Chapter 8
AFS Formal Concept and AFS Fuzzy Formal Concept Analysis

In this chapter, based on the original idea of Wille of formal concept analysis and the AFS (Axiomatic Fuzzy Set) theory, we presents a rigorous mathematical treatment of fuzzy formal concept analysis referred to as an AFS Formal Concept Analysis (AFSFCA). It naturally augments the existing formal concepts to fuzzy formal concepts, with the aim of deriving their mathematical properties and applying them in the exploration and development of knowledge representation. Compared with other fuzzy formal concept approaches such as the L-concept [1, 2] and the fuzzy concept [48], the main advantages of AFSFCA are twofold. One is that the original data and facts are the only ones required to generate AFSFCA lattices thus human interpretation is not required to define the fuzzy relation or the fuzzy set on $G \times M$ to describe the uncertainty dependencies between the objects in G and the attributes in M. Another advantage comes with the fact that is that AFSFCA is more expedient and practical to be directly applied to real world applications.

FCA(Formal Concept Analysis) was introduced by Rudolf Wille in 1980s [10]. In the past two decades, FCA has been a topic of interest both from the conceptual as well as applied perspective. In artificial intelligence community, FCA is used as a knowledge representation mechanism [15, 50, 51] as well as it can support the ideas of a conceptual clustering [4, 40] for Boolean concepts. Traditional FCA-based approaches are hardly able to represent vague information. To tackle with this problem, fuzzy logic can be incorporated into FCA to facilitate handling uncertainty information for conceptual clustering and concept hierarchy generation. Pollandt [42], Burusco and Fuentes-Gonza lez [3], Huynh and Nakamori [16], and Belohlavek [1, 2] have proposed the use of the L-Fuzzy context as an attempt to combine fuzzy logic with the FCA. The primary notion in this investigation is that of a fuzzy context (L-context): it comes as a triple $(G, M, \mathbb{I})$, where G and M are sets interpreted as the set of objects (G) and the set of attributes (M), and $\mathbb{I} \in L^{G\times M}$ is a fuzzy relation between G and M. The value $\mathbb{I}(g,m) \in L$ (L is a lattice) is interpreted as the truth value of the fact "the object $g \in G$ has the attribute $m \in M$". In accordance with the Port-Royal definition, a (formal) fuzzy concept (L-concept) is a pair (A,B), $A \in L^G, B \in L^M$, A plays the role of the extent (fuzzy set of objects which determine the concept), B plays the role of the intent (fuzzy set of attributes

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which determine the concept). The L-Fuzzy context uses linguistic variables, which are linguistic terms associated with fuzzy sets, to represent uncertainty in the context. However, human interpretation is required to define the linguistic variables and the fuzzy relation between G and M (i.e., $\mathbb{I} \in L^{G\times M}$). Moreover, the fuzzy concept lattice generated from the L-fuzzy context usually causes a combinatorial explosion of concepts as compared to the traditional concept lattice.

Tho, Hui, Fong, and Cao [48] proposed a technique that combines fuzzy logic and FCA giving rise to the idea of the Fuzzy Formal Concept Analysis (FFCA), in which the uncertainty information is directly represented by membership grades. The primary notion is that of a fuzzy context: it is a triple $(G,M,\mathbb{I})$, where G is a set of objects, M is a set of attributes, and $\mathbb{I}$ is a fuzzy set on domain $G\times M$. Each relation $(g,m) \in \mathbb{I}$ has a membership value $\mu_{\mathbb{I}}(g,m)$ in $[0,1]$. Compared to the fuzzy concept lattice generated from the L-fuzzy context, the fuzzy concept lattice generated by using FFCA is simpler in terms of the number of formal concepts. However, human interpretation is still referring to it as the required to define the membership function of the fuzzy set $\mathbb{I}$ for FFCA. In real world applications, just based on human interpretation, it is very difficult to properly define the fuzzy set $\mathbb{I}$ to describe the uncertainty relations between the objects and the attributes.

In order to cope with the above problems, we propose a new framework of fuzzy formal concept analysis based on the AFS (Axiomatic Fuzzy Set) theory [18, 54] referring to it as the AFS Formal Concept Analysis (AFSFCA,for brief). In the proposed AFSFCA, each fuzzy complex attribute in EM, which plays the role of the intent of an AFS formal concept, corresponds to a fuzzy set, which is automatically determined by the AFS structure and the AFS algebra via what we have discussed in Chapter 4, 5, and plays the role of the extent of the AFS formal concept. Thus the original data and facts are only required to generate AFSFCA lattices and human interpretation is not required to define the fuzzy relation or the fuzzy set $\mathbb{I}$ on $G\times M$ to describe the uncertainty relations between the objects and the attributes. Compared with the fuzzy concept lattices based on L-fuzzy context, the fuzzy concept lattice generated using AFSFCA will be simpler in terms of the number of formal concepts. Compared with FFCA, the fuzzy concept lattice generated using AFSFCA will be richer in expression, more relevant and practical.

8.1 Concept Lattices and AFS Algebras

In Chapter 4, 5, various kinds of representations and logic operations for fuzzy concepts in EM have been extensively discussed in the framework of AFS theory, in which the membership functions and their logic operations are automatically determined in an algorithmic fashion by taking advantage of the existing distribution of the original data. The purpose of this section is to extend these approaches by combining the AFS and FCA theories.

Let us briefly recall the Wille's notion of formal concept [57]: The basic notions of FCA are those of a formal context and a formal concept. A *formal context* is a triple (G,M,I) where G is a set of objects, M is a set of features or attributes, and I is a binary relation from G to M, i.e., $I \subseteq G\times M$. gIm, which is also written as

$(g,m) \in I$, denotes that the object g possesses the feature m. An example of a context (G,M,I) is shown in Table 8.1, where $G = \{g_1, g_2, ..., g_6\}$ and $M = \{m_1, m_2, ..., m_5\}$. An "$\times$" is placed in the ith row and jth column to indicate that $(g_i, m_j) \in I$. For a set of objects $A \subseteq G$, $\beta(A)$ is defined as the set of features shared by all the objects in A, that is,

$$\beta(A) = \{m \in M | (g,m) \in I, \forall g \in A\}. \tag{8.1}$$

Similarly, for $B \subseteq M$, $\alpha(B)$ is defined as the set of objects that possesses all the features in B, that is,

$$\alpha(B) = \{g \in G | (g,m) \in I, \forall m \in B\}. \tag{8.2}$$

The pair (β, α) is a *Galois connection* between the power sets of G and M. For more information on Galois connections, interested readers are referred to [57]. In this chapter, the symbols α, β always denote the Galois connection defined by (8.1) and (8.2). In the FCA, concept lattice, or Galois lattice is the core of its mathematical theory and can be used as an effective tool for symbolic data analysis and knowledge acquisition.

Table 8.1 Example of a context

	m_1	m_2	m_3	m_4	m_5
g_1	$\times$		$\times$	$\times$	$\times$
g_2	$\times$	$\times$	$\times$	$\times$	
g_3	$\times$	$\times$	$\times$	$\times$	
g_4	$\times$				$\times$
g_5	$\times$				$\times$
g_6	$\times$				$\times$

Lemma 8.1. *Let (G,M,I) be a context. Then the following assertions hold:*

(1) for $A_1, A_2 \subseteq G$, $A_1 \subseteq A_2$ implies $\beta(A_1) \supseteq \beta(A_2)$ and for $B_1, B_2 \subseteq M$, $B_1 \subseteq B_2$ implies $\alpha(B_1) \supseteq \alpha(B_2)$;

(2) $A \subseteq \alpha(\beta(A))$ and $\beta(A) = \beta(\alpha(\beta(A)))$ for all $A \subseteq G$, and $B \subseteq \beta(\alpha(B))$ and $\alpha(B) = \alpha(\beta(\alpha(B)))$ for all $B \subseteq M$.

Its proof is left to the reader.

Definition 8.1. ([51]) A *formal concept* in the context (G,M,I) is a pair (A,B) such that $\beta(A) = B$ and $\alpha(B) = A$, where $A \subseteq G$ and $B \subseteq M$.

In other words, a formal concept is a pair (A,B) of two sets $A \subseteq G$ and $B \subseteq M$, where A is the set of objects that possesses all the features in B and B is the set of features common to all the objects in A. In what follows, a formal concept (A,B) in (G,M,I) briefly noticed as $(A,B) \in (G,M,I)$. The set A is called the *extent of the concept* and B is called its *intent*. If we review $B \subseteq M$ as a new attribute generated by the "*and*" of all attributes in B like that in [28], then A is the set of objects that possess the new attribute B. The adjective "*formal*" in formal concept means that the concept is a rigorously defined mathematical object [8]. From the point of view of logic, the intent of a formal concept can be seen as a conjunct of features that each object of the extent must possess. For any given context (G,M,I), neither every subset of G nor every subset of M corresponds to a concept.

Definition 8.2. ([51]) A set $B \subseteq M$ is called a *feasible intent* if set B is the intent of the unique formal concept $(\alpha(B),B)$. Similarly, a set $A \subseteq G$ is called a *feasible extent* if A is the extent of the unique formal concept $(A,\beta(A))$. A set X is called a *feasible set* if it is either a feasible extent or a feasible intent. Otherwise, X is called *non-feasible*.

An important notion in FCA is that of a concept lattice, which makes it possible to depict the information represented in a context as a complete lattice. Let $\mathscr{L}(G,M,I)$ denote the set of all formal concepts of the context (G,M,I). An order relation on $\mathscr{L}(G,M,I)$ is defined as follows [51]. Let (A_1,B_1) and (A_2,B_2) be two concepts in $\mathscr{L}(G,M,I)$, then $(A_1,B_1) \leq (A_2,B_2)$ if and only if $A_1 \subseteq A_2$ (or equivalently $B_1 \supseteq B_2$). The formal concept (A_1,B_1) is called a *sub formal concept of the formal concept* (A_2,B_2) and (A_2,B_2) is called a *super formal concept* of (A_1,B_1). The fundamental theorem of Wille about concept lattices, states that $(\mathscr{L}(G,M,I),\vee,\wedge)$ is a complete lattice called the *concept lattice of the context* (G,M,I).

Lemma 8.2. *(Will's Lemma) Let (G,M,I) be a context and $\mathscr{L}(G,M,I)$ denote the set of all formal concepts of the context (G,M,I). Then*

$$\mathscr{L}(G,M,I) = \{(\alpha(B),\beta(\alpha(B))) \mid B \subseteq M\}. \tag{8.3}$$

Proposition 8.1. *Let (G,M,I) be a context. Then for any $A_i \subseteq G, i \in I$, $B_j \subseteq M$, $j \in J$,*

$$\alpha\left(\bigcup_{j\in J} B_j\right) = \bigcap_{j\in J} \alpha(B_j),$$
$$\beta\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} \beta(A_i).$$

Proof. By the definitions, for any $g \in \alpha(\bigcup_{j\in J} B_j)$, we have

$$\begin{aligned} g \in \alpha\left(\bigcup_{j\in J} B_j\right) &\Leftrightarrow \forall m \in \bigcup_{j\in J} B_j,\ (g,m) \in I \\ &\Leftrightarrow \forall j \in J,\ \forall m \in B_j,\ (g,m) \in I \\ &\Leftrightarrow \forall j \in J,\ g \in \alpha(B_j) \\ &\Leftrightarrow g \in \bigcap_{j\in J} \alpha(B_j). \end{aligned}$$

Similarly, we can prove that $\beta\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} \beta(A_i)$. □

Theorem 8.1. *(Fundamental Theorem of FCA) Let (G,M,I) be a context. Then $(\mathscr{L}(G,M,I),\vee,\wedge)$ is a complete lattice in which suprema and infima are given as follows: for any formal concepts $(A_j,B_j)\in\mathscr{L}(G,M,I)$, $j\in J$,*

$$\bigvee_{j\in J}(A_j,B_j)=\left(\gamma_G\left(\bigcup_{j\in J}A_j\right),\ \bigcap_{j\in J}B_j\right), \tag{8.4}$$

$$\bigwedge_{j\in J}(A_j,B_j)=\left(\bigcap_{j\in J}A_j,\ \gamma_M\left(\bigcup_{j\in J}B_j\right)\right), \tag{8.5}$$

where $\gamma_G=\alpha\cdot\beta$, $\gamma_M=\beta\cdot\alpha$.

Proof. First, let us explain the formula for the infimum. Since $A_j=\alpha(B_j)$, for each $j\in J$,

$$\left(\bigcap_{j\in J}A_j,\gamma_M\left(\bigcup_{j\in J}B_j\right)\right)$$

by Proposition 8.1 it can be transformed into

$$\left(\alpha\left(\bigcup_{j\in J}B_j\right),\gamma_M\left(\bigcup_{j\in J}B_j\right)\right),$$

i.e., it has the form $(\alpha(X),\gamma_M(X))$ and is therefore a concept. That this can only be the infimum, i.e., the largest common subconcept of the concepts (A_j,B_j), follows immediately from the fact that the extent of this concept is exactly the intersection of the extents of (A_j,B_j). The formula for the supremum is substantiated correspondingly. Thus, we have proven that $(\mathscr{L}(G,M,I),\vee,\wedge)$ is a complete lattice. □

In what follows, we denote the subsets of G with small letters and the subsets of M with capital letters in order to distinguish subsets of objects in G from subsets of attributes in M.

By sets G,M, we can establish the *EII* algebra over G,M and $(EGM,\vee,\wedge)$ is a completely distributivity lattice. Now, we study the relationship between the lattice $(\mathscr{L}(G,M,I),\vee,\wedge)$ and the lattice $(EGM,\vee,\wedge)$. We define $\alpha(EM)$ a sub sets of EGM as follows

$$\alpha(EM)=\left\{\gamma\in EGM\mid\gamma=\sum_{i\in I}b_iB_i,\ \forall i\in I,\ b_i=\alpha(B_i)\right\}. \tag{8.6}$$

Lemma 8.3. *Let (G,M,I) be a context. Then $\alpha(EM)$ is a sub EII algebra of EGM, i.e. $k\in K$, $\zeta_k=\sum_{i\in I_k}b_{ki}B_{ki}\in\alpha(EM)$, $\bigvee_{k\in K}\zeta_k,\bigwedge_{k\in K}\zeta_k\in\alpha(EM)$, and $(\alpha(EM),\vee,\wedge)$ is also a completely distributivity lattice.*

Proof. It could be easily verified that $\bigvee_{k\in K}\zeta_k\in\alpha(EM)$. Since EGM is a completely distributivity lattice, hence

$$\bigwedge_{k\in K}\zeta_k=\sum_{f\in\Pi_{k\in K}I_k}(\bigcap_{k\in K}b_{kf(k)}\bigcup_{k\in K}B_{kf(k)}).$$

By Proposition 8.1 and $\alpha(B_{kj})=b_{kj}$, for any $k\in K, j\in I_j$, we have

$$\alpha\left(\bigcup_{k\in K}B_{kf(k)}\right)=\bigcap_{k\in K}\alpha(B_{kf(k)})=\bigcap_{k\in K}b_{kf(k)}.$$

Therefore $\bigwedge_{k\in K}\zeta_k\in\alpha(EM)$. Because $(EGM,\vee,\wedge)$ is a completely distributivity lattice, $(\alpha(EM),\vee,\wedge)$ is also a completely distributivity lattice. □

Theorem 8.2. *Let (G,M,I) be a context. p_I is a homomorphism from lattice $(EM,\vee,\wedge)$ to lattice $(\mathscr{L}(G,M,I),\vee,\wedge)$ provided p_I is defined as follows: for any $\sum_{i\in I}B_i\in EM$,*

$$p_I\left(\sum_{i\in I}B_i\right)=\bigvee_{i\in I}(\alpha(B_i),\beta\cdot\alpha(B_i))=\left(\alpha\cdot\beta(\bigcup_{i\in I}\alpha(B_i)),\ \bigcap_{i\in I}\beta\cdot\alpha(B_i)\right).\tag{8.7}$$

Proof. By Lemma 8.2, for any $\sum_{i\in I}B_i\in EM$, one knows that $\forall i\in I,(\alpha(B_i),\beta\cdot\alpha(B_i))\in\mathscr{L}(G,M,I)$. Since lattice $(\mathscr{L}(G,M,I),\leq)$ is a complete lattice, hence $\forall\sum_{i\in I}B_i\in EM$,

$$\begin{aligned}p_I\left(\sum_{i\in I}B_i\right)&=\left(\alpha\cdot\beta(\bigcup_{i\in I}\alpha(B_i)),\ \bigcap_{i\in I}\beta(\alpha(B_i))\right)\\&=\bigvee_{i\in I}(\alpha(B_i),\ \beta\cdot\alpha(B_i))\in\mathscr{L}(G,M,I).\end{aligned}$$

Next, we prove that p_I is a map from EM to $\mathscr{L}(G,M,I)$. Suppose $\sum_{i\in I_1}B_{1i}=\sum_{i\in I_2}B_{2i}\in EM$. By Lemma 8.1, one has $\forall i\in I_1,\exists k\in I_2$ such that $B_{1i}\supseteq B_{2k}\Rightarrow\alpha(B_{1i})\subseteq\alpha(B_{2k})$ and $\forall j\in I_2,\exists l\in I_1$ such that $B_{2j}\supseteq B_{1l}\Rightarrow\alpha(B_{2j})\subseteq\alpha(B_{1l})$. Therefore $\cup_{i\in I_1}\alpha(B_{1i})=\cup_{j\in I_2}\alpha(B_{2i})$ and

$$\alpha\cdot\beta\left(\bigcup_{i\in I_1}\alpha(B_{1i})\right)=\alpha\cdot\beta\left(\bigcup_{j\in I_1}\alpha(B_{2j})\right).$$

Since both

$$\left((\alpha\cdot\beta\bigcup_{i\in I_1}\alpha(B_{1i})),\bigcap_{i\in I_1}\beta\cdot\alpha(B_{1i})\right)$$

and

$$\left(\alpha\cdot\beta(\bigcup_{i\in I_2}\alpha(B_{2i})),\bigcap_{i\in I_2}\beta\cdot\alpha(B_{2i})\right)$$

are formal concepts in (G,M,I), hence

$$(\alpha\cdot\beta(\cup_{i\in I_1}\alpha(B_{1i})),\cap_{i\in I_1}\beta\cdot\alpha(B_{1i})) = (\alpha\cdot\beta(\cup_{i\in I_2}\alpha(B_{2i})),\cap_{i\in I_2}\beta\cdot\alpha(B_{2i})),$$
$$p_I\left(\sum_{i\in I_1}B_{1i}\right) = p_I\left(\sum_{i\in I_2}B_{2i}\right).$$

For any $\zeta=\sum_{i\in I}A_i, \eta=\sum_{j\in J}B_j\in EM$, by (8.7), (8.4) and Proposition 8.1, we have

$$\begin{aligned}
&p_I(\zeta\vee\eta)\\
&=(\alpha\cdot\beta[(\cup_{i\in I}\alpha(A_i))\cup(\cup_{j\in J}\alpha(B_j))],[(\cap_{i\in I}\beta\cdot\alpha(A_i))\cap(\cap_{j\in J}\beta\cdot\alpha(B_j))]).\\
&p_I(\zeta)\vee p_I(\eta)\\
&=(\alpha\cdot\beta(\cup_{i\in I}\alpha(A_i)),\cap_{i\in I}\beta\cdot\alpha(A_i))\vee(\alpha\cdot\beta(\cup_{j\in J}\alpha(B_j)),\cap_{j\in J}\beta\cdot\alpha(B_j))\\
&=(\alpha\cdot\beta[\alpha\cdot\beta(\cup_{i\in I}\alpha(A_i))\cup\alpha\cdot\beta(\cup_{j\in J}\alpha(B_j))],[(\cap_{i\in I}\beta\cdot\alpha(A_i))\cap(\cap_{j\in J}\beta\cdot\alpha(B_j))])
\end{aligned}$$

Since both $p_I(\zeta\vee\eta)$ and $p_I(\zeta)\vee p_I(\eta)$ are formal concepts of the context (G,M,I), hence $p_I(\zeta\vee\eta)=p_I(\zeta)\vee p_I(\eta)$. By (8.7), we have

$$p_I(\zeta\wedge\eta)=p_I\left(\sum_{i\in I,j\in J}A_i\cup B_j\right)=\bigvee_{i\in I,j\in J}(\alpha(A_i\cup B_j),\beta\cdot\alpha(A_i\cup B_j)).$$

In addition, for any $i\in I, j\in J$, it follows by (8.5)

$$(\alpha(A_i),\beta\cdot\alpha(A_i))\wedge(\alpha(B_i),\beta\cdot\alpha(B_i))=(\alpha(A_i)\cap\alpha(B_j),\beta\cdot\alpha[\beta\cdot\alpha(A_i)\cup\beta\cdot\alpha(B_j)]).$$

By Proposition 8.1, we have $\alpha(A_i)\cap\alpha(B_j)=\alpha(A_i\cup B_j)$ and

$$\begin{aligned}
\beta\cdot\alpha[\beta\cdot\alpha(A_i)\cup\beta\cdot\alpha(B_j)] &= \beta\cdot\alpha(\beta(\alpha(A_i))\cup\beta(\alpha(B_j)))\\
&=\beta(\alpha(\beta(\alpha(A_i)))\cap\alpha(\beta(\alpha(B_j))))\\
&=\beta(\alpha(A_i)\cap\alpha(B_j))\\
&=\beta\cdot\alpha(A_i\cup B_j)).
\end{aligned}$$

Therefore for any $i\in I, j\in J$,

$$(\alpha(A_i\cup B_j),\beta\cdot\alpha(A_i\cup B_j))=(\alpha(A_i),\beta\cdot\alpha(A_i))\wedge(\alpha(B_i),\beta\cdot\alpha(B_i)).$$

and

$$\begin{aligned}
p_I(\zeta\wedge\eta) &= \bigvee_{i\in I,j\in J}(\alpha(A_i\cup B_j),\beta\cdot\alpha(A_i\cup B_j))\\
&=\bigvee_{i\in I,j\in J}[(\alpha(A_i),\beta\cdot\alpha(A_i))\wedge(\alpha(B_i),\beta\cdot\alpha(B_i))]\\
&=\left[\bigvee_{i\in I}(\alpha(A_i),\beta\cdot\alpha(A_i))\right]\wedge\left[\bigvee_{j\in J}(\alpha(B_i),\beta\cdot\alpha(B_i))\right]\\
&=p_I(\zeta)\wedge p_I(\eta).
\end{aligned}$$

This demonstrates that p_I is homomorphism. □

Theorem 8.3. *Let (G,M,I) be a context. p_I is homomorphism from lattice $(\alpha(EM), \vee, \wedge)$ to lattice $(\mathscr{L}(G,M,I), \vee, \wedge)$, if for any $\sum_{i\in I} b_iB_i \in \alpha(EM)$, p_I is defined as*

$$p_I\left(\sum_{i\in I} b_iB_i\right) = \bigvee_{i\in I}(b_i,\beta(b_i)) = \left(\alpha\cdot\beta(\bigvee_{i\in I} b_i),\ \bigcap_{i\in I}\beta(b_i)\right). \tag{8.8}$$

Proof. By Lemma 8.2 and (8.6), for any $\sum_{i\in I} b_iB_i \in \alpha(EM)$, one knows that $\forall i \in I$, $(b_i,\beta(b_i)) = (\alpha(B_i),\beta(\alpha(B_i))) \in \mathscr{L}(G,M,I)$. This implies that

$$(\alpha\cdot\beta(\cup_{i\in I} b_i), \cap_{i\in I}\beta(b_i)) = \bigvee_{i\in I}(b_i,\beta(b_i)) \in \mathscr{L}(G,M,I).$$

Now, we prove that p_I is a map from $\alpha(EM)$ to $\mathscr{L}(G,M,I)$. Suppose $\sum_{i\in I_1} b_{1i}B_{1i} = \sum_{i\in I_2} b_{2i}B_{2i} \in \alpha(EM)$, i.e., $\forall i \in I_1, \exists k \in I_2$ such that $B_{1i} \supseteq B_{2k}$, $b_{2k} \supseteq b_{1i} \Rightarrow \beta(b_{2k}) \subseteq \beta(b_{1i})$ and $\forall j \in I_2, \exists l \in I_1$ such that $B_{2j} \supseteq B_{1l}$, $b_{2j} \subseteq b_{1l}$, $\beta(b_{2j}) \supseteq \beta(b_{1l})$. This implies that

$$\cup_{i\in I_1} b_{1i} = \cup_{j\in I_2} b_{2j},\ \ \cap_{i\in I_1}\beta(b_{1i}) = \cap_{i\in I_2}\beta(b_{2i}).$$

Therefore $p_I(\sum_{i\in I_1} b_{1i}B_{1i}) = p_I(\sum_{i\in I_2} b_{2i}B_{2i})$, i.e., p_I is a map. Then for any $\zeta = \sum_{i\in I} a_iA_i, \eta = \sum_{j\in J} b_jB_j \in \alpha(EM)$, by (8.4) and (8.8), we have

$$\begin{aligned}
p_I(\zeta\vee\eta) &= (\alpha\cdot\beta[(\cup_{i\in I}a_i)\cup(\cup_{j\in J}b_j)],\ \ [(\cap_{i\in I}\beta(a_i))\cap(\cap_{j\in J}\beta(b_j))]) \\
p_I(\zeta)\vee p_I(\eta) &= (\alpha\cdot\beta(\cup_{i\in I}a_i),\ \ \cap_{i\in I}\beta(a_i))\vee(\alpha\cdot\beta(\cup_{j\in J}b_j),\ \ \cap_{j\in J}\beta(b_j)) \\
&= (\alpha\cdot\beta[\alpha\cdot\beta(\cup_{i\in I}a_i)\cup\alpha\cdot\beta(\cup_{j\in J}b_j)],[(\cap_{i\in I}\beta(a_i))\cap(\cap_{j\in J}\beta(b_j))])
\end{aligned}$$

Since both $p_I(\zeta\vee\eta)$ and $p_I(\zeta)\vee p_I(\eta)$ are formal concepts of the context (G,M,I), hence $p_I(\zeta\vee\eta) = p_I(\zeta)\vee p_I(\eta)$. By (8.5) and (8.8), we have

$$\begin{aligned}
p_I(\zeta\wedge\eta) &= p_I\left(\sum_{i\in I, j\in J} a_i\cap b_jA_i\cup B_j\right) \\
&= \left(\alpha\cdot\beta\left(\bigcup_{i\in I, j\in J} a_i\bigcap b_j\right),\ \bigcap_{i\in I, j\in J}\beta(a_i\cap b_i)\right) \\
&= \bigvee_{i\in I, j\in J}(a_i\cap b_j, \beta(a_i\cap b_i)).
\end{aligned}$$

In addition, for any $i \in I, j \in J$, it follows by (8.5)

$$(a_i,\beta(a_i))\wedge(b_j,\beta(b_j)) = (a_i\cap b_j, \beta\cdot\alpha[\beta(a_i)\cup\beta(b_j)]).$$

By Proposition 8.1 and Lemma 8.2, for any $i \in I, j \in J$, we have

$$\begin{aligned}
\beta\cdot\alpha[\beta(a_i)\cup\beta(b_j)] &= \beta\cdot\alpha(\beta(a_i)\cup\beta(b_j)) \\
&= \beta(\alpha(\beta(a_i))\cap\alpha(\beta(b_j)))
\end{aligned}$$

$$\begin{aligned}
&= \beta(\alpha(\beta(\alpha(A_i))) \cap \alpha(\beta(\alpha(B_j)))) \\
&= \beta(\alpha(A_i) \cap \alpha(B_j)) \\
&= \beta(a_i \cap b_j).
\end{aligned}$$

Therefore

$$(a_i, \beta(a_i)) \wedge (b_j, \beta(b_j)) = (a_i \cap b_j, \beta(a_i \cap b_i))$$

and

$$\begin{aligned}
p_I(\zeta \wedge \eta) &= \bigvee_{i \in I, j \in J} (a_i \cap b_j, \beta(a_i \cap b_i)) \\
&= \bigvee_{i \in I, j \in J} [(a_i, \beta(a_i)) \wedge (b_j, \beta(b_j))] \\
&= \left[\bigvee_{i \in I} (a_i, \beta(a_i)) \right] \wedge \left[\bigvee_{j \in J} (b_j, \beta(b_j)) \right] \\
&= p_I(\zeta) \wedge p_I(\eta).
\end{aligned}$$

Therefore p_I is homomorphism. □

By Theorem 8.2, 8.3, we know that concept lattice $\mathscr{L}(G,M,I)$ has similar algebraic properties to EI algebra and EII algebra. $\mathscr{L}(G,M,I)$ as a lattice is finer than the lattices $\alpha(EM)$ and $\mathscr{L}(G,M,I)$ as an algebra structure is more rigorous than EI, EII algebras. EI, EII algebras can be applied to study fuzzy attributes while $\mathscr{L}(G,M,I)$ can only be applied to Boolean attributes.

Theorem 8.4. *Let (G,M,I) be a context and $\mathscr{L}(G,M,I)$ be a concept lattice of the context (G,M,I). Let EGM be the EI^2 algebra over the sets G,M. If the map $h : \mathscr{L}(G,M,I) \to EGM$ is defined as follows: for any formal concept $(b,B) \in \mathscr{L}(G,M,I)$, $h(b,B) = bB \in EGM$, then the following assertions hold.*

(1) If $(a,A),(b,B) \in \mathscr{L}(G,M,I)$, $(a,A) \le (b,B)$, then $h(a,A) \le h(b,B)$;
(2) For $(a,A),(b,B) \in \mathscr{L}(G,M,I)$,

$$\begin{aligned}
h((a,A) \vee (b,B)) &\ge h(a,A) \vee h(b,B), \\
h((a,A) \wedge (b,B)) &\le h(a,A) \wedge h(b,B).
\end{aligned}$$

Proof. (1) $(a,A) \le (b,B) \Rightarrow a \subseteq b$, $A \supseteq B$. By Definition 5.2 and Theorem 5.1, one has

$$h(a,A) \vee h(b,B) = aA + bB = bB = h(b,B).$$

This implies that $h(a,A) \le h(b,B)$ in the lattice EGM.

(2) By the definition of the map h and (8.4), (8.5), we have

$$h((a,A) \vee (b,B)) = h(\alpha \cdot \beta(a \cup b), A \cap B) = \alpha \cdot \beta(a \cup b) A \cap B.$$

By Proposition 8.1 and Lemma 8.2, we have

$$\begin{aligned}\alpha\cdot\beta(a\cup b) &= \alpha\cdot\beta(\alpha(A)\cup\alpha(B)) = \alpha\cdot\beta(\alpha(A\cap B))\\ &= \alpha(\beta(\alpha(A\cap B))) = \alpha(A\cap B).\end{aligned}$$

Thus

$$\begin{aligned}\alpha\cdot\beta(a\cup b) &= \alpha(A\cap B)\supseteq\alpha(A) = a,\\ \alpha\cdot\beta(a\cup b) &= \alpha(A\cap B)\supseteq\alpha(B) = b.\end{aligned}$$

Similarly, we can prove $\beta\cdot\alpha(A\cup B)\supseteq A\cup B$. Then by Theorem 5.1, we have

$$\begin{aligned}h((a,A)\vee(b,B)) &= \alpha\cdot\beta(a\cup b)A\cap B\geq aA+bB = h(a,A)\vee h(b,B).\\ h((a,A)\wedge(b,B)) &= a\cap b[\beta\cdot\alpha(A\cup B)]\leq aA\wedge bB = h(a,A)\wedge h(b,B).\end{aligned}$$ □

8.2 Some AFS Algebraic Properties of Formal Concept Lattices

In order to explore some algebraic properties of formal concept lattices, we define a new algebra class $E^C II$ for a context (G,M,I), which is a new family of AFS algebra different from the AFS algebras discussed in some other chapters.

Definition 8.3. Let G and M be sets and (G,M,I) be a context, EGM^I is a set defined as follows:

$$EGM^I = \left\{\sum_{u\in U} a_uA_u \mid A_u\subseteq M, a_u = \alpha(A_u),\ u\in U,\ U \textit{ is a non-empty indexing set}\right\}.$$

Where each $\sum_{u\in U} a_uA_u$ as an element of EGM^I is the "formal sum" of terms a_uA_u. $\sum_{u\in U} a_uA_u$ and $\sum_{u\in U} a_{p(u)}A_{p(u)}$ are the same elements of EGM^I if p is a bijection from I to I. R is a binary relation on EGM^I defined as follows: $\sum_{u\in U} a_uA_u$, $\sum_{v\in V} b_vB_v\in EGM^I$, $(\sum_{u\in U} a_uA_u, \sum_{v\in V} b_vB_v)\in R \Leftrightarrow$ (i) $\forall a_uA_u\ (u\in U)\ \exists b_kB_k\ (k\in V)$ such that $a_u\subseteq b_k$, $A_u\subseteq B_k$, (ii) $\forall b_vB_v\ (v\in V)\ \exists a_lA_l\ (l\in U)$ such that $b_v\subseteq a_l$, $B_v\subseteq A_l$.

It is obvious that R is an equivalence relation on EGM^I. The quotient set EGM^I/R is denoted as E^IGM. $\sum_{u\in U} a_uA_u = \sum_{v\in V} b_vB_v$ means that $\sum_{u\in U} a_uA_u$ and $\sum_{v\in V} b_vB_v$ are equivalent under the equivalence relation R.

Proposition 8.2. *Let G and M be sets, (G,M,I) be a context and E^IGM be defined as Definition 8.3. For $\sum_{u\in U} a_uA_u\in E^IGM$, if $a_q\subseteq a_w$, $A_q\subseteq A_w$, $w,q\in U$, $w\neq q$, then*

$$\sum_{u\in U} a_uA_u = \sum_{u\in U, u\neq q} a_uA_u.$$

Its proof remains as an exercise.

Definition 8.4. Let G and M be sets, (G,M,I) be a context and E^IGM be the set defined as Definition 8.3. We introduce the following definitions.

(1) For $\sum_{u\in U} a_uA_u \in E^IGM$, $\sum_{u\in U} a_uA_u$ is called *E^CII irreducible* if $\forall w \in U$, $\sum_{u\in U} a_uA_u \neq \sum_{u\in U, u\neq w} a_uA_u$.
(2) For any $\sum_{u\in U} a_uA_u \in E^IGM$, $|\sum_{u\in U} a_uA_u|$, the *set of all E^CII irreducible items in* $\sum_{u\in U} a_uA_u$, is defined as follows.

$$|\sum_{u\in U} a_uA_u| \triangleq \left\{a_uA_u \mid u\in U, a_u \nsubseteq a_j, A_u \nsubseteq A_j \; for\ any\ j\in U\right\}.$$

$||\sum_{u\in U} a_uA_u||$, the length of $\sum_{u\in U} a_uA_u$, is defined as follows

$$||\sum_{u\in U} a_uA_u|| \triangleq |\left\{a_uA_u \mid u\in U, a_u \nsubseteq a_j, A_u \nsubseteq A_j \; for\ any\ j\in U\right\}|.$$

Proposition 8.3. *Let G and M be sets, (G,M,I) be a context and E^IGM be the set defined as Definition 8.3. The binary relation $\leq$ is a partial order relation if $\sum_{u\in U} a_uA_u$, $\sum_{v\in V} b_vB_v \in E^IGM$, $\sum_{u\in U} a_uA_u \leq \sum_{v\in V} b_vB_v \Leftrightarrow \forall a_uA_u$ $(u\in U)$ $\exists b_kB_k$ $(k\in V)$ such that $a_u \subseteq b_k$, $A_u \subseteq B_k$.*

Its proof remains as an exercise.

Proposition 8.4. *Let G and M be sets, (G,M,I) be a context and E^IGM be defined as Definition 8.3. Then for any $\Gamma \subseteq \{A\in 2^M \mid A = \beta\cdot\alpha(A)\} \subseteq M$, $\varnothing \neq \Gamma$, $\sum_{B\in\Gamma} \alpha(B)B$ is E^CII irreducible.*

Proof. Suppose there exists $A\in\Gamma$ such that $\sum_{B\in\Gamma}\alpha(B)B = \sum_{B\in\Gamma, B\neq A}\alpha(B)B$. By Definition 8.3, for $\alpha(A)A$ standing on the left side of the equation, we know that $\exists E\in\Gamma$, $E\neq A$ such that $\alpha(A)\subseteq\alpha(E), A\subseteq E$. By the properties of the Galois connection α,β shown in Lemma 8.1 and $A\subseteq E$, we have $\alpha(A)\supseteq\alpha(E)$. This implies that $\alpha(A)=\alpha(E)$ and $A=\beta\cdot\alpha(A)=\beta\cdot\alpha(E)=E$. It contradicts that $E\neq A$. Therefore $\sum_{B\in\Gamma}\alpha(B)B$ is E^CII irreducible. □

Proposition 8.5. *Let (G,M,I) be a context and E^IGM be defined as Definition 8.3. If for any $\sum_{u\in U} a_uA_u$, $\sum_{v\in V} b_vB_v \in E^IGM$, we define*

$$(\sum_{u\in U} a_uA_u) * (\sum_{v\in V} b_vB_v) = \sum_{u\in U, v\in V} a_u\cap b_v A_u\cup B_v, \tag{8.9}$$

$$(\sum_{u\in U} a_uA_u) + (\sum_{v\in V} b_vB_v) = \sum_{u\in U\sqcup V} c_uC_u, \tag{8.10}$$

where $u\in U\sqcup V$ (the disjoint union of indexing sets U, V), $c_u = a_u$, $C_u = A_u$, if $u\in U$; $c_u = b_u$, $C_u = B_u$, if $u\in U$. Then "+" and "∗" are binary compositions on E^IGM.

Its proof remains as an exercise.

The algebra system $(E^I GM, *, +, \leq)$ is called the *$E^C II$ algebra of context* (G,M,I) and denoted as $E^I GM$, where $*$ and $+$ are defined by (8.9) and (8.10), and $\leq$ is defined by Proposition 8.3. For $\sum_{u\in U} a_u A_u \in E^I GM$, let

$$\left(\sum_{u\in U} a_u A_u\right)^h = \overbrace{\sum_{u\in U} a_u A_u * \ldots * \sum_{u\in U} a_u A_u}^{h}.$$

The algebra system $(E^I GM, *, +, \leq)$ has the following properties which can be further applied to study the formal concept lattice.

Proposition 8.6. *Let G and M be finite sets, (G,M,I) be a context and $(E^I GM, *, +, \leq)$ be the $E^C II$ algebra of context (G,M,I). Then the following assertions hold. For any ψ, ϑ, γ, $\eta \in E^I GM$,*

(1) $\psi + \vartheta = \vartheta + \psi, \quad \psi * \vartheta = \vartheta * \psi$;
(2) $(\psi + \vartheta) + \gamma = \psi + (\vartheta + \gamma), \quad (\psi * \vartheta) * \gamma = \psi * (\vartheta * \gamma)$;
(3) $(\psi + \vartheta) * \gamma = (\psi * \gamma) + (\vartheta * \gamma), \quad \psi * (\varnothing M) = (\varnothing M), \quad \psi * (X\varnothing) = \psi$;
(4) *If* $\psi \leq \vartheta$, $\gamma \leq \eta$, *then* $\psi + \gamma \leq \vartheta + \gamma, \quad \psi * \gamma \leq \vartheta * \gamma$;
(5) *For any $\zeta \in E^I GM$, any positive integer n,*

$$\zeta \leq \zeta^n, \ \ (\zeta + \varnothing M)^n = \zeta^n + \varnothing M.$$

(6) *Let $A_j \subseteq M, j \in J$, J be any non-empty indexing set. For any $A \subseteq M$, $U(A)$ the set of all intents containing A is defined as follows.*

$$U(A) = \{B \mid A \subseteq B \subseteq M, B = \beta \cdot \alpha(B)\}.$$

Then the following assertions hold.

$$\left(\sum_{j\in J}\sum_{A\in U(A_j)} \alpha(A)A\right)^2 = \sum_{j\in J}\sum_{A\in U(A_j)} \alpha(A)A.$$

(7) *For any $\sum_{j\in J} a_j A_j \in E^I GM$ and any positive integer l,*

$$\left(\sum_{j\in J} a_j A_j\right)^l \leq \sum_{j\in J}\sum_{A\in U(A_j)} \alpha(A)A.$$

(8) *If $\gamma = \sum_{m\in M} \alpha(\{m\})\{m\}$, then there exists an positive integer h such that $(\gamma^h)^2 = \gamma^h$, $|\gamma^h|$ is the set of all concepts of context (G,M,I) except $(X,\varnothing)$ ($|\gamma^h|$ defined by Definition 8.4).*

Proof. (1), (2), (3) and (4) can be directly proved by using the definitions.

Now we prove (5). Let $\zeta = \sum_{u\in U} a_uA_u \in E^IGM$.

$$\left(\sum_{u\in U} a_uA_u\right) * \left(\sum_{u\in U} a_uA_u\right) = \sum_{u,v\in U} a_u \cap a_vA_u \cup A_v$$
$$= \sum_{u\in U} a_uA_u + \sum_{u,v\in U,u\neq v} a_u \cap a_vA_u \cup A_v$$
$$\geq \sum_{u\in U} a_uA_u.$$

Thus $\zeta \leq \zeta^2$. By (4), one has $\zeta \leq \zeta^2 \leq \zeta^3 \ \Rightarrow \ \zeta \leq \zeta^2 \leq \zeta^3 \leq \zeta^4 \ \Rightarrow ... \Rightarrow \zeta \leq \zeta^2 ... \leq \zeta^n$. From (1), (2), (3), we have

$(\sum_{u\in U} a_uA_u + \varnothing M) * (\sum_{u\in U} a_uA_u + \varnothing M)$
$= \sum_{u\in U} a_uA_u * \sum_{u\in U} a_uA_u + \varnothing M * \sum_{u\in U} a_uA_u + \sum_{u\in U} a_uA_u * \varnothing M + \varnothing M * \varnothing M$
$= (\sum_{u\in U} a_uA_u)^2 + \varnothing M.$

Now we prove it by induction with respect on n. Suppose

$$(\sum_{u\in U} a_uA_u + \varnothing M)^{n-1} = (\sum_{u\in U} a_uA_u)^{n-1} + \varnothing M.$$

We have

$(\sum_{u\in U} a_uA_u + \varnothing M)^n$
$= ((\sum_{u\in U} a_uA_u)^{n-1} + \varnothing M) * (\sum_{u\in U} a_uA_u + \varnothing M)$
$= (\sum_{u\in U} a_uA_u)^n + (\sum_{u\in U} a_uA_u)^{n-1} * \varnothing M + \varnothing M * \sum_{u\in U} a_uA_u + \varnothing M$
$= (\sum_{u\in U} a_uA_u)^n + \varnothing M.$

Therefore the assertion holds.

(6) Let $\sum_{j\in J} a_jA_j \in E^IGM$. For any $u,v \in J$,

$$\left(\sum_{A\in U(A_u)} \alpha(A)A\right) * \left(\sum_{A\in U(A_v)} \alpha(A)A\right) = \sum_{A\in U(A_u),B\in U(A_v)} \alpha(A)\cap\alpha(B)A\cup B.$$

For any $A \in U(A_u), B \in U(A_v)$, if $A\cup B$ is an intent of a concept of context (G,M,I), then $A\cup B \in U(A_u)\cap U(A_v)$. If $A\cup B$ is not an intent of a concept of context (G,M,I), then $A\cup B \subset \beta\cdot\alpha(A\cup B) \in U(A_u)\cap U(A_v)$ and $\alpha\cdot\beta\cdot\alpha(A\cup B) = \alpha(A\cup B) = \alpha(A)\cap\alpha(B)$. Thus in any case, for any $A \in U(A_u), B \in U(A_v)$ there exists $D \in U(A_v)$ such that $\alpha(A)\cap\alpha(B) \subseteq \alpha(D)$ and $A\cup B \subseteq D$ (e.g., $D = \beta\cdot\alpha(A\cup B)$). By Proposition 8.3, one has

$$\left(\sum_{A\in U(A_u)} \alpha(A)A\right) * \left(\sum_{A\in U(A_u)} \alpha(A)A\right) \leq \sum_{A\in U(A_u)} \alpha(A)A$$

From (5), we have

$$\left(\sum_{A\in U(A_u)} \alpha(A)A\right) * \left(\sum_{A\in U(A_u)} \alpha(A)A\right) = \sum_{A\in U(A_u)} \alpha(A)A$$

It follows from (1), (2) and (3),

$$\begin{aligned}\left(\sum_{u\in U}\sum_{A\in U(A_u)}\alpha(A)A\right)^2 &= \sum_{u,v\in U}\left(\sum_{A\in U(A_u)}\alpha(A)A * \sum_{A\in U(A_u)}\alpha(A)A\right)\\ &= \sum_{u,v\in U}\left(\sum_{A\in U(A_u)}\alpha(A)A\right)\\ &= \sum_{u\in U}\sum_{A\in U(A_u)}\alpha(A)A\end{aligned}$$

(7) Let $\sum_{j\in J} a_j A_j \in E^I GM$. It is obvious that $\forall u \in J, A_u \subseteq \beta\cdot\alpha(A_u) \in U(A_u)$ and $a_u = \alpha(A_u) = \alpha\cdot\beta\cdot\alpha(A_u)$. This implies that $\sum_{j\in J} a_j A_j \leq \sum_{j\in J}\sum_{A\in U(A_j)}\alpha(A)A$. By (4), (5) and (6), for any integer l, we have

$$\left(\sum_{j\in J} a_j A_j\right)^l \leq \left(\sum_{j\in J}\sum_{A\in U(A_j)}\alpha(A)A\right)^l = \sum_{j\in J}\sum_{A\in U(A_j)}\alpha(A)A.$$

(8) By (5), we know that

$$\sum_{m\in M}\alpha(\{m\})\{m\} \leq \left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^2 \leq ... \leq \left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^r.$$

Since both G and M are finite sets, hence there are finite number of elements in $E^I GM$ and there exists an integer h such that

$$\left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^h = \left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^{2h}.$$

From (7), we know that for any integer r,

$$\left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^r \leq \left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^h \leq \sum_{m\in M}\sum_{A\in U(\{m\})}\alpha(A)A.$$

Then for any $m \in M, A \in U(m)$, there exists an item $\alpha(B)B$ in $(\sum_{m\in M}\alpha(\{m\})\{m\})^{|M|}$ such that $\alpha(B) = \cap_{m\in A}\alpha(\{m\}) \supseteq \alpha(A), B = \cup_{m\in A}\{m\} \supseteq A$. This implies that

$$\left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^{|M|} \geq \sum_{m\in M}\sum_{A\in U(\{m\})}\alpha(A)A$$

Therefore

$$\left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^{|M|}=\left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^{h}=\sum_{m\in M}\sum_{A\in U(\{m\})}\alpha(A)A.$$

Let

$$\left(\sum_{m\in M}\alpha(\{m\})\{m\}\right)^{h}=\sum_{m\in M}\sum_{A\in U(\{m\})}\alpha(A)A=\sum_{j\in J}a_jA_j,$$

and $\sum_{j\in J}a_jA_j$ is E^CII irreducible. Since for any concept $(\alpha(A),A)$ of the context (G,M,I), $\alpha(A)A$ is an item in $\sum_{m\in M}\sum_{A\in U(\{m\})}\alpha(A)A$ and $\sum_{m\in M}\sum_{A\in U(\{m\})}\alpha(A)A=\sum_{j\in J}a_jA_j$. Then for any concept $(\alpha(A),A)$ there exists $j\in J$, such that $\alpha(A)\subseteq a_j=\alpha(A_j)$, $A\subseteq A_j$. By the properties of Galois connection α,β in Lemma 8.1 and $A\subseteq A_j$, we have $\alpha(A)\supseteq\alpha(A_j)=a_j$, $\alpha(A)=a_j=\alpha(A_j)$, $A=\beta\cdot\alpha(A)=\beta\cdot\alpha(A_j)=A_j$. Thus $(\alpha(A),A)\in\{(a_j,A_j)|j\in J\}$. For any $w\in J$, if (a_w,A_w) is not a concept of the context (G,M,I), then A_w is a proper subset of $\beta(a_w)=\beta\cdot\alpha(A_w)$ and $(a_w,\beta(a_w))$ is a concept of the context (G,M,I). This implies that $a_w\beta(a_w)$ is an item in $\sum_{m\in M}\sum_{A\in U(\{m\})}\alpha(A)A$. By Proposition 8.2, we know that item a_wA_w will be reduced and it cannot appear in $\sum_{j\in J}a_jA_j$. It is a contradiction. Therefore $\{(a_j,A_j)|j\in J\}$ is the set of all concepts of context (G,M,I) except $(X,\varnothing)$. □

Theorem 8.5. *Let G and M be finite sets, (G,M,I) be a context and $(E^IGM,*,+,\leq)$ be the E^CII algebra of context (G,M,I). Let $\gamma=\sum_{m\in M}\alpha(\{m\})\{m\}$. For an item $aA\in|\gamma^k|$ ($|\gamma^k|$ defined by Definition 8.4), if $|A|<k$, then (a,A) is a formal concept of the context (G,M,I), i.e., $\beta(a)=A,\ \alpha(A)=a$, where k is any positive integer.*

Proof. Assume that there exists an item $aA\in|\gamma^k|$ with $|A|<k$ in $|\gamma^k|$ such that (a,A) isn't a formal concept of the context (G,M,I). This implies that there exist $B\subseteq M$, $A\subseteq B$ and $|B|=|A|+1$ such that $a=\alpha(B)$. It is obvious that aB is an item in $\gamma^{|A|+1}$. By 5 of Proposition 8.6, one knows that $\gamma^{|A|+1}\leq\gamma^k$. From Proposition 8.3, we know there exist an item cC in γ^k such that $a=\alpha(B)\subseteq c,\ \ A\subseteq B\subseteq C$. By Proposition 8.2, item aA can be reduced and it contradicts $aA\in|\gamma^k|$. Therefore (a,A) is a concept of the context (G,M,I). □

In what follows, we discuss how to find concepts of a context using the above results. The following theorem gives a very simple way to compute the power of an E^CII element.

Theorem 8.6. *Let G and M be finite sets, (G,M,I) be a context and $(E^IGM,*,+,\leq)$ be the E^CII algebra of the context (G,M,I). For any nonempty set $C\subseteq M$, let $\gamma=\sum_{m\in C}\alpha(\{m\})\{m\}$ and $\gamma^k=\sum_{i\in I}\alpha(A_i)A_i$, where k is a positive integer, $k\leq|C|$. Then*

$$\gamma^{2k}=\sum_{i\in I,|A_i|<k}\alpha(A_i)A_i+\left(\sum_{i\in I,|A_i|=k}\alpha(A_i)A_i\right)^2,$$

where $\sum_{i\in I,|A_i|=k}\alpha(A_i)A_i\triangleq\varnothing C$, if there does not exist $i\in I$ such that $|A_i|=k$.

Proof. Let $\gamma^k = \sum_{i\in I} \alpha(A_i)A_i = \sum_{i\in I,|A_i|<k} \alpha(A_i)A_i + \sum_{i\in I,|A_i|=k} \alpha(A_i)A_i$. Then

$$\begin{aligned}\gamma^{2k} &= \left(\sum_{i\in I,|A_i|<k} \alpha(A_i)A_i + \sum_{i\in I,|A_i|=k} \alpha(A_i)A_i\right)^2 \\ &= \left(\sum_{i\in I,|A_i|<k} \alpha(A_i)A_i\right)^2 + \left(\sum_{i\in I,|A_i|<k} \alpha(A_i)A_i\right) * \left(\sum_{i\in I,|A_i|=k} \alpha(A_i)A_i\right) \\ &\quad + \left(\sum_{i\in I,|A_i|=k} \alpha(A_i)A_i\right)^2 \end{aligned} \tag{8.11}$$

Because $\gamma = \sum_{m\in C} \alpha(\{m\})\{m\}$. For any set $B \subseteq C$, $|B| < k$, $\alpha(B)B$ is an item in $\sum_{i\in I,|A_i|<k} \alpha(A_i)A_i$. Although $\alpha(B)B$ may be reduced by other items in $\sum_{i\in I,|A_i|<k} \alpha(A_i)A_i$, we always have $\alpha(B)B \leq \sum_{i\in I,|A_i|<k} \alpha(A_i)A_i$ and

$$\alpha(B)B + \sum_{i\in I,|A_i|<k} \alpha(A_i)A_i = \sum_{i\in I,|A_i|<k} \alpha(A_i)A_i. \tag{8.12}$$

Similarly, since for every set $B \subseteq C$, $|B| = k$, $B'B \leq \sum_{i\in I,|A_i|=k} A_i'A_i$ and for any set $E \subseteq C$, $k \leq |E| \leq 2k$, there exist $F,H \subseteq C$, $|F| = |H| = k$ such that $E = F \cup H$. By 4 of Proposition 8.3 and the facts $\alpha(F)F \leq \sum_{i\in I,|A_i|=k} \alpha(A_i)A_i$ and $\alpha(H)H \leq \sum_{i\in I,|A_i|=k} \alpha(A_i)A_i$, we have

$$\alpha(E)E = \alpha(F) \cap \alpha(H) F \cup H \leq \left(\sum_{i\in I,|A_i|=k} \alpha(A_i)A_i\right)^2,$$

$$\alpha(E)E + \left(\sum_{i\in I,|A_i|=k} \alpha(A_i)A_i\right)^2 = \left(\sum_{i\in I,|A_i|=k} \alpha(A_i)A_i\right)^2. \tag{8.13}$$

According to (8.11), (8.12) and (8.13), we have

$$\begin{aligned}\gamma^{2k} &= \left(\sum_{i\in I,|A_i|<k} \alpha(A_i)A_i\right)^2 + \left(\sum_{i\in I,|A_i|=k} \alpha(A_i)A_i\right)^2 \\ &= \sum_{i\in I,|A_i|<k} \alpha(A_i)A_i + \left(\sum_{i\in I,|A_i|=k} A_i'A_i\right)^2. \end{aligned}$$ □

The above discussion shows that the algebra characteristics of the formal concepts of a context can be explored by the $E^C II$ algebra of the context. For example, Theorem 8.5, Theorem 8.6 and Proposition 8.6 can be applied to identify all formal concepts of a context. For any context (G,M,I), let $\gamma = \sum_{m\in M} \alpha(\{m\})\{m\}$. By 8 of Proposition 8.6, we know that $|\gamma^{|M|}|$ defined by Definition 8.4 is the set of all formal concepts of the context (G,M,I). Notice that there is only one formal concept whose extent is $\varnothing$, i.e., $\varnothing M$ for (G,M,I). In order to simplify the computation of $\gamma^{|M|}$, we compute $(\gamma+\varnothing M)^2$, $(\gamma+\varnothing M)^4$, $(\gamma+\varnothing M)^8$,...,$(\gamma+\varnothing M)^{2^k}$, until $2^k \geq |M|$. By 5 of Proposition 8.6, one knows that $(\gamma+\varnothing M)^n = \gamma^n + \varnothing M$ for any positive integer n. So each item $\varnothing A$ in γ^n can be reduced by $\varnothing M$ and the number of items of $(\gamma+\varnothing M)^n$ is much lower than γ^n. Let $\gamma^k = \sum_{i\in I}\alpha(A_i)A_i = \sum_{i\in I,|A_i|<k}\alpha(A_i)A_i + \sum_{i\in I,|A_i|=k}\alpha(A_i)A_i$. According to Theorem 8.5 and Theorem 8.6, we know that for any formal concept (a,A), aA is in $|\sum_{i\in I,|A_i|<k}\alpha(A_i)A_i|$ if $|A|<k$ and aA is in $|(\sum_{i\in I,|A_i|=k}\alpha(A_i)A_i)^2|$ if $k \leq |A| \leq 2k$, where k is a positive integer. This fact and the following equation can further facilitate the computing,

$$\gamma^{2k} = \sum_{i\in I,|A_i|<k}\alpha(A_i)A_i + \left(\sum_{i\in I,|A_i|=k}\alpha(A_i)A_i\right)^2.$$

Example 8.1 demonstrates how the detailed calculations are carried out.

Table 8.2 The Reduced Mushroom

	m_1	m_2	m_3	m_4	m_5
Mushroom 1	×		×		
Mushroom 2	×		×		×
Mushroom 3	×			×	×
Mushroom 4	×			×	×
Mushroom 5	×		×		×
Mushroom 6	×		×		
Mushroom 7		×		×	×
Mushroom 8		×		×	×
Mushroom 9		×		×	×
Mushroom 10		×		×	

Example 8.1. The Table 8.2 shows the reduced mushroom example database from the UCI KDD Archive (http://kdd.ics.uci.edu) in [43]. Where m_1: *edible*, m_2: *poisonous*, m_3: *cap_shape:convex*, m_4: *cap-shape: flat*, m_5: *cap-surface:fibrous*. Let (G,M,I) be the context of Table 8.2, $G=\{1,2,...,10\}$, $M=\{m_1, m_2, m_3, m_4, m_5\}$. Let us find all formal concepts of the context (G,M,I) by E^CII algebra via the computing on the power of the following γ.

$$\gamma = \{1,2,3,4,5,6\}\{m_1\}+\{7,8,9,10\}\{m_2\}+\{1,2,5,6\}\{m_3\}$$
$$+\{3,4,7,8,9,10\}\{m_4\}+\{2,3,4,5,7,8,9\}\{m_5\} \in E^IGM.$$

For any positive integer $k>1$, let $\gamma^k = \sum_{i\in I}\alpha(A_i)A_i = \underline{\gamma}^k + \bar{\gamma}^k$. Where $\underline{\gamma}^k = \sum_{i\in I|A_i|<k}\alpha(A_i)A_i$, $\bar{\gamma}^k = \sum_{i\in I|A_i|=k}\alpha(A_i)A_i$.

$$
\begin{aligned}
(\gamma+\varnothing M)^2 = &\{1,2,3,4,5,6\}\{m_1\}+\{3,4,7,8,9,10\}\{m_4\}+\{2,3,4,5,7,8,9\}\{m_5\}\\
&+\{1,2,5,6\}\{m_1m_3\}+\{2,3,4,5\}\{m_1m_5\}+\{3,4\}\{m_1m_4\}\\
&+\{7,8,9,10\}\{m_2m_4\}+\{7,8,9\}\{m_2m_5\}+\{2,5\}\{m_3m_5\}\\
&+\{3,4,7,8,9\}\{m_4m_5\}+\varnothing M.
\end{aligned}
$$

$$
\begin{aligned}
(\gamma+\varnothing M)^4 &= \gamma^4+\varnothing M = \underline{\gamma}^2+(\bar{\gamma}^2)^2+\varnothing M\\
&= \{1,2,3,4,5,6\}\{m_1\}+\{3,4,7,8,9,10\}\{m_4\}+\{2,3,4,5,7,8,9\}\{m_5\}\\
&\quad+\{1,2,5,6\}\{m_1m_3\}+\{2,3,4,5\}\{m_1m_5\}+\{7,8,9,10\}\{m_2m_4\}+\\
&\quad\{3,4,7,8,9\}\{m_4m_5\}+\{2,5\}\{m_1m_3m_5\}+\{3,4\}\{m_1m_4m_5\}+\\
&\quad\{7,8,9\}\{m_2m_4m_5\}+\varnothing M
\end{aligned}
$$

Since there does not exist item aA in γ^4 such that $|A|=4$, hence $\overline{\gamma}^4=\varnothing M$ and

$$
(\gamma+\varnothing M)^8=\gamma^8+\varnothing M=\underline{\gamma}^4+(\overline{\gamma}^4)^2+\varnothing M=\underline{\gamma}^2+(\bar{\gamma}^2)^2+\varnothing M=(\gamma+\varnothing M)^4.
$$

According to Theorem 8.5 and 8 of Proposition 8.6, all concepts of the context for Table 8.2 are the items shown in the above $(\gamma+\varnothing M)^4$ except $(X,\varnothing)$. It is the same result as what has been obtained by the TITANIC algorithm presented in [43].

8.3 Concept Analysis via Rough Set and AFS Algebra

In this section, combining formal concept analysis (FCA) and AFS algebra, we propose AFS formal concept, which can be viewed as the generalization and development of monotone concept proposed by Deogun and Saquer (2003) [8]. Moreover, we show that the set of all AFS formal concepts forms a complete lattice. AFS formal concept can be applied to represent the logic operations of queries in information retrieval. Furthermore, we present an approach to find the AFS formal concepts whose intents (extents) approximate any fuzzy concepts in EM by virtue of rough set theory.

The characteristic of concept lattice theory lies in reasoning on the possible attributes of data sets [66]. Currently, FCA has been extended to other types for requirements of real word applications, such as fuzzy concept lattice [2, 46], triadic concept [57], monotone concept [8], variable threshold concept lattice [65], rough formal concept [66], etc.

Rough set and FCA are related and complementary. In recent years, many efforts have been made to compare and combine these two theories [61, 62, 64, 65]. The combination of FCA and rough set theory provides some new approaches for data analysis and knowledge discovery [44, 45, 55, 66].

In [8], Deogun and Saquer discussed some of limitations of Wille's formal concept [10] and proposed monotone concept. In Wille's notation of concepts, only one set is allowed as extent (intent). For many applications, it is necessary to allow intents to be disjunction expression. Monotone concept is a generalization of Wille's notion of concept where disjunctions are allowed in the intent and set unions are allowed in the extent. This generalization allows an information retrieval query containing disjunctions to be understood as the intent of a monotone concept whose answer is the extent of that concept. In [44], by using rough set theory, Saquer and Deogun formulated a general solution to find monotone concepts whose intents are close to the query, and show how to find monotone concepts whose extents approximate any given set of objects.

In this section, we propose AFS formal concept, which extend the Galois connection α, β of a context (X, M, I) to the connection between two AFS algebra systems $(EM, \vee, \wedge)$ and $(E^{\#}X, \vee, \wedge)$. The intent of an AFS formal concept is an element of the *EI* algebra $(EM, \vee, \wedge)$—a kind of AFS algebra over *M*; correspondingly, the extent of the AFS formal concept is an element of the $E^{\#}I$ algebra $(E^{\#}X, \vee, \wedge)$—another kind of AFS algebra over *X*. Where *M* is a set of elementary attributes on *X*, *EM* is the set of attributes logically compounded by some elementary attributes in *M* under logic operations $\vee$ and $\wedge$ (i.e., "*and*" and "*or*"). Each element of *EM* is called a complex attribute (or a fuzzy concept), and has definitely semantic interpretation. The extent and intent of an AFS formal concept can uniquely determine each other. Thus, the intent of an AFS formal concept not only generalizes that of the formal concept, but also has a well-defined semantic interpretation.

In an information retrieval system, the logic relationships between queries are usually expressed by logic connectives such as "*and*" and "*or*". AFS formal concepts can be used to represent the query with complex logic operations. When using the information retrieval system, we often find that not all queries are exactly contained in database, but some items close to those are enough to satisfy user's need. Thus, it is necessary to investigate how to approximate a complex attribute by AFS formal concepts such that the intents of lower and upper approximating concept are closely to the complex attribute underlying semantics.

In this section, first, FCA and rough set are briefly summarized. Monotone concept is also introduced and studied. Second, AFS formal concept is proposed and the mathematical properties of AFS formal concepts are discussed. Third, we show that the set of all AFS formal concepts forms a complete lattice. Fourth, an approach to approximate the element of the *EM* ($E^{\#}X$) is proposed.

8.3.1 Monotone Concept

Let us first recall monotone concept [8] and study the aspects which should be improved in concept representation and approximation. In [8], Deogun and Saquer introduced some notations as follows: (X, M, I) is a context. Associating with every set $B \subseteq M$, a Boolean conjunctive expression $\hat{B}$ is the conjunction of the elements of B. For example, if $B = \{a, b, c\}$, then the associated Boolean conjunctive expression is $\hat{B} = a \wedge b \wedge c$. A disjunction of Boolean conjunctive expressions is referred to as a *monotone Boolean formula*. If $\hat{B}_1$, $\hat{B}_2, \ldots, \hat{B}_n$ are Boolean conjunctive expressions, then $F = \hat{B}_1 \vee \hat{B}_2 \vee \ldots \vee \hat{B}_n = \bigvee_{i=1}^{n} \hat{B}_i$ is *monotone formula*. For example, let $B_1 = \{a, b, c\}$, $B_2 = \{a, d\}$, then $\hat{B}_1 = a \wedge b \wedge c$, $\hat{B}_2 = a \wedge d$, $F = \hat{B}_1 \vee \hat{B}_2 = (a \wedge b \wedge c) \vee (a \wedge d)$ is monotone formula. For simplicity, F would be written as $abc \vee ad$ [44].

Definition 8.5. ([8]) Let (X,M,I) be a context. For the monotone formula $F = \vee_{i=1}^{n}\hat{B}_i$, $B_i \subseteq M$, $\delta(F)$ is defined as the set of all objects that satisfy F, that is, $\delta(F) = \cup_{i=1}^{n}\delta(\hat{B}_i) \subseteq X$, where $\delta(\hat{B}_i)$ is the *set of all objects that satisfy* $\hat{B}_i$. For $A = \cup_{j=1}^{n}A_j$, define $\gamma(A)$ to be $\vee_{j=1}^{n}\gamma(A_j) \subseteq M$, where $A_j \subseteq X$, $\gamma(A_j)$ is defined to be the *Boolean conjunctive expression associated with* $\beta(A_j)$, "β" is Galois connection.

Example 8.2. Let $X = \{1,2,...,13\}$ and $M = \{a,b,c,d,e,f,h,i,j,l,x\}$ be the set of attributes on X. The context (X,M,I) is shown as Table 8.3. Assume that the monotone formula $F = \hat{B}_1 \vee \hat{B}_2 = abc \vee lx$, $A = \cup_{i=1}^{3}A_i = \{4,6\} \cup \{6,7\} \cup \{5\}$. By Definition 8.5, we have the following $\delta(F)$ and $\gamma(A)$.

$$\delta(F) = \delta(\hat{B}_1) \cup \delta(\hat{B}_2) = \{4\} \cup \{6\} = \{4,6\},$$
$$\gamma(A) = \vee_{i=1}^{3}\gamma(A_i) = efhl \vee fhijx \vee cdefhix.$$

Table 8.3 Relationship between objects and attributes [44]

	a	b	c	d	e	f	h	i	j	k	l	x
1				×								
2				×		×	×					
3						×	×					
4	×	×	×	×	×	×	×				×	
5			×	×	×	×	×	×				×
6					×	×	×	×	×	×	×	×
7						×	×	×	×			×
8			×	×	×	×	×					
9			×	×		×					×	
10			×	×	×	×					×	
11						×	×					
12			×	×	×	×	×	×				×
13			×	×	×	×	×	×				×

Definition 8.6. ([8]) Let (X,M,I) be a context, $A_i \subseteq X, B_j \subseteq M$, $1 \le i,j \le n$. A pair (A,F) where $A = \cup_{i=1}^{n}A_i$, $F = \vee_{j=1}^{n}\hat{B}_j$ is *monotone concept* if $\delta(F) = A$, $\gamma(A) = F$. A is called its *extent of the monotone concept* (A,F), F its *intent of the monotone concept* (A,F). Where B_j is the set of features associated with $\hat{B}_j$, and for each A_i, there exists a B_j such that (A_i,B_j) is a formal concept.

A monotone formula F is called *feasible* if it is the intent of a monotone concept; otherwise, F is called *non-feasible*. Similarly, $A \subseteq X$ is called *feasible* if it is the extent of a monotone concept; otherwise, A is called *non-feasible*. For instance, assume $F = ef \vee fhix$, $A = \{4,5,6,8,10,12,13\} \cup \{5,6,7,12,13\}$ in Table 8.3 of Example 8.2. One can verify that $\delta(F) = \{4,5,6,8,10,12,13\} \cup \{5,6,7,12,13\} = A$, and $\gamma(A) = ef \vee fhix = F$. Hence (A,F) is a monotone concept, and A, F are feasible. If $F = \hat{B}_1 \vee \hat{B}_2 = abc \vee lx$, $A = \{4,6,7\}$. According to Table 8.3, we have

$\delta(F) = \delta(\hat{B_1}) \cup \delta(\hat{B_2}) = \{4,6\} \neq A$, $\gamma(\delta(F)) \neq F$, $\delta(\gamma(A)) \neq A$. (A,F) is not a monotone concept, and A, and F are non-feasible.

Although the monotone concept overcomes some limitations of the Wille's formal concept [10], there remain two aspects that could be improved:

1) In monotone concept, intent and extent may not uniquely determine each other. In Example 8.2, according to Table 8.3 and Definition 8.6, we know that $(\{4,5,6,7,8,10,12,13\}, abcdefhl \vee ef \vee fhix)$ is a monotone concept which is different from $(\{4,5,6,7,8,10,12,13\}, ef \vee fhix)$, but their extents are identical.

2) Consider $\{4\},\{4,5,6,8,10,12,13\},\{5,6,7,12,13\}$ and $abcdefhl, fhix, ef$ in Table 8.3. It is easy to verify that $\{4\} \cup \{4,5,6,8,10,12,13\} \cup \{5,6,7,12,13\} = \{4,5,6,7,8,10,12,13\}$, and $(\{4,5,6,7,8,10,12,13\}, abcdefhl \vee ef \vee fhix)$ is a monotone concept. If considering $abcdefhl$, ef and $fhix$ as query words in an information retrieval system, we can find that $abcdefhl \vee ef \vee fhix$ represents the logical relations "*or*" among them. Notice $\{e,f\} \subset \{a,b,c,d,e,f,h,l\}$. Thus, if one object satisfies the condition expressed by $abcdefhl$, then it must satisfy that expressed by ef, i.e., $abcdefhl$ is redundant when $abcdefhl \vee ef \vee fhix$ forms a query. In other words, the queries $abcdefhl \vee ef \vee fhix$ and $ef \vee fhix$ are equivalent in semantics. However, they are intents of different monotone concepts defined by Definition 8.6.

In [44], Saquer and Deogun gave a general solution to find monotone concepts whose intents are close to the queries, and show how to find monotone concepts whose extents approximate any given set of objects. However, it seems that the following aspects of extents and intents approximations could be developed.

i) Let D be a set of objects. In [44], D is written as the union of the maximal extents of formal concepts that are contained in D and, possibly, a subset containing whatever elements remain in D. For example, the non-feasible object set $\{4,5,6,7\}$ is written as $\{4,6\} \cup \{6,7\} \cup \{5\}$. But it is also reasonable in practice to write it down as $\{4\} \cup \{4,6\} \cup \{6,7\} \cup \{5\}$. For instance, similar expressions have existed in [44] (see $L(\psi)$ in Example 8.3). Accordingly, both $(\{4,5,6,7,12,13\}, efhl \vee fhijx \vee cdefhix)$ and $(\{4,5,6,7,12,13\}, abcdefhl \vee efhl \vee fhijx \vee cdefhix)$ could be the upper approximation monotone concepts of $\{4,5,6,7\}$ in Table 8.3.

ii) When approximating $D = \{2,3,4,5,6,7,8,9,10,11,12,13\}$ in Table 8.3, one can get an approximation of the monotone concept (D, f) by using the approximation method presented in [44]. However, we can verify that $(D, cdf \vee ef \vee fh)$ is also a monotone concept which is another approximation monotone concept of D.

In order to deal with these problems, in the sequel we propose the AFS formal concept.

8.3.2 AFS Formal Concept

In this section, we propose AFS formal concept in which the Galois connection "α, β" of context (X,M,I) [10] can be extended to the connection between the EI algebra $(EM,\vee,\wedge)$ and the $E^{\#}I$ algebra $(E^{\#}X,\vee,\wedge)$ as follows: for any $\sum_{i\in I}(\prod_{m\in A_i} m)\in EM$, $\sum_{j\in J} a_j\in E^{\#}X$,

$$\alpha\left(\sum_{i\in I}\left(\prod_{m\in A_i} m\right)\right)=\sum_{i\in I}\alpha(A_i)\in E^{\#}X, \tag{8.14}$$

$$\beta\left(\sum_{j\in J} a_j\right)=\sum_{j\in J}\left(\prod_{m\in\beta(a_j)} m\right)\in EM. \tag{8.15}$$

For any $A\subseteq M, a\subseteq X$, we notice that $\alpha(\prod_{m\in A} m)=\alpha(A), \beta(a)=\prod_{m\in\beta(a)} m$, which are the same as the Galois connection "α,β" defined by (8.1) and (8.2). Thus the conventional formal concept lattice [10] can be explored in a more general mathematical framework—the AFS formal concept lattice.

In what follows, we denote the subsets of X with the lower case letters and the subsets of M with the capital letters, in order to distinguish the subsets of X from those of M.

Theorem 8.7. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over M and $E^{\#}X$ be the $E^{\#}I$ algebra over X. Then the following assertions hold:*

(1) α, β are maps, where α,β are defined by (8.14) and (8.15).

(2) For any ζ, $\eta\in EM$, ν, $\varsigma\in E^{\#}X$,

$$\alpha(\zeta\vee\eta)=\alpha(\zeta)\vee\alpha(\eta),\quad \alpha(\zeta\wedge\eta)=\alpha(\zeta)\wedge\alpha(\eta),$$
$$\beta(\nu\vee\varsigma)=\beta(\nu)\vee\beta(\varsigma),\quad \beta(\nu\wedge\varsigma)\leq\beta(\nu)\wedge\beta(\varsigma).$$

(3) For any ζ, $\eta\in EM$, ν, $\varsigma\in E^{\#}X$,

$$\zeta\leq\eta\Rightarrow\alpha(\zeta)\leq\alpha(\eta),$$
$$\nu\leq\varsigma\Rightarrow\beta(\nu)\leq\beta(\varsigma).$$

(4) For any $\zeta\in EM$, $\varsigma\in E^{\#}X$,

$$\zeta\geq\beta(\alpha(\zeta)),\quad \alpha(\zeta)=\alpha(\beta(\alpha(\zeta))),$$
$$\varsigma\leq\alpha(\beta(\varsigma)),\quad \beta(\varsigma)=\beta(\alpha(\beta(\varsigma))).$$

Proof. (1) Suppose $\zeta=\sum_{i\in I}(\prod_{m\in A_i} m)$, $\eta=\sum_{j\in J}(\prod_{m\in B_j} m)\in EM$, $\zeta=\eta$. That is, $\forall A_i$ ($i\in I$), $\exists B_k$ ($k\in J$) such that $A_i\supseteq B_k$ and $\forall B_j$ ($j\in J$), $\exists A_l$ ($l\in I$) such that $B_j\supseteq A_l$. This implies that $\forall\alpha(A_i)$ ($i\in I$), $\exists\alpha(B_k)$ ($k\in J$) such that $\alpha(A_i)\subseteq\alpha(B_k)$ and $\forall\alpha(B_j)$ ($j\in J$), $\exists\alpha(A_l)$ ($l\in I$) such that $\alpha(B_j)\subseteq\alpha(A_l)$. Therefore

$$\alpha\left(\sum_{i\in I}\left(\prod_{m\in A_i} m\right)\right)=\sum_{i\in I}\alpha(A_i)=\sum_{j\in J}\alpha(B_j)=\alpha\left(\sum_{j\in J}\left(\prod_{m\in B_j} m\right)\right)$$

and α is a map. Similarly, we can prove that β is also a map.

(2) $\alpha(\zeta \vee \eta) = \alpha(\zeta) \vee \alpha(\eta)$ and $\beta(\nu \vee \varsigma) = \beta(\nu) \vee \beta(\varsigma)$ can be directly verified by (8.14) and (8.15). Let $\zeta = \sum_{i\in I}(\prod_{m\in A_i} m)$, $\eta = \sum_{j\in J}(\prod_{m\in B_j} m) \in EM$, $\nu = \sum_{i\in I} a_i$, $\varsigma = \sum_{j\in J} b_j \in E^{\#}X$.

$$\begin{aligned}\alpha(\zeta \wedge \eta) &= \alpha\left(\sum_{i\in I, j\in J}\left(\prod_{m\in A_i\cup B_j} m\right)\right) = \sum_{i\in I, j\in J} \alpha(A_i \cup B_j)\\ &= \sum_{i\in I, j\in J} \alpha(A_i) \cap \alpha(B_j) = \alpha(\zeta) \wedge \alpha(\eta).\end{aligned}$$

$$\beta(\nu \wedge \varsigma) = \beta\left(\sum_{i\in I, j\in J} a_i \cap b_j\right) = \sum_{i\in I, j\in J} \beta(a_i \cap b_j).$$

For any $i \in I$, $j \in J$, since $\beta(a_i \cap b_j) \supseteq \beta(a_i)$, $\beta(a_i \cap b_j) \supseteq \beta(b_j)$, hence $\beta(a_i \cap b_j) \supseteq \beta(a_i) \cup \beta(b_j)$. This implies that

$$\begin{aligned}\beta(\nu \wedge \varsigma) &= \sum_{i\in I, j\in J}(\prod_{m\in\beta(a_i\cap b_j)} m) \le \sum_{i\in I, j\in J}(\prod_{m\in\beta(a_i)\cup\beta(b_j)} m)\\ &= \left(\sum_{i\in I}(\prod_{m\in\beta(a_i)} m)\right) \wedge \left(\sum_{j\in J}(\prod_{m\in\beta(b_j)} m)\right) = \beta\left(\sum_{i\in I} a_i\right) \wedge \beta\left(\sum_{j\in J} b_j\right).\end{aligned}$$

(3) It can be directly verified by Theorem 4.1 and Theorem 5.24 and the properties of the Galois connection in Proposition 8.1.

(4) For $\zeta = \sum_{i\in I}(\prod_{m\in A_i} m) \in EM$, since for any $i \in I$, $A_i \subseteq \beta \cdot \alpha(A_i)$, $\alpha(A_i) = \alpha \cdot \beta \cdot \alpha(A_i)$, hence

$$\begin{aligned}\beta(\alpha(\zeta)) &= \beta\left(\sum_{i\in I} \alpha(A_i)\right) = \sum_{i\in I}(\prod_{m\in\beta\cdot\alpha(A_i)} m) \le \sum_{i\in I}(\prod_{m\in A_i} m),\\ \alpha(\beta(\alpha(\zeta))) &= \sum_{i\in I} \alpha \cdot \beta \cdot \alpha(A_i) = \sum_{i\in I} \alpha(A_i) = \alpha(\zeta).\end{aligned}$$

For $\nu = \sum_{i\in I} a_i \in E^{\#}X$, since for any $i \in I$, $a_i \subseteq \alpha \cdot \beta(a_i)$, $\beta(a_i) = \beta \cdot \alpha \cdot \beta(a_i)$, hence

$$\alpha(\beta(\nu)) = \alpha\left(\sum_{i\in I}(\prod_{m\in\beta(a_i)} m)\right) = \sum_{i\in I} \alpha \cdot \beta(a_i) \ge \sum_{i\in I} a_i,$$

$$\beta(\alpha(\beta(\nu))) = \sum_{i\in I} \beta \cdot \alpha \cdot \beta(a_i) = \sum_{i\in I} \beta(a_i) = \beta(\nu).$$

The proof is complete. □

Definition 8.7. Let X be a set and M be a set of attributes on X. Let (X, M, I) be a context, EM be the EI algebra over M and $E^{\#}X$ be the $E^{\#}I$ algebra over X. Let $\zeta = \sum_{i\in I}(\prod_{m\in A_i} m) \in EM$, $\nu \in \sum_{j\in J} a_j \in E^{\#}X$. (ν, ζ) is called an *AFS formal concept of the context* (X, M, I), if $\alpha(\zeta) = \nu$, $\beta(\nu) = \zeta$. ν is called the *extent of the AFS* *formal concept* (ν, ζ) and ζ is called the *intent of the AFS formal concept* (ν, ζ). $\mathscr{L}(E^{\#}X, EM, I)$ is the *set of all AFS formal concepts of the context* (X, M, I).

In virtue of the semantics of each element in EM demonstrated in the previous chapters, we know the complex attributes in EM are much richer in expressions than the attributes in 2^M. In real world situations, many phenomena can be described by AFS formal concepts. For example, it is necessary to allow a query containing few search conditions when we use an information retrieval system. The relationships among the search conditions are usually "*or*" and "*and*" logic expression. Thus the query can be represented by the intent of an AFS formal concept. For example, $ab + bcd + e + hi$ in Table 8.3 can be used to represent the query "*ab* OR *bcd* OR *e* OR *hi*". The answer to query can be represented by the extent of an AFS formal concept.

Definition 8.8. Let X be a set and M be a set of attributes on X. Let (X, M, I) be a context, EM be the EI algebra over M and $E^{\#}X$ be the $E^{\#}I$ algebra over X. $\zeta = \sum_{i \in I}(\prod_{m \in A_i} m) \in EM$, $\nu = \sum_{j \in J} a_j \in E^{\#}X$, if $\beta(\alpha(\zeta)) \neq \zeta$, ζ is called a *non-feasible fuzzy concept*. If $\alpha(\beta(\nu)) \neq \nu$, ν is called a *non-feasible $E^{\#}I$ element*.

For example, let $\zeta = ab + f$ in Table 8.3. Due to $\beta \cdot \alpha(\{a, b\}) = \beta(\{4\}) = \{a, b, c, d, e, f, h, l\} \neq \{a, b\}$, $\beta \cdot \alpha(\{f\}) = f$. Then, $\beta(\alpha(\zeta)) \neq \zeta$, ζ is non-feasible.

Lemma 8.4. *Let X be a set and M be a set of attributes on X. Let (X, M, I) be a context. Then the following assertions hold:*

(1) For any $(\nu, \zeta) \in \mathscr{L}(E^{\#}X, EM, I)$, let $\nu = \sum_{i \in I} a_i$, $\zeta = \sum_{j \in J}(\prod_{m \in A_j} m)$. If $\sum_{j \in J}(\prod_{m \in A_j} m)$ and $\sum_{i \in I} a_i$ are irreducible, then $|I| = |J|$ ($|I|$ denotes the cardinality of I) and for any $i \in I$, $j \in J$, A_j is the intent of a formal concept of (X, M, I), a_i is the extent of a formal concept of (X, M, I).

(2) Let $\nu = \sum_{i \in I} a_i \in E^{\#}X$, $\zeta = \sum_{j \in J}(\prod_{m \in A_j} m) \in EM$, and $\sum_{i \in I} a_i$, $\sum_{j \in J}(\prod_{m \in A_j} m)$ be irreducible. If for any $j \in J$, A_j is the intent of a formal concept of context (X, M, I), then $(\alpha(\zeta), \zeta) \in \mathscr{L}(E^{\#}X, EM, I)$. If for any $i \in I$, a_i is the extent of a formal concept of context (X, M, I), then $(\nu, \beta(\nu)) \in \mathscr{L}(E^{\#}X, EM, I)$.

Proof. (1) Assume $|I| \neq |J|$. Without loss of generality, let $|I| < |J|$. By the fact that (ν, ζ) is an AFS formal concept (Definition 8.7), we know that $\beta(\nu) = \zeta$ and the cardinality of $\beta(\nu)$ is $|I|$. Since $|I| < |J|$, hence $\sum_{j \in J}(\prod_{m \in A_j} m)$ is not irreducible, which contradicts the fact that $\sum_{j \in J}(\prod_{m \in A_j} m)$ is irreducible.

Next, we will prove for any $j \in J$, A_j is the intent of a formal concept of (X, M, I) and a_i is an extent of some formal concept with an intent in $\{A_j \mid j \in J\}$. By Definition 8.7, we have $\beta(\alpha(\zeta)) = \zeta$, $\alpha(\beta(\nu)) = \nu$. This implies that there exists a bijection p from I to J such that for any $i \in I$, $\beta(a_i) = A_{p(i)}$. Since $\alpha(\beta(\nu)) = \nu$, then

$$\alpha(\beta(\nu)) = \alpha\left(\sum_{i \in I}(\prod_{m \in A_{p(i)}} m)\right) = \sum_{i \in I} \alpha(A_{p(i)}) = \sum_{i \in I} a_i.$$

If there exists $i \in I$ such that $\alpha(A_{p(i)}) = \alpha(\beta(a_i)) \neq a_i$, which means there exists a_k, $i \neq k$, such that $\alpha(A_{p(i)}) = \alpha(\beta(a_i)) = a_k$. By the properties of Galois connection "α, β", we have $a_k = \alpha(\beta(a_i)) \supseteq a_i$. It contradicts the fact that $\sum_{i\in I} a_i$ is irreducible. Thus, $\alpha(A_{p(i)}) = \alpha(\beta(a_i)) = a_i$ and $(a_i, A_{p(i)})$ is a concept of context (X,M,I).

(2) One can directly verify that $(\alpha(\zeta), \zeta)$ is an AFS formal concept of the context (X,M,I) by Definition 8.7, (8.14) and (8.15). Similarly, the second conclusion holds as well. □

Theorem 8.8. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context and $\mathscr{L}(E^{\#}X,EM,I)$ be the set of all AFS formal concepts of the context (X,M,I). Then, for any $(\nu,\zeta) \in \mathscr{L}(E^{\#}X,EM,I)$, ν and ζ are uniquely determined by each other.*

Proof. Let $\nu = \sum_{i\in I} a_i \in E^{\#}X$, $\zeta = \sum_{j\in J}(\prod_{m\in A_j} m) \in EM$. Without loss of generality, let $\sum_{i\in I} a_i$ and $\sum_{j\in J}(\prod_{m\in A_j} m)$ be irreducible. By the Lemma 8.4, we get $|I| = |J|$. For simplicity, let $I = J$. Assume that ν and ζ are not uniquely determined by each other. Then, for ν, there exists $\rho = \sum_{k\in I}(\prod_{m\in B_k} m) \in EM$ ($\rho \neq \zeta$) such that $(\nu,\rho) \in \mathscr{L}(E^{\#}X,EM,I)$. Thus, there is at least one $i_0 \in I$ such that $A_{i_0} \neq B_i$ for any $i \in I$. From the Lemma 8.4 and Definition 8.7, we get that there exist $k \in I$, $j \in I$ such that (a_k, A_{i_0}), (a_k, B_j) are formal concepts of the context (X,M,I), then a_{i_0} is not an extent of a formal concept, which contradicts to (ν,ζ) is an AFS formal concept (by Lemma 8.4). Similarly, for ζ, there exists unique ν such that $(\nu,\zeta) \in \mathscr{L}(E^{\#}X,EM,I)$. □

Definition 8.9. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context and $\mathscr{L}(E^{\#}X,EM,I)$ be the set of all AFS formal concepts of the context (X,M,I). Let $(\nu_1,\zeta_1), (\nu_2,\zeta_2) \in \mathscr{L}(E^{\#}X,EM,I)$. Define $(\nu_1,\zeta_1) \leq (\nu_2,\zeta_2)$ if and only if $\nu_1 \leq \nu_2$ in lattice $E^{\#}X$ (or equivalently $\zeta_1 \leq \zeta_2$ in lattice EM).

It is obvious that $\leq$ defined by Definition 8.9 is a partial order on $\mathscr{L}(E^{\#}X,EM,I)$. The following theorem shows that the set $\mathscr{L}(E^{\#}X,EM,I)$ forms a complete lattice.

Theorem 8.9. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context and $\mathscr{L}(E^{\#}X,EM,I)$ be the set of all AFS formal concepts of the context (X,M,I). Then $\mathscr{L}(E^{\#}X,EM,I,\leq)$ is a complete lattice in which suprema and infima are given as follows: for any $(\nu_k,\zeta_k) \in \mathscr{L}(E^{\#}X,EM,I)$,*

$$\bigvee_{k\in K}(\nu_k,\zeta_k) = \left(\bigvee_{k\in K}\alpha(\zeta_k),\ \beta\left(\bigvee_{k\in K}\alpha(\zeta_k)\right)\right), \tag{8.16}$$

$$\bigwedge_{k\in K}(\nu_k,\zeta_k) = \left(\bigwedge_{k\in K}\alpha(\zeta_k),\ \beta\left(\bigwedge_{k\in K}\alpha(\zeta_k)\right)\right), \tag{8.17}$$

where $k \in K$, K is any non-empty indexing set.

Proof. In order to show that $\mathscr{L}(E^{\#}X,EM,I,\leq)$ is a complete lattice, we need to show that any subset of $\mathscr{L}(E^{\#}X,EM,I)$ has a least upper bound (suprema) and a greatest lower bound (infima). Let $S = \{(\nu_k, \zeta_k) \mid k \in K\}$ be any subset of $\mathscr{L}(E^{\#}X, EM, I)$. Let $\zeta_k = \sum_{s_k \in J_k} (\prod_{m \in A_{ks_k}} m), k \in K$, J_k be the indexing set associating to ζ_k. We claim,

$$suprema = \left(\bigvee_{k\in K} \alpha(\zeta_k),\ \beta\left(\bigvee_{k\in K} \alpha(\zeta_k) \right) \right),$$

$$infima = \left(\bigwedge_{k\in K} \alpha(\zeta_k),\ \beta\left(\bigwedge_{k\in K} \alpha(\zeta_k) \right) \right).$$

First, we show that $suprema = (\bigvee_{k\in K} \alpha(\zeta_k),\ \beta(\bigvee_{k\in K} \alpha(\zeta_k)))$. By Theorem 8.7, we have

$$\begin{aligned}
&\alpha\left(\beta\left(\bigvee_{k\in K}\alpha(\zeta_k)\right)\right)\\
&= \alpha\left(\beta\left(\sum_{k\in K}\alpha\left(\sum_{s_k\in J_k}(\prod_{m\in A_{ks_k}} m)\right)\right)\right) = \alpha\left(\sum_{k\in K}\beta\left(\alpha\left(\sum_{s_k\in J_k}(\prod_{m\in A_{ks_k}} m)\right)\right)\right)\\
&= \sum_{k\in K}\sum_{s_k\in J_k}\alpha\left(\beta\left(\alpha\left(\prod_{m\in A_{ks_k}} m\right)\right)\right) = \sum_{k\in K}\sum_{s_k\in J_k}\alpha\left(\prod_{m\in A_{ks_k}} m\right)\\
&= \sum_{k\in K}\alpha\left(\sum_{s_k\in J_k}\prod_{m\in A_{ks_k}} m\right) = \bigvee_{k\in K}\alpha(\zeta_k).
\end{aligned}$$

This implies $(\bigvee_{k\in K} \alpha(\zeta_k), \beta(\bigvee_{k\in K} \alpha(\zeta_k))) \in \mathscr{L}(E^{\#}X, EM, I)$, i.e., it is an AFS formal concept. Moreover, for any $k \in K$, $\nu_k = \alpha(\zeta_k) \le \bigvee_{k\in K} \alpha(\zeta_k)$ holds. Furthermore, $(\bigvee_{k\in K} \alpha(\zeta_k),\ \beta(\bigvee_{k\in K} \alpha(\zeta_k)))$ is an upper bound for S. Let $(\nu, \zeta) \in \mathscr{L}(E^{\#}X, EM, I)$ and for any $k \in K$, $(\nu_k, \zeta_k) \le (\nu, \zeta)$, i.e., (ν, ζ) is another upper bound for S. It is easy to get $\nu_k = \alpha(\zeta_k) \le \nu$ for any $k \in K$. Therefore, $\vee_{k\in K} \alpha(\zeta_k) \le \nu$ and $(\bigvee_{k\in K} \alpha(\zeta_k),\ \beta(\bigvee_{k\in K} \alpha(\zeta_k))) \le (\nu, \zeta)$, i.e.,

$$suprema = \left(\bigvee_{k\in K} \alpha(\zeta_k),\ \beta\left(\bigvee_{k\in K} \alpha(\zeta_k) \right) \right).$$

Next, we show that $infima = (\bigwedge_{k\in K} \alpha(\zeta_k),\ \beta(\bigwedge_{k\in K} \alpha(\zeta_k)))$. Since $E^{\#}X$ is a complete distributive lattice according to Theorem 5.2. Hence for any $k \in K$, one has

$$\bigwedge_{k\in K} \alpha(\zeta_k) = \sum_{f\in\Theta} \bigcap_{k\in K} \alpha(A_{kf(k)})$$

where $\Theta = \{f \mid f : K \to \bigcup_{k\in K} J_k \ s.t.\ f(k) \in J_k\}$. Thus by the definitions of α, β (i.e., (8.14) and (8.15)), we have

$$\alpha\left(\beta\left(\bigwedge_{k\in K}\alpha(\zeta_k)\right)\right)$$
$$=\alpha\left(\beta\left(\sum_{f\in\Theta}\bigcap_{k\in K}\alpha(A_{kf(k)})\right)\right)=\alpha\left(\beta\left(\sum_{f\in\Theta}\alpha\left(\bigcup_{k\in K}A_{kf(k)}\right)\right)\right)$$
$$=\alpha\left(\sum_{f\in\Theta}\prod_{m\in\beta\cdot\alpha\left(\bigcup_{k\in K}A_{kf(k)}\right)}m\right)=\sum_{f\in\Theta}\alpha\cdot\beta\cdot\alpha\left(\bigcup_{k\in K}A_{kf(k)}\right)$$
$$=\sum_{f\in\Theta}\alpha\left(\bigcup_{k\in K}A_{kf(k)}\right)=\sum_{f\in\Theta}\bigcap_{k\in K}\alpha(A_{kf(k)})=\bigwedge_{k\in K}\alpha(\zeta_k).$$

This shows $(\bigwedge_{k\in K}\alpha(\zeta_k),\ \beta(\bigwedge_{k\in K}\alpha(\zeta_k)))\in\mathscr{L}(E^{\#}X,EM,I)$, i.e., it is an AFS formal concept. Moreover, for any $k\in K$, $\bigwedge_{k\in K}\alpha(\zeta_k)\le\nu_k=\alpha(\zeta_k)$ holds, so $(\bigwedge_{k\in K}\alpha(\zeta_k),\beta(\bigwedge_{k\in K}\alpha(\zeta_k)))$ is a lower bound for S. Let $(\nu,\zeta)\in\mathscr{L}(E^{\#}X,EM,I)$ and for any $k\in K$, $(\nu_k,\zeta_k)\ge(\nu,\zeta)$, i.e., (ν,ζ) is another lower bound for S. This implies that for any $k\in K$, $\nu_k=\alpha(\zeta_k)\ge\nu$. Therefore, both $\bigwedge_{k\in K}\alpha(\zeta_k)\ge\nu$ and

$$\left(\bigwedge_{k\in K}\alpha(\zeta_k),\ \beta\left(\bigwedge_{k\in K}\alpha(\zeta_k)\right)\right)\ge(\nu,\zeta)$$

hold, i.e., $infima=(\bigwedge_{k\in K}\alpha(\zeta_k),\beta(\bigwedge_{k\in K}\alpha(\zeta_k)))$. □

In an AFS formal concept, its intent is a complex attribute of EM; correspondingly, its extent is an element of $E^{\#}X$. The extent and intent of an AFS formal concept can uniquely determine each other. Given some extents $a_1,a_2,...,a_n$ of formal concept [10], we can find a unique $\zeta\in EM$, as the intent of the AFS concept with extent $\sum_{i=1}^n a_i$. ζ is a semantic description of $\sum_{i=1}^n a_i$. On the contrary, given some intents $A_1,A_2,...,A_n$ of formal concept [10], we can find a unique $\nu\in E^{\#}X$, as the extent of the AFS concept with intent $\sum_{i=1}^n(\prod_{m\in A_i}m)$. ν is uniquely suitable to the description of $\sum_{i=1}^n(\prod_{m\in A_i}m)$.

Remark 8.1. By using AFS formal concepts, we can avoid the following two issues discussed above.

1. In the AFS formal concept, intent and extent can be uniquely determined by each other (Theorem 8.8), and there exists a bijection between each item of intent and each item of extent (Lemma 8.4).
2. AFS formal concept is based on the EI algebra $(EM,\vee,\wedge)$ and the $E^{\#}I$ algebra $(E^{\#}X,\vee,\wedge)$. In $(EM,\vee,\wedge)$, we can consider whether two complex attributes are equivalent or not under the semantics (Definition 4.1)existing in an information table. Thus we can filter some complex attributes without loss of main information. For instance, let us continue discussing items $\{4\}$, $\{4,5,6,8,10,12,13\}$, $\{5,6,7,12,13\}$ and $abcdefhl,ef,fhix$ in Table 8.3. In terms of the AFS algebra,

$abcdefhl + ef + fhix = ef + fhix$ (Definition 4.1). Thus items {4}, {4, 5, 6, 8, 10, 12, 13}, $\{5,6,7,12,13\}$ and $abcdefhl, ef, fhix$ can consist of an AFS formal concept ({4, 5, 6, 8, 10, 12, 13}+ $\{5,6,7,12,13\}$, $ef + fhix$). Moreover, $\{5,6,7,12,13\} \cup \{4,5,6,8,10,12,13\}$ is just identical with extent of ({4, 5, 6, 7, 8, 10, 12, 13}, $abcdefhl \vee ef \vee fhix$). Then, AFS formal concept have not lost a crucial original information, although the intents of AFS formal concepts are usually simpler than those of monotone concepts. Thus AFS formal concept constitutes an improvement of the monotone concept.

In general, not all queries are exactly contained in an information system, but there exist many words (or phrases) close to those. For example, in Example 8.3, there does not exist an AFS formal concept with intent f+cd, but AFS formal concepts with intent $ef + cdf$ and $fh + cdf$ exist in information Table 8.3. Accordingly, we study how to approximate a complex attribute in EM (or an element in $E^{\#}X$) by AFS formal concepts. In next section, we will investigate this issue in terms of rough set theory.

8.3.3 Rough Set Theory Approach to Concept Approximation

Let (X, M, I) be a context. Inspired by [44], for each $m \in M$, denote set

$$Im = \{x \in X \mid (x, m) \in I\}$$

represent all objects that possess the attribute m. Define a binary relation R_I over M as follows, for any $m_i, m_j \in M$,

$$(m_i, m_j) \in R_I \Leftrightarrow Im_i = Im_j. \tag{8.18}$$

That is to say, two attributes are related under R_I if and only if they are possessed by the same object set. It is easy to demonstrate that R_I is an equivalence relation over M. Denote M/R_I to be the set of all equivalence classes deduced by R_I over M, i.e., $M/R_I = \{[m_i] \mid m_i \in M\}$, where $[m_i] = \{m_j \mid (m_i, m_j) \in R_I\} = \{m_j \mid Im_i = Im_j\}$.

Similarity, we can define an equivalence relation T_I over X:

$$(x_i, x_j) \in T_I \Leftrightarrow x_i I = x_j I, \tag{8.19}$$

where $x_i, x_j \in X$, $x_i I = \{m \in M \mid (x_i, m) \in I\}$ represent all attributes which are possessed by the object x_i. X/T_I be the set of all equivalence classes deduced by T_I over X, i.e., $X/T_I = \{[x_i] \mid x_i \in X\}$, where $[x_i] = \{x_j \mid (x_i, x_j) \in T_I\} = \{x_j \mid x_i I = x_j I\}$.

The lower and upper approximations of subset B of M in the approximation space $\mathscr{A} = (M, R_I)$ defined by (6.1) are listed as follows:

$$A_*(B) = \{m \in M \mid [m] \subseteq B\} = \bigcup \{Y \in M/R_I \mid Y \subseteq B\}, \tag{8.20}$$

$$A^*(B) = \{m \in M \mid [m] \cap B \neq \varnothing\} = \bigcup \{Y \in M/R_I \mid Y \cap B \neq \varnothing\}. \tag{8.21}$$

Similarity, the lower and upper approximations of subset a of X in the approximation space $\mathscr{A} = (X, T_I)$ defined by (6.1) are listed as follows:

$$A_*(a) = \{x \in X \mid [x] \subseteq X\} = \bigcup\{z \in X/T_I \mid z \subseteq a\}, \tag{8.22}$$

$$A^*(a) = \{x \in X \mid [x] \cap a \neq \varnothing\} = \bigcup\{z \in X/T_I \mid z \cap a \neq \varnothing\}. \tag{8.23}$$

Definition 8.10. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. For any $\psi = \sum_{i\in I}(\prod_{m\in B_i} m) \in EM$, $\underline{\psi}$ the *lower approximation* and $\overline{\psi}$ the *upper approximation of the fuzzy concept* ψ are given in the form:

$$\underline{\psi} = \sum_{i\in I}(\prod_{m\in A^*(B_i)} m) \in EM, \quad \overline{\psi} = \sum_{i\in I}(\prod_{m\in A_*(B_i)} m) \in EM. \tag{8.24}$$

For any $\theta = \sum_{i\in I} a_i \in E^{\#}M$, $\underline{\theta}$ the *lower approximation* and $\overline{\theta}$ the *upper approximation of the $E^{\#}I$ algebra element* θ are defined as follows.

$$\underline{\theta} = \sum_{i\in I} A_*(a_i) \in E^{\#}M, \quad \overline{\theta} = \sum_{i\in I} A^*(a_i)) \in E^{\#}M. \tag{8.25}$$

Proposition 8.7. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. Then the following assertions hold.*

(1) for any $\psi_1, \psi_2, \gamma \in EM$,

$$\underline{\gamma} \leq \gamma \leq \overline{\gamma},$$
$$\overline{(\psi_1 \vee \psi_2)} = \overline{(\psi_1)} \vee \overline{(\psi_2)}\ ,\ \underline{(\psi_1 \vee \psi_2)} = \underline{(\psi_1)} \vee \underline{(\psi_2)},$$
$$\overline{(\psi_1 \wedge \psi_2)} \leq \overline{(\psi_1)} \wedge \overline{(\psi_2)}\ ,\ \underline{(\psi_1 \wedge \psi_2)} = \underline{(\psi_1)} \wedge \underline{(\psi_2)}.$$

(2) for any $\theta_1, \theta_2, \vartheta \in E^{\#}X$,

$$\underline{\vartheta} \leq \vartheta \leq \overline{\vartheta},$$
$$\overline{(\theta_1 \vee \theta_2)} = \overline{(\theta_1)} \vee \overline{(\theta_2)}\ ,\ \underline{(\theta_1 \vee \theta_2)} = \underline{(\theta_1)} \vee \underline{(\theta_2)},$$
$$\overline{(\theta_1 \wedge \theta_2)} \leq \overline{(\theta_1)} \wedge \overline{(\theta_2)}\ ,\ \underline{(\theta_1 \wedge \theta_2)} = \underline{(\theta_1)} \wedge \underline{(\theta_2)}.$$

Its proof is left to the reader. Whether the upper and lower approximations defined by (8.24) and (8.25) have the same properties as the upper and lower approximation defined by (6.1) remains an open problem.

Let (X,M,I) be a context, M be set of elementary attributes, $B_i \subseteq M$, $\psi = \sum_{i\in I}(\prod_{m\in B_i} m) \in EM$ be non-feasible, i.e., $\beta(\alpha(\psi)) \neq \psi$ (Definition 8.8). We are interested in finding AFS formal concepts whose intents approximate ψ. Let $L(\psi)$ and $U(\psi)$ be two AFS formal concepts, whose intents are the *lower and upper approximations of* ψ respectively, as follows:

$$L(\psi)=\left(\sum_{i\in I}\alpha(A^*(B_i)),\ \sum_{i\in I}\prod_{m\in\beta\cdot\alpha(A^*(B_i))}m\right)\in\mathscr{L}(E^\#X,EM,I),\quad(8.26)$$

$$U(\psi)=\left(\sum_{i\in I}\alpha(A_*(B_i)),\ \sum_{i\in I}\prod_{m\in\beta\cdot\alpha(A_*(B_i))}m\right)\in\mathscr{L}(E^\#X,EM,I).\quad(8.27)$$

where $A_*(B_i)$, $A^*(B_i)$ defined by (8.20) and (8.21), respectively. "α,β" is Galois connection defined by (8.1) and (8.2). The following Proposition 8.8 shows that $L(\psi)$ and $U(\psi)$ are AFS formal concepts of the context (X,M,I). $L(\psi)$ is called the *lower AFS formal concept approximation of the fuzzy concept* ψ and $U(\psi)$ is called the *upper AFS formal concept approximation of the fuzzy concept* ψ.

Proposition 8.8. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^\#X$ be the $E^\#I$ algebra over the set X. Then for any $\psi=\sum_{i\in I}(\prod_{m\in B_i}m)\in EM$, the following assertions hold for the lower and upper AFS formal concept approximations of the fuzzy concept ψ:*

$$L(\psi)=\left(\sum_{i\in I}\alpha(A^*(B_i))),\ \sum_{i\in I}\prod_{m\in\beta\cdot\alpha(A^*(B_i))}m\right)=(\alpha(\underline{\psi}),\beta\cdot\alpha(\underline{\psi})),$$

$$U(\psi)=\left(\sum_{i\in I}\alpha(A_*(B_i)),\ \sum_{i\in I}\prod_{m\in\beta\cdot\alpha(A_*(B_i))}m\right)=(\alpha(\overline{\psi}),\beta\cdot\alpha(\overline{\psi})).$$

where α and β defined by (8.14) and (8.15), respectively. $\underline{\psi}$ and $\overline{\psi}$ defined by (8.24).

The proof of this proposition remains as an exercise. By Proposition 8.8, Definition 8.7 and Theorem 8.7, we know that both $L(\psi)$ and $U(\psi)$ are AFS formal concepts of the context (X,M,I).

Proposition 8.9. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^\#X$ be the $E^\#I$ algebra over the set X. Then the following assertions hold:*

(1) For any $\psi\in EM$, $L(\psi)\le(\alpha(\psi),\beta(\alpha(\psi)))\le U(\psi)$, where α,β defined by (8.14) and (8.15);
(2) For $\psi_1,\psi_2\in EM$, $\psi_1\le\psi_2\Rightarrow L(\psi_1)\le L(\psi_2)$, $U(\psi_1)\le U(\psi_2)$,

where $L(.)$ and $U(.)$ are defined by (8.26) and (8.27), respectively.

Proof. (1) Let $\psi=\sum_{i\in I}(\prod_{m\in A_i}m)\in EM$. For any $i\in I$, we can get $A_*(A_i)\subseteq A_i\subseteq A^*(A_i)$ from the formulas (8.20) and (8.21). By using properties of the Galois connection "α,β" Proposition 8.1, we have $\alpha(A_*(A_i))\supseteq\alpha(A_i)\supseteq\alpha(A^*(A_i))$. From the definition of AFS formal concept (Definition 8.7) and the formulas (8.26)–(8.27), we get $L(\psi)\le(\alpha(\psi),\beta(\alpha(\psi)))\le U(\psi)$.

(2) Let $\psi_1=\sum_{j\in J}(\prod_{m\in B_j}m)$, $\psi_2=\sum_{i\in I}(\prod_{m\in A_i}m)\in EM$. Since $\psi_1\le\psi_2$, hence for any $j\in J$, there exists an $i\in I$ such that $A_i\subseteq B_j$. By proposition 6.1, $A_*(A_i)\subseteq$

$A_*(B_j)$. From the definition of AFS formal concept (Definition 8.7) and the formulas (8.26)–(8.27), we get $L(\psi_2) \le L(\psi_1)$. Similarly, we obtain $U(\psi_2) \le U(\psi_1)$. □

In Example 8.3, we compare AFS formal concept approximations with results in [44].

Example 8.3. Let X be a set and M be a set of attributes on X. Consider the context (X,M,I) given in Table 8.3. An "×" is placed in the p-th row and q-th column to indicate that object p has attribute q. Let $B = \{f,h,i\}$, from (8.18), (8.20) and (8.21), one can get that $M/R_I = \{ab,c,d,e,f,h,ix,j,k,l\}$. In the approximation space $\mathscr{A} = (X,R_I)$, $A_*(B) = \{f,h\}$, $A^*(B) = \{f,h,i,x\}$. Let $\varphi = fhi \in EM$. Then owing to formulas (8.26)–(8.27), we have

$$L(\varphi) = (\{5,6,7,12,13\}, fhix),$$
$$U(\varphi) = (\{2,3,4,5,6,7,8,11,12,13\}, fh).$$

The authors in [44] gave an example on approximating a non-feasible monotone formula in which $\psi = ab \vee bcd \vee e \vee hi \vee fhi$. Due to $\alpha \cdot \beta(\{a,b\}) = \alpha(\{4\}) = \{a,b,c,d,e,f,h,l\} \ne \{a,b\}$, ψ is non-feasible. $L(\psi)$ and $U(\psi)$ are computed as illustrated in Table 8.4.

Table 8.4 The lower and upper approximation of ψ [44]

i	B_i	$A^*(B_i)$	$A_*(B_i)$	$L(B_i)$	$U(B_i)$
1	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$(\{4\}, abcdefhl)$	$(\{4\}, abcdefhl)$
2	$\{b,c,d\}$	$\{a,b,c,d\}$	$\{c,d\}$	$(\{4\}, abcdefhl)$	$(\{4,5,8,9,10,12,13\}, cdf)$
3	$\{e\}$	$\{e\}$	$\{e\}$	$(\{4,5,6,8,10,12,13\}, ef)$	$(\{4,5,6,8,10,12,13\}, ef)$
4	$\{h,i\}$	$\{h,i,x\}$	$\{h\}$	$(\{5,6,7,12,13\}, fhix)$	$(\{2,3,4,5,6,7,8,11,12,13\}, fh)$
5	$\{f,h,i\}$	$\{f,h,i,x\}$	$\{f,h\}$	$(\{5,6,7,12,13\}, fhix)$	$(\{2,3,4,5,6,7,8,11,12,13\}, fh)$

The authors concluded that

$$\begin{aligned} L(\psi) &= (\{4\} \cup \{4,5,6,8,10,12,13\} \cup \{5,6,7,12,13\}, \\ &\qquad\qquad abcdefhl \vee ef \vee fhix) \\ &= (\{4,5,6,7,8,10,12,13\}, abcdefhl \vee ef \vee fhix), \\ U(\psi) &= (\{4\} \cup \{4,5,8,9,10,12,13\} \cup \{4,5,6,8,10,12,13\} \\ &\qquad \cup \{2,3,4,5,6,7,8,11,12,13\}, abcdefhl \vee cdf \vee ef \vee fh) \\ &= (\{2,3,4,5,6,7,8,9,10,11,12,13\}, abcdefhl \vee cdf \vee ef \vee fh). \end{aligned}$$

However, by using Definition 4.1, we find that in EM

$$abcdefhl + ef + fhix = ef + fhix,$$
$$abcdefhl + cdf + ef + fh = cdf + ef + fh.$$

By Definition 5.3, we find that in $E^{\#}X$

$$\begin{aligned}
&\{4\}+\{4,5,6,8,10,12,13\}+\{5,6,7,12,13\}\\
&\quad=\{4,5,6,8,10,12,13\}+\{5,6,7,12,13\},\\
&\{4\}+\{4,5,8,9,10,12,13\}+\{4,5,6,8,10,12,13\}+\{2,3,4,5,6,7,8,11,12,13\}\\
&\quad=\{4,5,8,9,10,12,13\}+\{4,5,6,8,10,12,13\}+\{2,3,4,5,6,7,8,11,12,13\}.
\end{aligned}$$

By the formulas $L(\psi)$ and $U(\psi)$, we can get that

$$\begin{aligned}
L(\psi) &= (\{4\}+\{4,5,6,8,10,12,13\}+\{5,6,7,12,13\},\\
&\qquad\qquad abcdefhl+ef+fhix)\\
&= (\{4,5,6,8,10,12,13\}+\{5,6,7,12,13\}, ef+fhix),
\end{aligned}$$

and

$$\begin{aligned}
U(\psi) &= (\{4\}+\{4,5,8,9,10,12,13\}+\{4,5,6,8,10,12,13\}\\
&\qquad +\{2,3,4,5,6,7,8,11,12,13\}, abcdefhl+cdf+ef+fh)\\
&= (\{4,5,8,9,10,12,13\}+\{4,5,6,8,10,12,13\}\\
&\qquad +\{2,3,4,5,6,7,8,11,12,13\}, cdf+ef+fh).
\end{aligned}$$

It is easy to verify that ψ, $L(\psi)$, $U(\psi)$ satisfy (1) of Proposition 8.9.

Remark 8.2. From Example 8.3, one can observe that the semantics of the intents of the lower and upper approximations of ψ by AFS formal concepts are equivalent to those of ψ by monotone concepts. However, the extents of them are different, and the extents of AFS formal concepts preserve more information than those of monotone concepts. In addition, the semantic equivalence and logic operations are introduced in AFS formal concepts. These are more conveniently to represent the logic operations of queries in information retrieval.

Let (X,M,I) be a context, $a_i \subseteq X$, $\theta=\sum_{i\in I} a_i \in E^{\#}X$. We are interested in finding AFS formal concepts whose extents approximate θ. Let $L(\theta)$ and $U(\theta)$ be two AFS formal concepts, whose extents represent the lower and upper approximations of θ, respectively, as follows

$$L(\theta)=\left(\sum_{i\in I}\alpha\cdot\beta(A_*(a_i)),\ \sum_{i\in I}\prod_{m\in\beta(A_*(a_i))} m\right)\in\mathscr{L}(E^{\#}X,EM,I), \tag{8.28}$$

$$U(\theta)=\left(\sum_{i\in I}\alpha\cdot\beta(A^*(a_i)),\ \sum_{i\in I}\prod_{m\in\beta(A^*(a_i))} m\right)\in\mathscr{L}(E^{\#}X,EM,I). \tag{8.29}$$

where $A_*(a_i)$ and $A^*(a_i)$ defined by (8.22) and (8.23), respectively. "α,β" are Galois connection defined by (8.14) and (8.15). The following Proposition 8.10 shows that

$L(\theta)$ and $U(\theta)$ are AFS formal concepts of the context (X,M,I). $L(\theta)$ is called the *lower AFS formal concept approximation of the $E^{\#}I$ algebra element θ* and $U(\psi)$ is called the *upper AFS formal concept approximation of the $E^{\#}I$ algebra element θ*.

Proposition 8.10. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. Then for any $\theta = \sum_{i\in I} a_i \in E^{\#}X$, the following assertions hold for the lower and upper AFS formal concept approximations of θ:*

$$L(\psi) = \left(\sum_{i\in I}\alpha\cdot\beta(A_*(a_i)),\ \sum_{i\in I}\prod_{m\in\beta(A_*(a_i))} m\right) = (\alpha\cdot\beta(\underline{\theta}),\ \ \beta(\underline{\theta})),$$

$$U(\psi) = \left(\sum_{i\in I}\alpha\cdot\beta(A^*(a_i)),\ \sum_{i\in I}\prod_{m\in\beta(A^*(a_i))} m\right) = (\alpha\cdot\beta(\overline{\theta}),\ \ \beta(\overline{\theta})).$$

where α and β defined by (8.14) and (8.15), respectively. $\underline{\theta}$ and $\overline{\theta}$ defined by (8.25).

The proof of this proposition remains as an exercise. By Proposition 8.10, Definition 8.7 and Theorem 8.7, we know that both $L(\theta)$ and $U(\theta)$ are AFS formal concepts of the context (X,M,I).

Proposition 8.11. *Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. Then the following assertions hold:*

(1) For any $\theta \in E^{\#}X$, $L(\theta) \le (\alpha\cdot\beta(\theta),\ \ \beta(\theta)) \le U(\theta)$, where α,β defined by (8.14) and (8.15);
(2) For $\theta_1,\theta_2 \in EM$, $\theta_1 \le \theta_2 \Rightarrow L(\theta_1) \le L(\theta_2),\ U(\theta_1) \le U(\theta_2)$,

where $L(.)$ and $U(.)$ defined by (8.28) and (8.29), respectively.

Example 8.4. Let X be a set and M be a set of attributes on X. Consider the context (X,M,I) given in Table 8.3. From formula (8.19), one obtains

$$X/T_I = \{\{1\},\{2\},\{3,11\},\{4\},\{5,12,13\},\{6\},\{7\},\{8\},\{9\},\{10\}\}.$$

Let $\theta = \sum_{i\in I} a_i = \{2,3\}+\{4\}+\{5,6,7\} \in E^{\#}X$. From formulas (8.22) and (8.23), $L(\theta)$ and $U(\theta)$ are computed as presented in Table 8.5.

Table 8.5 The lower and upper approximation of θ

i	a_i	$A_*(a_i)$	$A^*(a_i)$	$L(a_i)$	$U(a_i)$
1	{2,3}	{2}	{2,3,11}	({2,4,5,8,12,13}, dfh)	({2,3,4,5,6,7,8,11,12,13}, fh)
2	{4}	{4}	{4}	({4}, $abcdefhl$)	({4}, $abcdefhl$)
3	{5,6,7}	{6,7}	{5,6,7,12,13}	({6,7}, $fhijx$)	({5,6,7,12,13}, $fhix$)

Therefore, from properties of EI, $E^{\#}I$ algebra and the formulas (8.28)-(8.29), we get

$$\begin{aligned} L(\theta) &= (\{2,4,5,8,12,13\}+\{4\}+\{6,7\}, dfh+abcdefhl+fhijx) \\ &= (\{2,4,5,8,12,13\}+\{6,7\}, dfh+fhijx), \end{aligned}$$

and

$$\begin{aligned} U(\theta) &= (\{2,3,4,5,6,7,8,11,12,13\}+\{4\}+\{5,6,7,12,13\}, \\ &\qquad fh+abcdefhl+fhix) \\ &= (\{2,3,4,5,6,7,8,11,12,13\}, fh). \end{aligned}$$

Remark 8.3. The extent and intent of AFS formal concept can uniquely determine each other. Thus, concept approximation by AFS formal concepts can avoid the issues i) and ii) stated in the abvoe section of monotone concepts and is more conveniently for query. When approximating $\{2,3,4,5,6,7,8,11,12,13\}+\{4,5,8,9,10,12,13\}+\{4,5,6,8,10,12,13\}$ by AFS formal concepts, we get that $(\{4,5,8,9,10,12,13\}+\{4,5,6,8,10,12,13\}+\{2,3,4,5,6,7,8,11,12,13\}, cdf+ef+fh)$ instead of (D,f), where $D=\{2,3,4,5,6,7,8,9,10,11,12,13\}$. When approximating D by AFS formal concepts, one also can obtain that (D,f) by union all of the items of extent of the AFS formal concept, which is the same as the approximation realized by monotone concepts.

In this section, the AFS formal concept is proposed, which can be more conveniently applied to represent query in information retrieval systems than both the monotone concept and the formal concept. The set of all AFS formal concepts forms a complete lattice. Furthermore, by virtue of rough set theory, we discuss how to find AFS formal concepts whose intents (extents) approximate a fuzzy concept in EM (or an element of $E^{\#}X$). The examples and remarks demonstrate that not only the forms of approximation results by using AFS formal concepts may be concise, but they do not lead to any loss of crucial information. In this way, the AFS formal concepts can be viewed as the generalization of the monotone concept and the formal concept.

8.4 AFS Fuzzy Formal Concept Analysis

In the above sections, the set M in any context (G,M,I) is a set of Boolean attributes on X. However, in the real world applications the set M often represents a set of fuzzy or Boolean attributes. Given this, in the this section, we show that any context $(G,M,\mathbb{I})$ with fuzzy attributes in M, where $\mathbb{I}$ stresses that there are fuzzy attributes in M, can be described by an AFS structure. Let G be a set of objects and M be a set of fuzzy or Boolean attributes. $\forall g_1,g_2 \in G$, τ is defined by

$$\tau(g_1,g_2) = \{m | m \in M, (g_1,g_2) \in R_m\},$$

where $(g_1,g_2) \in R_m$ (refer to Definition 4.2)$\Leftrightarrow$ g_1 belongs to attribute m at some degree and the degree of g_1 belonging to m is larger than or equal to that of g_2, or

g_1 belongs to m at some degree and g_2 does not at all. For a given context (G,M,I), we can establish an AFS structure (M,τ_I,G) according to (G,M,I) in the following manner.

$$\tau_I(g_1,g_2)=\{m\in M|(g_1,g_2)\in R_m\},$$

where for $m\in M$ and binary relation $I\subseteq G\times M$, g_1 belongs to attribute m at some degree which means that $(g_1,m)\in I$. Since each $m\in M$, m is a Boolean attribute, hence $(g_1,m)\in I$ implies that the degree of g_1 belonging to m is larger than or equal to that of g_2 for any $g_2\in G$. Therefore

$$\tau_I(g_1,g_2)=\{m\in M|(g_1,g_2)\in R_m\}=\{m\in M|(g_1,m)\in I\}.$$

Now, we discuss the AFS formal concept analysis, in which M is a set of fuzzy or Boolean attributes on X.

Definition 8.11. Let X, M be sets and (M,τ,X) be an AFS structure. A binary relation $\mathbb{I}_\tau$ from $X\times X$ to M is defined as follows: for $(x,y)\in X\times X, m\in M$,

$$((x,y),m)\in\mathbb{I}_\tau\Leftrightarrow m\in\tau(x,y). \tag{8.30}$$

It is clear that $(X\times X,\, M,\, \mathbb{I}_\tau)$ is a formal context defined by [10]. The formal context $(X\times X,M,\mathbb{I}_\tau)$ is called the *fuzzy context associating with the AFS structure* (M,τ,X).

Definition 8.12. Let X be a set and $E^\#(X\times X)$ be the $E^\#I$ algebra on $X\times X$. For any $a\subseteq X\times X$, any $x\in X$, we define

$$a^R(x)=\{y\in X\mid (x,y)\in a\}\subseteq X. \tag{8.31}$$

For any $\gamma=\sum_{i\in I}a_i\in E^\#(X\times X)$, the $E^\#I$ algebra valued membership function γ^R : $X\to E^\#X$ is defined as follows: for any $x\in X$,

$$\gamma^R(x)=\sum_{i\in I}a_i^R(x)\in E^\#X. \tag{8.32}$$

By the fuzzy norm (5.24) with $\mathscr{M}_\rho$ the measure shown as (5.16) for the function $\rho: X\to[0,+\infty)$, the membership function $\mu_{\gamma^R}(x)$ of γ^R is defined as follows: for any $x\in X$,

$$\mu_{\gamma^R}(x)=||\gamma^R(x)||_\rho=\sup_{i\in I}\{\mathscr{M}_\rho(a_i^R(x))\}\in[0,1]. \tag{8.33}$$

Thus every $\gamma\in E^\#(X\times X)$ can be regarded as a fuzzy set on X whose membership functions are defined by (8.32) or (8.33).

Since $E^\#X$ is a lattice, hence for each $\gamma\in E^\#(X\times X)$, $\gamma^R: X\to E^\#X$ defined by the formula (8.32) is a lattice valued fuzzy set. One can verify that for $\gamma,\eta\in E^\#(X\times X)$, if $\gamma\le\eta$ in lattice $E^\#(X\times X)$, then for any $x\in X$, $\gamma^R(x)\le\eta^R(x)$ in lattice $E^\#X$. Thus in $(X\times X,\, M,\, \mathbb{I}_\tau)$, the fuzzy context associated with the AFS structure (M,τ,X), for each attribute $\eta\in EM$, $\alpha(\eta)$ is a fuzzy set on X with the membership functions defined by (8.32) or (8.33), where "α" is the Galois connection defined by (8.15).

Contrastively, for any $\gamma \in E^{\#}(X \times X)$ as a fuzzy set defined by (8.32) or (8.33), $\beta(\gamma)$ is an attribute in EM, where "β" is the Galois connection defined by (8.14). If (γ, η) is an AFS formal concept defined by Definition 8.7, then the fuzzy set γ is the extent of (γ, η) and the attribute η, which is the AFS logic combination of the simple attributes in M and has a definitely semantic interpretation, is the intent of (γ, η).

Theorem 8.10. *Let X be a set and M be a set of simple attributes on X. Let (M, τ, X) be an AFS structure in which for any $x, y \in X, \tau(x,y) = \{m \in M \mid (x,y) \in R_m\}$ (refer to (4.26)) and $(X \times X,\ M,\ \mathbb{I}_\tau)$ be the fuzzy context associating with (M, τ, X). Then for $\zeta, \varsigma \in EM$, if $\beta(\alpha(\zeta)) = \beta(\alpha(\varsigma))$,i.e., both $\beta(\alpha(\zeta))$ and $\beta(\alpha(\varsigma))$ are the intent of an AFS formal concept, then $\forall x \in X$, $\zeta(x) = \varsigma(x)$ and $\mu_\zeta(x) = \mu_\varsigma(x)$, where "$\alpha, \beta$" are the Galois connections defined by (8.14) and (8.15); for any fuzzy attribute $\gamma = \sum_{u \in U}(\prod_{m \in C_u} m) \in EM, \gamma(x) = \sum_{u \in U} C_u^\tau(x) \in E^{\#}X$ is the $E^{\#}I$ valued membership function of γ defined by (5.13) and $\mu_\gamma(x) = ||\sum_{u \in U} C_u^\tau(x)||_\rho \in [0,1]$ is the membership function of γ defined by (5.25) for the fuzzy norm (5.24) with $\mathscr{M}_\rho$ the measure shown as (5.16) for the function $\rho : X \to [0, +\infty)$.*

Proof. According to the definitions of $(X \times X, M, \mathbb{I}_\tau)$ and the Galois connection α, for any $m \in M$, we have

$$\alpha(\{m\}) = \{(x,y) \in X \times X \mid m \in \tau(x,y)\}.$$

By Proposition 8.1 and (8.32), we can verify that for any $A \subseteq M$, any $x \in X$,

$$\begin{aligned}\alpha(A)^R(x) &= \left(\bigcap_{m \in A} \alpha(\{m\})\right)^R (x)\\ &= \left(\bigcap_{m \in A} \{(x,y) \in X \times X \mid m \in \tau(x,y)\}\right)^R (x)\\ &= (\{(x,y) \in X \times X \mid A \subseteq \tau(x,y)\})^R(x).\end{aligned} \tag{8.34}$$

By (4.27) and (8.31), we have

$$(\{(x,y) \in X \times X \mid A \subseteq \tau(x,y)\})^R(x) = A^\tau(x). \tag{8.35}$$

Furthermore for any $\gamma = \sum_{u \in U}(\prod_{m \in C_u} m) \in EM$ and any $x \in X$, from (8.35) and (8.32), one has

$$\alpha(\gamma)^R(x) = \sum_{u \in U} \alpha(C_u)^R(x) = \sum_{u \in U} C_u^\tau(x) = \gamma(x) \tag{8.36}$$

That is the $E^{\#}I$ valued membership function of the fuzzy attribute γ defined by (5.13). For $\zeta, \varsigma \in EM$, if $\beta(\alpha(\zeta)) = \beta(\alpha(\varsigma))$, then

$$\alpha(\zeta) = \alpha(\beta(\alpha(\zeta))) = \alpha(\beta(\alpha(\varsigma))) = \alpha(\varsigma).$$

It follows from (8.36) that for any $x \in X$,

$$\zeta(x) = \alpha(\zeta)^R(x) = \alpha(\varsigma)^R(x) = \varsigma(x),$$

Since $||.||$ is a fuzzy norm on the lattice $E^\# X$, we have

$$\mu_\zeta(x) = ||\zeta(x)||_\rho = ||\varsigma(x)||_\rho = \mu_\varsigma(x).$$ □

Assume that each attribute in M is a Boolean attribute on X. For any $m \in M$, let R_m be the binary relation of m defined by Definition 4.2. Since $m \in M$ is Boolean concept, hence for any $x \in X$, either $(x,y) \in R_m$ for any $y \in X$ or $(x,y) \notin R_m$ for any $y \in X$. By (8.31), one has that for any $m \in M$, any $x \in X$, either $\{m\}^R(x) = X$ or $\{m\}^R(x) = \varnothing$. This implies that for any $A \subseteq M$, any $x \in X$, either $A^R(x) = X$ or $A^R(x) = \varnothing$. Further, by (8.32) and (8.33), for any $\zeta \in E^\#(X \times X)$, any $x \in X$, either $\zeta^R(x) = X$ or $\zeta^R(x) = \varnothing$, and either $\mu_{\zeta^R}(x) = 1$ or $\mu_{\zeta^R}(x) = 0$, i.e., $\mu_{\zeta^R}(x)$ is the characteristic function of a Boolean set $C_\zeta \subseteq X$. Proposition 4.3 has showed that the AFS logic system $(EM, \vee, \wedge, ')$ will degenerate into Boolean logic system $(2^X, \cup, \cap, ')$ if every attribute in M is a Boolean attribute. Therefore if each $m \in M$ is a Boolean attribute, then the AFS formal concept lattice of an AFS structure (M, τ, X) will degenerate into the formal concept lattice of context (X, M, I), where for $x \in X$ and $m \in M$, $(x,m) \in I \Leftrightarrow ((x,y),m) \in \mathbb{I}_\tau$ for any $y \in X \Leftrightarrow (x,y) \in R_m$ for any $y \in X \Leftrightarrow x$ has attribute m (refer to Definition 4.2). For each AFS formal concept (γ, η), the intent $\eta = \sum_{i \in I}(\prod_{m \in A_i} m) \in EM$ corresponds to the disjunctive normal form of a monotone Boolean formula $\bigvee_{i \in I} A_i$, where each $\prod_{m \in A_i} m$ is a Boolean conjunctive expression $\bigwedge_{a \in A_i} a$, and the extent $\gamma \subseteq X$ is

$$\gamma = \alpha(\eta) = \bigcup_{i \in I} \bigcap_{a \in A_i} \alpha(a).$$

For instance, in Example 8.1, for instance, the attribute $\xi = m_1 + m_2 m_4 + m_4 m_5 \in EM$ read as "*edible*" or "*poisonous and cap-shape*" or "*cap-shape and cap-surface: fibrous*". According to Table 8.2, we know that $\{m_1\}$, $\{m_2, m_4\}$ and $\{m_4, m_5\}$ are all intents of some concepts of the context (G, M, I). From Lemma 8.4, one has that

$$\left(\alpha(\{m_1\}) + \left(\alpha(\{m_2\}) \bigcap \alpha(\{m_4\})\right) + \left(\alpha(\{m_4\}) \bigcap \alpha(\{m_5\})\right),\ \xi\right)$$

is an AFS formal concept of the context (X, M, I). The following Example 8.5 demonstrates how to implement AFS fuzzy formal concept analysis for a data with both fuzzy and Boolean attributes.

Example 8.5. Let $X = \{x_1, x_2, ..., x_{10}\}$ be a set of 10 people and their features (attributes) which are described by real numbers (age, height, weight, salary, estate), Boolean values (gender) and the ordered relations (hair black, hair white, hair yellow), see Table 8.6; there the number i in the "*hair color*" columns which corresponds to some $x \in X$ implies that the hair color of x has ordered ith following our perception of the color by our intuitive perception. Let $M = \{m_1, m_2,\ ...,\ m_{10}\}$ be

Table 8.6 Descriptions of features

	appearance			wealth		gender		hair color		
	age	*height*	*weigh*	*salary*	*estate*	*male*	*female*	*black*	*white*	*yellow*
x_1	20	1.9	90	1	0	1	0	6	1	4
x_2	13	1.2	32	0	0	0	1	4	3	1
x_3	50	1.7	67	140	34	0	1	6	1	4
x_4	80	1.8	73	20	80	1	0	3	4	2
x_5	34	1.4	54	15	2	1	0	5	2	2
x_6	37	1.6	80	80	28	0	1	6	1	4
x_7	45	1.7	78	268	90	1	0	1	6	4
x_8	70	1.65	70	30	45	1	0	3	4	2
x_9	60	1.82	83	25	98	0	1	4	3	1
x_{10}	3	1.1	21	0	0	0	1	2	5	3

the set of fuzzy or Boolean concepts on X and each $m \in M$ associate to a single feature. Where m_1 : *"old people"*, m_2 : *"tall people"*, m_3 : *"heavy people"*, m_4 : *"high salary"*, m_5 : *"more estate"*, m_6 :*"male"*, m_7 : *"female"*, m_8 : *"black hair people"*, m_9 : *"white hair people"*, m_{10} : *"yellow hair people"*.

Let (M, τ, X) be the AFS structure of the data shown in Table 8.6. For simplicity, let S=2^X be the σ-algebra over X and m_ρ be the measure defined by (5.16) for the a weight function $\rho(x)$=1, $\forall x \in X$. Let

$$\zeta = m_1 m_3 m_4 + m_1 m_3 m_7,\ \xi = m_1 m_2 m_3 m_4 + m_1 m_2 m_3 m_7$$

be two fuzzy attributes in EM. It is obvious that $\zeta \geq \xi$ and $\zeta \neq \xi$ in lattice EM. One can verify that

$$\beta(\alpha(\prod_{m \in \{m_1, m_3, m_4\}} m)) = \prod_{m \in \{m_1, m_2, m_3, m_4\}} m,$$
$$\beta(\alpha(\prod_{m \in \{m_1, m_3, m_7\}} m)) = \prod_{m \in \{m_1, m_2, m_3, m_7\}} m.$$

Although ζ and ξ are different attributes in EM, i.e., ζ and ξ capture different semantics, the fuzzy sets defined by (5.13) or the norm of the lattice $E^{\#}X$ defined by (5.24) are identical, i.e., their extents are equal as shown in Table 8.7.

Table 8.7 Membership functions of ζ and ξ defined by (8.33)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$\mu_\zeta(.) = \mu_\xi(.)$	0.3	0.2	0.4	0.4	0.3	0.4	0.4	0.4	0.7	0.1

Let $N = \{m_1, m_2, m_3, m_6, m_7\} \subseteq M$. Here, we study the AFS fuzzy formal concept lattice $\mathscr{L}(E^{\#}(X \times X), EN, \mathbb{I}_\tau)$. According to Lemma 8.4, we know that for any AFS formal concept $(\nu, \eta) \in \mathscr{L}(E^{\#}(X \times X), EN, \mathbb{I}_\tau)$ there exist $A_i \subseteq N, i \in I$, A_i is the intent of a formal concept of context $(X \times X, N, \mathbb{I}_\tau)$ which is the fuzzy context associating with the AFS structure (N, τ, X) (refer to Definition 8.11) such that

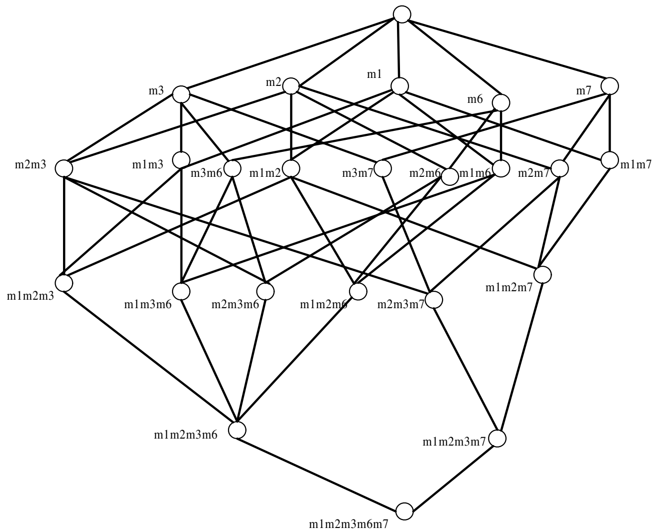

Fig. 8.1 Concept lattice of context $(X \times X, N, \mathbb{I}_\tau)$

Table 8.8 Membership functions of the extents of the formal concepts shown in Figure 8.1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$\mu_{m_1}(\cdot)$	0.3	0.2	0.7	1	0.4	0.5	0.6	0.9	0.8	0.1
$\mu_{m_1m_2}(\cdot)$	0.3	0.2	0.6	0.8	0.3	0.4	0.5	0.5	0.7	0.1
$\mu_{m_1m_3}(\cdot)$	0.3	0.2	0.4	0.6	0.3	0.4	0.4	0.5	0.7	0.1
$\mu_{m_1m_6}(\cdot)$	0.3	0	0	1	0.4	0	0.6	0.9	0	0
$\mu_{m_1m_7}(\cdot)$	0	0.2	0.7	0	0	0.5	0	0	0.8	0.1
$\mu_{m_1m_2m_3}(\cdot)$	0.3	0.2	0.4	0.6	0.3	0.4	0.4	0.4	0.7	0.1
$\mu_{m_1m_2m_6}(\cdot)$	0.3	0	0	0.8	0.3	0	0.5	0.5	0	0
$\mu_{m_1m_2m_7}(\cdot)$	0	0.2	0.6	0	0	0.4	0	0	0.7	0.1
$\mu_{m_1m_3m_6}(\cdot)$	0.3	0	0	0.6	0.3	0	0.4	0.5	0	0
$\mu_{m_1m_2m_3m_6}(\cdot)$	0.3	0	0	0.6	0.3	0	0.4	0.4	0	0
$\mu_{m_1m_2m_3m_7}(\cdot)$	0	0.2	0.4	0	0	0.4	0	0	0.7	0.1
$\mu_{m_2}(\cdot)$	1	0.2	0.7	0.8	0.3	0.4	0.7	0.5	0.9	0.1
$\mu_{m_2m_3}(\cdot)$	1	0.2	0.4	0.6	0.3	0.4	0.6	0.4	0.9	0.1
$\mu_{m_2m_6}(\cdot)$	1	0	0	0.8	0.3	0	0.7	0.5	0	0
$\mu_{m_2m_7}(\cdot)$	0	0.2	0.7	0	0	0.4	0	0	0.9	0.1
$\mu_{m_2m_3m_6}(\cdot)$	1	0	0	0.6	0.3	0	0.6	0.4	0	0

$\eta = \sum_{i \in I}(\prod_{m \in A_i} m)$. So we show the concept lattice generated by the fuzzy context $(X \times X, N, \mathbb{I}_\tau)$ in Figure 8.1 and the membership functions of the extents are shown in Table 8.8. Notice that although both the extents and the intents of the formal concepts in $(X \times X, N, \mathbb{I}_\tau)$ may be fuzzy sets and fuzzy attributes, $(X \times X, N, \mathbb{I}_\tau)$ is a traditional context [10]. This implies that its complexity is the same as a traditional context with $|X|^2$ objects and $|N|$ attributes.

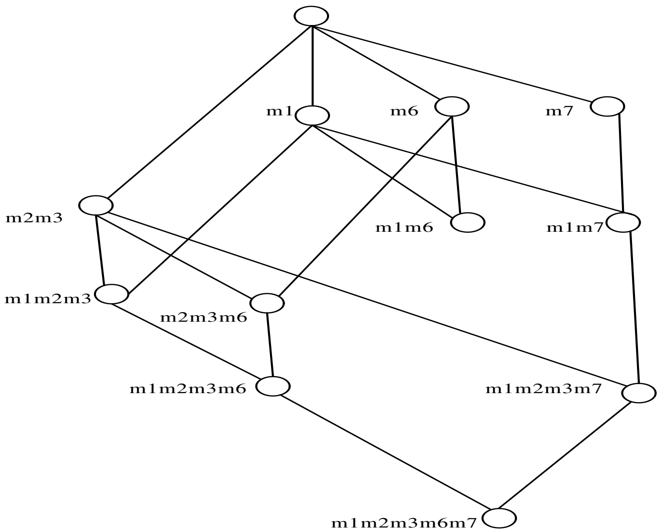

Fig. 8.2 Concept lattice of context $(X_1 \times X_1, N, \mathbb{I}_\tau)$

Table 8.9 Membership functions of the extents of the formal concept shown in Figure 8.2

	μ_{m_1}	$\mu_{m_1m_6}$	$\mu_{m_1m_7}$	$\mu_{m_1m_2m_3}$	$\mu_{m_1m_2m_3m_6}$	$\mu_{m_1m_2m_3m_7}$	$\mu_{m_2m_3}$	$\mu_{m_2m_3m_6}$	μ_{m_6}	μ_{m_7}
x_1	0.6	0.6	0	0.6	0.6	0	1	1	1	0
x_2	0.4	0	0.4	0.4	0	0.4	0.4	0	0	1
x_5	0.8	0.8	0	0.6	0.6	0	0.6	0.6	1	0
x_6	1	0	1	0.8	0	0.8	0.8	0	0	1
x_{10}	0.2	0	0.2	0.2	0	0.2	0.2	0	0	1

Let $X_1 = \{x_1, x_2, x_5, x_6, x_{10}\} \subseteq X$. Figure 8.2 shows the concept lattice generated by $(X_1 \times X_1, N, \mathbb{I}_\tau)$ and the membership functions of the extents are shown in Table 8.9. Although the intent and extent of an AFS formal concept are a fuzzy attribute in EM and a fuzzy set on X respectively, the context $(X \times X, M, \mathbb{I}_\tau)$ associating with an AFS structure (M, τ, X) is a traditional context which can be directly established by the original data without the use of the fuzzy set $\mathbb{I}$ to describe the uncertainty between the objects and the attributes. Thus the AFS formal concept lattices preserve more information contained in original data than the other fuzzy formal concept lattices. This observation stresses that the AFS formal concept analysis naturally extends the traditional formal concepts to the fuzzy formal concepts.

In order to cope with the data with various data types such as real numbers, Boolean value and even the human intuition description with sub-preferences, the AFS fuzzy formal concept analysis, which intuitively augments the traditional formal concepts to fuzzy formal concepts and overcomes the difficulties of other fuzzy formal concepts to define the fuzzy binary relation by human interpretations, is proposed and developed. The examples demonstrate that the AFS fuzzy formal concept analysis can be directly applied to the original data with both fuzzy and Boolean attributes and preserve more information contained in the original data than other fuzzy formal concepts. In the framework of AFS fuzzy formal concept analysis, the original data is only required to generate AFSFFCA lattices, human interpretation is not required to define the fuzzy binary relations and the fuzzy sets corresponding to all attributes in EM are automatically determined by a consistent algorithm according to the AFS structure and the AFS algebra. So AFSFFCA lattices are more objective and comprehensive representations of the knowledge contained in the original data than traditional and other fuzzy formal concepts. The theorems prove that AFS fuzzy formal concept lattices are more general mathematization of the traditional formal concept lattices. Many already existing mathematical tools such as topology, measure theory, combinatorics and algebras can be applied to the research of the AFS theory. These facts encourage us to derive mathematical properties of AFSFFCA and apply them to future research and development of knowledge representation schemes.

Exercises

Exercise 8.1. Let (G,M,I) be a context. Show that the following assertions hold:

(1) for $A_1,A_2 \subseteq G$, $A_1 \subseteq A_2$ implies $\beta(A_1) \supseteq \beta(A_2)$ and
for $B_1,B_2 \subseteq M$, $B_1 \subseteq B_2$ implies $\alpha(B_1) \supseteq \alpha(B_2)$;
(2) $A \subseteq \alpha(\beta(A))$ and $\beta(A) = \beta(\alpha(\beta(A)))$ for all $A \subseteq G$, and
$B \subseteq \beta(\alpha(B))$ and $\alpha(B) = \alpha(\beta(\alpha(B)))$ for all $B \subseteq M$.

Exercise 8.2. (Wille's Lemma) Let (G,M,I) be a context and $\mathscr{L}(G,M,I)$ denote the set of all formal concepts of the context (G,M,I). Show that

$$\mathscr{L}(G,M,I) = \{(\alpha(B),\beta(\alpha(B))) \mid B \subseteq M\}.$$

Exercise 8.3. (Fundamental Theorem of FCA) Let (G,M,I) be a context. Prove that $(\mathscr{L}(G,M,I),\vee,\wedge)$ is a complete lattice in which suprema and infima are given as follows: for any formal concepts $(A_j,B_j) \in \mathscr{L}(G,M,I)$, $j \in J$,

$$\bigvee_{j\in J}(A_j,B_j) = \left(\gamma_G\left(\bigcup_{j\in J}A_j\right),\ \bigcap_{j\in J}B_j\right),$$

$$\bigwedge_{j\in J}(A_j,B_j) = \left(\bigcap_{j\in J}A_j,\ \gamma_M\left(\bigcup_{j\in J}B_j\right)\right),$$

where $\gamma_G = \alpha\cdot\beta$, $\gamma_M = \beta\cdot\alpha$.

Exercise 8.4. Let X and M be sets, (G,M,I) be a context and E^IXM be defined as Definition 8.3. For $\sum_{u\in U} a_uA_u \in E^IXM$, if $a_q \subseteq a_w$, $A_q \subseteq A_w$, $w,q \in U$, $w \neq q$, prove that

$$\sum_{u\in U} a_uA_u = \sum_{u\in U, u\neq q} a_uA_u.$$

Exercise 8.5. Let X and M be sets, (G,M,I) be a context and E^IXM be the set defined as Definition 8.3. Prove that the binary relation $\leq$ is a partial order relation if $\sum_{u\in U} a_uA_u$, $\sum_{v\in V} b_vB_v \in E^IXM$, $\sum_{u\in U} a_uA_u \leq \sum_{v\in V} b_vB_v \Leftrightarrow \forall a_uA_u$ $(u\in U)$ $\exists b_kB_k$ $(k\in V)$ such that $a_u \subseteq b_k$, $A_u \subseteq B_k$.

Exercise 8.6. Let (G,M,I) be a context and E^IGM be defined as Definition 8.3. If for any $\sum_{u\in U} a_uA_u$, $\sum_{v\in V} b_vB_v \in E^IGM$, we define

$$\left(\sum_{u\in U} a_uA_u\right) * \left(\sum_{v\in V} b_vB_v\right) = \sum_{u\in U, v\in V} a_u \cap b_vA_u \cup B_v,$$
$$\left(\sum_{u\in U} a_uA_u\right) + \left(\sum_{v\in V} b_vB_v\right) = \sum_{u\in U\sqcup V} c_uC_u,$$

where $u\in U\sqcup V$ (the disjoint union of indexing sets U, V), $c_u = a_u$, $C_u = A_u$,if $u\in U$; $c_u = b_u$, $C_u = B_u$,if $u\in U$. Prove that "+" and "$*$" are binary compositions on E^IGM.

Exercise 8.7. Let G and M be finite sets, (G,M,I) be a context and $(E^IGM,*,+,\leq)$ be the E^CII algebra of context (G,M,I). Show that the following assertions hold. For any ψ, ϑ, γ, $\eta \in E^IGM$,

(1) $\psi+\vartheta = \vartheta+\psi, \quad \psi*\vartheta = \vartheta*\psi$;
(2) $(\psi+\vartheta)+\gamma = \psi+(\vartheta+\gamma), \quad (\psi*\vartheta)*\gamma = \psi*(\vartheta*\gamma)$;
(3) $(\psi+\vartheta)*\gamma = (\psi*\gamma)+(\vartheta*\gamma), \quad \psi*(\emptyset M) = (\emptyset M), \psi*(X\emptyset) = \psi$;
(4) If $\psi\leq\vartheta$, $\gamma\leq\eta$, then $\psi+\gamma\leq\vartheta+\gamma, \quad \psi*\gamma\leq\vartheta*\gamma$;

Exercise 8.8. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^\#X$ be the $E^\#I$ algebra over the set X. Show the validity of the following assertions hold.

(1) for any $\psi_1,\psi_2,\gamma \in EM$,

$$\underline{\gamma} \leq \gamma \leq \overline{\gamma},$$
$$\overline{(\psi_1\vee\psi_2)} = \overline{(\psi_1)}\vee\overline{(\psi_2)}\ ,\ \underline{(\psi_1\vee\psi_2)} = \underline{(\psi_1)}\vee\underline{(\psi_2)},$$
$$\overline{(\psi_1\wedge\psi_2)} \leq \overline{(\psi_1)}\wedge\overline{(\psi_2)}\ ,\ \underline{(\psi_1\wedge\psi_2)} = \underline{(\psi_1)}\wedge\underline{(\psi_2)}.$$

(2) for any $\theta_1,\theta_2,\vartheta \in E^\#X$,

$$\underline{\vartheta} \leq \vartheta \leq \overline{\vartheta},$$
$$\overline{(\theta_1\vee\theta_2)} = \overline{(\theta_1)}\vee\overline{(\theta_2)}\ ,\ \underline{(\theta_1\vee\theta_2)} = \underline{(\theta_1)}\vee\underline{(\theta_2)},$$
$$\overline{(\theta_1\wedge\theta_2)} \leq \overline{(\theta_1)}\wedge\overline{(\theta_2)}\ ,\ \underline{(\theta_1\wedge\theta_2)} = \underline{(\theta_1)}\wedge\underline{(\theta_2)}.$$

Exercise 8.9. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. Let (X,M,I) be a context, $B_i \subseteq M$, $\psi = \sum_{i\in I}(\prod_{m\in B_i} m) \in EM$ be a complex attribute. Prove that for any $\psi \in EM$, the lower and upper AFS formal concept approximations of the fuzzy concept ψ satisfy the relationships

$$L(\psi) = (\sum_{i\in I} \alpha(A^*(B_i))), \sum_{i\in I} \prod_{m\in\beta\cdot\alpha(A^*(B_i))} m) = (\alpha(\underline{\psi}), \beta(\alpha(\underline{\psi})))$$

$$U(\psi) = (\sum_{i\in I} \alpha(A_*(B_i)), \sum_{i\in I} \prod_{m\in\beta\cdot\alpha(A_*(B_i))} m) = (\alpha(\overline{\psi}), \beta(\alpha(\overline{\psi})))$$

where α and β are defined by (8.14) and (8.15), respectively. $\underline{\psi}$ and $\overline{\psi}$ are defined by (8.24).

Exercise 8.10. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. For any $\theta = \sum_{i\in I} a_i \in E^{\#}X$, show the following assertions hold for the lower and upper AFS formal concept approximations of θ:

$$L(\psi) = \left(\sum_{i\in I} \alpha\cdot\beta(A_*(a_i)),\ \sum_{i\in I} \prod_{m\in\beta(A_*(a_i))} m\right) = (\alpha\cdot\beta(\underline{\theta}),\ \beta(\underline{\theta})),$$

$$U(\psi) = \left(\sum_{i\in I} \alpha\cdot\beta(A^*(a_i)),\ \sum_{i\in I} \prod_{m\in\beta(A^*(a_i))} m\right) = (\alpha\cdot\beta(\overline{\theta}),\ \beta(\overline{\theta})).$$

where α and β are defined by (8.14) and (8.15), respectively. $\underline{\theta}$ and $\overline{\theta}$ are defined by (8.25).

Exercise 8.11. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. Show the following assertions hold:

(1) For any $\theta \in E^{\#}X$, $L(\theta) \le (\alpha\cdot\beta(\theta),\ \ \beta(\theta)) \le U(\theta)$, where α,β defined by (8.14) and (8.15);
(2) For $\theta_1,\theta_2 \in EM$, $\theta_1 \le \theta_2 \Rightarrow L(\theta_1) \le L(\theta_2)$, $U(\theta_1) \le U(\theta_2)$,

where $L(.)$ and $U(.)$ are defined by (8.28) and (8.29), respectively.

Open problems

Problem 8.1. Let X and M be sets, (G,M,I) be a context and $(E^I XM,\le)$ be the partially ordered set defined as Definition 8.3. Show whether $(E^I XM,\le)$ is a lattice. What are the lattice operations $\vee$ and $\wedge$?

Problem 8.2. Demonstrate whether the upper and lower approximations defined by (8.24) and (8.25) have the same properties as the upper and lower approximation defined by (6.1).

Problem 8.3. Discuss whether the upper and lower AFS formal concept approximations defined by (8.26) and (8.27) (or (8.28) and (8.29)) have the same properties as the upper and lower approximation defined by (6.1).

Problem 8.4. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. For any $\psi \in EM$, whether $L(\psi)$ is the maximal formal concept smaller than $(\alpha(\psi),\beta(\alpha(\psi)))$ and $U(\psi)$ is the minimal formal concept larger than $(\alpha(\psi),\beta(\alpha(\psi)))$? Here α,β are defined by (8.14) and (8.15), $L(.)$ and $U(.)$ are defined by (8.26) and (8.27), respectively.

Problem 8.5. Let X be a set and M be a set of attributes on X. Let (X,M,I) be a context, EM be the EI algebra over the set X and $E^{\#}X$ be the $E^{\#}I$ algebra over the set X. For any $\psi \in EM$ and $\theta \in E^{\#}X$, what are the relationships between the following pairs?

$$L(\psi) \ \textit{and} \ L(\alpha(\psi)), \ U(\psi) \ \textit{and} \ U(\alpha(\psi)),$$
$$L(\theta) \ \textit{and} \ L(\beta(\theta)), \ U(\theta) \ \textit{and} \ U(\beta(\theta)).$$

Here $L(.)$ is defined by (8.26) or (8.28), and $U(.)$ is defined by (8.27) or (8.29), respectively.

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