Chapter 1
Introduction

These lecture notes are mainly concerned with the study of conformal quantum field theories in two dimensions. Over the last 20 years, the understanding and mathematical formulation of such theories has developed to a very mature state, and conformal field theory (CFT) has influenced both mathematics and physics. As such, it can be considered a prototype example for a constructive interplay between these two subjects.

Compared to ordinary quantum field theories in four dimensions, conformal field theories in two dimensions can be defined in a rather abstract way via operator algebras and their representation theory. In fact, there are many examples of CFTs where the usual description in terms of a Lagrangian action with resulting perturbative expansion is not even known. Instead, following a so-called *boot-strap* approach, one can define these theories without making reference to an action and sometimes one can even solve them exactly. Such a procedure is possible because the algebra of infinitesimal conformal transformations in two dimensions is very special: in contrast to its higher dimensional counterparts, it is infinite dimensional and therefore highly constraining.

The main feature of a conformal field theory is the invariance under conformal transformations. Roughly speaking, these are transformations leaving angles invariant and a particular example is the scaling $\vec{x} \mapsto a\,\vec{x}$ of a point $\vec{x}$ by some constant a. A field theory exhibiting such a symmetry has no preferred scale and one can only expect a physical system to have this property, if there are no dimensionful scales involved.

At first sight, it seems hard to find examples for such systems. However, the field theory of a free boson encounters a conformal symmetry for the case of vanishing mass. And even for interacting theories it is known that at the fixed point of a renormalisation group flow, there are only long-range correlations. Therefore, the natural mass scale at this point, that is, the inverse of the correlation length, vanishes and a conformal field theory description might be available. Physical systems with a conformal symmetry are thus more common than one would have naively expected.

More concrete examples featuring a conformal symmetry are the following. For statistical models in two dimensions, the continuum description at a second-order

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phase transition is given by a conformal field theory. The prime example is the so-called Ising model which is a two-dimensional model of a ferromagnet. It has been shown to be integrable and to have a critical temperature where a second-order phase transition occurs.

Another important instance featuring conformal symmetry is string theory, which is a candidate theory for the unification of all interactions including gravity. Here, the CFT arises as a two-dimensional field theory living on the world-volume of a string which moves in some background space–time. The dynamics of this string is governed by a so-called non-linear sigma model whose condition for conformal invariance, that is, the vanishing of the β-functional, gives the string equations of motion. The sigma model perturbation theory is governed by an expansion in ℓ_s/R, where ℓ_s is the natural string length and R a typical length scale of the background geometry. With the help of CFT techniques, one can solve this theory exactly to all orders in perturbation theory and one can sum all contributions of so-called world-sheet instantons. Therefore, conformal field theory is a very powerful tool for string theory, not only in the perturbative regime $\ell_s/R \ll 1$ but also at small length scales $R \sim \ell_s$ where genuine string effects become important and geometric intuition often fails.

These lecture notes are based on a 30×1.5 hours of graduate course for master students and thus provide only a first introduction into the broad field of conformal field theory. In particular, the main emphasis of this course was on applications of CFT techniques to string theory and so we will neither attempt to give an axiomatic approach to CFTs nor are we giving a complete survey of the many advances in this field. Instead, we are going to present some topics important for string theory which are usually not covered in the standard CFT literature. This includes super-conformal field theories (SCFTs), a very powerful class of exactly solvable string compactifications known as Gepner models, and boundary conformal field theory (BCFT), which in string theory appears for the description of so-called D-branes. A more detailed overview is the following:

- In Chap. 2 of these notes, we study the basic properties of conformal field theories including the discussion of the conformal group, primary fields, radial quantisation, the operator product expansion, the operator algebra of chiral quasi-primary fields and the representation theory of the Virasoro algebra. However, due to our personal selection of priorities, not all mathematical details are proven in a rigorous way. Instead, we put more emphasis on providing computational techniques which have been proven to be useful in string theory.
- In Chap. 3, we discuss in more detail symmetries of conformal field theories which are crucial for their solvability. In particular, we study infinite-dimensional generalisations of Lie algebras known as Kač–Moody algebras, and we see how they define concrete examples of CFTs. This involves a presentation of the Sugawara and coset constructions. Moreover, we also explain non-linear extensions of the Virasoro algebra, the so-called $\mathcal{W}$ algebras.

- In Chap. 4, we move forward and study CFTs on the torus where new consistency conditions arise from the action of the modular group. We present some simple but important examples such as the free boson, the free fermion, orbifold CFTs and the parafermionic CFT. We also state the Verlinde formula and discuss the simple current construction which is important for string theory.
- In Chap. 5, we present the generalisations of our previous findings to Supersymmetric conformal field theories. In particular, $\mathcal{N} = 2$ SCFTs have important applications in string theory as they are the underlying structure for compactifications preserving supersymmetry in four space–time dimensions. We will discuss the spectral flow operator, the chiral ring and the so-called Gepner models which are exactly known backgrounds in string theory valid beyond the perturbative level.
- In Chap. 6, we finally discuss boundary conformal field theories which in string theory describe open strings ending on D-branes. We show that these BCFTs can be defined in an abstract two-dimensional way without referring to the space–time notion of D-branes, we discuss the computation of partition functions for BCFTs and we introduce CFTs defined on non-orientable surfaces. With all this structure available, as the last result of this lecture, we derive the condition that the orientifold of the bosonic string has gauge group $SO(8192)$ in 26 dimensions.