Uniform Consensus among Self-driven Particles

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Abstract. A nonconservative stability theory for switched linear systems is applied to the convergence analysis of consensus algorithms in the discrete-time domain. It is shown that the uniform-joint-connectedness condition for asymptotic consensus in distributed asynchronous algorithms and multi-particle models is in fact necessary and sufficient for uniform exponential consensus.

1 Introduction

We consider teams of mobile agents working together towards the common goal of reaching consensus asymptotically [1,2]. These agents are often modeled as spatially distributed self-driven particles whose states (e.g., positions and velocities) evolve according to the information received from their neighbors. Each agent has its own neighbor set, and the collection of such neighbor sets over all agents determines a communication topology of a team. As the agents' states evolve, their neighbor sets are updated over time, and the team's communication topology undergoes changes as well. Since the number of agents is finite, the number of all possible communication topologies is finite. Therefore, as argued in [3], the behavior of these mobile agents can be described by a switched, or hybrid, dynamical system whose mode of operation jumps from one to another within a finite set according to the underlying, possibly nondeterministic, switching structure [4,5,6,7].

The purpose of this paper is to use switched system stability theory and establish a condition for teams of mobile agents to reach consensus in the discrete-time domain. Existing work in the literature [8,3,9,10] builds on Markov chain and Lyapunov stability theories. However, despite the apparent connection between the area of switched systems and that of multi-agent teams, not much work has been done at the intersection of the two areas. This is partly because seeking a common quadratic Lyapunov function does not work for the latter [3], which is discouraging, and because a relevant nonconservative stability analysis for the former was discovered only very recently [11,12,13]. This paper presents a convergence analysis that fully exploits the connection between switched systems and multi-agent models.

One of the nice things that comes from the use of switched system theory is that the notion of uniform exponential consensus arises as a natural notion of

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convergence. Uniform exponential consensus requires the existence of a single rate at which the agents' states converge to a common value regardless of the initial time. This uniformity requirement guarantees that asymptotic consensus will occur against a disturbance that causes a sudden change in the agents' states at an arbitrary time instant. This robustness property against disturbance is not guaranteed under the notion of mere asymptotic consensus. Moreover, our convergence condition is equivalent to a well-known sufficient condition for asymptotic consensus (i.e., the uniform-joint-connectedness condition in [3, Theorem 2]), which turns out to be not only sufficient but also necessary for uniform exponential consensus.

In summary, the novelty of this work lies in the following aspects:

- The connection between switched systems and multi-agent models is fully exploited;
- The common notion of asymptotic consensus is replaced with the stronger but more useful notion of uniform exponential consensus;
- The condition that the communication topology be uniformly jointly connected is shown to be an exact condition for uniform exponential consensus.

The main result is presented in Section 2, and its proof is given in Section 3. Concluding remarks are made in Section 4.

Notation

The n-dimensional real Euclidean space is denoted by $\mathbb{R}^n$. The Euclidean vector norm $\|\cdot\|$ on $\mathbb{R}^n$ is defined by $\|x\| = \sqrt{x^{\mathrm{T}}x}$ for $x \in \mathbb{R}^n$. The spectral norm on $\mathbb{R}^{n\times n}$ is denoted by $\|\cdot\|$ as well, and is defined by

$$\|\mathbf{X}\| = \sup\left\{\sqrt{\lambda}\colon \lambda \text{ is an eigenvalue of } \mathbf{X}^{\mathrm{T}}\mathbf{X}\right\}$$

for $\mathbf{X} \in \mathbb{R}^{n\times n}$. If $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times n}$ are symmetric (i.e., $\mathbf{X} = \mathbf{X}^{\mathrm{T}}$ and $\mathbf{Y} = \mathbf{Y}^{\mathrm{T}}$) and $\mathbf{X} - \mathbf{Y}$ is negative definite (i.e., $x^{\mathrm{T}}(\mathbf{X} - \mathbf{Y})x < 0$ whenever $x \neq 0$), we write either $\mathbf{X} < \mathbf{Y}$ or $\mathbf{X} - \mathbf{Y} < \mathbf{0}$.

2 Main Result

Let $\mathbb{S}$ be the set of all symmetric stochastic matrices in $\mathbb{R}^{n\times n}$ with positive diagonal entries. (That is, $\mathbf{F} = (f_{ij}) \in \mathbb{S}$ if and only if $f_{ij} = f_{ji}$, $f_{ij} \geq 0$, $f_{ii} > 0$, and $\sum_{k=1}^{n} f_{ik} = 1$ for all i, $j \in \{1, \ldots, n\}$.) Associated with each $\mathbf{F} = (f_{ij}) \in \mathbb{S}$ is a graph $G \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that $(i, j) \in G$ if and only if $f_{ij} > 0$ and $i \neq j$. (Note that these graphs are identified with sets of edges as they share the common set of vertices given by $\{1, \ldots, n\}$.)

Definition 1. *A graph $G \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$ is said to be connected if between every pair of distinct vertices i, $j \in \{1, \ldots, n\}$ there exists a path $(i_0, i_1, \ldots, i_L) \in \{1, \ldots, n\}^{L+1}$ such that $i_0 = i$, $i_L = j$, and $(i_k, i_{k+1}) \in G$ for $k = 0, \ldots, L-1$. A set of graphs $\{G_j\colon j \in J\}$ is said to be jointly connected if its union $\bigcup_{j\in J} G_j$ is connected.*

A finite set

$$\mathcal{F} = \{\mathbf{F}_1, \ldots, \mathbf{F}_N\} \subset \mathbb{S} \tag{1}$$

defines a discrete linear inclusion (i.e., a discrete-time switched linear system under arbitrary switching) whose state-space representation is of the form

$$x(t+1) = \mathbf{F}_{\theta(t)} x(t) \tag{2}$$

for each switching sequence $\theta = (\theta(0), \theta(1), \ldots) \in \{1, \ldots, N\}^\infty$. For each $i \in \{1, \ldots, N\}$, let G_i be the graph associated with $\mathbf{F}_i$.

Definition 2. *Let $\mathcal{F}$ be as in* (1). *Let G_i be the graph associated with $\mathbf{F}_i$ for $i = 1, \ldots, N$. A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ is said to yield uniformly jointly connected graphs if there exists an integer $T \geq 0$ such that the set of graphs $\{G_{\theta(t)}, \ldots, G_{\theta(t+T)}\}$ is jointly connected for all $t = 0, 1, \ldots$.*

Associated with each $\mathbf{F}_i \in \mathcal{F}$ is a unique matrix $\mathbf{A}_i \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$\begin{bmatrix} 1 & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -1 \end{bmatrix} \mathbf{F}_i = \mathbf{A}_i \begin{bmatrix} 1 & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -1 \end{bmatrix}, \quad i = 1, \ldots, N.$$

Then

$$\mathcal{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_N\}$$

defines a discrete linear inclusion whose state-space description is given by

$$\hat{x}(t+1) = \mathbf{A}_{\theta(t)} \hat{x}(t) \tag{3}$$

for all switching sequences $\theta \in \{1, \ldots, N\}^\infty$. As argued in [3], the state equation (2) satisfies

$$\lim_{t\to\infty} x(t) = x_0 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{4}$$

for each $x(0) \in \mathbb{R}^n$, where $x_0 \in \mathbb{R}$ is a constant that depends on $x(0)$ (i.e., θ achieves asymptotic consensus for $\mathcal{F}$), if and only if the state equation (3) satisfies

$$\lim_{t\to\infty} \hat{x}(t) = 0 \tag{5}$$

for all $\hat{x}(0) \in \mathbb{R}^{n-1}$ (i.e., θ is asymptotically stabilizing for $\mathcal{A}$).

Definition 3. *Let $\mathcal{F}$ be as in* (1). *A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ is said to achieve uniform exponential consensus for $\mathcal{F}$ if there exist $c > 0$ and $\lambda \in (0,1)$ such that the state-space equation* (3) *satisfies*

$$\|\hat{x}(t)\| \leq c\lambda^{t-t_0} \|\hat{x}(t_0)\| \tag{6}$$

for all $t_0 \geq 0$, $t \geq t_0$, and $\hat{x}(t_0) \in \mathbb{R}^{n-1}$.

The following is the main result that establishes an exact condition under which a given switching sequence θ achieves uniform exponential consensus for $\mathcal{F}$.

Theorem 4. *Let $\mathcal{F}$ be as in* (1). *A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ achieves uniform exponential consensus for $\mathcal{F}$ if and only if it yields uniformly jointly connected graphs.*

The proof of this theorem is deferred to Section 3. The result is applicable to a large class of distributed algorithms and multi-agent networks; e.g., some of the linear discrete-time consensus algorithms studied in [8,3,9,10]. In particular, Vicsek et al.'s multi-particle model [14] employs a nearest neighbor rule with parameter $r > 0$ for n agents moving at a common speed. Here, a real-valued state $x_i(t)$ of the i-th agent (i.e., the heading of the i-th agent) is updated according to

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t), \quad i = 1,\ \ldots,\ N, \quad t = 0,\ 1,\ \ldots, \tag{7}$$

where

$$\begin{aligned} N_i(t) = \{j \in \{1, \ldots, n\} \colon & \text{position of agent } j \text{ at time } t \text{ is} \\ & \text{within radius } r \text{ from position of agent } i \text{ at time } t,\ j = 1, \ldots, n\} \end{aligned}$$

is the set of nearest neighbors of agent i (including agent i itself), and where $|N_i(t)|$ is the cardinality of $N_i(t)$. This update rule gives rise to a state equation of the form

$$x(t+1) = \mathbf{F}(t)x(t)$$

with $x(t) = [x_1(t) \ \cdots \ x_n(t)]^{\mathrm{T}}$ and $\mathbf{F}(t) \in \mathbb{S}$ for all t. Since the number N of distinct network topologies $\{N_1(t), \ldots, N_n(t)\}$ that can occur over all initial states $x(0) \in \mathbb{R}^n$ and over all time instants t is finite, we can label these topologies from 1 to N and obtain the state equation (2) with $\mathbf{F}(t) = \mathbf{F}_{\theta(t)}$, $\theta(t) \in \{1, \ldots, N\}$. Jadbabaie et al.'s sufficient condition [3] for asymptotic consensus states that, if there exists a τ and time instants $0 < t_1 < t_2 < \cdots$ such that $t_{k+1} - t_k \leq \tau$ for all k and such that the sets of graphs

$$\{G_{\theta(0)}, \ldots, G_{\theta(t_1-1)}\}, \quad \{G_{\theta(t_1)}, \ldots, G_{\theta(t_2-1)}\}, \quad \ldots$$

are all jointly connected, then the nearest neighbor rule (7) is guaranteed to yield asymptotic consensus; that is, all headings $x_i(t)$ approach a common value x_0 as $t \to \infty$. Putting $T = 2\tau$, this condition implies that the set $\{G_{\theta(t)}, \ldots, G_{\theta(t+T)}\}$ is jointly connected for all $t = 0,\ 1,\ \ldots$. Thus Theorem 4 asserts this sufficient condition for asymptotic consensus is in fact necessary and sufficient for uniform exponential consensus.

3 Proof of Main Result

3.1 Lemmas

There are a few lemmas required to prove Theorem 4. This subsection is devoted to summarizing them.

In the contexts of distributed asynchronous algorithms and multi-particle models, where each initial state leads to a deterministic switching sequence, it is known that asymptotic convergence of the state variables to a common value is guaranteed if the switching sequence yields uniformly jointly connected graphs [8,14,3,15].

Lemma 5. *If the switching sequence θ yields uniformly jointly connected graphs, then the state equation* (2) *satisfies* (4) *for each $x(0) \in \mathbb{R}^n$, where $x_0 \in \mathbb{R}$ is a constant that depends on $x(0)$.*

Proof. The result is due to $\mathcal{F}$ being a finite subset of $\mathbb{S}$. See, e.g., [3, Theorem 2].

On the other hand, in the context of discrete inclusions and switched systems under nondeterministic switching, it is known that the discrete linear inclusion $\mathcal{A}$ is asymptotically stable under arbitrary switching if and only if the generalized spectral radius of $\mathcal{A}$ is less than one, or equivalently, there exists a sub-multiplicative norm $\|\cdot\|_{\mathcal{A}}$ such that $\|\mathbf{A}_i\|_{\mathcal{A}} < 1$ for all $i \in \{1, \dots, N\}$ [16,17,18,19]. The following lemma is a simple consequence of this, and says that asymptotically stable discrete linear inclusions are in fact uniformly exponentially stable.

Lemma 6. *The state equation* (3) *satisfies* (5) *for all $\hat{x}(0) \in \mathbb{R}^{n-1}$ and $\theta \in \{1, \dots, N\}^\infty$ if and only if there exist $c > 0$ and $\lambda \in (0, 1)$ such that* (3) *satisfies* (6) *for all $t_0 \geq 0$, $t \geq t_0$, $\hat{x}(t_0) \in \mathbb{R}^{n-1}$, and $\theta \in \{1, \dots, N\}^\infty$.*

Proof. In fact, the result holds for any finite subset $\mathcal{A}$ of $\mathbb{R}^{(n-1)\times(n-1)}$. See, e.g., [11, Proposition 8].

Recent advances in the stability analysis of discrete-time switched linear systems give a characterization of uniformly exponentially stabilizing switching sequences. This characterization plays a crucial role in establishing our result, and hence is described here. For each integer $L \geq 0$, tuples of integers of the form $(i_0, \dots, i_L) \in \{1, \dots, N\}^{L+1}$ are called L-paths. Following the terminology used in [13], a finite set $\mathcal{N}$ of L-paths shall be said to be admissible if for each $(i_0, \dots, i_L) \in \mathcal{N}$ there exists an integer $M > 0$ such that $(i_0, \dots, i_L) = (i_{M-L}, \dots, i_M)$ and such that $(i_t, \dots, i_{t+L}) \in \mathcal{N}$ for all $t = 0, \dots, M - L$. Likewise, an admissible set $\mathcal{N}$ of L-paths shall be called $\mathcal{A}$-admissible if there exist symmetric positive definite matrices $\mathbf{X}_{(j_1,\dots,j_L)} \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfying the coupled Lyapunov inequalities

$$\mathbf{A}_{i_L}^{\mathrm{T}} \mathbf{X}_{(i_1,\dots,i_L)} \mathbf{A}_{i_L} - \mathbf{X}_{(i_0,\dots,i_{L-1})} < \mathbf{0}$$

for all L-paths $(i_0, \dots, i_L) \in \mathcal{N}$. Given a switching sequence $\theta \in \{1, \dots, N\}^\infty$ and an integer $L \geq 0$, let $\mathcal{N}_L(\theta)$ be the largest admissible subset of

$$\{(\theta(0), \dots, \theta(L)), (\theta(1), \dots, \theta(L+1)), \dots\}.$$

Then we have the following result:

Lemma 7. *There exist $c > 0$ and $\lambda \in (0, 1)$ such that the state equation* (3) *satisfies* (6) *for all $t_0 \geq 0$, $t \geq t_0$, and $\hat{x}(t_0) \in \mathbb{R}^{n-1}$, if and only if there exists an integer $L \geq 0$ such that $\mathcal{N}_L(\theta)$ is $\mathcal{A}$-admissible.*

Proof. As in the proof of Lemma 6, the result holds for any finite subset $\mathcal{A}$ of $\mathbb{R}^{(n-1)\times(n-1)}$. See [12, Corollary 3.4].

Suppose $\mathcal{N}$ is an $\mathcal{A}$-admissible set of L-paths. If the smallest $\mathcal{A}$-admissible subset of $\mathcal{N}$ is $\mathcal{N}$ itself, then $\mathcal{N}$ is called $\mathcal{A}$-minimal. As argued in [13], associated with each $\mathcal{A}$-minimal set of L-paths is a periodic uniformly exponentially stabilizing switching sequence for $\mathcal{A}$; moreover, each $\mathcal{A}$-admissible set is a finite union of $\mathcal{A}$-minimal sets. For switching sequences $\theta \in \{1, \ldots, N\}^\infty$ and integers $L \geq 0$, define $\mathcal{N}_L^\infty(\theta)$ as the set of L-paths $(i_0, \ldots, i_L)$ such that for any $t_0 \geq 0$ there exists a $t > t_0$ satisfying $(\theta(t), \ldots, \theta(t+L)) = (i_0, \ldots, i_L)$. Then $\mathcal{N}_L^\infty(\theta)$ contains the L-paths that occur infinitely many times in θ; it is nonempty because the set $\{1, \ldots, N\}^{L+1}$ of all L-paths is finite. In summary, we have the following lemma:

Lemma 8. *Suppose that there exists an integer $L \geq 0$ such that $\mathcal{N}_L(\theta)$ is $\mathcal{A}$-admissible. Then the following hold:*

(a) The set $\mathcal{N}_L^\infty(\theta)$ is $\mathcal{A}$-admissible and is identical to $\mathcal{N}_L((\theta(t_0), \theta(t_0+1), \ldots))$ for some integer $t_0 \geq 0$.
(b) The set $\mathcal{N}_L^\infty(\theta)$ is a finite union of $\mathcal{A}$-minimal sets of L-paths.

Proof. Part (b) is an immediate consequence of part (a), so it suffices to show part (a) holds true. Suppose $\mathcal{N}_L^\infty(\theta)$ is not admissible. Then, there exists an L-path $(i_0, \ldots, i_L) \in \mathcal{N}_L^\infty(\theta)$ such that, whenever $M > 0$ and $i_{L+1}, \ldots, i_M \in \{1, \ldots, N\}$ satisfy $(i_{M-L}, \ldots, i_M) = (i_0, \ldots, i_L)$, there exists a $t \in \{0, \ldots, M - L\}$ such that $(i_t, \ldots, i_{t+L})$ does not belong to $\mathcal{N}_L^\infty(\theta)$. That is, whenever we form a cycle of L-paths that contains $(i_0, \ldots, i_L)$, the cycle contains an L-path that does not occur infinitely many times in θ. Therefore, $(i_0, \ldots, i_L)$ cannot occur infinitely many times in θ. This contradicts the fact that $(i_0, \ldots, i_L) \in \mathcal{N}_L^\infty(\theta)$. Thus $\mathcal{N}_L^\infty(\theta)$ is admissible. Moreover, $\mathcal{N}_L^\infty(\theta)$ is $\mathcal{A}$-admissible because $\mathcal{N}_L^\infty(\theta)$ is an admissible subset of $\mathcal{N}_L(\theta)$, which is $\mathcal{A}$-admissible. To complete the proof, it remains to show that $\mathcal{N}_L^\infty(\theta) = \mathcal{N}_L((\theta(t_0), \theta(t_0+1), \ldots))$ for some $t_0 \geq 0$. Since $\mathcal{N}_L(\theta)$ is finite, the set difference $\mathcal{N}_L(\theta) \setminus \mathcal{N}_L^\infty(\theta)$ is finite. For each $(i_0, \ldots, i_L)$ in $\mathcal{N}_L(\theta) \setminus \mathcal{N}_L^\infty(\theta)$, let τ be the largest integer such that $(\theta(\tau-1), \ldots, \theta(\tau+L-1)) = (i_0, \ldots, i_L)$. Then letting t_0 be the maximum of such τ's over all L-paths in the finite set $\mathcal{N}_L(\theta) \setminus \mathcal{N}_L^\infty(\theta)$ yields the desired result.

3.2 Sufficiency

To prove sufficiency of Theorem 4, suppose a switching sequence θ yields uniformly jointly connected graphs. If G_i are the graphs associated with $\mathbf{F}_i$ for $i = 1, \ldots, N$, then there exists a $T \geq 0$ such that $\{G_{\theta(t)}, \ldots, G_{\theta(t+T)}\}$ is jointly connected for all $t = 0, 1, \ldots$. Given such a T, define

$$\mathcal{S} = \left\{ (i_0, \ldots, i_T) \in \{1, \ldots, N\}^{T+1} \colon \bigcup\nolimits_{t=0}^{T} G_{i_t} \text{ is connected} \right\},$$

so that $\mathcal{S}$ is the set of all T-paths over which the associated graphs are jointly connected. Define

$$\widetilde{\mathbf{F}}_{(i_0,\ldots,i_T)} = \mathbf{F}_{i_T} \cdots \mathbf{F}_{i_0} \quad \text{and} \quad \widetilde{\mathbf{A}}_{(i_0,\ldots,i_T)} = \mathbf{A}_{i_T} \cdots \mathbf{A}_{i_0}$$

for $(i_0, \ldots, i_T) \in \mathcal{S}$, and let

$$\widetilde{\mathcal{F}} = \{\widetilde{\mathbf{F}}_{(i_0,\ldots,i_T)} \colon (i_0, \ldots, i_T) \in \mathcal{S}\},$$
$$\widetilde{\mathcal{A}} = \{\widetilde{\mathbf{A}}_{(i_0,\ldots,i_T)} \colon (i_0, \ldots, i_T) \in \mathcal{S}\}.$$

By construction, $\widetilde{\mathcal{F}}$ forms a discrete linear inclusion whose elements $\widetilde{\mathbf{F}}_{(i_0,\ldots,i_T)}$ are associated with connected graphs

$$\widetilde{G}_{(i_0,\ldots,i_T)} = \bigcup\nolimits_{t=0}^{T} G_{i_t}, \quad (i_0, \ldots, i_T) \in \mathcal{S}.$$

By Lemma 5 we have that, for every sequence of T-paths $\tilde{\theta} = (\tilde{\theta}(0), \tilde{\theta}(1), \ldots)$ such that $\tilde{\theta}(t) \in \mathcal{S}$, $t = 0, 1, \ldots$, the state equation

$$\bar{x}(t+1) = \widetilde{\mathbf{F}}_{\tilde{\theta}(t)} \bar{x}(t)$$

satisfies $\lim_{t\to\infty} \bar{x}(t) = \bar{x}_0 \, [1 \ \cdots \ 1]^{\mathrm{T}}$ for each $\bar{x}(0) \in \mathbb{R}^n$, with some constant $\bar{x}_0$ depending on $\bar{x}(0)$. That is, the state equation

$$\tilde{x}(t+1) = \widetilde{\mathbf{A}}_{\tilde{\theta}(t)} \tilde{x}(t) \tag{8}$$

satisfies $\lim_{t\to\infty} \tilde{x}(t) = 0$ for all $\tilde{x}(0) \in \mathbb{R}^{n-1}$ and for all $\tilde{\theta} = (\tilde{\theta}(0), \tilde{\theta}(1), \ldots)$ with $\tilde{\theta}(t) \in \mathcal{S}$, $t = 0, 1, \ldots$. Then, by Lemma 6, there exist $\tilde{c} > 0$ and $\tilde{\lambda} \in (0, 1)$ such that the state equation (8) satisfies $\|\tilde{x}(t)\| \le \tilde{c}\tilde{\lambda}^{t-t_0} \|\tilde{x}(t_0)\|$ for all $t_0 \ge 0$, $t \ge t_0$, $\tilde{x}(t_0) \in \mathbb{R}^{n-1}$, and $\tilde{\theta} = (\tilde{\theta}(0), \tilde{\theta}(1), \ldots)$ with $\tilde{\theta}(s) \in \mathcal{S}$, $s = 0, 1, \ldots$. In particular, the given switching sequence $\theta = (\theta(0), \theta(1), \ldots)$ can be identified with a sequence of T-paths $\tilde{\theta} = (\tilde{\theta}(0), \tilde{\theta}(1), \ldots)$ via

$$\tilde{\theta}(t) = \big(\theta(t(T+1)), \ldots, \theta(t(T+1)+T)\big), \quad t = 0, 1, \ldots,$$

and it yields a state equation of the form (3) that satisfies

$$\|\hat{x}(\tau(T+1))\| \le \tilde{c}\tilde{\lambda}^{\tau-\tau_0} \|\hat{x}(\tau_0(T+1))\| \tag{9}$$

whenever $\tau \ge \tau_0 \ge 0$ and $\hat{x}(\tau_0(T+1)) \in \mathbb{R}^{n-1}$.

It remains to convert (9) to an inequality of the form (6). Let $\lambda \in (0, 1)$ be such that $\tilde{\lambda} = \lambda^{T+1}$, and let $M = \max_{1\le i\le N} \|\mathbf{A}_i\|/\lambda$. Whenever $t \ge t_0 \ge 0$, let τ be the largest integer such that $t \ge \tau(T+1)$, and let τ_0 be the smallest integer such that $\tau_0(T+1) \ge t_0$. Then it follows from (9) that

$$\|\hat{x}(t)\| \le \begin{cases} M^{t-t_0} \lambda^{t-t_0} \|\hat{x}(t_0)\| & \text{if } \tau_0 > \tau; \\ \tilde{c} M^{(t-\tau(T+1))+(\tau_0(T+1)-t_0)} \lambda^{t-t_0} \|\hat{x}(t_0)\| & \text{if } \tau_0 \le \tau. \end{cases}$$

If $\tau_0 > \tau$, then $t - t_0 \leq T$. Similarly, if $\tau_0 \leq \tau$, then $t - \tau(T+1) \leq T$ and $\tau_0(T+1) - t_0 \leq T$. Thus

$$\|\hat{x}(t)\| \leq \begin{cases} \max\{1, M\}^T \lambda^{t-t_0} \|\hat{x}(t_0)\| & \text{if } \tau_0 > \tau; \\ \tilde{c} \max\{1, M\}^{2T} \lambda^{t-t_0} \|\hat{x}(t_0)\| & \text{if } \tau_0 \leq \tau. \end{cases}$$

Now, letting $c = \max\{1, \tilde{c}\} \max\{1, M\}^{2T}$ yields that (6) holds for all $t_0 \geq 0$, $t \geq t_0$, and $\hat{x}(t_0) \in \mathbb{R}^{n-1}$. Therefore, θ achieves uniform exponential consensus for $\mathcal{F}$. This completes the proof of the sufficiency part of Theorem 4.

3.3 Necessity

To prove necessity of Theorem 4, suppose a switching sequence θ achieves uniform exponential consensus for $\mathcal{F}$. Then the state equation (3) satisfies (6) whenever $t \geq t_0 \geq 0$ and $\hat{x}(t_0) \in \mathbb{R}^{n-1}$. By Lemma 7 there exists a nonnegative integer L such that $\mathcal{N}_L(\theta)$ is $\mathcal{A}$-admissible, and hence by Lemma 8 the set $\mathcal{N}_L^\infty(\theta)$ is $\mathcal{A}$-admissible and is a finite union of $\mathcal{A}$-minimal sets of L-paths.

Choose an $\mathcal{A}$-minimal set $\mathcal{N}_{\min}$ of L-paths and the associated periodic switching sequence

$$\theta_{\min} = (i_0, \ldots, i_M, i_0, \ldots, i_M, \ldots),$$

where the period $M+1$ equals the cardinality of $\mathcal{N}_{\min}$. We will first show that $\theta_{\min}$ yields uniformly jointly connected graphs. Since $\mathcal{N}_{\min}$ is an $\mathcal{A}$-admissible set of L-paths, by Lemma 7 there exist $c > 0$ and $\lambda \in (0,1)$ such that the state equation (3), with θ replaced by $\theta_{\min}$, satisfies (6) whenever $t \geq t_0 \geq 0$ and $\hat{x}(t_0) \in \mathbb{R}^{n-1}$. That is, $\theta_{\min}$ achieves uniform exponential consensus for $\mathcal{F}$. Suppose $\theta_{\min}$ does not yield uniformly jointly connected graphs. Then, since $\theta_{\min}$ is periodic with period $M+1$, we have that the union $G = \bigcup_{t=0}^{M} G_{i_t}$, where G_i is the graph of $\mathcal{F}_i$, is not connected. That is, we can partition the set of vertices $\{1, \ldots, n\}$ into two disjoint sets $V_1, V_2 \subset \{1, \ldots, n\}$ such that $(i,j) \notin G$ whenever $(i,j) \in V_1 \times V_2$. Now, choose two distinct $x_1, x_2 \in \mathbb{R}$, and let $x(0) = (x_1(0), \ldots, x_n(0)) \in \mathbb{R}^n$ be such that

$$x_i(0) = \begin{cases} x_1 & \text{if } i \in V_1; \\ x_2 & \text{if } i \in V_2. \end{cases}$$

Because V_1 and V_2 remain disconnected under $\theta_{\min}$, and because the matrices $\mathbf{F}_i$ are stochastic, the state equation (2) will have that $x(t) = x(0)$ for all t under $\theta_{\min}$. This contradicts $\theta_{\min}$ achieving uniform exponential consensus for $\mathcal{F}$, and hence proves that $\theta_{\min}$ indeed yields uniformly jointly connected graphs.

Now that we have shown each $\mathcal{A}$-minimal set leads to a periodic switching sequence that yields uniformly jointly connected graphs, we are ready to show that the given θ, which achieves uniform exponential consensus for $\mathcal{F}$, yields uniformly jointly connected graphs. Let τ be the cardinality of $\mathcal{N}_L^\infty(\theta)$. By Lemma 8, there exists a t_0 such that, for each $t \geq t_0$, there exists an L-path $(i_0, \ldots, i_L)$ that occur more than once in the switching path $(\theta(t), \ldots, \theta(t+\tau+L))$; that is, for some $t_1, t_2 \in \{t, \ldots, t+\tau\}$ such that $t_1 < t_2$, we have

$$(\theta(t_1), \ldots, \theta(t_1+L)) = (\theta(t_2), \ldots, \theta(t_2+L)) = (i_0, \ldots, i_L).$$

Then it is clear that the set

$$\mathcal{N} = \{(\theta(t_1), \ldots, \theta(t_1 + L)), \ldots, (\theta(t_2 - 1), \ldots, \theta(t_2 + L - 1))\} \quad (10)$$

forms an $\mathcal{A}$-admissible set of L-paths. Since $\mathcal{N}$ contains an $\mathcal{A}$-minimal set of L-paths, we have that the union $\bigcup_{t=t_1}^{t_2-1} G_{\theta(t)}$ is connected. This is true for each $t \geq t_0$, and so the union $\bigcup_{s=t}^{t+\tau-1} G_{\theta(s)}$ is connected for all $t \geq t_0$. Therefore, putting $T = t_0 + \tau$ gives that the set of graphs $\{G_{\theta(t)}, \ldots, G_{\theta(t+T)}\}$ is jointly connected for all $t = 0, 1, \ldots$. This concludes the proof of the necessity part of Theorem 4.

4 Conclusions

Multi-agent consensus algorithms were studied via a nonconservative stability theory for switched systems, and a well-known sufficient condition for asymptotic consensus was shown to be necessary and sufficient for uniform exponential consensus. Possible extensions of this work include consideration of more general classes of consensus algorithms and incorporation of the state-dependent switching structure.

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