3, 4,\ 1 \le i < m.$$

Now consider the component $\mathbf{q}^{[j]}$ with $1 \le j < m$. It is the solution of

$$-E_j \frac{d^2\mathbf{q}^{[j]}}{dx^2} + \int_0^1 J(x, \sum_{i=j+1}^{m} \mathbf{q}^{[i]} + s\mathbf{q}^{[j]}) ds\, \mathbf{q}^{[j]} = \varepsilon_j^2 \sum_{i=j+1}^{m} (\mathbf{q}^{[i]})_j\, \mathbf{e_j},\ x \in \Omega,$$
$$(\mathbf{q}^{[j]})_i(0) = (\mathbf{q}^{[j]})_i(1) = 0,\ j \le i \le m.$$

The maximum principle yields $\|\mathbf{q}^{[j]}\|_\infty \le C\varepsilon_j^2$, and then $\|d^2(\mathbf{q}^{[j]})_i/dx^2\|_\infty \le C(\varepsilon_j/\varepsilon_i)^2 \le C,\ j \le i \le m$. Then, $\|d(\mathbf{q}^{[j]})_i/dx\|_\infty \le C,\ j \le i \le m$, and hence (if $j > 1$) we have $\|d(\mathbf{q}^{[j]})_i/dx\|_\infty \le C,\ \|d^2(\mathbf{q}^{[j]})_i/dx^2\|_\infty \le C,\ 1 \le i \le j-1$.

Differentiating the differential equation (7) twice, using the bounds for $\mathbf{q}^{[j]}$ and its derivatives, we deduce that $\|d^4(\mathbf{q}^{[j]})_i/dx^4\|_\infty \le C\varepsilon_i^{-2},\ i = 1,\dots,m$. Hence $\|d^3(\mathbf{q}^{[j]})_i/dx^3\|_\infty \le C\varepsilon_i^{-1},\ i = 1,\dots,m$. □

To establish first order error bounds in the case of an arbitrary number of equations, we consider a further decomposition of the singular component $\mathbf{w}$, which is similar to that used in [1] for linear systems. For simplicity, we present the main ideas for the particular case of two equations and these decompositions can be extended to the general case of $m$ semilinear equations using the arguments in [1,3].

In the case of $m = 2$, consider the following decomposition of the left singular component $\mathbf{w}$

$$\mathbf{w} = \mathbf{w}^{[1]} + \mathbf{w}^{[2]}, \tag{8a}$$

where $\mathbf{w}^{[2]}(1) = \mathbf{w}^{[1]}(1) = \mathbf{0}$, and

$$-\mathbf{E}\frac{d^2\mathbf{w}^{[2]}}{dx^2} + (\mathbf{b}(x,\mathbf{v}+\mathbf{w}^{[2]}) - \mathbf{b}(x,\mathbf{v})) = \mathbf{0}, \quad x \in \Omega, \tag{8b}$$

$$b_1(0,\mathbf{v}(0)+\mathbf{w}^{[2]}(0)) - b_1(0,\mathbf{v}(0)) = 0,\ w_2^{[2]}(0) = w_2(0), \tag{8c}$$

$$-\mathbf{E}\frac{d^2\mathbf{w}^{[1]}}{dx^2} + (\mathbf{b}(x,\mathbf{v}+\mathbf{w}) - \mathbf{b}(x,\mathbf{v}+\mathbf{w}^{[2]})) = \mathbf{0},\ x \in \Omega, \tag{8d}$$

$$w_1^{[1]}(0) = w_1(0) - w_1^{[2]}(0),\ w_2^{[1]}(0) = 0. \tag{8e}$$

Below we see that the components $\mathbf{w}^{[2]}$ depend weakly on $\varepsilon_1$ and the appearance of $\mathbf{w}^{[1]}$ requires that $w_1(0) - w_1^{[2]}(0) \ne 0$. Moreover, if $\varepsilon_1 = \varepsilon_2$, it is not necessary to decompose $\mathbf{w}$ into these subcomponents to perform the numerical analysis. We introduce the following notation

$$B_\varepsilon(x) := e^{-x\beta/\varepsilon},\ \text{where } \beta \text{ is defined by (1c).}$$

**Lemma 2.** *For any $x \in \Omega$, the component $\mathbf{w}^{[2]}$, satisfies the bounds*

$$\left|\frac{d^k\mathbf{w}^{[2]}}{dx^k}(x)\right| \le \mathbf{C}\varepsilon_2^{-k}B_{\varepsilon_2}(x),\ k = 0,1,2,$$
$$\left|\frac{d^3\mathbf{w}^{[2]}}{dx^3}(x)\right| \le C\left(\varepsilon_1^{-2},\varepsilon_2^{-2}\right)^T \varepsilon_2^{-1}B_{\varepsilon_2}(x).$$

*Proof.* Note that

$$-\mathbf{E}\frac{d^2\mathbf{w}^{[2]}}{dx^2} + \int_{s=0}^{1} J\left(x,\mathbf{v}+s\mathbf{w}^{[2]}\right)ds\ \mathbf{w}^{[2]} = \mathbf{0},$$

from which it follows that $|\mathbf{w}^{[2]}(x)| \le \mathbf{C}B_{\varepsilon_2}(x)$. Then, from the second equation in (8b) we deduce that

$$\left|\frac{d^k w_2^{[2]}}{dx^k}(x)\right| \le C\varepsilon_2^{-k} B_{\varepsilon_2}(x),\ k = 0, 1, 2. \tag{9}$$

To obtain bounds for the first component, consider the decomposition $\mathbf{w}^{[2]} = \mathbf{p}^{[2]} + \mathbf{r}^{[2]},\ r_2^{[2]} \equiv 0$, where

$$b_1\left(x, \mathbf{v} + \mathbf{p}^{[2]}\right) - b_1(x, \mathbf{v}) = 0, \tag{10a}$$

$$-\varepsilon_1^2 \frac{d^2 r_1^{[2]}}{dx^2} + b_1\left(x, \mathbf{v} + \mathbf{p}^{[2]} + \mathbf{r}^{[2]}\right) - b_1\left(x, \mathbf{v} + \mathbf{p}^{[2]}\right) = \varepsilon_1^2 \frac{d^2 p_1^{[2]}}{dx^2}. \tag{10b}$$

As $p_2^{[2]} \equiv w_2^{[2]}$ this is simply a decomposition of the first component $w_1^{[2]}$. Note that the condition on the coefficients (1c) means that $p_1^{[2]}(0) = w_1^{[2]}(0)$, and $p_1^{[2]}(1) = w_1^{[2]}(1)$. Therefore $r_1^{[2]}(0) = r_1^{[2]}(0) = 0$. Writing (10a) in the form

$$\sum_{i=1}^{2} \int_0^1 \frac{\partial b_1}{\partial u_i}\left(x, \mathbf{v} + s\mathbf{p}^{[2]}\right) p_i^{[2]} ds = 0,$$

and using (1b) and (9), we deduce that $|p_1^{[2]}(x)| \le CB_{\varepsilon_2}(x)$ for any $x \in \Omega$.

Differentiating (10a) and grouping terms, we have

$$\frac{\partial}{\partial x}\left(b_1(x, \mathbf{v} + \mathbf{p}^{[2]}) - b_1(x, \mathbf{v})\right) + \left(\nabla_u b_1(x, \mathbf{v} + \mathbf{p}^{[2]}) - \nabla_u b_1(x, \mathbf{v})\right) \frac{d\mathbf{v}}{dx}$$
$$+ \nabla_u b_1\left(x, \mathbf{v} + \mathbf{p}^{[2]}\right) \frac{d\mathbf{p}^{[2]}}{dx} = 0, \quad \text{where} \quad \nabla_u b_1 := \left(\frac{\partial b_1}{\partial u_1}, \frac{\partial b_1}{\partial u_2}\right)^T.$$

Note if $b_1(x, \mathbf{u} + \mathbf{v}) - b_1(x, \mathbf{u}) = Q(x)$, then

$$\frac{\partial}{\partial x}\left[b_1(x, \mathbf{u} + \mathbf{v}) - b_1(x, \mathbf{u})\right] = \frac{\partial}{\partial x}\left[\sum_{i=1}^{2} \int_0^1 \frac{\partial b_1}{\partial u_i}(x, \mathbf{u} + t\mathbf{v})v_i\, dt\right],$$

which implies that $\left|\frac{\partial v_1}{\partial x}\right| \le C\left|\frac{\partial Q}{\partial x}\right| + C\left|\frac{\partial v_2}{\partial x}\right| + C|v_1| + C|v_2|$.

From these expressions, (1b) and (9), we have that $\left|\frac{dp_1^{[2]}}{dx}(x)\right| \le C\varepsilon_2^{-1} B_{\varepsilon_2}(x)$. Use the same argument to prove $\left|\frac{d^2 p_1^{[2]}}{dx^2}(x)\right| \le C\varepsilon_2^{-2} B_{\varepsilon_2}(x)$.

The remainder is the solution of the following problem

$$-\varepsilon_1^2 \frac{d^2 r_1^{[2]}}{dx^2} + \left( \int_0^1 \frac{\partial b_1}{\partial u_1}(x, \mathbf{v} + \mathbf{p}^{[2]} + s\mathbf{r}_1^{[2]}) ds \right) r_1^{[2]}$$
$$= \varepsilon_1^2 \frac{d^2 p_1^{[2]}}{dx^2}, \; r_1^{[2]}(0) = r_1^{[2]}(1) = 0.$$

The maximum principle proves that $|r_1^{[2]}(x)| \leq C\varepsilon_1^2 \varepsilon_2^{-2} B_{\varepsilon_2}(x)$. Hence,

$$\left| \frac{d^k r_1^{[2]}}{dx^k}(x) \right| \leq C\varepsilon_2^{-k} B_{\varepsilon_2}(x), \; k = 1, 2.$$

To obtain the bound on the third derivatives, differentiate (8b) and use the bounds on the lower derivatives. □

**Lemma 3.** *For any $x \in \Omega$ and for $i = 1, 2$, the component $\mathbf{w}^{[1]}$ satisfies the bounds*

$$\left| w_1^{[1]}(x) \right| \leq C(B_{\varepsilon_1}(x) + \frac{\varepsilon_1^2}{\varepsilon_2^2} B_{\varepsilon_2}(x)), \quad \left| w_2^{[1]}(x) \right| \leq C \frac{\varepsilon_1^2}{\varepsilon_2^2} B_{\varepsilon_2}(x),$$

$$\left| \frac{dw_i^{[1]}}{dx}(x) \right| \leq C(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2^{-1} B_{\varepsilon_2}(x)),$$

$$\varepsilon_i^2 \left| \frac{d^2 w_i^{[1]}}{dx^2}(x) \right| \leq C(B_{\varepsilon_1}(x) + \frac{\varepsilon_1^2}{\varepsilon_2^2} B_{\varepsilon_2}(x)),$$

$$\varepsilon_i^2 \left| \frac{d^3 w_i^{[1]}}{dx^3}(x) \right| \leq C(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2^{-1} B_{\varepsilon_2}(x)).$$

*Proof.* Decompose $\mathbf{w}^{[1]}$ further into the following sum $\mathbf{w}^{[1]} = \mathbf{z}^{[1]} + \mathbf{s}^{[1]}$, where $\mathbf{z}^{[1]}(0) = \mathbf{w}^{[1]}(0)$, $\mathbf{z}^{[1]}(1) = \mathbf{w}^{[1]}(1) = \mathbf{s}^{[1]}(0) = \mathbf{s}^{[1]}(1) = \mathbf{0}$, and for $x \in \Omega$

$$-\varepsilon_1^2 \frac{d^2 z_1^{[1]}}{dx^2} + \left( \int_0^1 \frac{\partial b_1}{\partial u_1}(x, \mathbf{v} + \mathbf{w}^{[2]} + s(z_1^{[1]}, 0)^T) ds \right) z_1^{[1]} = 0,$$

$$-\varepsilon_2^2 \frac{d^2 z_2^{[1]}}{dx^2} + \left( \int_0^1 \frac{\partial b_2}{\partial u_2}(x, \mathbf{v} + \mathbf{w}^{[2]} + t\mathbf{z}^{[1]}) dt \right) z_2^{[1]}$$
$$= -\left( \int_0^1 \frac{\partial b_2}{\partial u_1}(x, \mathbf{v} + \mathbf{w}^{[2]} + t\mathbf{z}^{[1]}) dt \right) z_1^{[1]},$$

$$-\mathbf{E} \frac{d^2 \mathbf{s}^{[1]}}{dx^2} + \int_0^1 J(x, \mathbf{v} + \mathbf{w}^{[2]} + \mathbf{z}^{[1]} + t\mathbf{s}^{[1]}) dt \; \mathbf{s}^{[1]}$$
$$= \left( b_1(x, \mathbf{v} + \mathbf{w}^{[2]} + (z_1^{[1]}, 0)^T) - b_1(x, \mathbf{v} + \mathbf{w}^{[2]} + \mathbf{z}^{[1]}), 0 \right)^T.$$

From the maximum principle, we have $|\frac{d^k z_1^{[1]}}{dx^k}| \leq C\varepsilon_1^{-k} B_{\varepsilon_1}(x),\ k = 0,1,2$. If $\varepsilon_2^2 \leq 2\varepsilon_1^2$, then the maximum principle proves $|z_2^{[1]}(x)| \leq CB_{\varepsilon_2}(x)$. For the case $\varepsilon_2^2 \geq 2\varepsilon_1^2$, to obtain appropriate bounds of $z_2^{[1]}$, we observe that

$$\left|\int_0^1 \frac{\partial b_2}{\partial u_1}\left(x, \mathbf{v} + \mathbf{w}^{[2]} + t\mathbf{z}^{[1]}\right) dt z_1^{[1]}\right| \leq C_1 B_{\varepsilon_1}(x).$$

Consider the barrier function $Z$ [1], which is the solution of the problem

$$-\varepsilon_2^2 Z'' + \beta^2 Z = C_1 B_{\varepsilon_1}(x),\ Z(0) = Z(1) = 0.$$

This allows one to prove that $|z_2^{[1]}(x)| \leq Z(x) \leq C\varepsilon_1^2\varepsilon_2^{-2} B_{\varepsilon_2}(x)$, if $2\varepsilon_1^2 \leq \varepsilon_2^2$. Thus, for all $\varepsilon_1 \leq \varepsilon_2$, we have $|z_2^{[1]}(x)| \leq C\varepsilon_1^2\varepsilon_2^{-2} B_{\varepsilon_2}(x)$. Hence,

$$\begin{aligned}
\varepsilon_2^2 |\frac{d^2 z_2^{[1]}}{dx^2}(x)| &\leq C(B_{\varepsilon_1}(x) + \varepsilon_1^2\varepsilon_2^{-2} B_{\varepsilon_2}(x)) \\
&\leq CB_{\varepsilon_2}(x), |\frac{dz_2^{[1]}}{dx}(x)| \leq C\varepsilon_2^{-1} B_{\varepsilon_2}(x).
\end{aligned}$$

To obtain bounds for the remainder $\mathbf{s}^{[1]}$, note that the first component of the right–hand–side can be written as

$$\begin{aligned}
&b_1(x, \mathbf{v} + \mathbf{w}^{[2]} + (z_1^{[1]}, 0)^T) - b_1(x, \mathbf{v} + \mathbf{w}^{[2]} + \mathbf{z}^{[1]}) \\
&= -\int_0^1 \frac{\partial b_1}{\partial u_2}(x, \mathbf{v} + \mathbf{w}^{[2]} + (z_1^{[1]}, tz_2^{[1]})^T) dt z_2^{[1]}.
\end{aligned}$$

Then, the maximum principle proves that $|\mathbf{s}^{[1]}(x)| \leq \mathbf{C}\varepsilon_1^2\varepsilon_2^{-2} B_{\varepsilon_2}(x)$. Hence,

$$|\frac{d^k \mathbf{s}^{[1]}}{dx^k}(x)| \leq C(\varepsilon_1^{-k}, \varepsilon_2^{-k})^T \frac{\varepsilon_1^2}{\varepsilon_2^2} B_{\varepsilon_2}(x),\ k = 0,1,2.$$

Differentiate (8d) and use above arguments to bound the third derivatives. □

## 3 Discrete Problem and Analysis of Uniform Convergence

The domain is divided into the subintervals $[0, \tau_{\varepsilon_1}], [\tau_{\varepsilon_1}, \tau_{\varepsilon_2}], \ldots, [\tau_{\varepsilon_m}, 1-\tau_{\varepsilon_m}], \ldots, [1-\tau_{\varepsilon_1}, 1]$. Distribute half the mesh points uniformly within $(\tau_{\varepsilon_m}, 1-\tau_{\varepsilon_m}]$ and the other half in the remaining intervals, distributing $N/(4m)+1$ mesh points uniformly in each $(\tau_{\varepsilon_i}, \tau_{\varepsilon_{i+1}}]$. The transition points are defined as

$$\tau_{\varepsilon_m} = \min\{0.25, 2\varepsilon_m/\beta \ln N\},\ \tau_{\varepsilon_i} = \min\left\{0.5\tau_{\varepsilon_{i+1}}, 2\varepsilon_i/\beta \ln N\right\},\ 1 \leq i < m. \tag{11}$$

On the mesh $\bar{\Omega}^N = \{x_i\}_{i=0}^N$, consider the following finite difference scheme

$$(\mathbf{T}_N\mathbf{U})(x_j) := -(\mathbf{E}\delta^2\mathbf{U})(x_j) + \mathbf{b}(x_j, \mathbf{U}(x_j)) = \mathbf{0},\ x_j \in \Omega^N = \bar{\Omega}^N \cap \Omega, \quad (12)$$

with $\mathbf{U}(0) = \mathbf{u}(0)$, $\mathbf{U}(1) = \mathbf{u}(1)$ and $\delta^2$ is the classical three–point finite difference approximation of the second derivative on a non–uniform mesh.

From (1c), the Frechet–derivative $\mathbf{T}'_N$ is an $M$–matrix and then for any two mesh functions $\mathbf{Y}$ and $\mathbf{Z}$ with $\mathbf{Y}(0) = \mathbf{Z}(0)$ and $\mathbf{Y}(1) = \mathbf{Z}(1)$, we have that

$$\|\mathbf{Y} - \mathbf{Z}\|_\infty \le \|(\mathbf{T}'_N)^{-1}\|_\infty \|\mathbf{T}_N\mathbf{Y} - \mathbf{T}_N\mathbf{Z}\|_\infty \le \frac{1}{\min\{1, \beta^2\}} \|\mathbf{T}_N\mathbf{Y} - \mathbf{T}_N\mathbf{Z}\|_\infty.$$

This implies the uniqueness of the solution to problem (12). In bounding the truncation error, we must bound the same terms

$$|\mathbf{T}_N\mathbf{u}(x)| = |\mathbf{T}_N\mathbf{u}(x) - \mathbf{T}\mathbf{u}(x)| \le \mathbf{E}|(\delta^2\mathbf{v} - \mathbf{v}'')(x)| + \mathbf{E}|(\delta^2\mathbf{w} - \mathbf{w}'')(x)|,$$

as in the linear problem [1]. The derivatives of both the regular and singular components have a similar behaviour to their linear counterparts, and thus we can deduce that $\|\mathbf{T}_N\mathbf{u}\|_\infty \le CN^{-1}$.

**Theorem 1.** *Let* $\mathbf{u}$ *be the solution of the problem (1) and* $\mathbf{U}$ *the solution of problem (12) on the Shishkin mesh* $\bar{\Omega}^N$. *Then,*

$$\|\mathbf{U} - \mathbf{u}\|_\infty \le CN^{-1}.$$

*Remark 1.* In the particular case of equal diffusion parameters $\varepsilon_i = \varepsilon, i = 1, \dots m$, it is possible [2] to prove essentially second order uniform convergence. In the linear case of $m = 2$, Linss and Madden [4] have established second order (up to logarithmic factors). To achieve this higher order, Linss and Madden [4] employ a decomposition (based on the decomposition in Madden and Stynes [6]) of the solutions, which is different to the decomposition presented in this paper. In the linear case of $m \ge 2$ Linß and Madden [5] have established second order convergence for arbitrary $\varepsilon_i$, under the assumption that the elements in the coefficient matrix $B(x)$ of the zero order terms satisfy

$$b_{ii}(x) > 0,\ \sum_{k \ne i}^{m} \|b_{ik}(x)/b_{ii}(x)\| < 1,\ 1 \le i \le m.$$

For variable coefficients and $m > 2$, these conditions will only be satisfied by a subset of problems from the class (1). Hence, the question of proving second order convergence for the class of problems in (1) for $m > 2$ and arbitrary $\varepsilon_i$ remains open.

## 4 Numerical Experiments

*Example 1.* Consider a nonlinear problem of type (1) where $m = 2$, $\mathbf{u}(0) = \mathbf{u}(1) = (0, 0)^T$, and

$$b_1(x,\mathbf{u}) = u_1-1-(1-u_1)^3+e^{u_1-u_2},\ b_2(x,\mathbf{u}) = u_2-0.5-(0.5-u_2)^5+e^{u_2-u_1}.$$

The corresponding nonlinear systems of equations associated with the discrete problem are solved using Newton's method with zero as an initial guess. We iteratively compute $\mathbf{U}^k(x_j)$, for $k = 1, 2, \ldots, K$, until

$$\|\mathbf{U}^K(x_j) - \mathbf{U}^{K-1}(x_j)\|_\infty \le N^{-2}.$$

To estimate the pointwise errors $|\mathbf{U}^K(x_j) - \mathbf{u}(x_j)|$ we calculate a new approximation $\{\hat{\mathbf{U}}^K(x_j)\}$ on the mesh $\{\hat{x}_j\}$ that contains the mesh points of the original mesh and its midpoints. At the coarse mesh points we calculate the uniform two-mesh differences and the orders of convergence

$$d_i^{N,K} = \max_{S_\varepsilon} \max_{0\le j\le N} |U_i^K(x_j) - \hat{U}_i^K(x_{2j})|,\ p_{i,uni}^{N,K} = \log_2(d_i^{N,K}/d_i^{2N,K}),\ i = 1, 2,$$

where the singular perturbation parameters take values in the set

$$S_\varepsilon = \{(\varepsilon_1, \varepsilon_2) \mid \varepsilon_2^2 = 2^0, 2^{-1}, \ldots, 2^{-30},\ \varepsilon_1^2 = \varepsilon_2^2, 2^{-1}\varepsilon_2^2, \ldots, 2^{-59}, 2^{-60}\}.$$

In Table 1 we display the uniform two-mesh differences and the approximate orders of convergence for both components $u_1$ and $u_2$. Finally, we report that $K \le 4$ for all $(\varepsilon_1, \varepsilon_2) \in S_\varepsilon$ and all $N = 2^{-j},\ j = 5, \ldots, 12$.

*Example 2*. Consider a linear problem of the type (1) where $m = 3,\ \mathbf{u}(0) = \mathbf{u}(1) = (1, 1, 1)^T$, and

$$\begin{aligned} b_1(x,\mathbf{u}) &= 2.1u_1 - (1-x)u_2 - (1+x)u_3 - x, \\ b_2(x,\mathbf{u}) &= -xu_1 + (1.1+x)u_2 - xu_3 + x, \\ b_3(x,\mathbf{u}) &= -(2+x)u_1 - (1-x)u_2 + (3.1+x)u_3 - 1. \end{aligned}$$

This linear problem is not covered by the theory in [5], but is covered by the theory in this paper. In Table 2 the uniform two-mesh differences and the approximate orders of uniform convergence are displayed, where the values of the singular perturbation parameters vary over the range

$$\varepsilon_3 = 2^0, 2^{-2}, \ldots, 2^{-30},\ \varepsilon_2 = \varepsilon_3, 2^{-2}\varepsilon_3, \ldots, 2^{-40},\ \varepsilon_1 = \varepsilon_2, 2^{-2}\varepsilon_2, \ldots, 2^{-60}.$$

**Table 1** Uniform two-mesh differences $\mathbf{d}^{N,K}$ and orders of convergence $\mathbf{p}_{uni}^{N,K}$ for Example 1

| $(\varepsilon_1, \varepsilon_2) \in S_\varepsilon$ | N = 32 | N = 64 | N = 128 | N = 256 | N = 512 | N = 1,024 | N = 2,048 | N = 4,096 |
|---|---|---|---|---|---|---|---|---|
| $d_1^{N,K}$ | 6.861E−3 | 6.222E−3 | 3.568E−3 | 1.313E−3 | 4.486E−4 | 1.423E−4 | 4.327E−5 | 1.291E−5 |
| $p_{1,uni}^{N,K}$ | 0.141 | 0.802 | 1.443 | 1.549 | 1.656 | 1.718 | 1.745 | |
| $d_2^{N,K}$ | 8.130E−3 | 3.915E−3 | 1.523E−3 | 5.343E−4 | 1.736E−4 | 5.375E−5 | 1.644E-5 | 4.943E−6 |
| $p_{2,uni}^{N,K}$ | 1.054 | 1.362 | 1.511 | 1.622 | 1.691 | 1.709 | 1.733 | |

**Table 2** Uniform two-mesh differences $\mathbf{d}^N$ and approximate uniform orders of convergence $\mathbf{p}_{uni}^N$ for Example 2

| | N = 16 | N = 32 | N = 54 | N = 128 | N = 256 | N = 512 | N = 1,024 | N = 2,048 |
|---|---|---|---|---|---|---|---|---|
| $[\mathbf{d}^N]_1$ | 0.151E+00 | 0.135E+00 | 0.113E+00 | 0.747E−01 | 0.378E−01 | 0.145E−01 | 0.484E−02 | 0.154E−02 |
| $[\mathbf{p}_{uni}^N]_1$ | 0.159 | 0.256 | 0.599 | 0.982 | 1.381 | 1.586 | 1.655 | |
| $[\mathbf{d}^N]_2$ | 0.159E+00 | 0.147E+00 | 0.119E+00 | 0.778E−01 | 0.381E−01 | 0.145E−01 | 0.472E−02 | 0.150E−02 |
| $[\mathbf{p}_{uni}^N]_2$ | 0.115 | 0.303 | 0.613 | 1.030 | 1.391 | 1.620 | 1.656 | |
| $[\mathbf{d}^N]_3$ | 0.158E+00 | 0.142E+00 | 0.119E+00 | 0.784E−01 | 0.397E−01 | 0.152E−01 | 0.508E−02 | 0.161E-02 |
| $[\mathbf{p}_{uni}^N]_3$ | 0.157 | 0.256 | 0.598 | 0.982 | 1.381 | 1.586 | 1.655 | |

For both examples, we observe uniform convergence of the finite difference approximations, which is in agreement with Theorem 1. However, orders greater than one are observed in both Tables.

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## References

1. J.L. Gracia, F.J. Lisbona, E. O'Riordan, A system of singularly perturbed reaction-diffusion equations, Dublin City University School of Mathematical Sciences preprint MS-07-10, (2007).
2. J.L. Gracia, F.J. Lisbona, M. Madaune-Tort, E. O'Riordan, A coupled system of singularly perturbed semilinear reaction-diffusion equations, Dublin City University School of Mathematical Sciences Preprint, MS-08-11, (2008).
3. J.L. Gracia, F.J. Lisbona, E. O'Riordan, A coupled system of singularly perturbed parabolic reaction-diffusion equations, *Adv. Comp. Math.*, DOI 10.1007/s10444-008-9086-3.
4. T. Linß, N. Madden, Accurate solution of a system of coupled singularly perturbed reaction-diffusion equations, *Computing*, **73** (2004), 121–133.
5. T. Linß, N. Madden, Layer-adapted meshes for a system of coupled singularly perturbed reaction-diffusion equations, *IMA J. Numer. Anal.*, **29** (2009), 109–125.
6. N. Madden, M. Stynes, A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems, *IMA J. Numer. Anal.*, **23** (2003), 627–644.