Epsilon-Tubes and Generalized Skorokhod Metrics for Hybrid Paths Spaces

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Abstract. We develop several generalized Skorokhod pseudo-metrics for hybrid path spaces, cast in a quite general setting, where the basic open sets are epsilon-tubes around paths that, intuitively, allow for some "wiggle room" in both time and space via set-valued retiming maps between the time domains of paths. We then determine necessary and sufficient conditions under which these topologies are Hausdorff and their distance functions are metrics. On spaces of paths with closed time domains, our metric topology of generalized Skorokhod uniform convergence on finite prefixes is equivalent to the implicit topology of graphical convergence of hybrid paths, currently used extensively by Teel and co-workers.

1 Introduction

A basic problem in the foundations of hybrid systems is that of giving useful quantitative measures of closeness between trajectories that may differ in their time domains, in virtue of variations in timing of discrete transition events, or in their way of letting time "run to infinity"; for example, how do we compare a Zeno trajectory with one that exhibits finite-escape time after finitely-many discrete transitions? Topological – and preferably metric – structure on spaces of hybrid trajectories, and on spaces of paths discretely simulating or approximating hybrid trajectories, is a necessary prelude to addressing questions of robustness, or of the accuracy of discrete simulations or approximations.

One approach addressing several of these issues (proposed independently by Teel and co-workers in [1] and by Collins in [2], and employed in [3,4,5,6]) is to model the time domain of a hybrid path as a linearly-ordered subset of the partially-ordered structure $\mathbb{R} \times \mathbb{Z}$; the coordinate in $\mathbb{R}$ gives the "normal" time and the coordinate in $\mathbb{Z}$ is incremented with each discrete transition.[1] In developing topological structure on hybrid path spaces, the papers [1,3,4,6], and also [2,5], take an indirect route: the convergence of a sequence of hybrid paths is formulated in terms of the set-convergence of the graphs of those paths as subsets of $\mathbb{R} \times \mathbb{Z} \times \mathbb{R}^n$, with set-convergence as in [16]. A more direct approach is taken in [17,18] and also in [19], which use variants of the *Skorokhod metric* (originally from stochastic processes with right-continuous sample paths

[1] This approach is equivalent to the so-called "hybrid time trajectories" used in [7,8]. Two-dimensional time structures linearly-ordered by the lexicographic order are also used in earlier work on hybrid trajectories in the context of logics and formal methods for hybrid systems in [9,10,11,12,13], and in behavioural systems approaches to hybrid systems, in [14,15].

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[20]) to structure the space of infinite non-Zeno hybrid trajectories modeled as functions with real-time domain $\mathbb{R}^+ = [0, \infty)$. These Skorokhod-type metrics accommodate trajectories with different transition times by using retiming maps which are strictly order-preserving, bijective functions from one time domain to the other. Intuitively, Skorokhod-type metrics allow us to "wiggle space and time a bit" – in contrast with the topology of uniform convergence of continuous functions over a common time domain, which only allows us to "wiggle space a bit". However, a significant limitation of the original Skorokhod-type metrics (discussed in [19]) is that strictly order-preserving, single-valued retiming maps are too inflexible and restrictive in a hybrid setting.

The first contribution of the present paper is to develop generalized Skorokhod pseudo-metrics for hybrid path spaces in a quite general setting, and to determine necessary and sufficient conditions under which these topologies are Hausdorff and their distance functions are metrics. We start with spaces of finite-length paths (including those with finite-escape time), where the key notion is that of ε-tolerance relations which pair finite-length paths that can be viewed as ε-close via a set-valued retiming map between their domains; the generalized Skorokhod distance between two paths is then the infimum of all such ε for that pair of paths. We then extend up to spaces of arbitrary-length paths by considering the ε-closeness of longer and longer finite prefixes. The generalized Skorokhod distance between two arbitrary-length paths is given as an infinite sum weighted by 2^{-n} of the distances between length-n finite prefixes. For arbitrary-length paths, we identify two distinct topologies, that of generalized Skorokhod uniform convergence, and that of generalized Skorokhod uniform convergence on finite prefixes, and determine distinct metrics for them. For the Hausdorff property, we give an easy-to-satisfy sufficient condition, as well as a more technical necessary and sufficient condition to mark the limits of metrizability. We also show that, restricted to spaces of arbitrary-length paths with closed time domains, the implicit topology of graph-convergence for hybrid paths from [1,3,4] is equivalent to the weaker of the two generalized Skorokhod metrics. The metric and convergence notions developed here are illustrated on spaces of solution paths of hybrid systems, under the standing assumptions used by Teel and co. in [1,3,4] in addressing questions of asymptotic stability.

The paper is a substantial advance on [21], which introduces set-valued retiming maps in order to accommodate various hybrid phenomena, and uses them in developing several (2- and 3-parameter) uniform topologies on hybrid path spaces, but without developing a pseudo-metric or characterizing the Hausdorff property, as is done here.

On notation: we write $R\colon X \rightsquigarrow Y$ to mean R is a set-valued map, with (possibly empty) values $R(x) \subseteq Y$; *domain* $\mathrm{dom}(R) := \{x \in X \mid R(x) \neq \varnothing\}$; *inverse* $R^{-1}\colon Y \rightsquigarrow X$ with $x \in R^{-1}(y)$ iff $y \in R(x)$; and *range* $\mathrm{ran}(R) := \mathrm{dom}(R^{-1})$. We do not distinguish between a set-valued map and a relation/set of ordered pairs $R \subseteq X \times Y$. For any set $A \subseteq X$, the direct or post-image is the set $R(A) := \{y \in Y \mid R^{-1}(y) \cap A \neq \varnothing\}$. If a map R is a *partial function*, we write $R\colon X \dashrightarrow Y$ to mean R is single-valued on its domain, with values $R(x) = y$ (rather than $R(x) = \{y\}$). As usual, $R\colon X \to Y$ means R is single-valued with $\mathrm{dom}(R) = X$ and $\mathrm{ran}(R) \subseteq Y$. We write $\mathbb{R}^+$ for $[0, \infty)$, $\mathbb{R}^{>0}$ for $(0, \infty)$, $\mathbb{R}^{+\infty}$ for $\mathbb{R}^+ \cup \{\infty\}$, and $\mathbb{N}^{>0}$ for $\{n \in \mathbb{N} \mid n > 0\}$.

2 Time Structures and Their Topologies

A structure $(S, \leqslant, 0, +, -)$ is an *partially-ordered abelian group* [22] if $(S, \leqslant)$ is a partial order, $(S, 0, +, -)$ is an abelian group, and the strict ordering $<$ is *shift-invariant*: $s < t$ implies $s + r < t + r$, for all $s, t, r \in S$. An element $u > 0$ is called an *order-unit* for the partially-ordered group S if for every $s \in S$, there exists an $m \in \mathbb{N}^+$ (depending on s) such that $s \leqslant m\,u$, where integer multiplication is just iterated addition. An order-unit uniquely determines a *pseudo-norm* $\|\cdot\|: S \to \mathbb{R}^+$ that assigns $\|u\| = 1$ and is such that for all $s, t \in S$, if $t \geq 0$ and $-t \leqslant s \leqslant t$ then $\|s\| \leq \|t\|$. As first identified by Stone [23], the order-unit pseudo-norm $\|\cdot\|$ from u has the explicit description:

$$(\forall s \in S) \quad \|s\| := \inf\left\{ \frac{m}{n} \in \mathbb{Q}^+ \mid m, n \in \mathbb{N}^+ \wedge -m\,u \leqslant n\,s \leqslant m\,u \right\} . \tag{1}$$

The pseudo-norm $\|\cdot\|$ is a norm (satisfying $\|s\| = 0$ iff $s = 0$, for all $s \in S$) when S is *archimedean*, which means that if $ks \leqslant t$ for all $k \in \mathbb{N}$, then $s \leqslant 0$.

Definition 1. [Time structures [21]]
A time structure $(S, \leqslant, 0, +, -, u)$ *is an archimedean partially-ordered abelian group with a distinguished order-unit* $u > 0$ *that determines an order-unit norm* $\|\cdot\|$. *A* future time structure T *is the positive cone of a time structure, so* $T = S^+ := \{s \in S \mid 0 \leqslant s\}$ *for some* S. *A time structure* S *is* finite-dimensional *iff for some integer* $n \geq 1$, S *is isomorphic with a partially-ordered abelian sub-group of* $(\mathbb{R}^n, \mathbf{1}^n)$ *with order-unit* $\mathbf{1}^n = (1, 1, \ldots, 1)$ *(hence* S *is lattice-ordered), where the embedding is a strictly order-preserving group isomorphism that is a continuous function w.r.t. the norm topologies and maps order-unit to order-unit and positive elements to positive elements.*

The continuous time structure $\mathbb{R}$ and the discrete time structure $\mathbb{Z}$ are both linearly-ordered abelian groups, and both are Dedekind-complete and archimedean; taking 1 as the order-unit gives the usual absolute-value $\|s\| = |s| = \max\{s, -s\}$. The basic hybrid time structure $\mathbb{Z}\times\mathbb{R}$ is a 2-dimensional abelian group with pair-wise addition and group identity $(0, 0)$, partially-ordered by the product order, $(i, t) \leqslant (i', t')$ iff $i \leq i'$ and $t \leq t'$; it is also Dedekind-complete and archimedean. The basic hybrid future time structure $\mathbb{H} := \mathbb{N} \times \mathbb{R}^+$ is the positive cone (and positive quadrant) of $\mathbb{Z} \times \mathbb{R}$. For the order-unit, we can take $u = (1, 1)$, and the Stone order-unit-norm is $\|(i, t)\| = \max\{\,|i|, |t|\,\}$. An equivalent norm, implicitly used in [3,4], is $\|(i, t)\|' := \frac{1}{2}(\,|i| + |t|\,)$, which satisfies $\frac{1}{2}\|(i, t)\| \leq \|(i, t)\|' \leq \|(i, t)\|$. For modeling and analysis of discrete-time simulations of hybrid systems, one uses $\mathbb{Z} \times \mathbb{Z}$, with future cone $\mathbb{N} \times \mathbb{N}$.

For each $r \in S$ in a time structure, the *r-shift function* $\sigma^r : S \to S$ is strictly order-preserving, where $\sigma^r(s) := s + r$ for all $s \in S$. In partial orders (as in linear orders) the basic sets are the *intervals* between points: sets $[a, b] := \{s \in S \mid a \leqslant s \leqslant b\}$ and $(a, b) := \{s \in S \mid a < s < b\}$; the *up-sets* above a given point: $[a\uparrow) := \{s \in S \mid a \leqslant s\}$ and $(a\uparrow) := \{s \in S \mid a < s\}$; symmetrically, the *down-sets* $(\downarrow a]$ and $(\downarrow a)$; and the *incomparability set*: $(a\bot) := S \setminus ([a\uparrow) \cup (a\downarrow])$, which is empty for all $a \in S$ iff the ordering is linear. In general, intervals, up-sets and down-sets are only partially-ordered.

In a time structure S with order-unit u, the *unit interval* is $[0, u]$, and the *granularity* of the norm $\|\cdot\|$ is defined by $\mathsf{gr}(S) := \inf\{\|s\| \in \mathbb{R}^+ \mid s \in (0, u]\}$. A time structure S

is *discrete* iff $\mathsf{gr}(S) > 0$, and is *dense* iff $\mathsf{gr}(S) = 0$. For example, $\mathbb{R}$, $\mathbb{Z} \times \mathbb{R}$, and $\mathbb{Q}_{\mathbb{B}} \times \mathbb{R}$ all have granularity 0, while $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ have granularity 1.

On a time structure S, let $\mathcal{T}_{\leqslant}$ be the order topology on S which has as a basis the family $\mathcal{B}_{\leqslant}$ of all strict up-sets and down-sets, and their intersections, the strict open intervals. Let $\mathcal{T}_{\text{norm}}$ be the norm topology on S determined by $\|\cdot\|$ which has as a basis the family $\mathcal{B}_{\text{norm}}$ of all norm-balls $B_\delta(s) := \{t \in S \mid \|t - s\| < \delta\}$, for $s \in S$ and real $\delta > 0$; $\mathcal{T}_{\text{norm}}$ is also the coarsest topology on S w.r.t. which $\|\cdot\| : S \to \mathbb{R}^+$ is continuous. From [21], some key properties of finite-dimensional time structures S are as follows:

(1) The norm topology is refined by the order topology; that is: $\mathcal{T}_{\text{norm}} \subseteq \mathcal{T}_{\leqslant}$, with $\mathcal{T}_{\text{norm}} = \mathcal{T}_{\leqslant}$ if $\leqslant$ is a linear-ordering.

(2) For all $s, t \in S$, intervals $[s, t]$, up-sets $[s\uparrow)$, and down-sets $(s\downarrow]$, are closed in $\mathcal{T}_{\text{norm}}$; if $s \leqslant t$, then $[s, t]$ is compact in $\mathcal{T}_{\text{norm}}$.

(3) For any subset $A \subseteq S$, A is norm-bounded iff A is order-bounded; if S is also Dedekind-complete, then A is compact in $\mathcal{T}_{\text{norm}}$ iff A is closed and bounded in $\mathcal{T}_{\text{norm}}$.

3 Compact Paths and Their Maximal Extensions

Definition 2. [Compact time domains [21]]
Given a time structure S with future time T, let $\mathsf{Lin}(T)$ *be the set of all non-empty linearly-ordered subsets L of T; i.e. the partial-order $\leqslant$ restricted to L is a linear-order. A* compact time domain *in T is any set $L \in \mathsf{Lin}(T)$ such that $0 \in L$ and L is compact in $\mathcal{T}_{\text{norm}}$. Let $\mathsf{CD}(T)$ be the set of all compact time domains in T.*

If S is finite-dimensional and Dedekind-complete and $L \in \mathsf{Lin}(T)$, then $L \in \mathsf{CD}(T)$ iff L contains 0, $L \subset [0, t]$ for some $t \in T$ and L is closed $\mathcal{T}_{\text{norm}}$. As a special case, all finite sample-time sets $L = \{0, t_1, \ldots, t_{N-1}\}$ are compact time domains. For any L in $\mathsf{CD}(T)$, either L is a single linearly-ordered and densely-ordered subset of T (including the one-point set $\{0\} = [0, 0]$), or else there exist one or more pairs of *discrete-successor points* $t_i, t'_i \in L$ such that $t_i < t'_i$ and $(t_i, t'_i) \cap L = \emptyset$.

Definition 3. [Compact continuous paths [21]]
Given a time structure S with future time T, let the signal value-space be a non-empty metric space (X, d_X). Define the set $\mathsf{CP}(T, X)$ *of* compact continuous T-paths in X *by:*

$$\mathsf{CP}(T, X) := \{\gamma : T \dashrightarrow X \mid \mathrm{dom}(\gamma) \in \mathsf{CD}(T) \wedge \gamma \textit{ is continuous on } \mathrm{dom}(\gamma)\}.$$

For $\gamma \in \mathsf{CP}(T, X)$, define the end-time *of γ by $b_\gamma := \max(\mathrm{dom}(\gamma))$, and the* length *of γ by $\mathsf{len}(\gamma) := \|b_\gamma\|_T$. Define a partial-ordering on $\mathsf{CP}(T, X)$ using subset-inclusion (on sets of ordered pairs) and the partial-ordering on T: $\gamma < \gamma'$ iff $\gamma \subset \gamma'$ and $t < t'$ for all $t \in \mathrm{dom}(\gamma)$ and all $t' \in \mathrm{dom}(\gamma') \setminus \mathrm{dom}(\gamma)$, in which case the path γ' is a* strict extension *of γ, and γ is a* strict prefix *of γ'; as usual, $\gamma \leqslant \gamma'$ iff $\gamma < \gamma'$ or $\gamma = \gamma'$.*

Being continuous on compact domains, all paths $\gamma \in \mathsf{CP}(T, X)$ are uniformly continuous. From [21] (differing slightly from [12,13]), the following three operations on paths are well-defined partial functions on $\mathsf{CP}(T, X)$: for $\gamma \in \mathsf{CP}(T, X)$, $t \in T$ and $b_\gamma = \max(\mathrm{dom}(\gamma))$:

- the *t-prefix* $\gamma|_t$, with $\mathrm{dom}(\gamma|_t) := [0, t] \cap \mathrm{dom}(\gamma)$ and $\gamma|_t(s) := \gamma(s)$ for all $s \in \mathrm{dom}(\gamma|_t)$;
- the *t-suffix* ${}_t|\gamma$, which is defined only when $t \in \mathrm{dom}(\gamma)$, with $\mathrm{dom}({}_t|\gamma) := [0, b_\gamma - t] \cap \sigma^{-t}(\mathrm{dom}(\gamma))$ where ${}_t|\gamma(s) := \gamma(s + t)$ for all $s \in \mathrm{dom}({}_t|\gamma)$;

- the *t-fusion* $\gamma *_t \gamma'$, which is defined only when $t \in \mathrm{dom}(\gamma)$ and $\gamma(t) = \gamma'(0)$, and which has $\mathrm{dom}(\gamma *_t \gamma') := \mathrm{dom}(\gamma|_t) \cup \sigma^{+t}(\mathrm{dom}(\gamma'))$ and $(\gamma *_t \gamma')(s) := \gamma(s)$ if $s \in \mathrm{dom}(\gamma|_t)$ and $(\gamma *_t \gamma')(s) := \gamma'(s-t)$ if $s \in \sigma^{+t}(\mathrm{dom}(\gamma'))$.

This prefix operation is well-defined for *all* times $t \in T$, not just $t \in \mathrm{dom}(\gamma)$, and $\gamma|_t \leqslant \gamma$ for all $t \in T$; in particular, $\gamma|_t < \gamma$ if $t \not\geqslant b_\gamma$, while $\gamma|_t = \gamma$ if $t \geqslant b_\gamma$. Moreover, for any $t \in T$, if $\|t\|_T > \mathsf{len}(\gamma)$, then $\gamma|_t = \gamma$. For all $t \in T$ and compact γ, the set $[0,t] \cap \mathrm{dom}(\gamma) = \mathrm{dom}(\gamma|_t)$ is compact and linearly-ordered, with maximum $t_0 = \max\{\, s \in \mathrm{dom}(\gamma) \mid s \leqslant t \,\}$. A set $P \subseteq \mathsf{CP}(T,X)$ is *prefix-closed* iff for all $\gamma \in P$ and all $t \in T$, the path $\gamma|_t \in P$. A set $P \subseteq \mathsf{CP}(T,X)$ is *deadlock-free* iff for all $\gamma \in P$, there exists $\gamma' \in P$ such that $\gamma < \gamma'$. From [12,13], a *general flow system* is a set-valued map $\Phi\colon X \rightsquigarrow \mathsf{CP}(T,X)$ such that for all $x \in \mathrm{dom}(\Phi)$, for all $\gamma \in \Phi(x)$, and all $t \in \mathrm{dom}(\gamma)$: (GF0) $x = \gamma(0)$; (GF1) suffix-closure ${}_t|\gamma \in \Phi(\gamma(t))$; and (GF2) fusion-closure $(\gamma *_t \gamma') \in \Phi(x)$ for all $\gamma' \in \Phi(\gamma(t))$.

We take finite-length compact paths as the basic objects precisely because in multi-dimensional time structures, there are multiple ways of "letting time go to infinity". However, for the asymptotic analysis of dynamics, as well as for the semantics of temporal logics of such systems [12,13] we do need to determine the *maximal extensions* of compact paths. When S is finite-dimensional, any $L \in \mathsf{Lin}(T)$ will have cardinality at most that of the reals, so we only need to consider extending sequences of paths of ordinal length at most ω_1, the first uncountable ordinal. Let CLO be the set of all countable limit ordinals ν with $\omega \le \nu < \omega_1$, where ω is the ordinal length of $\mathbb{N}$. Given any set $P \subseteq \mathsf{CP}(T,X)$, and a $\nu \in \mathrm{CLO}$, a ν-length sequence $\{\gamma_m\}_{m<\nu}$ is a *P-chain* if $\gamma_m < \gamma_{m'}$ for all $m < m' < \nu$. The *asymptotic limit* of a P-chain is the partial function $\eta\colon T \dashrightarrow X$ such that $\eta = \bigcup_{m<\nu} \gamma_m$ (considered as sets of ordered-pairs), with the *length* $\mathsf{len}(\eta) := \sup_{m<\nu} \mathsf{len}(\gamma_m)$, possibly infinite.

Definition 4. [Limit extension and maximal extension of path sets [12,13]] *Let T be the future of a finite-dimensional time structure. For any set $P \subseteq \mathsf{CP}(T,X)$ of compact paths, define the* limit extension $\mathsf{L}(P)$*, the* maximal extension $\mathsf{M}(P) \subseteq \mathsf{L}(P)$*, and the* maximal infinite-length extension $\mathsf{M}^\infty(P) \subseteq \mathsf{M}(P)$*, as follows:*

$$\begin{aligned}\mathsf{L}(P) := \{\, \eta \in [T \dashrightarrow X] \mid (\exists \nu \in \mathrm{CLO})\big(\exists \overline{\boldsymbol{\gamma}} \in [\nu \to \mathsf{CP}(T,X)]\big)(\forall m < \nu)\\ \gamma_m := \overline{\boldsymbol{\gamma}}(m) \in P \;\wedge\; (\forall m' < \nu)\,(m < m' \;\Rightarrow\; \gamma_m < \gamma_{m'}) \;\wedge\; \eta = \textstyle\bigcup_{m<\nu} \gamma_m \,\};\\ \mathsf{M}(P) := \{\, \eta \in \mathsf{L}(P) \mid (\forall \gamma \in P)\; \eta \nless \gamma \,\} \quad \textit{and} \quad \mathsf{M}^\infty(P) := \{\, \eta \in \mathsf{M}(P) \mid \mathsf{len}(\eta) = \infty \,\}.\end{aligned}$$

A set of compact paths P is called maximally-extendible *iff for all $\gamma \in P$, there exists $\eta \in \mathsf{M}(P)$ such that $\gamma < \eta$, and P is* forward-complete *iff P is maximally-extendible and $\mathsf{M}(P) = \mathsf{M}^\infty(P)$. Set* $\mathsf{LCP}(T,X) := \mathsf{L}(\mathsf{CP}(T,X))$.

The extension partial order on compact paths readily extends to limit paths: $\eta < \eta'$ iff $\eta \subset \eta'$ and $t < t'$ for all $t \in \mathrm{dom}(\eta)$ and $t' \in \mathrm{dom}(\eta') \setminus \mathrm{dom}(\eta)$. The prefix, suffix and fusion operations also extend to limit paths in the straight-forward way, with the strict prefix of a limit path always a compact path. It is also readily established that every limit path $\eta \in \mathsf{LCP}(T,X)$ is continuous (but may fail to be uniformly continuous).

Given a general flow system $\Phi\colon X \rightsquigarrow \mathsf{CP}(T,X)$, the *maximal extension* of Φ is the set-valued map $\mathsf{M}\Phi\colon X \rightsquigarrow \mathsf{LCP}(T,X)$ given by $\mathsf{M}\Phi(x) := \mathsf{M}(\Phi(x))$ for all $x \in X$, with $\mathsf{M}\Phi(x) = \emptyset$ if $x \notin \mathrm{dom}(\Phi)$. A general flow Φ is *maximally-extendible* (*forward-complete*) iff for all $x \in \mathrm{dom}(\Phi)$, the path set $\Phi(x)$ is maximally-extendible (forward-complete). From [12,13], a core result (using the Axiom of Choice/Zorn's Lemma) is that a set of paths $P \subseteq \mathsf{CP}(T, X)$ is maximally-extendible iff P is deadlock-free, and hence, for general flow $\Phi\colon X \rightsquigarrow \mathsf{CP}(T, X)$, Φ is maximally-extendible iff Φ is deadlock-free.

We will subsequently be interested in: $\mathsf{CP}^\star(T, X) := \mathsf{CP}(T, X) \cup \mathsf{LCP}(T, X)$, the combined path set of both compact and limit continuous paths under the path-extension ordering, of finite or infinite length, and also the distinguished subsets:

$$\mathsf{CP}^\star_{\mathrm{cl}}(T, X) := \mathsf{CP}(T, X) \;\cup\; \{\, \eta \in \mathsf{LCP}(T, X) \mid \mathrm{dom}(\eta) \textit{ is norm-closed in } T \,\}$$
$$\mathsf{CP}^\star_{\mathrm{fin}}(T, X) := \{\, \eta \in \mathsf{CP}^\star(T, X) \mid \mathsf{len}(\eta) < \infty \,\}$$
$$= \mathsf{CP}(T, X) \;\cup\; (\mathsf{CP}^\star(T, X) \setminus \mathsf{CP}^\star_{\mathrm{cl}}(T, X))\,.$$

The basic fact being used here is that a path $\eta \in \mathsf{CP}^\star(T, X) \setminus \mathsf{CP}^\star_{\mathrm{cl}}(T, X)$ exactly when η is a limit path with $\mathrm{dom}(\eta)$ failing to be norm-closed, which is the case if and only if $\mathrm{dom}(\eta)$ is norm-bounded with finite length. Given a set of compact paths $P \subseteq \mathsf{CP}(T, X)$, we say a limit path $\eta \in \mathsf{L}(P)$ has *finite-escape time* w.r.t. P iff $\eta \in \mathsf{M}(P)$ and $\mathsf{len}(\eta) < \infty$, and so $\eta \notin \mathsf{M}^\infty(P)$, and $\mathrm{dom}(\eta)$ will be norm-bounded but not norm-closed in T.

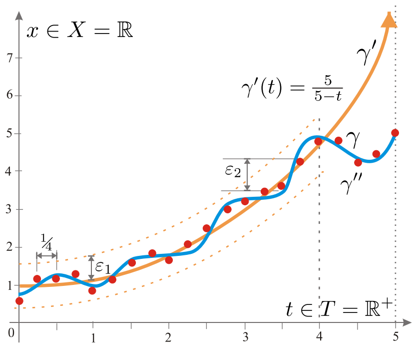

Fig. 1. Three finite-length real-time paths, with differing time domains

Example 1. Consider three finite-length paths $\gamma, \gamma', \gamma'' \in Z \subseteq \mathsf{CP}^\star_{\mathrm{fin}}(T, X)$ in Figure 1 where $T = \mathbb{R}^+$ and $X = \mathbb{R}$, and P the set of all $\gamma \in \mathsf{CP}(T, X)$ with either $\mathrm{dom}(\gamma) = [0, b]$, or $\mathrm{dom}(\gamma) = \{0, s_1, s_2, \ldots, s_N\}$, and $Z = (P \cup \mathsf{M}(P)) \cap \mathsf{CP}^\star_{\mathrm{fin}}(T, X)$. Here, γ is a compact path with $\mathrm{dom}(\gamma) = [0, 5]$, while γ' is a limit path in $\mathsf{M}(P)$ having $\mathrm{dom}(\gamma') = [0, 5)$ and $\gamma'(t) = \frac{5}{5-t}$ for all $t \in [0, 5)$, with escape to infinity at time 5. The third path γ'' is also compact (and uniformly continuous!), with $\mathrm{dom}(\gamma'') = \{\, \frac{k}{4} \mid 0 \leq k \leq 20 \,\}$, giving a time-sampling of the interval $[0, 5]$ with a (rather coarse) sampling period $d = \frac{1}{4}$.

4 Path Spaces and General Flows of Hybrid Systems

For a fixed metric space X, let $P_{\mathrm{hyb}}(X) \subset \mathsf{CP}(\mathbb{H}, X)$ be the set of regular compact hybrid paths γ whose time domains within $\mathbb{H} = \mathbb{N} \times \mathbb{R}^+$ are finite unions of the form

$\mathrm{dom}(\gamma) = \bigcup_{i<m} \{i\} \times [s_i, s_{i+1}] \cup \{m\} \times [s_m, b_\gamma]$, where $m \in \mathbb{N}$ in the number of discrete transitions, $s_0 := 0$ and for each $i < m$, $s_{i+1} \in \mathbb{R}^+$ is the real time at the $(i+1)^{\mathrm{st}}$ switching or discrete transition, with $s_{i+1} \geq s_i$. For maximal paths $\eta \in \mathsf{M}(P_{\mathrm{hyb}}(X))$, we have $\mathsf{len}(\eta) < \infty$ iff $\mathrm{dom}(\eta)$ fails to be norm-closed and $\mathrm{dom}(\eta) \subset [(0,0),(i,c))$ for some $(i,c) \in \mathbb{H}$, which will be the case exactly when the last continuous time evolution has finite-escape time. A hybrid limit path $\eta \in \mathsf{M}^\infty(P_{\mathrm{hyb}}(X))$ is *Zeno* iff $\mathsf{len}(\eta) = \infty$ and $\mathrm{dom}(\eta) \subset \mathbb{N} \times [0,c)$ for some $c < \infty$, in which case the length of η is infinite but the total real-time duration is finite and bounded by c. The non-Zeno infinite-length hybrid paths are those of infinite real-time duration, and such paths $\eta \in \mathsf{M}^\infty(P_{\mathrm{hyb}}(X))$ may have either an infinite or a finite number of discrete transitions; in the latter case, $\mathrm{dom}(\eta) = \bigcup_{i<m}\{i\} \times [s_i, s_{i+1}] \cup \{m\} \times [s_m, \infty)$ for some $m \in \mathbb{N}^{>0}$, while in the former case, $\mathrm{dom}(\eta) = \bigcup_{i\in\mathbb{N}}\{i\} \times [s_i, s_{i+1}]$.

Formulated within the framework of differential and difference inclusions [3,7], a *hybrid system* is a structure $H = (X, F, G, C, D)$ where:

- $X \subseteq \mathbb{R}^n$ is the state space, with $(C \cup D) \subseteq X$;
- $F\colon X \rightsquigarrow \mathbb{R}^n$ describes the continuous dynamics $\dot{x} \in F(x)$;
- $G\colon X \rightsquigarrow X$ describes the discrete dynamics $x' \in G(x)$;
- $C \subseteq (X \cap \mathrm{dom}(F))$ is the region of continuous flow; and
- $D \subseteq (X \cap \mathrm{dom}(G))$ is the discrete switching, jump or transition guard region.

The trajectories of H determine a prefix-closed general flow $\Phi_H\colon X \rightsquigarrow \mathsf{CP}(\mathbb{H}, X)$ such that a compact-domain hybrid path $\gamma \in \Phi_H(x)$ exactly when: (i) $x \in \mathrm{dom}(\Phi_H) := C \cup D$, and $x = \gamma(0,0)$; (ii) $\gamma \in P_{\mathrm{hyb}}(X)$ is a regular hybrid path, with end-time $(m, b_\gamma) := \max(\mathrm{dom}(\gamma))$, and switching times $\{s_{i+1}\}_{i\leq m}$ with $s_0 = 0$; (iii) for each $(i,t) \in \mathrm{dom}(\gamma)$, (a) if $t = s_{i+1}$, a switching time, then $\gamma(i,t) \in D$, and $\gamma(i+1,t) \in G(\gamma(i,t))$, and (b) if $i < m$ and $t \in [s_i, s_{i+1}]$, or if $i = m$ and $t \in [s_m, b_\gamma]$, then $\gamma(i,t) \in C$ and $\frac{d}{d\tau}\gamma(i,\tau) \in F(\gamma(i,\tau))$ for almost all $\tau \in [s_i, s_{i+1}]$, taking $s_{m+1} := b_\gamma$ when $i = m$, where the real-time curve segment $\xi_i\colon [s_i, s_{i+1}] \to X$ given by $\xi_i(\tau) := \eta(i,\tau)$ for all $\tau \in [s_i, s_{i+1}]$, is absolutely continuous on the interval $[s_i, s_{i+1}]$.

If any of the vector coordinates, say x_1 of $x \in X$, is designated *discrete*, as is the case for the locations in *hybrid automata*, then the first component $F_1\colon X \rightsquigarrow \mathbb{R}$ has $F_1(x) = \{0\}$ for all $x \in C$, and $x_1 \in Q$ for all $x \in C \cup D$, with Q a finite subset of $\mathbb{R}$, so that x_1 only changes value under G. If x_j is an (accurate) *clock*, then $F_j(x) = \{1\}$.

5 Generalized Skorokhod Topologies and Metrics on Path Spaces

When two paths η and η' in $\mathsf{CP}(T,X)$ or $\mathsf{LCP}(T,X)$ have *the same time domain*, we can use the metric d_X on X to determine if they spatially ε-close for their whole length by taking $d_\infty(\eta,\eta') := \sup_{t\in L} d_X(\eta(t),\eta'(t))$ for $L = \mathrm{dom}(\eta) = \mathrm{dom}(\eta')$. The infinity-metric d_∞ intuitively allows for some "wiggle in space" between the paths η and η'. In order to compare paths with *different time domains*, we need a notion of *retimings* between the time domains of paths, that allow for some "wiggle in time" as well as in space.

The *Skorokhod metric* allows for the comparison of real-time piecewise-continuous signals with differing points of discontinuity by using retimings that are *strictly order-preserving* functions between the time domains. Let $\mathsf{SRet}(\mathbb{R}^+)$ be the set of all strictly order-preserving and surjective $\rho : [0,b] \to [0,b']$, for $b, b' \in \mathbb{R}^+$, and for each $\rho \in$

$\mathsf{SRet}(\mathbb{R}^+)$, the *temporal deviation* is $\mathsf{dev}(\rho) := \sup_{t\in\mathrm{dom}(\rho)} |\, t - \rho(t)\,|$, possibly infinite, and applied to two signals with $\mathrm{dom}(\eta) = [0,b]$ and $\mathrm{dom}(\eta') = [0,b']$, the *spatial variation* is $\mathsf{var}(\eta,\eta',\rho) := \sup_{t\in\mathrm{dom}(\rho)} d_X(\eta(t),\eta'(\rho(t)))$, possibly infinite. For two finite-length interval-domain paths $\eta,\eta' : \mathbb{R}^+ \dashrightarrow X$ (one of several variants of) the Skorokhod distance between them is:

$$d_{\mathrm{Skor}}(\eta,\eta') := \begin{cases} \infty & \text{if there does not exist } \varepsilon \in \mathbb{R}^{>0} \text{ and } \rho \in \mathsf{SRet}(\mathbb{R}^+) \\ & \quad \text{such that } \mathsf{dev}(\rho) < \varepsilon \ \wedge\ \mathsf{var}(\eta,\eta',\rho) < \varepsilon\,, \\ \inf\{\,\varepsilon > 0 \mid (\exists\rho \in \mathsf{SRet}(\mathbb{R}^+))\ \mathsf{dev}(\rho) < \varepsilon \ \wedge\ \mathsf{var}(\eta,\eta',\rho) < \varepsilon\,\} \\ & \text{otherwise.} \end{cases} \tag{2}$$

The limitation of the Skorokhod metric when time T is hybrid is that too often, there will not be any strictly order-preserving functions between the domains of “close” paths. Non-strictly order-preserving single-valued maps are not invertible, so symmetry is lost. This motivates our relaxation to retiming maps that are order-preserving in a set-valued sense, are readily invertible and composable (like bijections), and include all order-preserving single-valued maps, strict and non-strict (the latter with set-valued inverses).

Definition 5. [The earlier-than relation on linearly-ordered sets, and retimings [21]] *Given a time structure S with future time T, the* earlier-than *relation $\trianglelefteq$ on the set $\mathsf{Lin}(T)$ of non-empty linearly-ordered subsets of T, is defined by:*

$$L \trianglelefteq L' \quad\Leftrightarrow\quad (\forall t \in (L \setminus L'))(\forall t' \in L')\ \ t < t' \ \ \wedge\ \ (\forall t \in L)(\forall t' \in (L' \setminus L))\ \ t < t'\,.$$

for all $L, L' \in \mathsf{Lin}(T)$. A set-valued map $\rho\colon T \rightsquigarrow T$ will be called order-preserving *iff $t_1 < t_2$ implies $\rho(t_1) \trianglelefteq \rho(t_2)$, for all $t_1,t_2 \in \mathrm{dom}(\rho)$. Given sets $L, L' \in \mathsf{Lin}(T)$, a set-valued map $\rho\colon T \rightsquigarrow T$ will be called a* retiming *from L to L' iff the following hold:*
(i) $\mathrm{dom}(\rho) = L$ *and* $\mathrm{ran}(\rho) = L'$*;*
(ii) *for all $t \in L$, $\rho(t) \in \mathsf{Lin}(T)$, and for all $t' \in L'$, $\rho^{-1}(t') \in \mathsf{Lin}(T)$; and*
(iii) *ρ and ρ^{-1} are both order-preserving.*
For a retiming $\rho\colon L \rightsquigarrow L'$, define the deviation $\mathsf{dev}(\rho) \in \mathbb{R}^{+\infty}$ *as follows:*

$$\mathsf{dev}(\rho) := \sup\{\, \|t - s\| \in \mathbb{R}^+ \mid t \in \mathrm{dom}(\rho) \ \wedge\ s \in \rho(t)\,\}\,.$$

Let $\mathsf{Ret}(L,L')$ denote the set of all retimings $\rho\colon L \rightsquigarrow L'$ together with all retimings $\rho'\colon L' \rightsquigarrow L$, so that $\mathsf{Ret}(L,L') = \mathsf{Ret}(L',L)$.

The key facts from [21] are: $\trianglelefteq$ is a partial-order on $\mathsf{Lin}(T)$; $\mathsf{Ret}(T)$ is closed under relational inverses and compositions of retimings, with $\mathsf{dev}(\rho^{-1}) = \mathsf{dev}(\rho)$ and $\mathsf{dev}(\rho \circ \rho') \le \mathsf{dev}(\rho) + \mathsf{dev}(\rho')$. In [21], we worked with a finer 2-parameter uniform structure on the space $\mathsf{CP}(T,X)$, with one parameter $\delta \in \mathbb{R}^{>0}$ bounding the temporal deviation $\mathsf{dev}(\rho)$ and the second $\varepsilon \in \mathbb{R}^{>0}$ bounding the spatial variation $\mathsf{var}(\gamma,\gamma',\rho)$. Here, we work on the larger space $\mathsf{CP}^\star_{\mathrm{fin}}(T,X)$ of all finite-length continuous paths, and with a view to developing a pseudo-metric and metric, we combine those two parameters into one by effectively taking their maximum. For the rest of the paper, we assume the time structure S is finite-dimensional, and (X,d_X) is a metric space.

Definition 6. [Uniform relations and generalized Skorokhod distance: finite-length] *Let $Z \subseteq \mathsf{CP}^\star_{\mathrm{fin}}(T,X)$ be any set of finite-length paths. For each pair $(\gamma,\gamma') \in Z \times Z$, let* $\mathsf{Ret}(\gamma,\gamma') := \mathsf{Ret}(\mathrm{dom}(\gamma),\mathrm{dom}(\gamma'))$*, and let* $\mathsf{Ret}(Z)$ *be the union of all* $\mathsf{Ret}(\gamma,\gamma')$ *for*

$\gamma, \gamma' \in Z$. *Then define the* variation function $\mathsf{var}\colon (Z \times Z \times \mathsf{Ret}(Z)) \to \mathbb{R}^{+\infty}$ *such that* $\mathsf{var}(\gamma, \gamma', \rho) := \infty$ *if* $\rho \notin \mathsf{Ret}(\gamma, \gamma')$*, and otherwise,* $\mathsf{var}(\gamma', \gamma, \rho^{-1}) = \mathsf{var}(\gamma, \gamma', \rho)$*, and assuming that* $\mathrm{dom}(\rho) = \mathrm{dom}(\gamma)$ *and* $\mathrm{ran}(\rho) = \mathrm{dom}(\gamma')$*, we have:*

$$\mathsf{var}(\gamma, \gamma', \rho) := \sup \{ d_x(\gamma(t), \gamma'(t')) \mid t \in \mathrm{dom}(\gamma) \wedge t' \in \mathrm{dom}(\gamma') \wedge (t, t') \in \rho \}.$$

For each strictly positive real $\varepsilon \in \mathbb{R}^{>0}$*, define the relation* $V_\varepsilon\colon Z \rightsquigarrow Z$ *as follows:*

$$V_\varepsilon := \{ (\gamma, \gamma') \in Z \times Z \mid (\exists \rho \in \mathsf{Ret}(\gamma, \gamma'))\ \mathsf{dev}(\rho) < \varepsilon \wedge \mathsf{var}(\gamma, \gamma', \rho) < \varepsilon \}.$$

The finite-length-paths generalized Skorokhod distance *function* $d_{\mathrm{fgS}} : Z \times Z \to \mathbb{R}^{+\infty}$ *is defined for all* $\gamma, \gamma' \in Z$ *by:*

$$d_{\mathrm{fgS}}(\gamma, \gamma') := \begin{cases} \inf \{ \varepsilon \in \mathbb{R}^{>0} \mid (\gamma, \gamma') \in V_\varepsilon \} & \textit{if } (\exists \varepsilon \in \mathbb{R}^{>0})\ (\gamma, \gamma') \in V_\varepsilon \\ \infty & \textit{otherwise.} \end{cases} \tag{3}$$

As with the original Skorokhod metric $\mathsf{var}(\gamma, \gamma', \rho)$ bounds the "wiggle in space" variation between γ and γ' under a retiming ρ, while $\mathsf{dev}(\rho)$ bounds the "wiggle in time" allowed by ρ. The (reflexive, symmetric) relation V_ε is one of ε-*tolerance* between paths γ and γ', and the ε-*tube* $V_\varepsilon(\gamma)$ around γ is the set of all paths $\gamma' \in Z$ that are ε-close, and contains only paths of length within ε of that of γ. For brevity, we will usually write "gS-" for the adjectival phrase "generalized Skorokhod", and "fgS-" for "finite-length-paths generalized Skorokhod".

Proposition 1. [Generalized Skorokhod uniform topology on finite-length paths]
Let $Z \subseteq \mathsf{CP}^\star_{\mathrm{fin}}(T, X)$ *be any set of finite-length continuous paths. For all* $\varepsilon, \varepsilon_1, \varepsilon_2 \in \mathbb{R}^{>0}$*:*

$$V_{\varepsilon_1} \subseteq V_{\varepsilon_2} \text{ when } \varepsilon_1 \le \varepsilon_2 \qquad V_\varepsilon \subseteq V_{\varepsilon_1} \cap V_{\varepsilon_2} \text{ when } \varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$$

$$V_{\varepsilon_1} \circ V_{\varepsilon_2} \subseteq V_\varepsilon \text{ when } \varepsilon_1 + \varepsilon_2 \le \varepsilon \qquad V_\varepsilon \circ V_\varepsilon \subseteq V_{\varepsilon_1} \text{ when } \varepsilon \le \tfrac{1}{2}\varepsilon_1,$$

and for all $\gamma, \gamma' \in Z$*,* $\quad d_{\mathrm{fgS}}(\gamma, \gamma') < \varepsilon$ *iff* $(\gamma, \gamma') \in V_\varepsilon$*.*
Hence the family $\mathcal{V}_{\mathrm{fgS}} := \{ V_\varepsilon\colon Z \rightsquigarrow Z \mid \varepsilon \in \mathbb{R}^{>0} \}$ *constitutes a basis for a uniformity on the path set Z, and the* fgS-uniform topology $\mathcal{T}_{\mathrm{fgS}}$ *generated by* $\mathcal{V}_{\mathrm{fgS}}$ *has as its basic open sets the* ε*-tubes* $V_\varepsilon(\gamma)$ *around paths* $\gamma \in Z$*. Furthermore, the fgS-distance function* $d_{\mathrm{fgS}} : Z \times Z \to \mathbb{R}^{+\infty}$ *is a pseudo-metric, and the topology generated by* d_{fgS} *is the same as the uniform topology* $\mathcal{T}_{\mathrm{fgS}}$.

Example 1 revisited. (See Fig. 1) For the example of the compact path γ with $\mathrm{dom}(\gamma) = [0, 5]$ and the spatially-unbounded limit path γ' with $\mathrm{dom}(\gamma') = [0, 5)$, the fgS-distance d_{fgS} comes out as $d_{\mathrm{fgS}}(\gamma, \gamma') = \infty$ because the distance $d_x(\gamma(5), \gamma'(t'))$ becomes arbitrarily large as $t' \to 5$, so no retiming of finite variation exists. However, from Fig. 1, the prefixes $\gamma|_4$ and $\gamma'|_4$ are quite close, with $d_{\mathrm{fgS}}(\gamma|_4, \gamma'|_4) < \varepsilon_1$ witnessed by the identity retiming; to be concrete, take $\varepsilon_1 \le 0.65$. To determine the fgS-distance between the (coarsely) sampled+quantized path γ'', and the original γ, three quantities come into play: (a) the sampling period, here $d = 0.25$; (b) the quantization error, here bounded by 0.2; and (c) the quantity labeled ε_2 in Fig. 1 from the uniform continuity of γ, such that for all $t, s \in \mathrm{dom}(\gamma)$, if $|\, t - s \,| \le 0.25$ then $d_{\mathbb{R}}(\gamma(t), \gamma(s)) \le \varepsilon_2$. Say $\varepsilon_2 \le 0.75$. The sampling retiming map $\rho_d : \mathrm{dom}(\gamma) \rightsquigarrow \mathrm{dom}(\gamma'')$ is given by $\rho_d(0) := \{0\}$ and $\rho_d(t) := \{\frac{k+1}{4}\}$ for all $t \in (\frac{k}{4}, \frac{k+1}{4}]$ and $k < 20$, so that $\mathsf{dev}(\rho_d) = d = 0.25$. Via the triangle inequality, the retiming ρ_d gives $d_{\mathrm{fgS}}(\gamma, \gamma'') \le \varepsilon_2 + 0.2 \le 0.95$.

Having established we have a uniform topology generated by ε-tubes, the further, more substantial task, is to identify sets $Z \subseteq \mathsf{CP}^\star_{\mathrm{fin}}(T, X)$ of finite-length paths for which this uniform topology is Hausdorff, as in this case, the fgS-pseudo-metric d_{fgS} is actually a metric. For a set $Z \subseteq \mathsf{CP}^\star(T,X)$ of arbitrary-length paths, we call Z *highly discerning* iff $Z \subseteq (P \cup \mathsf{M}(P))$ for some set $P \subseteq \mathsf{CP}(T,X)$ of compact paths. In particular, Z contains no limit paths $\gamma \in \mathsf{L}(P) \setminus \mathsf{M}(P)$ that are not maximal w.r.t. P. We will show that the highly discerning property is sufficient for the Hausdorff property. In seeking a necessary and sufficient characterization of the Hausdorff property, we weaken the condition on path sets $Z \subseteq \mathsf{CP}^\star(T,X)$ to isolate the problem cases. Call a set Z *discerning* iff for all paths $\gamma, \gamma' \in Z$, if $\gamma < \gamma'$ and $\gamma \notin \mathsf{CP}(T,X)$, then the set difference $\text{dom}(\gamma') \setminus \text{dom}(\gamma)$ is not a singleton set. The fact that, for a set Z of finite-length paths, highly discerning implies discerning, will be a corollary of the following main result. Note that both properties are trivially satisfied by all sets $Z \subseteq \mathsf{CP}^\star(T,X) = \mathsf{CP}(T,X) \cup \mathsf{M}^\infty(\mathsf{CP}(T,X))$ if T is discrete, and by all sets $Z \subseteq \mathsf{CP}^\star_{\text{cl}}(T,X)$, for arbitrary T.

Proposition 2. [Properties of fgS-uniform topology and pseudo-metric]
Let $Z \subseteq \mathsf{CP}^\star_{\text{fin}}(T,X)$ be equipped with the uniform topology $\mathcal{T}_{\text{fgS}}$.

1. *The topology $\mathcal{T}_{\text{fgS}}$ on Z has a countable sub-basis for its uniformity.*
2. *The topology $\mathcal{T}_{\text{fgS}}$ on Z is Hausdorff if Z is highly discerning.*
3. *The topology $\mathcal{T}_{\text{fgS}}$ on Z is Hausdorff if and only if Z is discerning.*
4. *The topology $\mathcal{T}_{\text{fgS}}$ on Z is Hausdorff if and only if the fgS-pseudo-metric d_{fgS} is a an extended-valued metric on Z.*
5. *If the topology $\mathcal{T}_{\text{fgS}}$ on Z is Hausdorff, then for all sequences $\{\gamma_k\}_{k\in\mathbb{N}}$ in Z and all paths $\gamma \in Z$, $\gamma = \lim_{k\to\infty} \gamma_k$ iff $\lim_{k\to\infty} d_{\text{fgS}}(\eta, \eta_k) = 0$.*
6. *Restricted to the subset $P := Z \cap \mathsf{CP}(T,X)$ of compact paths, the uniform topology $\mathcal{T}_{\text{fgS}}$ on P is always Hausdorff, and the fgS-metric is always finite-valued.*

The difficult part of the proof of Proposition 2 is *Part 3*, in establishing that the discerning property is sufficient for the topology to be Hausdorff. Most parts of the proof make essential use of the paths being continuous on their domains.

In "lifting up" the uniform structure of the V_ε relations on finite-length paths, in order to use it on spaces $Z \subseteq \mathsf{CP}^\star(T,X) = \mathsf{CP}(T,X) \cup \mathsf{LCP}(T,X)$ of paths of arbitrary length, the key idea is that since a limit path is just the union of a chain of longer and longer compact prexes, we should look at closeness of longer and longer compact prexes. This motivates the introduction of a second parameter $v \in \mathbb{R}^+$ which references the length up to which two paths are ε-close. (In [21], we used a time position parameter $t \in T$, which turned out to be sub-optimal when looking for a metric). As parameter sets, let $A_2 := \mathbb{R}^{>0} \times \mathbb{R}^+$ and $A_2^\infty = \mathbb{R}^{>0} \times \mathbb{R}^{+\infty} = A_2 \cup \{(\varepsilon, \infty) \mid \varepsilon \in \mathbb{R}^{>0}\}$. We need the following key technical result.

Proposition 3. [Path operations within fgS-uniform topology]
For any paths $\gamma, \gamma' \in \mathsf{CP}^\star_{\text{fin}}(T,X)$ and for any parameter $\varepsilon \in \mathbb{R}^{>0}$, if $d_{\text{fgS}}(\gamma, \gamma') < \varepsilon$ with witness $\rho \in \mathsf{Ret}(\gamma, \gamma')$ with $\mathsf{dev}(\rho) < \varepsilon$ and $\mathsf{var}(\gamma, \gamma', \rho) < \varepsilon$, then for all pairs of time points $(t, t') \in \rho$ related by ρ, we have $d_{\text{fgS}}(\gamma|_t, \gamma'|_{t'}) < \varepsilon$, $d_{\text{fgS}}({}_t|\gamma, {}_{t'}|\gamma') < \varepsilon$, and for all $\eta, \eta' \in \mathsf{CP}^\star_{\text{fin}}(T,X)$, $d_{\text{fgS}}(\gamma *_t \eta, \gamma' *_{t'} \eta') < \varepsilon$ if $\gamma(t) = \eta(0)$, $\gamma'(t') = \eta'(0)$ and $d_{\text{fgS}}(\eta, \eta') < \varepsilon$.

Definition 7. [Uniform relations and gS-distances: arbitrary-length]
Let $Z \subseteq \mathsf{CP}^\star(T,X)$ be any set of continuous paths of arbitrary length, and for each pair $(\varepsilon, v) \in A_2$, let $U_{\varepsilon,v} \colon Z \rightsquigarrow Z$ be the relation defined as follows:

$$U_{\varepsilon,v} := \{ (\eta,\eta') \in Z\times Z \mid \Big(\max\{\mathsf{len}(\eta), \mathsf{len}(\eta')\} \le v+\varepsilon \ \wedge\ d_{\mathrm{fgS}}(\eta,\eta') < \varepsilon\Big) \vee ((\exists\, t\in \mathrm{dom}(\eta))(\exists\, t'\in\mathrm{dom}(\eta')) \ \min\{\|t\|_T, \|t'\|_T\} \ge v \ \wedge\ d_{\mathrm{fgS}}(\eta|_t, \eta'|_{t'}) < \varepsilon\Big)\Big\}$$

and for each $\varepsilon \in \mathbb{R}^{>0}$*, let* $U_{\varepsilon,\infty} : Z \rightsquigarrow Z$ *be the relation defined by:*

$$U_{\varepsilon,\infty} \ := \ \bigcap_{v\in\mathbb{R}^+} U_{\varepsilon,v} \ = \ \{ (\eta,\eta')\in Z\times Z \mid (\forall v\in\mathbb{R}^+)\ (\eta,\eta')\in U_{\varepsilon,v} \} .$$

For each $v \in \mathbb{R}^{+\infty}$*, define the* length-$v$ gS-distance *function* $d^v_{\mathrm{gS}} : (Z\times Z) \to \mathbb{R}^{+\infty}$ *by:*

$$d^v_{\mathrm{gS}}(\eta,\eta') \ := \ \begin{cases} \infty & \text{if } (\forall \varepsilon\in\mathbb{R}^{>0})\ (\eta,\eta')\notin U_{\varepsilon,v} \\ \inf\{ \varepsilon\in\mathbb{R}^{>0} \mid (\eta,\eta')\in U_{\varepsilon,v} \} & \text{otherwise}. \end{cases} \tag{4}$$

Define the weak gS-distance $d_{\mathrm{wgS}} : Z\times Z \to [0,1]$*, for all* $\eta,\eta' \in Z$*, by:*

$$d_{\mathrm{wgS}}(\eta,\eta') := \sum_{n=1}^{\infty} 2^{-n}\min\{1, d^n_{\mathrm{gS}}(\eta,\eta')\}, \tag{5}$$

and the gS-distance $d_{\mathrm{gS}} : Z\times Z \to [0,1]$*, for all* $\eta,\eta'\in Z$*, by:*

$$d_{\mathrm{gS}}(\eta,\eta') := \tfrac{1}{2}\Big(\min\{1, d^\infty_{\mathrm{gS}}(\eta,\eta')\} \ + \ d_{\mathrm{wgS}}(\eta,\eta')\Big) \tag{6}$$

In defining the length-v tolerance relation $U_{\varepsilon,v}$ and, from that, the length-v gS-distance d^v_{gS} in equation (4), *either* the paths η and η' are both of length less than $v{+}\varepsilon$, and they are ε-close in the fgS metric, *or else* there is a pair of time points $(t,t') \in \mathrm{dom}(\eta)\times\mathrm{dom}(\eta')$ with both of *at least* length v and the compact prefixes $\eta|_t$ and $\eta'|_{t'}$ are ε-close in the fgS metric; the latter entails that $\|t - t'\| < \varepsilon$ from the witnessing retiming, without requiring the overly-strong condition that $t' = t$.

Proposition 4. [gS-uniform topologies and pseudo-metrics on arbitrary-length paths] *Let* $Z \subseteq \mathsf{CP}^\star(T,X)$ *be any set of continuous paths. Then for all* $\varepsilon\in\mathbb{R}^{>0}$ *and for all paths* $\eta,\eta'\in Z$*, and all* $v\in\mathbb{R}^{+\infty}$*,*

$$d^v_{\mathrm{gS}}(\eta,\eta') < \varepsilon \quad \text{iff} \quad (\eta,\eta')\in U_{\varepsilon,v};$$

and

$$U_{\varepsilon,\infty}(\eta) = V_\varepsilon(\eta) \ \text{ iff } \ \mathsf{len}(\eta) < \infty; \quad \text{and} \quad U_{\varepsilon,v}(\eta) = V_\varepsilon(\eta) \ \text{ if } \ \mathsf{len}(\eta) < v.$$

Each of the length-v *distance functions* d^v_{gS} *are pseudo-metrics on Z, as are both the gS-distance* d_{gS} *and the weak gS-distance* d_{wgS}*, and both families:*

$$\mathcal{U}_{\mathrm{gS}} := \{ U_{\varepsilon,v} : Z \rightsquigarrow Z \mid (\varepsilon,v)\in A_2^\infty \} \quad \text{and} \quad \mathcal{U}_{\mathrm{wgS}} := \{ U_{\varepsilon,v} : Z\rightsquigarrow Z \mid (\varepsilon,v)\in A_2 \}$$

constitute bases for uniformities on the path set Z. The uniform topology $\mathcal{T}_{\mathrm{wgS}}$ *on Z generated by* $\mathcal{U}_{\mathrm{wgS}}$ *has as its basic opens the* (ε,v)*-tubes* $U_{\varepsilon,v}(\eta)$ *for all finite pairs* $(\varepsilon,v)\in A_2$*, and is equivalently described by the family* $\{ d^v_{\mathrm{gS}} \mid v\in\mathbb{R}^+\}$ *of pseudo-metrics. The uniform topology* $\mathcal{T}_{\mathrm{gS}}$ *on Z generated by* $\mathcal{U}_{\mathrm{gS}}$ *has as its basic opens the* (ε,v)*-tubes* $U_{\varepsilon,v}(\eta)$ *around* $\eta\in Z$*, for all* $(\varepsilon,v)\in A_2^\infty$*; it is equivalently described by the family* $\{ d^v_{\mathrm{gS}} \mid v\in\mathbb{R}^{+\infty}\}$ *of pseudo-metrics; and it contains* $\mathcal{T}_{\mathrm{fgS}}$ *and* $\mathcal{T}_{\mathrm{wgS}}$ *as sub-topologies.*

We call $\mathcal{T}_{\mathrm{wgS}}$ the *topology of weak gS-uniform convergence*, and $\mathcal{T}_{\mathrm{gS}}$ the *topology of gS-uniform convergence*. For the Hausdorff property and metricizability, we can re-use the same notions developed for finite-length paths: the *discerning* and *highly discerning* properties. As for Proposition 2, by far the hardest part of Proposition 5 is that the discerning property implies the topology is Hausdorff. Verifying the equivalence of metric convergence and convergence in the uniform structures also takes some effort.

Proposition 5. [Properties of the 2-parameter gS-uniform topologies]
Let $Z \subseteq \mathsf{CP}^\star(T, X)$ be any set of continuous paths, of finite or infinite length.

1. *The uniform topologies $\mathcal{T}_{\mathrm{gS}}$ and $\mathcal{T}_{\mathrm{wgS}}$ on Z both have countable sub-bases.*
2. *The topologies $\mathcal{T}_{\mathrm{gS}}$ and $\mathcal{T}_{\mathrm{wgS}}$ on Z are both Hausdorff if Z is highly discerning.*
3. *The following five conditions are equivalent:*
   - *the path set Z is discerning;*
   - *the topology $\mathcal{T}_{\mathrm{gS}}$ on Z is Hausdorff;*
   - *the generalized Skorokhod pseudo-metric d_{gS} is a metric on Z;*
   - *the topology $\mathcal{T}_{\mathrm{wgS}}$ on Z is Hausdorff;*
   - *the weak generalized Skorokhod pseudo-metric d_{wgS} is a metric on Z.*
4. *If the path set Z is discerning, then for all sequences $\{\eta_k\}_{k\in\mathbb{N}}$ in Z and $\eta \in Z$:*
   (a) *$\{\eta_k\}_{k\in\mathbb{N}}$ converges gS-uniformly to η iff $\lim_{k\to\infty} d_{\mathrm{gS}}(\eta, \eta_k) = \lim_{k\to\infty} d^{\infty}_{\mathrm{gS}}(\eta, \eta_k) = 0$;*
   (b) *$\{\eta_k\}_{k\in\mathbb{N}}$ converges wgS-uniformly to η iff $\lim_{k\to\infty} d_{\mathrm{wgS}}(\eta, \eta_k) = 0$;*
   (c) *if all but finitely-many of the paths η_k, for $k \in \mathbb{N}$, as well as the path η, have finite length, then the following conditions on $\{\eta_k\}_{k\in\mathbb{N}}$ are equivalent:*
      - *it converges to η in the finite-length paths topology $\mathcal{T}_{\mathrm{fgS}}$;*
      - *it converges gS-uniformly to η ; and*
      - *it converges wgS-uniformly to η .*

   Hence when the path set Z is discerning, the topology $\mathcal{T}_{\mathrm{gS}}$ is metricized by d_{gS}, and the topology $\mathcal{T}_{\mathrm{wgS}}$ is metricized by d_{wgS} .

For $T = \mathbb{R}^+$ and $T = \mathbb{H}$, it is easy to find examples of sequences of paths $\{\eta_k\}_{k\in\mathbb{N}}$ that converge wgS-uniformly to an infinite-length path η, but do not converge in the stronger metrics d_{gS} and d^{∞}_{gS}. So the metrics and topologies are quite distinct, with $\mathcal{T}_{\mathrm{wgS}} \subsetneq \mathcal{T}_{\mathrm{gS}}$.

Example 1 revisited. Taking $\varepsilon_1 \leq 0.65$, we can compute rough numerical bounds of $d_{\mathrm{gS}}(\gamma, \gamma') < 0.84$ and $d_{\mathrm{wgS}}(\gamma, \gamma') < 0.61$ for gS-distances between the compact path γ and the spatially-unbounded path γ' with finite-escape time. Compare these with bounds of $d_{\mathrm{wgS}}(\eta, \eta'') \leq d_{\mathrm{gS}}(\gamma, \gamma'') \leq d_{\mathrm{fgS}}(\gamma, \gamma'') \leq 0.95$ for the sampling+quantization, with all three distances about the same. As depicted in Fig. 1, this makes sense: the path γ' is *closer* to γ than the coarsely sampled+quantized path γ''.

6 Relationship with Graphical Set-Convergence of Paths

Goebel and Teel in [3] develop a notion of convergence for sequences of hybrid paths (compact or limit) for the case of Euclidean space $X \subseteq \mathbb{R}^n$ and $T = \mathbb{H} \subset \mathbb{R}^2$ by employing the machinery of *set-convergence* for sequences of subsets Euclidean space, applied to paths $\eta \in \mathsf{CP}^\star(T, X)$ considered via their graphs as subsets of $T \times X \subset \mathbb{R}^{n+2}$; the text [16] is a standard reference on set-convergence. For any sequence $\{A_k\}_{k\in\mathbb{N}}$ of non-empty subsets of a metric space, in general, $\liminf_{k\to\infty} A_k \subseteq \limsup_{k\to\infty} A_k$, and the sequence $\{A_k\}_{k\in\mathbb{N}}$ *set-converges* to a set A if $\limsup_{k\to\infty} A_k = A = \liminf_{k\to\infty} A_k$, in which case A must be closed in the metric, and we write $A = \mathrm{setlim}_{k\to\infty} A_k$.

Proposition 6. [Equivalence of concepts of convergence]
Let S be a finite-dimensional time structure with future T, let (X, d_X) be a metric space, and let $Z \subseteq \mathsf{CP}^\star_{\mathrm{cl}}(T, X)$ be any set of paths with norm-closed time domains. Then for all paths $\eta \in Z$ and for all sequences of paths $\{\eta_k\}_{k\in\mathbb{N}}$ within Z, the following are equivalent:

(1) *the sequence* $\{\eta_k\}_{k\in\mathbb{N}}$ *converges wgS-uniformly to* η ;

(2) $\lim_{k\to\infty} d_{\text{wgS}}(\eta, \eta_k) = 0$;

(3) $\eta = \operatorname{setlim}_{k\to\infty} \eta_k$ *as graphs in the product topology on* $T \times X$; *and*

(4) $\forall$ *open sets* O *in* $T \times X$, *if* $\eta \cap O \neq \emptyset$ *then* $(\exists\, m_1 \in \mathbb{N})(\forall k \geq m_1)\quad \eta_k \cap O \neq \emptyset$, *and* $\forall$ *compact sets* K *in* $T \times X$, *if* $\eta \cap K = \emptyset$ *then* $(\exists\, m_2 \in \mathbb{N})(\forall k \geq m_2)\quad \eta_k \cap K = \emptyset$.

7 Application: Completeness and Semi-continuity of Hybrid Flows

A key result of [3] (subsequently used in [4,6] and elsewhere) is their Theorem 4.4 on a type of sequential compactness; it identifies conditions on the components of a hybrid system $H = (X, F, G, C, D)$ such that for $P := \operatorname{ran}(\Phi_{\text{H}})$, the path set $Z := P \cup \mathsf{M}(P)$ is such that for every locally eventually bounded sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in Z, there exists a path $\eta \in Z$ and a sub-sequence $\{\eta_{k_m}\}_{m\in\mathbb{N}}$ with $\eta = \operatorname{setlim}_{m\to\infty} \eta_{k_m}$. A sequence $\{\eta_k\}_{k\in\mathbb{N}}$ is *locally eventually bounded* iff for all length-bounds $b \in \mathbb{R}^{>0}$, there exists $m_b \in \mathbb{N}$ and a compact set $K_b \subseteq X$ such that for all $k \geq m_b$ and all $(i, t) \in \operatorname{dom}(\eta_k)$, if $\|\,(i, t)\,\|_{\mathbb{H}} \leq b$ then $\eta_k(i, t) \in K_b$. In the result below, we take the same conditions as identified in [3,4,6], and derive stronger conclusions cast in terms of the metrics d_{fgS}, d_{gS} and d_{wgS} on the spaces spaces Z_{fin} and Z.

For metric spaces X and Y, a set-valued map $R\colon X \rightsquigarrow Y$ is *locally-bounded* iff for every compact set $K \subseteq X$, the set-image $R(K)$ is bounded in Y. If $Y \subseteq \mathbb{R}^n$, then $R\colon X \rightsquigarrow Y$ is locally-bounded and outer semi-continuous iff R is upper semi-continuous and has compact values $R(x) \subseteq Y$. For $x \in \mathbb{R}^n$ and a set $C \subset \mathbb{R}^n$, the *tangent cone to* C *at* x is the set $\mathrm{TC}_C(x)$ of all vectors $v \in \mathbb{R}^n$ for which there exists a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ of positive reals converging monotonically to 0, together with a sequence $\{v_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to v, such that $v + \alpha_k v_k \in C$ for all $k \in \mathbb{N}$; see [7,16].

Proposition 7. [Cauchy-completeness, and semi-continuity of hybrid trajectories] *Let* $H = (X, F, G, C, D)$ *be a hybrid system as described in Section 4, with general flow map* $\Phi_H\colon X \rightsquigarrow P_{\text{hyb}}(X)$ *giving the compact-domain trajectories of* H *from any initial state* $x \in \operatorname{dom}(\Phi_H) = (C \cup D) \subseteq X$. *From* [3,4,6], *assume:*

(A0) $X \subseteq \mathbb{R}^n$ *is an open set;*

(A1) C *and* D *are relatively closed sets in* X*;*

(A2) $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *is outer semi-continuous and locally-bounded, and* $F(x)$ *is convex and compact in* $\mathbb{R}^n$ *for each* $x \in C$*;*

(A3) $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *is outer semi-continuous;*

(VC) *for all* $x \in C \setminus D$, *there exists an* $\varepsilon > 0$ *such that* $\mathrm{TC}_C(x') \cap F(x') \neq \emptyset$ *for every* ε*-close state* $x' \in B_\varepsilon(x) \cap C$; *and*

(VD) $G(x) \subseteq (C \cup D)$ *for all* $x \in D$.

Then let $P := \operatorname{ran}(\Phi_H) = \{\,\gamma \in \Phi_H(x) \mid x \in C \cup D\,\}$, *let* $Z := P \cup \mathsf{M}(P)$, *let* $Z_{\text{inf}} := \mathsf{M}^\infty(P)$, *let* $Z_{\text{fe}} := \mathsf{M}(P) \setminus \mathsf{M}^\infty(P)$, *and let* $Z_{\text{fin}} := P \cup Z_{\text{fe}}$, *so* $Z = P \cup Z_{\text{fe}} \cup Z_{\text{inf}} = Z_{\text{fin}} \cup Z_{\text{inf}}$, *with the unions disjoint. Further partition* Z_{inf} *as* $Z_{\text{inf}} = Z_0 \cup Z_1 \cup Z_\infty$, *where* $\eta \in Z_0$ *iff* η *is Zeno;* $\eta \in Z_1$ *iff* η *has finitely-many discrete transitions and* $\mathsf{len}(\eta) = \infty$*; and* $\eta \in Z_\infty$ *iff* η *has infinitely-many discrete transitions and* $\mathsf{len}(\eta) = \infty$. *Then:*

1. *P is prefix-closed and maximally-extendible, and for all* $\eta \in \mathsf{M}(P)$, *either* η *has infinite length or* η *is spatially-unbounded in* X.

2. *Both the sets P and Z_{fe}, as well as the space Z_{fin}, are both open and closed, and Cauchy-complete, in the metric d_{fgS} on Z_{fin}.*
3. *Each of the five path sets P, Z_{fe}, Z_0, Z_1 and Z_∞, as well as the whole space Z, are both open and closed, and Cauchy-complete, in the metrics d_{gS} and d^{∞}_{gS} on Z.*
4. *Each of the sets Z_{inf}, $P \cup Z_{\mathrm{inf}}$, and $\mathsf{M}(P)$, as well as the whole space Z, are closed and Cauchy-complete in the metric d_{wgS} on Z, while P and Z_{fe} are both open.*
5. *Additionally assume* (A4) : *the map $G : X \rightsquigarrow X$ is locally-bounded. Then:*
   (a) *The flow map $\Phi_H : X \rightsquigarrow P_{\mathrm{hyb}}(X)$ is (globally) outer semi-continuous w.r.t. the metric d_X on X and the metric d_{fgS} on $P_{\mathrm{hyb}}(X)$, and the map $\mathsf{M}\Phi_H : X \rightsquigarrow Z$ is (globally) outer semi-continuous w.r.t. each of d_{wgS}, d_{gS} and d^{∞}_{gS} on Z.*
   (b) *For each $x \in \mathrm{dom}(\Phi_H) = C \cup D$, the set $\Phi_H(x)$ of compact paths is closed and Cauchy-complete in the metric d_{fgS} on P.*
   (c) *For each $x \in C \cup D$, the set $\mathsf{M}\Phi_H(x)$ of maximal paths is closed and Cauchy-complete w.r.t. each of the metrics d_{gS}, d^{∞}_{gS} and d_{wgS} on Z.*
   (d) *For each $x \in C \cup D$, if $\mathsf{M}\Phi_H(x) \subset Z_{\mathrm{inf}}$, then for every $(\varepsilon, \nu) \in A_2$, there exists a real $\delta \in (0, \varepsilon]$ such that $\mathsf{M}\Phi_H(B_\delta(x)) \subseteq U_{\varepsilon,\nu}(\mathsf{M}\Phi_H(x))$, hence $\mathsf{M}\Phi_H : X \rightsquigarrow Z$ is locally upper semi-continuous at x w.r.t. the metric d_{wgS} on Z.*
   (e) *If $K \subseteq (C \cup D)$ is compact and $\mathsf{M}\Phi_H(K) \subseteq Z_{\mathrm{inf}}$, then for every $(\varepsilon, \nu) \in A_2$, there exists a real $\delta \in (0, \varepsilon]$ such that $\mathsf{M}\Phi_H(B_\delta(K)) \subseteq U_{\varepsilon,\nu}(\mathsf{M}\Phi_H(K))$.*

Theorem 4.4 of [3] can be used in proving part of *Part 4*, while *Part 5(e)* is an equivalent reformulation of Corollary 4.8 from that paper. From the viewpoint of stability, the upper semi-continuity of the map $\mathsf{M}\Phi_H : X \rightsquigarrow Z$ is highly desirable. The slightly stronger assumptions on the components of H used in [7] are sufficient to ensure that $\mathsf{M}\Phi_H$ is globally upper semi-continuous w.r.t. each of the metrics d_{wgS}, d_{gS} and d^{∞}_{gS}.

8 Conclusion

This paper develops several generalized Skorokhod pseudo-metrics for hybrid path spaces, cast in a quite general setting, where paths are continuous functions from a normed and partially-ordered time structure into a metric space, with the domains of paths linearly-ordered. The topologies generalize the original Skorokhod metric by allowing set-valued order-preserving retiming maps that are readily invertible and composable, are in practice quite easy to work with, and they include single-valued order-preserving maps as special cases. We determine necessary and sufficient conditions under which these topologies are Hausdorff and the distances are metrics. One of these metrics on arbitrary-length paths, that of weak gS-uniform convergence, is shown to be equivalent to the implicit topology of graphical convergence of hybrid paths, currently used extensively by Teel and co-workers. We apply the framework to investigate topological properties of hybrid general flows in the metrics d_{fgS}, d_{wgS} and d_{gS}.

The original motivation for this work was to develop topological and metric foundations as a prequel to giving a *robust semantics* for the temporal logic **GFL*** [12,13], which generalizes computational tree logic **CTL*** to semantics over general flow systems, uniformly for arbitrary time structures – discrete, continuous or hybrid. The key idea is that if a system satisfies a performance specification robustly, with the specification given by a logic formula, then a path η satisfies the specification only when all the paths in some ε-tube around η also satisfy the specification. With those foundations in place, that research project is under way.

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