Chapter 7
AFS Topology and Its Applications

In this chapter, first we construct some topologies on the AFS structures, discuss the topological molecular lattice structures on EI, $^*EI$, EII, $^*EII$ algebras, and elaborate on the main relations between these topological structures. Second, we apply the topology derived by a family of fuzzy concepts in EM, where M is a set of simple concepts, to analyze the relations among the fuzzy concepts. Thirdly, we propose the differential degrees and fuzzy similarity relations based on the topological molecular lattices generated by the fuzzy concepts on some features. Furthermore, the fuzzy clustering problems are explored using the proposed differential degrees and fuzzy similarity relations. Compared with other fuzzy clustering algorithms such as the Fuzzy C-Means and k-nearest-neighbor fuzzy clustering algorithms, the proposed fuzzy clustering algorithm can be applied to data sets with mixed feature variables such as numeric, Boolean, linguistic rating scale, sub-preference relations, and even descriptors associated with human intuition. Finally, some illustrative examples show that the proposed differential degrees are very effective in pattern recognition problems whose data sets do not form a subset of a metric space such as the Eculidean one. This approach offers a promising avenue that could be helpful in understanding mechanisms of human recognition.

7.1 Topology on AFS Structures and Topological Molecular Lattice on $^*EI^n$ Algebras

In this section, we first study the topological molecular lattice on the $^*EI$ algebra over a set M, i.e.,($^*EM, \vee, \wedge$), in which the lattice operators $\vee$, $\wedge$ are defined as follows: for any $\sum_{i\in I} A_i$, $\sum_{j\in J} B_j \in EM$,

$$\sum_{i\in I} A_i \vee \sum_{j\in J} B_j = \sum_{i\in I, j\in J} A_i \cup B_j, \tag{7.1}$$

$$\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j = \sum_{i\in I} A_i + \sum_{j\in J} B_j. \tag{7.2}$$

M is the maximum element of the lattice $^*EM$ and $\varnothing$ is the minimum element of this lattice. That is, the above lattice $^*EM$ is a dual lattice of EM. In the lattice $^*EM$,

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for $\sum_{i\in I} A_i, \sum_{j\in J} B_j \in EM$, $\sum_{i\in I} A_i \le \sum_{j\in J} B_j$ if and only if for any B_j $(j \in J)$ there exists A_k $(i \in I)$ such that $B_j \supseteq A_k$ (refer to Theorem 5.24). Secondly, we study the topology on the universe of discourse X induced by the topological molecular lattice of some fuzzy concepts in EM. Finally the topological molecular lattice on the $^*EI^2$ algebra over the sets X, M, i.e.,$(^*EXM, \vee, \wedge)$, in which the lattice operators $\vee$, $\wedge$ are defined as follows: for any $\sum_{i\in I} a_i A_i$, $\sum_{j\in J} b_j B_j \in EXM$,

$$\sum_{i\in I} a_i A_i \vee \sum_{j\in J} b_j B_j = \sum_{i\in I, j\in J} a_i \cap b_j A_i \cup B_j, \tag{7.3}$$

$$\sum_{i\in I} a_i A_i \wedge \sum_{j\in J} b_j B_j = \sum_{i\in I} a_i A_i + \sum_{j\in J} b_j B_j. \tag{7.4}$$

$\emptyset M$ is the maximum element of the lattice $^*EM$ and $X\emptyset$ is the minimum element of the lattice $^*EM$. That is, the lattice $^*EXM$ is a dual lattice of EXM. In the lattice $^*EXM$, for $\sum_{i\in I} a_i A_i, \sum_{j\in J} b_j B_j \in EM$, $\sum_{i\in I} a_i A_i \le \sum_{j\in J} b_j B_j$ if and only if for any $b_j B_j$ $(j \in J)$ there exists $a_k A_k$ $(i \in I)$ such that $B_j \supseteq A_k$ and $a_k \supseteq b_j$ (refer to Theorem 5.1).

Lemma 7.1. *Let M be a set and EM be the $^*EI$ algebra over M. For $A \subseteq M$, $\sum_{i\in I} A_i$, $\sum_{j\in J} B_j \in EM$, the following assertions hold:*

(1) $A \ge \sum_{i\in I} A_i$ and $A \ge \sum_{j\in J} B_j \Leftrightarrow A \ge \sum_{i\in I} A_i \vee \sum_{j\in J} B_j$;
(2) $A \ge \sum_{i\in I} A_i$ or $A \ge \sum_{j\in J} B_j \Leftrightarrow A \ge \sum_{i\in I} A_i \wedge \sum_{j\in J} B_j$.

Its proof is left as an exercise.

Definition 7.1. Let M be a set and $(^*EM, \vee, \wedge)$ be the $^*EI$ algebra over M defined by (7.1) and (7.2). Let $\eta \subseteq {}^*EM$. If $\emptyset, M \in \eta$ and η is closed under finite unions (i.e., $\vee$) and arbitrary intersections (i.e., $\wedge$), then η is called a *topological molecular lattice on the lattice* $^*EM$, denoted as $(^*EM, \eta)$. Let η be a topological molecular lattice on the lattice $^*EM$. If for any $\sum_{i\in I} A_i \in \eta$, $A_i \in \eta$ for any $i \in I$, then η is called an *elementary topological molecular lattice on the lattice* $^*EM$.

It is easy proved that if η is a topological molecular on the lattice $^*EM$ and η is a dual idea of the lattice $^*EM$, then η is an elementary topological molecular lattice on the lattice $^*EM$. In what follows, we apply the elementary topological molecular lattice on the lattice $^*EM$ to induce some topological structures on X via the AFS structure (M, τ, X) of a data. Thus the pattern recognition problem can by explored in the setting of these topological structures on X.

Definition 7.2. Let X and M be sets and (M, τ, X) be an AFS structure. Let $(^*EM, \eta)$ be a topological molecular lattice on $^*EI$ algebra over M. For any $x \in X$, $\sum_{i\in I} A_i \in \eta \subseteq {}^*EM$, the set $N^{\tau}_{\sum_{i\in I} A_i}(x) \subseteq X$ is defined as follows.

$$N^{\tau}_{\sum_{i\in I} A_i}(x) = \left\{ y \in X \ \middle|\ \tau(x, y) \ge \sum_{i\in I} A_i \right\}, \tag{7.5}$$

and it is called the *neighborhood of x induced by the fuzzy concept* $\sum_{i\in I} A_i$ in the AFS structure (M,τ,X). The set $N^{\tau}_{\eta}(x) \subseteq 2^X$ is defined as follows.

$$N^{\tau}_{\eta}(x) = \left\{ N_{\sum_{i\in I} A_i}(x) \;\middle|\; \sum_{i\in I} A_i \in \eta \right\}, \tag{7.6}$$

and it is called the *neighborhood of x induced by the topological molecular lattice* η in the AFS structure (M,τ,X).

Since $\tau(x,y) \subseteq M$, hence $\tau(x,y)$ is an element in EM and $\tau(x,y) \geq \sum_{i\in I} A_i$ in (7.5) is well-defined.

Definition 7.3. Let X and M be sets and (M,τ,X) be an AFS structure. (M,τ,X) is called a *strong relative AFS structure* if $\forall (x,y) \in X\times X, \tau(x,y)\cup\tau(y,x) = M$.

Since in a strong relative AFS structure (M,τ,X), $\forall x \in X$, $\tau(x,x) = M$, hence $\forall x \in X, \forall m \in M$, x belongs to the simple concept m to some extent.

Proposition 7.1. *Let X and M be sets and* (M,τ,X) *be a strong relative AFS structure. Let* η *be a topological molecular lattice on* $^*EI$ *algebra over M. For any* $\sum_{i\in I} A_i$, $\sum_{j\in J} B_j \in EM$*, the following assertions hold: for any* $x \in X$

(1) *If* $\sum_{i\in I} A_i \geq \sum_{j\in J} B_j$ *in* $^*EM$*, then* $N^{\tau}_{\sum_{i\in I} A_i}(x) \subseteq N^{\tau}_{\sum_{j\in J} B_j}(x)$;
(2) $N^{\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\tau}_{\sum_{j\in J} B_j}(x) = N^{\tau}_{\sum_{i\in I} A_i \vee \sum_{j\in J} B_j}(x)$;
(3) $N^{\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\tau}_{\sum_{j\in J} B_j}(x) = N^{\tau}_{\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j}(x)$.

Proof. (1) Let $y \in N^{\tau}_{\sum_{i\in I} A_i}(x)$. Then there exists $A_k, k \in I$ such that $\tau(x,y) \supseteq A_k$. On the other hand, since $\sum_{i\in I} A_i \geq \sum_{j\in J} B_j$, hence for A_k there exists $B_j, j \in J$ such that $\tau(x,y) \supseteq A_k \supseteq B_j$. This implies that $y \in N^{\tau}_{\sum_{j\in J} B_j}(x)$. It follows that $N^{\tau}_{\sum_{i\in I} A_i}(x) \subseteq N^{\tau}_{\sum_{j\in J} B_j}(x)$.

(2) For any $y \in N^{\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\tau}_{\sum_{j\in J} B_j}(x)$, in virtue of Lemma 7.1, we have

$$\begin{aligned}
y \in N^{\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\tau}_{\sum_{j\in J} B_j}(x) &\Leftrightarrow y \in N^{\tau}_{\sum_{i\in I} A_i}(x) \;\; \textit{and} \;\; y \in N^{\tau}_{\sum_{j\in J} B_j}(x) \\
&\Leftrightarrow \tau(x,y) \geq \sum_{i\in I} A_i \;\; \textit{and} \;\; \tau(x,y) \geq \sum_{j\in J} B_j \\
&\Leftrightarrow \tau(x,y) \geq \sum_{i\in I, j\in J} A_i \cup B_j \\
&\Leftrightarrow \tau(x,y) \geq \left(\sum_{i\in I} A_i\right) \vee \left(\sum_{j\in J} B_j\right) \\
&\Leftrightarrow y \in N^{\tau}_{\sum_{i\in I} A_i \vee \sum_{j\in J} B_j}(x).
\end{aligned}$$

So we have showed that (2) holds.

(3) For any $y \in N^{\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\tau}_{\sum_{j\in J} B_j}(x)$, from Lemma 7.1, we have

$$\begin{aligned} y \in N^{\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\tau}_{\sum_{j\in J} B_j}(x) &\Leftrightarrow y \in N^{\tau}_{\sum_{i\in I} A_i}(x) \;\; or \;\; y \in N^{\tau}_{\sum_{j\in J} B_j}(x) \\ &\Leftrightarrow \tau(x,y) \geq \sum_{i\in I} A_i \;\; or \;\; \tau(x,y) \geq \sum_{j\in J} B_j \\ &\Leftrightarrow \tau(x,y) \geq \sum_{i\in I} A_i + \sum_{j\in J} B_j \\ &\Leftrightarrow \tau(x,y) \geq \left(\sum_{i\in I} A_i\right) \wedge \left(\sum_{j\in J} B_j\right) \\ &\Leftrightarrow y \in N^{\tau}_{\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j}(x). \end{aligned}$$

This implies that (3) is satisfied. □

Theorem 7.1. *Let X and M be sets and (M,τ,X) be a strong relative AFS structure. Let η be a topological molecular lattice on the lattice $^*EM$. If η is an elementary topological molecular lattice on the lattice $^*EM$ and we define*

$$\mathscr{B}_\eta = \left\{ N^{\tau}_{\sum_{i\in I} A_i}(x) \;\middle|\; x \in X, \; \sum_{i\in I} A_i \in \eta \right\},$$

then $\mathscr{B}_\eta$ is a base for some topology of X.

Proof. Firstly, because (M,τ,X) is a strong relative AFS structure, for any $x \in X$, $\tau(x,x) = M$. M is the maximum element of the lattice $^*EM$. This implies that for any $\sum_{i\in I} A_i \in \eta$, $\tau(x,x) \geq \sum_{i\in I} A_i$ so that $x \in N^{\tau}_{\sum_{i\in I} A_i}(x)$ and $X = \bigcup_{\beta\in\mathscr{B}_\eta} \beta$.

Secondly, suppose $x \in X, U, V \in \mathscr{B}_\eta$, and $x \in U \cap V$. We will prove there exists $W \in \mathscr{B}_\eta$ such that $x \in W \subseteq U \cap V$. By the hypothesis, we know there exists $\sum_{i\in I} A_i, \sum_{j\in J} B_j \in \eta$ such that $U = N^{\tau}_{\sum_{i\in I} A_i}(u)$, $V = N^{\tau}_{\sum_{j\in J} B_j}(v)$ for some $u, v \in X$ and $\exists l \in I, \exists k \in J$, $\tau(u,x) \supseteq A_l$ and $\tau(v,x) \supseteq B_k$. Since (M,τ,X) is a strong relative, hence $x \in N^{\tau}_{A_l}(x)$ and $x \in N^{\tau}_{B_k}(x)$. For any $y \in N^{\tau}_{A_l}(x)$, i.e., $\tau(x,y) \supseteq A_l$, by Definition 4.5, we have $\tau(u,y) \supseteq \tau(u,x) \cap \tau(x,y) \supseteq A_l$, that is $y \in U$. It follows $N^{\tau}_{A_l}(x) \subseteq U$. For the same reason, $N^{\tau}_{B_k}(x) \subseteq V$. By Proposition 7.1, we have

$$x \in N^{\tau}_{A_l}(x) \cap N^{\tau}_{B_k}(x) = N^{\tau}_{A_l \vee B_k}(x) \subseteq U \cap V.$$

Since η is an elementary topological molecular lattice on the lattice $^*EM$, hence $A_l, B_k \in \eta$ and we have $x \in W = N^{\tau}_{A_l \vee B_k}(x) \in \mathscr{B}_\eta$ such that $W \subseteq U \cap V$. Now by Theorem 1.21, $\mathscr{B}_\eta$ is a base for some topology on X. □

The topological space $(X, \mathscr{T}_\eta)$, in which $\mathscr{B}_\eta$ is the base for $\mathscr{T}_\eta$, is called the *topology of X induced by the topological molecular lattice η.*

Theorem 7.2. *Let X and M be sets and (M,τ,X) be a strong relative AFS structure. Let η be a topological molecular lattice on the lattice $^*EM$ and*

$$\mathscr{L}_\eta = \Big\{ \sum_{i\in I} a_i A_i \in EXM \mid \sum_{i\in I} A_i \in \eta, a_i \in \mathscr{T}_\eta \ for\ any\ i \in I,$$
$$I\ is\ any\ non-empty\ indexing\ set \Big\}. \qquad (7.7)$$

Then $\mathscr{L}_\eta$ is a topological molecular lattice on the lattice ${}^*EXM$. It is called the ${}^*EI^2$ topological molecular lattice induced by the ${}^*EI$ the topological molecular lattice η.

Proof. For any finite integer n, let $\lambda_j = \sum_{i\in I_j} a_{ij} A_{ij} \in EXM$, $j = 1,2,...,n$. Because for any $f \in \prod_{1\le j\le n} I_j$, $\bigcap_{1\le j\le n} a_{f(j)j} \in \mathscr{T}_\eta$ and

$$\bigvee_{1\le j\le n} \left(\sum_{i\in I_j} A_{ij} \right) = \sum_{f\in\prod_{1\le j\le n} I_j} \bigcup_{1\le j\le n} A_{f(j)j} \in \eta,$$

then we have

$$\bigvee_{1\le j\le n} \lambda_j = \sum_{f\in\prod_{1\le j\le n} I_j} \left(\bigcap_{1\le j\le n} a_{f(j)j} \bigcup_{1\le j\le n} A_{f(j)j} \right) \in \mathscr{L}_\eta.$$

This implies that $\mathscr{L}_\eta$ is closed under finite unions (i.e.,$\vee$). It is obvious that $\wedge$ is closed under arbitrary intersection. Therefore $({}^*EXM, \mathscr{L}_\eta)$ is a ${}^*EI^2$ topological molecular lattice on the lattice ${}^*EXM$. □

It is clear that $\mathscr{T}_\eta$ the topology on X is determined based on the distribution of raw data and the chosen set of fuzzy concepts $\eta \subseteq EM$ and it is an abstract geometry relation among the objects in X under the fuzzy concepts under consideration, i.e., η. What are the interpretations of the special topological structures on X obtained from given database? What are the topological structures associated with the essential nature of database? All these questions are related to the metric space of the topology. With a metric in the topological space on X, it will be possible to handle pattern recognition problems for the databases with various data types.

Let X be a set and M be a set of simple concepts on X. Let (M,τ,X) be an AFS structure and S be the σ-algebra over X. In real world applications, it is obvious that only some fuzzy concepts in EM are related with the problem under consideration. Let these fuzzy concepts form the set $\Lambda \subseteq EM$. Let η be the topological molecular lattice generated by Λ and $(X, \mathscr{T}_\eta)$ be the topology induced by η. Let S be the σ-algebra generated by $\mathscr{T}_\eta$, i.e., the Borel set corresponding to the topological space $(X, \mathscr{T}_\eta)$ and (S,m) be a measure space. For the fuzzy concept $\sum_{i\in I} A_i \in EM$, if for any $x \in X$, any $i \in I$, $A_i^\tau(x) \in S$, then $\sum_{i\in I} A_i$ is called a *measurable fuzzy concept under the σ-algebra S*. Thus the membership function of each measurable fuzzy concept in EM can be obtained by the norm of $E^\# I$ algebra via (5.13), (5.24) and (S,m).

Theorem 7.3. *Let X and M be sets. Let (M,τ,X) be a strong relative AFS structure and η be an elementary topological molecular lattice on the lattice ${}^*EM$. Let η be a topological molecular lattice on the lattice ${}^*EM$ and the topological space $(X, \mathscr{T}_\eta)$*

be the topology induced by η. Let S be the σ-algebra generated by $\mathscr{T}_\eta$ and $\mathscr{L}_\eta$ be the $^*EI^2$ topological molecular lattice on $^*EXM$ induced by η. Then the following assertions hold.

(1) For any fuzzy concept $\sum_{i\in I} A_i \in \eta$, $\sum_{i\in I} A_i$ is a measurable concept under S ;
(2) For each fuzzy concept $\gamma = \sum_{i\in I} A_i \in \eta$, let $\gamma : X \to EXM$ be the EI^2 algebra representation membership degrees defined by (5.10) as follows: for any $x \in X$,

$$\gamma(x) = \sum_{i\in I} A_i^\tau(x) A_i \in EXM. \tag{7.8}$$

Let D be a directed set and $\delta : D \to X$ be a net (i.e., $\{\delta(d) \mid d \in D\}$). If δ is converged to $x_0 \in X$ under topology $\mathscr{T}_\eta$, then the net of the composition $\gamma \cdot \delta : D \to EXM$ (i.e.,$\{\gamma(\delta(d)) \mid d \in D\}$) converges to $\gamma(x_0) = \sum_{i\in I} A_i^\tau(x_0)A_i$ under the topological molecular lattice $\mathscr{L}_\eta$. That is, the membership function of any fuzzy concept in EM defined by (7.8) is a continuous function from the topological space $(X, \mathscr{T}_\eta)$ to the topological molecular lattice $(^*EXM, \mathscr{L}_\eta)$.

Proof. (1) For any $\sum_{i\in I} A_i \in \eta$, since $A_i \geq \sum_{i\in I} A_i$ for all $i \in I$ and η is an elementary topological molecular lattice on the lattice $^*EM$, hence $A_i \in \eta$ for all $i \in I$ and

$$A_i^\tau(x) = N_{A_i}^\tau(x) \in \mathscr{T}_\eta \Rightarrow A_i^\tau(x) \in S, \;\; for\ any\ x \in X\ and\ any\ i \in I.$$

Therefore $\sum_{i\in I} A_i$ is a measurable concept under S.

(2) Suppose $\sum_{j\in J} p_j P_j \in \mathscr{L}_\eta$ and $\sum_{j\in J} p_j P_j$ is a R-neighborhood of $\sum_{i\in I} A_i^\tau(x_0)A_i$, i.e., $\sum_{i\in I} A_i^\tau(x_0)A_i \nleq \sum_{j\in J} p_j P_j$. This implies that there exists $p_l P_l$ $(l \in J)$ such that for any $i \in I$, either $A_i^\tau(x_0) \not\supseteq p_l$ or $P_l \not\supseteq A_i$. First, assume $\forall k \in I$, $P_l \not\supseteq A_k$. It follows, for any $d \in D$, $\sum_{i\in I} A_i^\tau(\delta(d))A_i \nleq \sum_{j\in J} p_j P_j$.

Second, assume that $k \in I$, $A_k^\tau(x_0) \not\supseteq p_l$. Since $x_0 \in A_k^\tau(x_0) \in \mathscr{T}_\eta$ and δ is converged to $x_0 \in X$ under $\mathscr{T}_\eta$, hence the exists $N \in D$ such that for any $d \in D$, $d \geq N$, $\delta(d) \in A_k^\tau(x_0) \not\supseteq p_l$. For any $y \in A_k^\tau(\delta(d))$, i.e., $\tau(\delta(d), y) \supseteq A_k$, since $\delta(d) \in A_k^\tau(x_0)$, i.e., $\tau(x_0, \delta(d)) \supseteq A_k$ and τ is an AFS structure, hence we have

$$\tau(x_0, y) \supseteq \tau(x_0, \delta(d)) \cap \tau(\delta(d), y) \supseteq A_k \Rightarrow y \in A_k^\tau(x_0) \Rightarrow A_k^\tau(x_0) \supseteq A_k^\tau(\delta(d)).$$

This implies that for $i \in I$ if $A_i^\tau(x_0) \not\supseteq p_l$, then exists $N \in D$ such that for any $d \in D, d \geq N$, $A_i^\tau(\delta(d)) \not\supseteq p_l$. Thus for any R-neighborhood of $\sum_{i\in I} A_i^\tau(x_0)A_i$, $\upsilon \in \mathscr{L}_\eta$, there exists $N \in D$ such that for any $d \in D, d \geq N$,

$$\sum_{i\in I} A_i^\tau(\delta(d))A_i \nleq \upsilon.$$

Therefore the net $\gamma \cdot \delta$ is converged to $\sum_{i\in I} A_i^\tau(x_0)A_i$ under the topological lattice $\mathscr{L}_\eta$. □

In a strong relative AFS structure (M, τ, X), $\forall x \in X$, $\tau(x, x) = M$, i.e. $\forall x \in X, \forall m \in M$, x belongs to the simple concept m at some extent. It is too strict to be exploited in the setting of real world applications. In order to offer an abstract description of the similar relation between the objects in X concerning some given concepts, Definition 7.2 should be modified as follows.

Definition 7.4. Let X and M be sets and (M,τ,X) be an AFS structure. Let η be a topological molecular lattice on the lattice $^*EM$. For any $x \in X$, $\sum_{i\in I} A_i \in \eta$, the set $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \subseteq X$ is defined as follows.

$$N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) = \left\{ y \in X \mid \tau(x,y) \cap \tau(y,y) \geq \sum_{i\in I} A_i \right\}, \tag{7.9}$$

and it is called the *limited neighborhood of x induced by the fuzzy concept* $\sum_{i\in I} A_i \in \eta$, if $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \neq \varnothing$. The set $N^{\triangle\tau}_{\eta}(x) \subseteq 2^X$ is defined as follows.

$$N^{\triangle\tau}_{\eta}(x) = \left\{ N_{\sum_{i\in I} A_i}(x) \neq \varnothing \mid \sum_{i\in I} A_i \in \eta \right\},$$

and it is called the *limited neighborhood of x induced by the topological molecular lattice η*.

By the definition of the AFS structure (refer to Definition 4.5), we know that for any $x,y \in X$,

$$\tau(x,x) \supseteq \tau(x,y) \supseteq \tau(x,y) \cap \tau(y,y).$$

Therefore $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \subseteq N^{\tau}_{\sum_{i\in I} A_i}(x)$ for any $x \in X$, any $\sum_{i\in I} A_i \in \eta$ and

$$N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \neq \varnothing \Leftrightarrow x \in N^{\tau}_{\sum_{i\in I} A_i}(x).$$

Proposition 7.2. *Let X and M be sets and (M,τ,X) be an AFS structure. Let η be a topological molecular lattice on the lattice $^*EM$. For any $x \in X$, $\sum_{i\in I} A_i, \sum_{j\in J} B_j \in EM$, the following assertions hold.*

(1) If $\sum_{i\in I} A_i \geq \sum_{j\in J} B_j$ in the lattice $^*EM$, then $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \subseteq N^{\triangle\tau}_{\sum_{j\in J} B_j}(x)$ for any $x \in X$;
(2) $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\triangle\tau}_{\sum_{j\in J} B_j}(x) = N^{\triangle\tau}_{\sum_{i\in I} A_i \vee \sum_{j\in J} B_j}(x)$ for any $x \in X$;
(3) $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\triangle\tau}_{\sum_{j\in J} B_j}(x) = N^{\triangle\tau}_{\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j}(x)$ for any $x \in X$.

Proof. (1) Suppose $y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(x)$, $x \in X$. By (7.9), we know that there exists A_k, $k \in I$ such that $\tau(x,y) \cap \tau(y,y) \supseteq A_k$. Since $\sum_{i\in I} A_i \geq \sum_{j\in J} B_j$, then for A_k, there exists B_l, $l \in J$ such that

$$\tau(x,y) \cap \tau(y,y) \supseteq A_k \supseteq B_l \Rightarrow \tau(x,y) \cap \tau(y,y) \geq \textstyle\sum_{j\in J} B_j.$$

This implies that $y \in N^{\triangle\tau}_{\sum_{j\in J} B_j}(x)$. It follows $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \subseteq N^{\triangle\tau}_{\sum_{j\in J} B_j}(x)$.

(2) For any $y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\triangle\tau}_{\sum_{j\in J} B_j}(x)$,

$$\begin{aligned} y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\triangle\tau}_{\sum_{j\in J} B_j}(x) &\Leftrightarrow y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \ \ and \ \ y \in N^{\triangle\tau}_{\sum_{j\in J} B_j}(x) \\ &\Leftrightarrow \tau(x,y) \cap \tau(y,y) \geq \sum_{i\in I} A_i \ and \ \tau(x,y) \cap \tau(y,y) \geq \sum_{j\in J} B_j \\ &\Leftrightarrow \ \tau(x,y) \cap \tau(y,y) \geq \sum_{i\in I} A_i \vee \sum_{j\in J} B_j \quad (by\ Lemma\ 7.1) \\ &\Leftrightarrow y \in N^{\triangle\tau}_{\sum_{i\in I} A_i \vee \sum_{j\in J} B_j}(x). \end{aligned}$$

Therefore $N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\triangle\tau}_{\sum_{j\in J} B_j}(x) = N^{\triangle\tau}_{\sum_{i\in I} A_i \vee \sum_{j\in J} B_j}(x)$.

(3) For any $y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\triangle\tau}_{\sum_{j\in J} B_j}(x)$,

$$\begin{aligned} y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\triangle\tau}_{\sum_{j\in J} B_j}(x) &\Leftrightarrow y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(x) \ \ or \ \ y \in N^{\triangle\tau}_{\sum_{j\in J} B_j}(x) \\ &\Leftrightarrow \tau(x,y) \cap \tau(y,y) \geq \sum_{i\in I} A_i \ \ or \ \ \tau(x,y) \cap \tau(y,y) \geq \sum_{j\in J} B_j \\ &\Leftrightarrow \tau(x,y) \cap \tau(y,y) \geq \sum_{i\in I} A_i \wedge \sum_{j\in J} B_j \quad (by\ Lemma\ 7.1) \\ &\Leftrightarrow y \in N^{\triangle\tau}_{\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j}(x). \end{aligned}$$

Subsequently (3) is satisfied. □

Theorem 7.4. *Let X and M be sets and (M,τ,X) be an AFS structure. Let η be a topological molecular lattice on the lattice $^*EM$. If η is an elementary topological molecular lattice on the lattice $^*EM$ and $\mathscr{B}^{\triangle}_{\eta}$ is defined as follows*

$$\mathscr{B}^{\triangle}_{\eta} = \{N_{\sum_{i\in I} A_i}(x) \ \mid \ x \in X, \sum_{i\in I} A_i \in \eta\}, \tag{7.10}$$

then $\mathscr{B}^{\triangle}_{\eta}$ is a base for some topology of X.

Proof. Firstly, for any $x \in X$, since $\varnothing \in \eta$, hence $\tau(x,x) \geq \varnothing$, i.e., $x \in N^{\triangle\tau}_{\varnothing}(x)$. This implies that $X = \bigcup_{N\in\mathscr{B}^{\triangle}_{\eta}} N$. Secondly, suppose $x \in X$, U, $V \in \mathscr{B}^{\triangle}_{\eta}$, and $x \in U \cap V$. We will prove that there exists $W \in \mathscr{B}^{\triangle}_{\eta}$ such that $x \in W \subseteq U \cap V$. By (7.10), we know that there exists $\sum_{i\in I} A$, $\sum_{j\in J} B_j \in \eta$, $u, v \in X$ such that there $U = N^{\triangle\tau}_{\sum_{i\in I} A_i}(u)$, $V = N^{\triangle\tau}_{\sum_{j\in J} B_j}(v)$. That is, $\exists l \in I$, $\exists k \in J$, $\tau(u,x) \cap \tau(x,x) \supseteq A_l$ and $\tau(v,x) \cap \tau(x,x) \supseteq B_k$. By $\tau(u,x) \cap \tau(x,x) \subseteq \tau(x,x)$ and $\tau(v,x) \cap \tau(x,x) \subseteq \tau(x,x)$, we have $x \in N^{\triangle\tau}_{A_l}(x)$ and $x \in N^{\triangle\tau}_{B_k}(x)$. For any $y \in N^{\triangle\tau}_{A_l}(x)$, i.e., $\tau(x,y) \cap \tau(y,y) \supseteq A_l$, by AX1 and AX2 in Definition 4.5, we have $\tau(x,y) \subseteq \tau(x,x)$ and $\tau(u,x) \cap \tau(x,y) \subseteq \tau(u,y)$. It follows

$$\tau(u,y) \cap \tau(y,y) \supseteq \tau(u,x) \cap \tau(x,x) \cap \tau(x,y) \cap \tau(y,y) \supseteq A_l.$$

This fact implies that $\tau(u,y) \cap \tau(y,y) \geq \sum_{i\in I} A$ and $y \in N^{\triangle\tau}_{\sum_{i\in I} A_i}(u)$. Thus we have $N^{\triangle\tau}_{A_l}(x) \subseteq U$. Similarly, we can prove $N^{\triangle\tau}_{B_k}(x) \subseteq V$. Since η is an elementary topological molecular lattice on the lattice $^*EM$, hence A_l, $B_k \in \eta$, and $A_l \cup B_k = A_l \vee B_k \in \eta$. In virtue of Proposition 7.2, one has $W = N^{\triangle\tau}_{A_l}(x) \cap N^{\triangle\tau}_{B_k}(x) = N^{\triangle\tau}_{A_l \vee B_k}(x) \in \mathscr{B}^{\triangle}_{\eta}$ such that $x \in W \subseteq U \cap V$. Therefore by Theorem 1.21 $\mathscr{B}^{\triangle}_{\eta}$ is a base for some topology on X. □

The topological space $(X, \mathscr{T}^{\triangle}_{\eta})$, in which $\mathscr{B}^{\triangle}_{\eta}$ is a base for $\mathscr{T}^{\triangle}_{\eta}$, is called the *limited topology of X induced by the topological molecular lattice* η.

In what follows, we look more carefully at these topological structures by discussing the following illustrative examples.

Example 7.1. Let $X = \{x_1, x_2, ..., x_5\}$ be a set of 5 persons. M = {old, heavy, tall, high salary, more estate, male, female } be a set of simple concepts on the attributes which are shown as Table 7.1.

Table 7.1 Description of attributes

	age	*heigh*	*weigh*	*salary*	*estate*	*male*	*female*
x_1	21	1.69	50	0	0	1	0
x_2	30	1.62	52	120	200,000	0	1
x_3	27	1.80	65	100	40,000	1	0
x_4	60	1.50	63	80	324,000	0	1
x_5	45	1.71	54	140	940,000	1	0

We can construct the AFS structure τ according to the data shown in Table 7.1 and the semantics of the simple concepts in M. τ is shown as the following Table 7.2. Here A: *old*, M: *male*, W: *female*, H: *tall*, We: *heavey*, S: *high salary*, Q: *more estate*.

Table 7.2 The AFS structure (M, τ, X) of data shown in Table 7.1

$\tau(.,.)$	x_1	x_2	x_3	x_4	x_5
x_1	{A,M,H,We,S }	{M,H }	{M }	{M,H }	{M }
x_2	{A,W,We,S,Q }	{A,W,H,We,S,Q }	{A,W,S,Q}	{W,H,S }	{W }
x_3	{A,M,H,We,S,Q }	{M,H,We }	{A,M,H,We,S,Q}	{M,H,We,S}	{M,H,We}
x_4	{A,W,We,Q}	{A,W,We,Q }	{A,W,Q }	{A,W,We,H,S,Q}	{A,W,We }
x_5	{A,M,S,Q}	{A,M,H,We,S,Q }	{A,M,S,Q }	{M,H,S,Q }	{A,M,H,S,We,Q }

We can verify that τ satisfies Definition 4.5 and (M, τ, X) is an AFS structure. Since for any $x \in X$, $\tau(x,x) \neq M$, hence (M, τ, X) is not an strong relative AFS structure.

If we consider some health problem and suppose the problem just involves the attributes *age, high and weight*. Thus we just consider simple concepts $A, H, We \in M$ and let $M_1 = \{A, H, We\}$. (M_1, τ_{M_1}, X) is an AFS structure if the map $\tau_{M_1} : X \times X \to 2^{M_1}$ is defined as follows: for any $x, y \in X$, $\tau_{M_1}(x,y) = \tau(x,y) \cap M_1$. Obviously,

(M_1, τ_{M_1}, X) is a strong relative AFS structure. Let $\eta \subseteq EM$ be the topological molecular lattice generated by the fuzzy concepts $\{A\}, \{H\}, \{We\} \in EM$ on the lattice $^*EM$. η consists of the following elements.

$\varnothing$, M, $\{A\}+\{H\}+\{We\}$, $\{A\}+\{H\}$, $\{A\}+\{We\}$, $\{H\}+\{We\}$, $\{A\}$, $\{H\}$, $\{We\}$, $\{A,H\}+\{A,We\}+\{We,H\}$, $\{A,H\}+\{A,We\}$, $\{A,H\}+\{We,H\}$, $\{A,We\}+\{We,H\}$, $\{A,H\}$, $\{A,We\}$, $\{We,H\}$, $\{A\}+\{We,H\}$, $\{H\}+\{A,We\}$, $\{We\}+\{A,H\}$, $\{A,H,We\}$.

It could be easily verified that η is an elementary topological molecular lattice on the lattice $^*EM$. Now we study $\mathscr{T}_\eta$ the topology on X induced by η via the AFS structure (M_1, τ_{M_1}, X). The neighborhood of x_1 induced by the fuzzy concepts in η, which is obtained by Definition 7.2, are listed as follows.

$N^{\tau}_{\{A\}+\{H\}+\{We\}}(x_1) = \{x_1,x_2,x_4\}$, $N^{\tau}_{\{A\}+\{H\}}(x_1) = \{x_1,x_2,x_4\}$, $N^{\eta}_{\{A\}+\{We\}}(x_1) = \{x_1\}$,
$N^{\tau}_{\{H\}+\{We\}}(x_1) = \{x_1,x_2,x_4\}$, $N^{\tau}_{\{A\}}(x_1) = \{x_1\}$, $N^{\tau}_{\{H\}}(x_1) = \{x_1,x_2,x_4\}$,
$N^{\eta}_{\{We\}}(x_1) = \{x_1\}$, $N^{\tau}_{\{A,H\}+\{A,We\}+\{We,H\}}(x_1) = \{x_1\}$, $N^{\tau}_{\{A,H\}+\{A,We\}}(x_1) = \{x_1\}$,
$N^{\tau}_{\{A,H\}+\{We,H\}}(x_1) = \{x1\}$, $N^{\tau}_{\{A,We\}+\{We,H\}}(x_1) = \{x_1\}$, $N^{\tau}_{\{A,H\}}(x_1) = \{x_1\}$,
$N^{\tau}_{\{A,We\}}(x_1) = \{x_1\}$, $N^{\tau}_{\{We,H\}}(x_1) = \{x_1\}$, $N^{\tau}_{\{A\}+\{We,H\}}(x_1) = \{x_1\}$, $N^{\tau}_{\varnothing} = X$
$N^{\tau}_{\{H\}+\{A,We\}}(x_1) = \{x_1,x_2,x_4\}$, $N^{\tau}_{\{We\}+\{A,H\}}(x_1) = \{x_1\}$, $N^{\tau}_{\{A,H,We\}}(x_1) = \{x_1\}$.

Therefore the neighborhood of x_1 induced by the fuzzy concepts in η comes as

$$N^{\tau}_{\eta}(x_1) = \{X,\ \{x_1,x_2,x_4\},\ \{x_1\}\}.$$

Similarly, we have the neighborhood of other elements in X as follows.

$$\begin{aligned}
N^{\tau}_{\eta}(x_2) &= \{X,\ \{x_1,x_2,x_3,x_4\},\ \{x_1,x_2,x_4\},\ \{x_1,x_2,x_3\},\ \{x_1,x_2\},\ \{x_2,x_4\},\ \{x_2\}\},\\
N^{\tau}_{\eta}(x_3) &= \{X,\ \{x_1,x_3\}\},\\
N^{\tau}_{\eta}(x_4) &= \{X,\ \{x_1,x_2,x_4,x_5\},\ \{x_4\}\},\\
N^{\tau}_{\eta}(x_5) &= \{X,\ \{x_1,x_2,x_3,x_5\},\ \{x_2,x_4,x_5\},\ \{x_2,x_5\}\}.
\end{aligned}$$

What is the interpretations of the above topological structure on X obtained from the given data shown as Table 7.1? This remains an open problem. How to establish a distance function according the above topology on X for a pattern recognition problem will be explored in Section 7.3. Here we just simply analyze it alluding to intuition. One can observe that x_1, x_2, x_4 are discrete points for the topology $\mathscr{T}_\eta$. Coincidentally, their membership degrees to the fuzzy concepts $\{A\}, \{H\}, \{We\} \in EM$ taken on minimal values, respectively. For any $U \in \mathscr{T}_\eta$, we can prove that if $x_5 \in U$ then $x_2 \in U$. This implies that the degree of x_5 belonging to any concept in EM_1 is always larger than or equal to that of x_2. Since $x_5 \notin \{x_1,x_3\} \in \mathscr{T}_\eta$, $x_3 \notin \{x_2,x_5\} \in \mathscr{T}_\eta$, i.e., the separation property of topology $\mathscr{T}_\eta$. This implies that there exist two fuzzy concepts in η such that x_5, x_3 can be distinguished by them.

7.2 Topology on AFS Structures and Topological Molecular Lattice on EI^n Algebras

Most of the results in this section can be proved by using the similar methods to those we exercised in the previous section, since the lattice of EI^n algebras is the dual lattice of $^*EI^n$ algebras. We first list the corresponding results of the topological molecular lattice on the EI algebra over a set M, in which the lattice operators $\vee$, $\wedge$ are defined as follows: for any $\sum_{i\in I} A_i,\ \sum_{j\in J} B_j \in EM$,

$$\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j = \sum_{i\in I, j\in J} A_i \cup B_j,$$
$$\sum_{i\in I} A_i \vee \sum_{j\in J} B_j = \sum_{i\in I} A_i + \sum_{j\in J} B_j.$$

$\varnothing$ is the maximum element of the lattice EM and M is the minimum element of the lattice EM. That is, the above lattice EM is a dual lattice of $^*EM$. In the lattice EM, for $\sum_{i\in I} A_i, \sum_{j\in J} B_j \in EM$, $\sum_{i\in I} A_i \geq \sum_{j\in J} B_j$ if and only if for any $B_j\ (j\in J)$ there exists $A_k\ (i\in I)$ such that $B_j \supseteq A_k$ (refer to Theorem 4.1). Secondly, we list the results for the topology on the universe of discourse X induced by the topological molecular lattice of some fuzzy concepts in EM. Finally, we present the results of the topological molecular lattice on the EI^2 algebra over the sets X, M, i.e.,$(EXM, \vee, \wedge)$, in which the lattice operators $\vee$, $\wedge$ are defined as follows: for any $\sum_{i\in I} a_i A_i,\ \sum_{j\in J} b_j B_j \in EXM$,

$$\sum_{i\in I} a_i A_i \wedge \sum_{j\in J} b_j B_j = \sum_{i\in I, j\in J} a_i \cap b_j A_i \cup B_j,$$
$$\sum_{i\in I} a_i A_i \vee \sum_{j\in J} b_j B_j = \sum_{i\in I} a_i A_i + \sum_{j\in J} b_j B_j.$$

$X\varnothing$ is the maximum element of the lattice EM and $\varnothing M$ is the minimum element of the lattice EM. That is, the lattice EXM is a dual lattice of $^*EXM$. In the lattice EXM, for $\sum_{i\in I} a_i A_i, \sum_{j\in J} b_j B_j \in EM$, $\sum_{i\in I} a_i A_i \geq \sum_{j\in J} b_j B_j$ if and only if for any $b_j B_j\ (j\in J)$ there exists $a_k A_k\ (i\in I)$ such that $B_j \supseteq A_k$ and $a_k \supseteq b_j$ (refer to Theorem 5.1).

Definition 7.5. Let M be set and $(EM, \vee, \wedge)$ be the EI algebra over M. Let $\eta \subseteq EM$. If $\varnothing, M \in \eta$ and η is closed under finite unions (i.e., $\vee$) and arbitrary intersections (i.e., $\wedge$), then η is called a *topological molecular lattice on the lattice EM*, denoted as (EM, η). Let η be a topological molecular lattice on the lattice EM. If for any $\sum_{i\in I} A_i \in \eta$, $A_i \in \eta$ for any $i \in I$, then η is called an *elementary topological molecular lattice on the lattice EM*.

In what follows, we apply the elementary topological molecular lattice on the lattice EM to induce some topological structures on X via the AFS structure (M, τ, X) of a data. Thus the pattern recognition problem can by explored under these topological structures on X.

Definition 7.6. Let X and M be sets and (M,τ,X) be an AFS structure. Let (EM,η) be a topological molecular lattice on EI algebra over M. For any $x \in X$, $\sum_{i\in I} A_i \in \eta \subseteq EM$, the set $N^{\tau}_{\sum_{i\in I} A_i}(x) \subseteq X$ is defined as follows.

$$N^{\tau}_{\sum_{i\in I} A_i}(x) = \left\{ y \in X \mid \tau(x,y) \leq \sum_{i\in I} A_i \right\}, \tag{7.11}$$

and it is called the *neighborhood of x induced by the fuzzy concept* $\sum_{i\in I} A_i$ in the AFS structure (M,τ,X). The set $N^{\tau}_{\eta}(x) \subseteq 2^X$ is defined as follows.

$$N^{\tau}_{\eta}(x) = \left\{ N_{\sum_{i\in I} A_i}(x) | \sum_{i\in I} A_i \in \eta \right\}, \tag{7.12}$$

and it is called the *neighborhood of x induced by the topological molecular lattice* η in the AFS structure (M,τ,X).

Since $\tau(x,y) \subseteq M$, hence $\tau(x,y)$ is an element in EM and $\tau(x,y) \leq \sum_{i\in I} A_i$ in (7.11) is well-defined.

Proposition 7.3. *Let X and M be sets and (M,τ,X) be an strong relative AFS structure. Let η be a topological molecular lattice on EI algebra over M. For any $x \in X$, $\sum_{i\in I} A_i$, $\sum_{j\in J} B_j \in EM$, the following assertions hold: for any $x \in X$*

(1) If $\sum_{i\in I} A_i \geq \sum_{j\in J} B_j$ in EM, then $N^{\tau}_{\sum_{i\in I} A_i}(x) \supseteq N^{\tau}_{\sum_{j\in J} B_j}(x)$;
(2) $N^{\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\tau}_{\sum_{j\in J} B_j}(x) = N^{\tau}_{\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j}(x)$;
(3) $N^{\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\tau}_{\sum_{j\in J} B_j}(x) = N^{\tau}_{\sum_{i\in I} A_i \vee \sum_{j\in J} B_j}(x)$.

Its proof, which is similar to the proof of Proposition 7.1, remains as an exercise.

Theorem 7.5. *Let X and M be sets and (M,τ,X) be a strong relative AFS structure. Let η be a topological molecular lattice on the lattice EM. If η is an elementary topological molecular lattice on the lattice EM and we define*

$$\mathscr{B}_{\eta} = \left\{ N^{\tau}_{\sum_{i\in I} A_i}(x) \mid x \in X,\ \sum_{i\in I} A_i \in \eta \right\},$$

then $\mathscr{B}_{\eta}$ is a base for some topology of X.

Its proof, which is similar to the proof of Theorem 7.1, is left to the reader.

Theorem 7.6. *Let X and M be sets, (M,τ,X) be a strong relative AFS structure. Let η be a topological molecular lattice on the lattice EM and*

$$\mathscr{L}_{\eta} = \left\{ \sum_{i\in I} a_i A_i \in EXM \ \mid \sum_{i\in I} A_i \in \eta,\ a_i \in \mathscr{T}_{\eta} \ for\ any\ i \in I \right\}. \tag{7.13}$$

Then $\mathscr{L}_\eta$ is a topological molecular lattice on the lattice EXM. It is called the EI^2 topological molecular lattice induced by the EI topological molecular lattice η.

Its proof (similar to the proof of Theorem 7.2) remains as an exercise.

Theorem 7.7. *Let X and M be sets. Let (M,τ,X) be a strong relative AFS structure and η be an elementary topological molecular lattice on the lattice EM. Let η be a topological molecular lattice on the lattice EM and the topological space $(X,\mathscr{T}_\eta)$ be the topology induced by η. Let S be the σ-algebra generated by $\mathscr{T}_\eta$ and $\mathscr{L}_\eta$ be the EI^2 topological molecular lattice on EXM induced by η. Then the following assertions hold.*

(1) For any fuzzy concept $\sum_{i\in I} A_i \in \eta$, $\sum_{i\in I} A_i$ is a measurable concept under S ;
(2) For each fuzzy concept $\gamma = \sum_{i\in I} A_i \in \eta$, let $\gamma : X \to EXM$ be the EI^2 algebra representation membership degrees defined by (5.10) as follows: for any $x \in X$,

$$\gamma(x) = \sum_{i\in I} A_i^\tau(x)A_i \in EXM. \tag{7.14}$$

Let D be a directed set and $\delta : D \to X$ be a net (i.e., $\{\delta(d) \mid d \in D\}$). If δ is converged to $x_0 \in X$ under topology $\mathscr{T}_\eta$, then the net of the composition $\gamma\cdot\delta : D \to EXM$ (i.e.,$\{\gamma(\delta(d)) \mid d \in D\}$) converges to $\gamma(x_0) = \sum_{i\in I} A_i^\tau(x_0)A_i$ under the topological molecular lattice $\mathscr{L}_\eta$. That is the membership function defined by (7.14) is a continuous function from the topological space $(X,\mathscr{T}_\eta)$ to the topological molecular lattice $(EXM,\mathscr{L}_\eta)$.

Its proof, which is similar to the proof of Theorem 7.3, can be treated as an exercise.

Example 7.2. Let us study the topological structures on the same AFS structure (M_1,τ_{M_1},X) of the same data we used in Example 7.1.

Let $\eta \subseteq EM$ be the topological molecular lattice generated by the fuzzy concepts $\{A\},\{H\},\{We\} \in EM$ on the lattice EM. η consists of the following elements which are the same as for η in Example 7.1.

$\varnothing$, M, $\{A\}+\{H\}+\{We\}$, $\{A\}+\{H\}$, $\{A\}+\{We\}$, $\{H\}+\{We\}$, $\{A\}$, $\{H\}$, $\{We\}$, $\{A,H\}+\{A,We\}+\{We,H\}$, $\{A,H\}+\{A,We\}$, $\{A,H\}+\{We,H\}$, $\{A,We\}+\{We,H\}$, $\{A,H\}$, $\{A,We\}$, $\{We,H\}$, $\{A\}+\{We,H\}$, $\{H\}+\{A,We\}$, $\{We\}+\{A,H\}$, $\{A,H,We\}$.

It can be easily to verify that η is an elementary topological molecular lattice on the lattice EM. Now we study $\mathscr{T}_\eta$-the topology on X induced by η via the AFS structure (M_1,τ_{M_1},X). The neighborhood of x_1 induced by the fuzzy concepts in η, which is obtained by Definition 7.6, is listed as follows.

$N^\tau_{\{A\}+\{H\}+\{We\}}(x_1) = \{x_1,x_2,x_3,x_4,x_5\}$, $N^\tau_{\{A\}+\{H\}}(x_1) = \{x_1,x_2,x_3,x_4,x_5\}$,
$N^\tau_{\{A\}+\{We\}}(x_1) = \{x_1,x_2,x_3,x_4,x_5\}$, $N^\tau_{\{H\}+\{We\}}(x_1) = \{x_1,x_2,x_3,x_4\}$,
$N^\tau_{\{A\}}(x_1) = \{x_1,x_2,x_3,x_4,x_5\}$, $N^\tau_{\{H\}}(x_1) = \{x_1,x_3\}$, $N^\tau_{\{We\}}(x_1) = \{x_1,x_2,x_3,x_4\}$,
$N^\tau_{\{A,H\}+\{A,We\}+\{We,H\}}(x_1) = \{x_1,x_2,x_3,x_4\}$, $N^\tau_{\{A,H\}+\{A,We\}}(x_1) = \{x_1,x_2,x_3,x_4\}$,

$N^{\tau}_{\{A,H\}+\{We,H\}}(x_1) = \{x_1,x_3\}$, $N^{\tau}_{\{A,We\}+\{We,H\}}(x_1) = \{x_1,x_2,x_3,x_4\}$,
$N^{\tau}_{\{A,H\}}(x_1) = \{x_1,x_3\}$, $N^{\tau}_{\{A,We\}}(x_1) = \{x_1,x_2,x_3,x_4\}$, $N^{\tau}_{\{We,H\}}(x_1) = \{x_1,x_3\}$,
$N^{\tau}_{\{A\}+\{We,H\}}(x_1) = \{x_1,x_2,x_3,x_4,x_5\}$, $N^{\tau}_{\{H\}+\{A,We\}}(x_1) = \{x_1,x_2,x_3,x_4,x_5\}$,
$N^{\tau}_{\{We\}+\{A,H\}}(x_1) = \{x_1,x_2,x_3,x_4\}$, $N^{\tau}_{\{A,H,We\}}(x_1) = \{x_1,x_3,x_5\}$, $N^{\tau}_{M}(x_1) = X$.

Therefore the neighborhood of x_1 induced by the fuzzy concepts in η is

$$N^{\tau}_{\eta}(x_1) = \{X,\ \{x_1,x_2,x_3,x_4\},\ \{x_1,x_3,x_5\},\ \{x_1,x_3\}\}.$$

Similarly, we have the neighborhood of other elements in X as follows.

$$\begin{aligned}
N^{\tau}_{\eta}(x_2) &= \{X,\ \{x_2,x_3,x_4,x_5\},\ \{x_1,x_2,x_3,x_4\},\ \{x_2,x_4,x_5\},\ \{x_2,x_3,x_5\},\{x_2,x_5\}\},\\
N^{\tau}_{\eta}(x_3) &= \{X,\ \{x_2,x_3,x_4,x_5\},\ \{x_3\}\},\\
N^{\tau}_{\eta}(x_4) &= \{X,\ \{x_3,x_4\},\ \{x_4\}\},\\
N^{\tau}_{\eta}(x_5) &= \{X,\ \{x_3,x_4,x_5\},\ \{x_4,x_5\},\ \{x_3,x_5\},\ \{x_5\}\}.
\end{aligned}$$

Here we just simply analyze the topology on X resorting ourselves to intuition. One can observe that x_3, x_4, x_5 are discrete points for the topology $\mathscr{T}_{\eta}$. Coincidentally, their membership degrees to the fuzzy concepts $\{A\},\{H\},\{We\}$ taken on the maximal values, respectively. For any $U \in \mathscr{T}_{\eta}$, we can prove that if $x_1 \in U$ then $x_3 \in U$. This implies that the degree of x_3 belonging to any fuzzy concept in EM_1 is always larger than or equal that of x_1. Since $x_2 \in \{x_1,x_3\} \in \mathscr{T}_{\eta}$, $x_1 \in \{x_2,x_5\} \in \mathscr{T}_{\eta}$ i.e., the separation property of topology $\mathscr{T}_{\eta}$, hence there exist two fuzzy concepts in η such that x_2,x_1 can be distinguished by them. Compared with the topological structure on X induced by the topological molecular lattice on $^*EI^n$ algebra, the above topological structure has many differences. What are the relationship between these topological structures still remains as an open problem.

7.3 Fuzzy Similarity Relations Based on Topological Molecular Lattices

In this section, by considering the AFS structure (M,τ,X) of a data, we apply $\mathscr{T}_{\eta}$ the topology on X induced by the topological molecular lattice η of some fuzzy concepts on $^*EM$ to study the fuzzy similarity relations on X for problems of pattern recognition. The topology $\mathscr{T}_{\eta}$ on X is determined by the original data and some selected fuzzy concepts in EM. It represents the abstract geometry relations among the objects in X. We study the interpretations of the induced topological structures on the AFS structures directly obtained by a given data set through the differential degrees between objects in X and the fuzzy similarity relations on X in the topological space $(X,\mathscr{T}_{\eta})$. We know that human can classify, cluster and recognize the objects in the set X without any metric in Euclidean space. What is human recognition based on if X is not a subset of some metric space in Euclidean space? For example, if you want to classify all your friends into two classes $\{close\,friends\}$ and $\{common\,friends\}$. The criteria/metric you are using in the process is very important though it may not be based on the Euclidean metric. By the fuzzy clustering analysis based on the topological spaces induced by the fuzzy concepts in EM, we hope find some clues for these challenge problems.

Theorem 7.8. *The following three conditions on a topological space are equivalent.*

(1) The space is metrizable;
(2) The space is T_1 and regular, and the topology has a σ-locally finite base;
(3) The space is T_1 and regular, and the topology has a σ-discrete base.

Here a topological space is a T_1 space if and only if each set which consists of a single point is closed, a topological space is regular if and only if for each point x and each neighborhood U of x there is a closed neighborhood V of x such that $V \subseteq U$, and a family is σ-locally finite (σ-discrete) if and only if it is the union of a countable number of locally finite (respectively, discrete) subfamilies.

Its proof (refer to Theorem 1.43) is left to the reader.

The topology $\mathscr{T}_\eta$ on X induced by the topological molecular lattice η of some fuzzy concepts in EM is a description of the abstract geometry relations among the objects determined by the semantic interpretations of the fuzzy concepts in η and the distributions of the original data. We can state the problem in mathematical ways as follows: Let X be a set of some objects and $\mathcal{F}$ be the set of all features, including features which are independent or irrelated to the problems under considering. M is the set of simple concepts on the features in $\mathcal{F}$. $\Lambda \subseteq EM$, Λ is the set of fuzzy concepts an individual considers crucial to his problem. η is the topological molecular lattice generated by Λ. If the topology $\mathscr{T}_\eta$ satisfies (2) or (3) in Theorem 7.8, then the topology space $(X, \mathscr{T}_\eta)$ is metrizable. Thus we can study the clustering and recognition problems by the metric induced by topology $\mathscr{T}_\eta$, i.e., the distance function d on the cartesian product $X \times X$ to the non-negative reals defined by Definition 1.33 as follows: for all points x, y, and z of X,

1. $d(x,y) = d(y,x)$,
2. $d(x,y) + d(y,z) \geq d(x,z)$, (*triangle inequality*)
3. $d(x,y) = 0$ if $x = y$, and
4. if $d(x,y) = 0$, then $x = y$.

However, for a real world applications, it is very difficult to satisfy the conditions of Theorem 7.8. In other words, this theorem cannot be directly applied to real world classification scenarios. By the analysis of the definition of metric in metrizable topology space $(X, \mathscr{T}_\eta)$ in mathematics (refer to Urysohn Lemma Lemma 1.1), we know that the more fuzzy concepts distinguish x from y are there in η, the larger the distance of x and y, i.e., $d(x,y)$. In practice, for X a set of objects and $\Lambda \subseteq EM$ a set of selected fuzzy or Boolean concepts, although $\mathscr{T}_\eta$ the topology induced by the topological molecular lattice η seldom satisfies (2) or (3) in Theorem 7.8, $\mathscr{T}_\eta$ also can reflect the similar relations between the objects in X determined by the concepts in Λ and the distributions of the original data. Thus we define the differential degree and the similarity degree of x, $y \in X$ based on the topology $\mathscr{T}_\eta$ as follows.

Definition 7.7. Let X and M be finite sets and (M,τ,X) be an AFS structure. Let η be a topological molecular lattice on the lattice $^*EM$ and $(X,\mathscr{T}_\eta)$ be the topology space on X induced by η. We define the *partial distance function* $D(x,y)$, the *differential degree* $d(x,y)$ and the *similarity degree* $s(x,y)$ in the topological space $(X,\mathscr{T}_\eta)$ as follows: for $x,y\in X$,

$$D(x,y) = \sum_{\delta\in\mathscr{T}_\eta, x\in\delta, y\notin\delta} |\delta|; \tag{7.15}$$

$$d(x,y) = D(x,y)+D(y,x); \tag{7.16}$$

$$s(x,y) = 1-\frac{d(x,y)}{\max_{z\in X}\{d(z,y)\}}. \tag{7.17}$$

Because there are too many fuzzy concepts in η, in practice, it is difficult or impossible to calculate $d(x,y)$ by Definition 7.7 for the topological molecular lattice η generated by Λ, if $|\Lambda|>4$. The following Definition 7.8 and Definition 7.9 introduce the differential degrees of x,y, $d(x,y)$ which are more expedient to compute than that in Definition 7.7, although they may loose some information compared with the concept captured by Definition 7.7. Definition 7.8 and Definition 7.9 are applicable to discuss real world problems while Definition 7.7 is more appealing from the theoretical perspective.

Definition 7.8. Let X and M be finite sets and (M,τ,X) be an AFS structure. Let η be an elementary topological molecular lattice on the lattice $^*EM$ and $(X,\mathscr{T}_\eta)$ be the topology space on X induced by η. We define $D_A(x,y)$, the *distance function on the molecular* A; $d_M(x,y)$, the *molecular differential degree*; and $s_M(x,y)$, the *molecular similarity degree* in the topological space $(X,\mathscr{T}_\eta)$ as follows: for $x,y\in X$, $A\subseteq M$, $A\in\eta$,

$$D_A(x,y) = \sum_{u\in X, x\in N_A^{\triangle\tau}(u), y\notin N_A^{\triangle\tau}(u)} |\, N_A^{\triangle\tau}(u)|; \tag{7.18}$$

$$d_M(x,y) = \sum_{A\subseteq M, A\in\eta} (D_A(x,y)+D_A(y,x)); \tag{7.19}$$

$$s_M(x,y) = 1-\frac{d_M(x,y)}{\max_{z\in X}\{d_M(z,y)\}}. \tag{7.20}$$

$D_A(x,y)$ in Definition 7.8 is considered under the fuzzy molecular concept $A\in\eta$ and $d_M(x,y)$, the molecular differential degree of x,y is the sum of the distances of x,y under all fuzzy molecular concepts in η.

Definition 7.9. Let X and M be finite sets and (M,τ,X) be an AFS structure. Let η be an elementary topological molecular lattice on the lattice $^*EM$. Let $(X,\mathscr{T}_\eta)$ be the topology space on X induced by η. We define the *elementary partial distance function* $D^e(x,y)$, the *elementary differential degree* $d^e(x,y)$ and the *elementary similarity degree* $s_e(x,y)$ in the topological space $(X,\mathscr{T}_\eta)$ as follows: for any $x,y\in X$,

$$D_e(x,y) = \sum_{\delta \in \mathscr{B}_M^{\triangle\tau}, x\in\delta, y\notin\delta} |\delta|; \tag{7.21}$$

$$d_e(x,y) = D_e(x,y) + D_e(y,x); \tag{7.22}$$

$$s_e(x,y) = 1 - \frac{d_e(x,y)}{\max_{z\in X}\{d_e(z,y)\}}. \tag{7.23}$$

Here

$$\mathscr{B}_M^{\triangle\tau} = \left\{ N_A^{\triangle\tau}(x) \mid A \subseteq M, A \in \eta, x \in X \right\}.$$

It is clear that $\mathscr{B}_M^{\triangle\tau} \subseteq \mathscr{T}_\eta$ is the set of all neighborhoods induced by the fuzzy molecular concepts in η which determine the distances and similarity degrees defined by Definition 7.9. However, in Definition 7.7, they are determined by all neighborhoods in $\mathscr{T}_\eta$. Since the number of the elements of $\mathscr{T}_\eta$ is much larger than that of the set $\mathscr{B}_M^{\triangle\tau}$, hence much time will save if Definition 7.9 or Definition 7.8 is applied to a pattern recognition problem. The problem is still open: are the similarity degrees defined by Definition 7.7, Definition 7.8 and Definition 7.9 equivalent?

Proposition 7.4. *Let X and M be finite sets and (M,τ,X) be an AFS structure. Let η be an elementary topological molecular lattice on the lattice $^*EM$ and $(X,\mathscr{T}_\eta)$ be the topology space on X induced by η. Then for any $x,y \in X$ the following assertions hold.*

(1) $d(x,x)=0$, $d(x,y)=d(y,x)$ *and* $s(x,y)=s(y,x)\le s(x,x)$;
(2) $d_M(x,x)=0$, $d_M(x,y)=d_M(y,x)$ *and* $s_M(x,y)=s_M(y,x)\le s_M(x,x)$;
(3) $d_e(x,x)=0$, $d_e(x,y)=d_e(y,x)$ *and* $s_e(x,y)=s_e(y,x)\le s_e(x,x)$.

Its proof is left to the reader.

7.4 Fuzzy Clustering Algorithms Based on Topological Molecular Lattices

Numerous mathematical tools, investigated for clustering, have been considered to detect similarities between objects inside a cluster. The two-valued clustering is described by a characteristic function. This function assigns each object to one and only one of the clusters, with a degree of membership equal to one. However, the boundaries between the clusters are not often well-defined and this description does not fully reflect the reality. The fuzzy clustering, founded upon fuzzy set theory [35], is meant to deal with not well-defined boundaries between clusters. Thus, in fuzzy clustering, the membership function is represented by grades located anywhere in-between zero and one. Therefore, this membership degree indicates how the object is classified (allocated) to each cluster. This can be advantageous for patterns located in the boundary region which may not be precisely defined. In particular, we could flag some patterns that are difficult to assign to a single cluster as being inherently positioned somewhere at the boundary of the clusters.

Many fuzzy clustering algorithms have been developed, but the most widely used is the Fuzzy C-Means algorithm (FCM) along with a significant number of their variants. Conceived by Dunn [2] and generalized by Bezdek [1], this family of algorithms is based on iterative optimization of a fuzzy objective function. The convergence of the algorithm, proved by Bezdek, shows that the method converges to some local minima [4]. Nevertheless, the results produced by these algorithms depend on some predefined distance formulated in a metric space, for instance Euclidean space R^n. However, in this section we will cluster the objects in ordinary data set $X \subsetneqq R^{p\times n}$ according to the fuzzy concepts or attributes on the features without using any kind of distance functions expressed in the Euclidean space.

In general, FCM is an objective function optimization approach to solve the following problem [1, 4]:

$$minimize: \quad J_m(U,V) = \sum_i \sum_k u_{ik}^m d^2(x_k, v_i)$$

with respect to $U = [u_{ik}] \in R^{c\times n}$, a fuzzy c-partition of n data set $X = \{x_1, ..., x_n\} \in R^{p\times n}$ and V, a set of c cluster centers $V = \{v_1, ..., v_c\} \in R^{p\times c}$. The parameter $m > 1$ is a fuzziness coefficient. $d(x_k, v_i)$ is a distance from x_k to the ith cluster center v_i. The performance of FCM is affected by different distances $d(.,.)$. In general, the distance is expressed in some metric space [4, 34], if data set X is a subset of a metric space. FCM fuzzy clustering algorithms are very efficient if the data set $X \subset R^{p\times n}$, as in this case there exists a distance function. Let c be a positive integer greater than one. $\mu = \{\mu_1, ..., \mu_c\}$ is called a fuzzy c-partition of X, if $\mu_i(x)$ is the membership functions in fuzzy sets μ_i on X assuming values in the $[0,1]$ such that $\sum_{i=1}^{c} \mu_i(x) = 1$ for all x in X. Thus, the Fuzzy C-Mean (FCM) objective function $J(\mu, V)$ is also defined as

$$J(\mu,V) = \sum_{i=1}^{c} \sum_{j=1}^{n} \mu_i^m(x_j)||x_j - v_i||^2, \tag{7.24}$$

where $\mu_i(x_j) = u_{ij} = \mu_{ij}$ and $d(x_k, v_i) = \|x_k - v_i\|$. The FCM clustering is an iterative algorithm where the update formulas for the prototypes and the partition matrix read as follows:

$$v_i = \frac{\sum_{j=1}^{n} \mu_{ij}^m x_j}{\sum_{j=1}^{n} \mu_{ij}^m}, i = 1, ..., c \tag{7.25}$$

and

$$\mu_{ij} = \mu_i(x_j) = (\sum_{k=1}^{c} \frac{||x_j - v_i||^{2/(m-1)}}{||x_j - v_k||^{2/(m-1)}})^{-1}, i = 1, ..., c, j = 1, ..., n. \tag{7.26}$$

If the feature vectors are numeric data in R^d, the FCM clustering algorithm is a suitable optimization tool. However, when applying the FCM to data set with mixed features such as Boolean, partial order and linguistic rating scale, we encounter some problems, because the conventional distance $||.||$ is not suitable any longer.

To overcome these problems, the differential degrees $d(x,y)$ (or $d_e(x,y)$, $d_M(x,y)$) defined in the above section can substitute the Euclidean distance $||.||$ and the FCM can be modified as follows.

$$\min_{\{v_1,...,v_c\}\subseteq X}: J(\mu,V)=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_i^m(x_j)d(x_j,v_i)^2 \tag{7.27}$$

subject to

$$\mu_i(x_j)=\left(\sum_{k=1}^{c}\frac{d(x_j,v_i)^{2/(m-1)}}{d(x_j,v_k)^{2/(m-1)}}\right)^{-1}, i=1,...,c, j=1,...,n.$$

This algorithm is called the *AFS fuzzy c-mean algorithm* (AFS_FCM).

In order to compare the differential functions defined in the above section with the Euclidean distance function, we directly apply the similarity matrix derived by the differential function and Euclidean distance function to the clustering problem. Let $X=\{x_1,x_2,...,x_n\}$ and the similarity matrix $S=(s_{ij})_{n\times n}$, where $s_{ij}=s_e(x_i,x_j)$ is elementary similarity degree of x,y defined by Definition 7.8. For the similarity matrix S, we know $s_{ij}=s_{ji}$ and $s_{ij}\le s_{ii}$, $1\le i,j\le n$ from Proposition 7.4, hence there exists an integer r such that $S\le S^2\le ...\le S^r=S^{r+1}$, where $S^2=(r_{ij})=SS$ is the fuzzy matrix product, i.e., $r_{ij}=\max_{1\le k\le n}\min\{s_{ik},s_{kj}\}$. Thus, $(S^r)^2=S^r$ (S^r is the transitive closure matrix of S) and the fuzzy equivalence relation matrix $Q=(q_{ij})=S^r$ can yield a partition tree with equivalence classes in which x_i and x_j are in the same cluster (i.e., in the same equivalence classe) under some threshold $\alpha\in[0,1]$ if and only if $q_{ij}\ge\alpha$.

7.5 Empirical Studies

In this section, we apply the similarity relations and the differential functions defined by Definition 7.8 to the conventional FCM and compare the elementary differential function with the Euclidean distance function in the clustering analysis of the Iris data. Furthermore, they are also applied to Taiwan airfreight forwarder data which is just described by means of linguistic terms. These examples show that the topology $\mathscr{T}_\eta$ on a universe of discourse X induced by the topological molecular lattice η of some fuzzy concepts in EM can be applied to the real world pattern recognition problems for the data set with mixed features on which the classical distance functions could not be defined.

7.5.1 Empirical Examples of Taiwan Airfreight Forwarder

In what follows, we apply the elementary differential degree and elementary similarity degree defined by Definition 7.8 to empirical examples of Taiwan airfreight forwarder for the clustering and analyzing current operation strategies in [27]. In [27], the authors gathered 28 strategic criteria from scholars, experts and proprietors. They select 30 companies of airfreight forwarder in Taiwan by random selection. Using Statistical Analysis System (SAS), they obtain seven factors: Factor1: *Core* *ability*, Factor2: *Organization management*, Factor3: *Pricing*, Factor4: *Competitive forces*, Factor5: *Finance*, Factor6: *Different advantage*, Factor7: *Information technology*. The decision-makers may tackle preference rating system by adopting one of various rating scales assumed in the literature [8, 28, 29] or may develop their own rating scales system by using trapezoidal fuzzy number to show the individual conception of the linguistic variable "*attention degree*". According to the preference ratings proposed by Liang and Wang [28], it is suggested that the decision-makers utilize the linguistic rating set

$$W = \{VL, B.VL\&L, L, B.L\&M, M, B.M\&H, H, B.H\&VH, VH\},$$

where VL: *Very Low*, $B.VL\&L$: *Between Very Low and Low*, L: *Low*, $B.L\&M$: *Between Low and Medium*, M: *Medium*, $B.M\&H$: *Between Medium and High*, H:*High*, $B.H\&VH$:*Between High and Very High*, VH:*Very High*, to assess the attention degree of subjects of companies under each of the management strategies. The decision-makers utilize the linguistic rating as above and obtain the evaluation results as Table 7.3. Let $X = \{C_1, ..., C_5\}$ and $M = \{m_1, m_2, ..., m_7\}$ be the set of simple concepts on the features *Factor1* to *Factor7*. Where m_i: *great attention degree of Factor i*, $i = 1, 2, ..., 7$. The following order relation of the elements in the linguistic rating set W is determined by their linguistic rating scales:

$$VL < B.VL\&L < L < B.L\&M < M < B.M\&H < H < B.H\&VH < VH \quad (7.28)$$

For each $m_i \in M$, we can define a binary relation R_{m_i} on X by Table 7.3 and the order relation shown as (7.28): $(C_k, C_k) \in R_i$, for any $k = 1, ..., 5$ and for any $k \neq l$, $(C_k, C_l) \in R_i \Leftrightarrow C_k$(Factor i)$\geq C_l$(Factor i), where C_j(Factor i) is the linguistic rating scale of C_j for Factor i. By Definition 4.3, one can verify that for each $m_i \in M$, R_i is a simple concept. (X, τ, M) is an AFS structure if τ is defined as follows: For any C_i, $C_j \in X$, $\tau(C_i, C_j) = \{m_k \in M | (C_i, C_j) \in R_k\}$ (refer to (4.26)). Let $\Lambda = \{\{m_1\}, ..., \{m_7\}\} \subseteq EM$ and η be the topological molecular lattice generated by Λ. Let $(X, \mathscr{T}_\eta)$ be the topology space on X induced by η. Let $d_e(C_i, C_j)$ be the elementary differential degree of C_i, C_j and $s_e(C_i, C_j)$ be the elementary similarity degree of x, y defined by Definition 7.8. We obtain the following fuzzy similar matrix $S = (s_{ij})_{n\times n}$, $s_{ij} = s_e(C_i, C_j)$ and the following elementary differential matrix $T = (t_{ij})_{n\times n}$, $t_{ij} = d_e(C_i, C_j)$.

Table 7.3 The evaluation results of five companies

Company	Factor						
	Factor1	Factor2	Factor3	Factor4	Factor5	Factor6	Factor7
C1	M	H	H	B.H & VH	VH	L	B.M & H
C2	H	B.L& M	M	B.M & H	H	B.M & H	VL
C3	H	H	B.M & H	H	H	VH	B.M & H
C4	VL	M	H	B.VL&L	H	B.L& M	M
C5	L	M	B.H & VH	H	B.H & VH	B.VL&L	B.M & H

$$T = \begin{bmatrix} 0 & 1513 & 1175 & 1112 & 666 \\ 1513 & 0 & 638 & 1067 & 1391 \\ 1175 & 638 & 0 & 1161 & 1263 \\ 1112 & 1067 & 1161 & 0 & 918 \\ 666 & 1391 & 1263 & 918 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1.0 & 0 & 0.2234 & 0.2650 & 0.5598 \\ 0 & 1.0 & 0.5783 & 0.2948 & 0.0806 \\ 0.2234 & 0.5783 & 1.0 & 0.2327 & 0.1652 \\ 0.2650 & 0.2948 & 0.2327 & 1.0 & 0.3933 \\ 0.5598 & 0.0806 & 0.1652 & 0.3933 & 1.0 \end{bmatrix}$$

Then, the transitive closure of similar matrix S is S^4, i.e.,

$$(S^4)^2 = S^4 = \begin{bmatrix} 1.0 & 0.2948 & 0.2948 & 0.3933 & 0.5598 \\ 0.2948 & 1.0 & 0.5783 & 0.2948 & 0.2948 \\ 0.2948 & 0.5783 & 1.0 & 0.2948 & 0.2948 \\ 0.3933 & 0.2948 & 0.2948 & 1.0 & 0.3933 \\ 0.5598 & 0.2948 & 0.2948 & 0.3933 & 1.0 \end{bmatrix}$$

Let the threshold $\alpha = 0.5$. Then the clusters are $\{$C1, C5$\}$, $\{$C2, C3$\}$ and $\{$C4$\}$. In [27], the transitive closure of the compatibility relation R_T of Table 7.3 is obtained as follows:

$$R_T = \begin{bmatrix} 1 & 0.389 & 0.415 & 0.590 & 0.679 \\ 0.389 & 1 & 0.389 & 0.389 & 0.389 \\ 0.415 & 0.389 & 1 & 0.415 & 0.415 \\ 0.590 & 0.389 & 0.415 & 1 & 0.590 \\ 0.679 & 0.389 & 0.415 & 0.590 & 1 \end{bmatrix}.$$

By taking $\lambda \in (0.590, 0.679]$, the authors in [27] obtained the clusters: $\{$C1,C5$\}$, $\{$C2$\}$, $\{$C3$\}$ and $\{$C4$\}$.

By the application of the AFS-FCM algorithm described by (7.27) with the elementary differential degree defined by Definition 7.8 to the data of the 30 companies shown in **Appendix A**, let the cluster number c be equal to 5, we obtain the clustering results

cluster1=$\{$C2, C3, C6,C7$\}$, cluster2=$\{$C1, C4, C5, C10, C16, C21, C23, C25, C28$\}$,
cluster3=$\{$C9, C11, C13, C17, C19, C27$\}$,
cluster4=$\{$C8, C18, C20, C24, C26, C29$\}$, cluster5=$\{$C12, C14, C15, C22, C30$\}$.

Figure 7.1, in which the x-axis is the re-order of the C1,..., C30 by the order cluster 1,...,cluster 5, i.e., 1:4 cluster 1; 5:13 cluster 2; 14:19 cluster 3; 20:25 cluster 4; 26:30 cluster 5, shows the membership functions of the fuzzy partition matrix of X, $\mu = \{\mu_1, ..., \mu_5\}$.

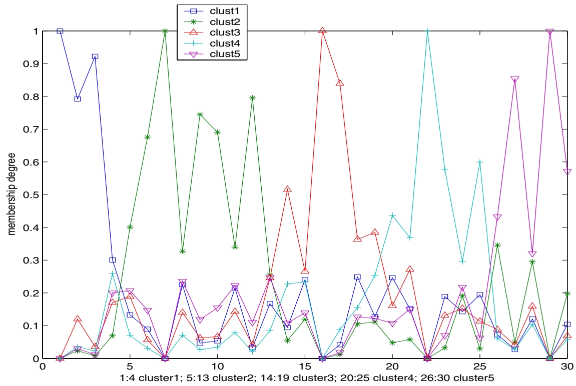


Fig. 7.1 The membership functions μ_i of the fuzzy 5-partition of X, $\mu = \{\mu_1, ..., \mu_5\}$

7.5.2 Experimental Studies on the Iris Data Set

The Iris data [30] have 150×4 matrix $W = (w_{ij})_{150\times 4}$ evenly distributed in three classes: iris-setosa, iris-versicolor, and iris-virginica. Vector of sample i, $(w_{i1}$, w_{i2}, w_{i3}, $w_{i4})$ has four features: sepal length and width, and petal length and width (all given in centimeters). Let $X = \{x_1, x_2, ..., x_{150}\}$ be the set of the 150 samples, where $x_i = (w_{i1}, w_{i2}, w_{i3}, w_{i4})$. Let $M = \{m_1, m_2, ..., m_8\}$ be the set of simple concepts on the features, where

m_1: *the sepal is long*, m_2: *sepal is wide*, m_3: *petal is long*, m_4: *petal is wide*; $m_5 = m_4'$: *petal is not wide*, $m_6 = m_3'$: *petal is not long*, $m_7 = m_2'$: *sepal is not wide*, $m_8 = m_1'$: *the sepal is not long.*

Given the original Iris data, we can verify that each concept $m \in M$ is a simple concept and (M, τ, X) is an AFS structure if for any $x, y \in X$, we define $\tau(x,y) = \{m | m \in M, (x,y) \in R_m\}$ (refer to (4.26)). For example, $\tau(x_1,x_1) = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$, since the sample x_1 has sepal length and width, and petal length and width. Similarly we can get $\tau(x_i,x_i)$, $i = 2, ..., 150$. For sample $x_4 = (4.6, 3.1, 1.5, 0.2)$ and sample $x_7 = (4.6, 3.4, 1.4, 0.3)$, we have $\tau(x_4,x_7) = \{m_1, m_3, m_5, m_7, m_8\}$, since the degrees of x_4 belonging to simple concepts *long sepal*, *long petal*, *not wide petal*, *not wide sepal*, *not long sepal* are larger than or equal to that of x_7. Similarly, we can determine $\tau(x_i,x_j)$ for any i, j according to the given feature values of the samples or the binary relation R_m of the simple concepts $m \in M$.

Let (M, τ, X) be the AFS structure of the Iris data set and η be the topological molecular lattice on the lattice $^*EM$ generated by all simple concepts in M,i.e., $\Lambda = \{\{m_1\}, ..., \{m_8\}\} \subseteq EM$. Let $(X, \mathscr{T}_\eta)$ be the topology space of X induced by the topological molecular lattice η. In order to compare the elementary differential degree of x, y in topology $\mathscr{T}_\eta$ with Euclidean distance in R^4. Let R_η be the fuzzy relation matrix derived by topology $\mathscr{T}_\eta$, where $R_\eta = S_\eta^r$, $(S_\eta^r)^2 = S_\eta^r$, $S_\eta = (s_{ij})$, $s_{ij} = s_e(x_i, x_j)$, the elementary similarity degree is defined by Definition 7.8. Let

R_E be the fuzzy relation matrix derived by the Euclidean distance where $R_E = S_E^k$, $(S_E^k)^2 = S_E^k$, $S_E = (e_{ij})$, $e_{ij} = 1 - (\sum_{1\leq k\leq 4}(x_{ik} - x_{jk})^2)^{\frac{1}{2}}$. Let fuzzy equivalence relation matrix $Q = (q_{ij}) = R_\eta$ or R_E, and for each threshold $\alpha \in [0,1]$, let Boolean matrix $Q_\alpha = (q_{ij}^a)$, $q_{ij}^a = 1 \Leftrightarrow q_{ij} \geq \alpha$. Since $R_\eta^2 = R_\eta$, $R_E^2 = R_E$, hence for each threshold $\alpha \in [0,1]$, Q_α is an equivalence relation Boolean matrix and it can yield a partition on X (refer to [23]). The following Figure 7.2 shows the clustering accuracy rates of fuzzy equivalence relation matrices R_η and R_E for threshold $\alpha \in [0.1,1]$. The accuracy is determined as follows: Suppose that the clusters C_1, C_2, ...,C_l are obtained by the fuzzy equivalence relation matrices R_η or R_E for some specific threshold α. Let $N_1 = \{1,2,...,50\}, N_2 = \{51,52,...,100\}, N_3 = \{101,102,...,150\}$. For $l \geq 3$, the clustering accuracy rate r is

$$r = \max_{1\leq i,j,k\leq l, i\neq j, i\neq k, j\neq i}\{\frac{|N_1\cap C_i|+|N_2\cap C_j|+|N_3\cap C_k|}{150}\};$$

For $l = 2$, let

$$|N_k \cap C_1| = \max_{1\leq u\leq 3}\{|N_u \cap C_1|\}, 1 \leq k \leq 3,$$
$$|N_l \cap C_2| = \max_{1\leq u\leq 3, u\neq k}\{|N_u \cap C_2|\}, 1 \leq l \leq 3, l \neq k,$$

$r = \frac{|N_k\cap C_1|+|N_l\cap C_2|}{150}$. For $l = 1$, let $r = \frac{1}{3}$. When threshold $\alpha = 0.8409$, the clustering accuracy rate of R_η is 90.67% (the best one), 9 clusters are obtained, the error clustering samples are $x_{23}, x_{42}, x_{69}, x_{71}, x_{73}, x_{78}, x_{84}, x_{88}, x_{107}, x_{109}, x_{110}, x_{118}, x_{132}, x_{135}$. When threshold $\alpha = 0.8905$, the clustering accuracy rate of R_E is 72.67% (the best one), and 29 clusters have been obtained. In Figures 7.2, we can observe that the elementary differential degrees defined by Definition 7.8 are better than those obtained for the Eculidean distance when it comes to the description of the difference of objects for this cluster analysis.

In order to compare the fuzzy equivalence relation matrices R_η with R_E, we show that the similar relation degrees of x_k to $\forall x \in X$,i.e., $R_\eta(x_k,x)$ and $R_E(x_k,x)$, $k = 71,72,...,130$ in Figures 7.3- 7.14 in Appendix B as examples. Since for Iris-data,

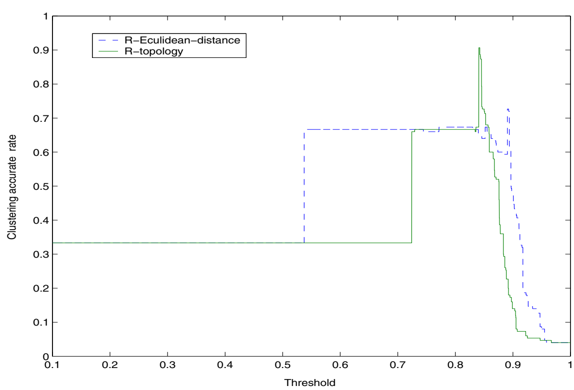


Fig. 7.2 The clustering accuracy rates of fuzzy equivalence relation matrices R_η and R_E for threshold $\alpha \in [0.1,1]$

the samples $x_{71},...,x_{130}$ are most difficult to be clustered, hence we show $R_\eta(x_k,x)$ and $R_E(x_k,x)$, $k=71,72,...,130$ in the figures. For Iris-data, samples 1:50 are cluster 1, i.e., iris-setosa; samples 51:100 are cluster 2, i.e., iris-versicolor; samples 101:150 are cluster 3, i.e., iris-virginica. In Figures 7.3 and Figures 7.4, for $x_{71},x_{72},...,x_{80}$ which are cluster 2, $R_\eta(x_k,x)$ and $R_E(x_k,x)$, $x\in X$ are shown. Compared with Figure 7.4, we can observe that in Figure 7.3, the similarity degrees of x_k to most samples in cluster 2 are larger than that of x_k to the samples in cluster 1,3. This implies that x_k are more similar to the samples in cluster 2 and $R_\eta(x_k,x)$, $k=71,...,80$, in Figure 7.3 are more clearly distinguish x_k from the samples in cluster 1, 3 than $R_E(x_k,x)$, $k=71,...,80$, in Figure 7.4. Similar phenomenon can be observed in Figures 7.4-7.14 and the others for $k=1,...,70,131,...,150$ which are not shown here. These examples show that the fuzzy equivalence relation matrix based on the topology is obviously better than that based on Eculidean distance for clustering of Iris data.

By the application of the AFS-FCM algorithm shown in (7.27) to the distance matrix $T=(t_{ij})_{150\times150}$, $t_{ij}=d_e(x_i,x_j)$ defined by Definition 7.8, the clustering accuracy rate is 86.67%. Using the function k means in MATLAB toolbox, which is based on the well known k-mean clustering algorithm [32], the clustering accuracy rate is 89.33%. And using the function FCM in MATLAB toolbox, which is based on the FCM clustering algorithm [1], the clustering accuracy rate is also 89.33%. Considering that the cluster centers of AFS-FCM must be the samples, i.e., $\{v_1,...,v_c\}\subseteq X$, while the cluster centers of FCM can be any vectors, i.e., $\{v_1,...,v_c\}\subseteq R^n$, the clustering accuracy rate of AFS-FCM is acceptable.

In some situations, it is difficult or impossible to describe some features of objects using real numbers, considering some inevitable errors and noise. For example, we do not describe a degree "*white hair*" of a person by counting the number of white hair on his head. But the order relations can be easily and accurately established by the simple comparisons of each pair of person's hair. In the framework of AFS theory, (M,τ,X) is determined by the binary relations R_m, $m\in M$ and the order relations are enough to establish the AFS structure of a data system. The membership functions and their logic operations of the fuzzy concepts in EM can be obtained by the AFS fuzzy logic system $(EM,\vee,\wedge,')$ and the AFS structure (M,τ,X). Therefore the AFS-FCM can be applied to the data set with the attributes described by mixed features such as numeric data, Boolean, order, even descriptors of human intuition, but FCM and k-mean can only be applied to the data set with the attributes described by numeric data.

The differential degrees and similarity degrees based on the topology induced by some fuzzy concepts are the criteria/metric human are using in their recognition process. This criteria/metric may not be the metric in the Euclidean space. The illustrative examples give some interpretations of the special topological structures on the AFS structures directly obtained by a given data set. Thus this approach also offers a new idea to data mining, artificial intelligence, pattern recognition,..., etc. Furthermore the real world examples demonstrate that this approach is promising.

Exercises

Exercise 7.1. Let M be a set and EM be the ${}^*EI$ algebra over M. For $A \subseteq M$, $\sum_{i\in I} A_i$, $\sum_{j\in J} B_j \in EM$, show the following assertions hold:

(1) $A \geq \sum_{i\in I} A_i$ and $A \geq \sum_{j\in J} B_j \Leftrightarrow A \geq \sum_{i\in I} A_i \vee \sum_{j\in J} B_j$.
(2) $A \geq \sum_{i\in I} A_i$ or $A \geq \sum_{j\in J} B_j \Leftrightarrow A \geq \sum_{i\in I} A_i \wedge \sum_{j\in J} B_j$.

Exercise 7.2. Let X and M be sets and (M,τ,X) be an strong relative AFS structure. Let η be a topological molecular lattice on EI algebra over M. For any $x \in X$, $\sum_{i\in I} A_i$, $\sum_{j\in J} B_j \in EM$, show the following assertions hold: for any $x \in X$

(1) If $\sum_{i\in I} A_i \geq \sum_{j\in J} B_j$ in EM, then $N^{\tau}_{\sum_{i\in I} A_i}(x) \supseteq N^{\tau}_{\sum_{j\in J} B_j}(x)$;
(2) $N^{\tau}_{\sum_{i\in I} A_i}(x) \cap N^{\tau}_{\sum_{j\in J} B_j}(x) = N^{\tau}_{\sum_{i\in I} A_i \wedge \sum_{j\in J} B_j}(x)$;
(3) $N^{\tau}_{\sum_{i\in I} A_i}(x) \cup N^{\tau}_{\sum_{j\in J} B_j}(x) = N^{\tau}_{\sum_{i\in I} A_i \vee \sum_{j\in J} B_j}(x)$.

Exercise 7.3. Proved that if η is a topological molecular on the lattice ${}^*EM$ and η is a dual idea of the lattice ${}^*EM$, then η is an elementary topological molecular lattice on the lattice ${}^*EM$.

Exercise 7.4. ([13]) Let X and M be sets and (M,τ,X) be a strong relative AFS structure. Let η be a topological molecular lattice on the lattice EM. If η is an elementary topological molecular lattice on the lattice EM and we define

$$\mathscr{B}_\eta = \left\{ N^{\tau}_{\sum_{i\in I} A_i}(x) \mid x \in X,\ \sum_{i\in I} A_i \in \eta \right\},$$

prove that $\mathscr{B}_\eta$ is a base for some topology of X.

Exercise 7.5. Let X and M be sets, (M,τ,X) be a strong relative AFS structure. Let η be a topological molecular lattice on the lattice EM and

$$\mathscr{L}_\eta = \left\{ \sum_{i\in I} a_i A_i \in EXM \mid \sum_{i\in I} A_i \in \eta, a_i \in \mathscr{T}_\eta \ for\ any\ i \in I \right\}. \tag{7.29}$$

Prove that $\mathscr{L}_\eta$ is a topological molecular lattice on the lattice EXM.

Exercise 7.6. Let X and M be sets. Let (M,τ,X) be a strong relative AFS structure and η be an elementary topological molecular lattice on the lattice EM. Let η be a topological molecular lattice on the lattice EM and the topological space $(X,\mathscr{T}_\eta)$ be the topology induced by η. Let S be the σ-algebra generated by $\mathscr{T}_\eta$ and $\mathscr{L}_\eta$ be the EI^2 topological molecular lattice on EXM induced by η. Show the following assertions hold.

(1) For any fuzzy concept $\sum_{i\in I} A_i \in \eta$, $\sum_{i\in I} A_i$ is a measurable concept under S ;
(2) The membership function defined by (7.14) is a continuous function from the topological space $(X,\mathscr{T}_\eta)$ to the topological molecular lattice $(EXM,\mathscr{L}_\eta)$.

Exercise 7.7. Prove that the following three conditions on a topological space are equivalent.

(1) The space is metrizable;
(2) The space is T_1 and regular, and the topology has a σ-locally finite base;
(3) The space is T_1 and regular, and the topology has a σ-discrete base.

Exercise 7.8. Let X and M be finite sets and (M,τ,X) be an AFS structure. Let η be an elementary topological molecular lattice on the lattice ${}^*EM$ and $(X,\mathscr{T}_\eta)$ be the topology space on X induced by η. Show for any $x,y\in X$ the following assertions hold.

(1) $d(x,x)=0$, $d(x,y)=d(y,x)$ and $s(x,y)=s(y,x)\leq s(x,x)$;
(2) $d_M(x,x)=0$, $d_M(x,y)=d_M(y,x)$ and $s_M(x,y)=s_M(y,x)\leq s_M(x,x)$;
(3) $d_e(x,x)=0$, $d_e(x,y)=d_e(y,x)$ and $s_e(x,y)=s_e(y,x)\leq s_e(x,x)$.

Open problems

Problem 7.1. Let X be a set and M be the set of simple concepts on X. Let (M,τ,X) be an AFS structure. If M is a finite set, then for any topological molecular lattice η on the EI algebra EM is also a topological molecular lattice on the ${}^*EI$ algebra ${}^*EM$. What are the relationships between the topological structures on X induced by η as a topological molecular on EM and that induced by η as a topological molecular on ${}^*EM$?

Problem 7.2. It is clear that $\mathscr{T}_\eta$ the topology on X is determined based on the distribution of raw data and the chosen set of fuzzy concepts $\eta\subseteq EM$ and it is an abstract geometry relation among the objects in X under the considering fuzzy concepts, i.e., η.

1. What are the interpretations of the special topological structures on X obtained from given database?
2. What are the topological structures associating with the essential nature of database?

Problem 7.3. Let X and M be sets. Let (M,τ,X) be a strong relative AFS structure and η be an elementary topological molecular lattice on the lattice EM. Let η be a topological molecular lattice on the lattice EM and the topological space $(X,\mathscr{T}_\eta)$ be the topology induced by η.

(1) How to induce a topological molecular lattice $\mathscr{L}_\eta^2$ on the lattices ${}^*EXMM$, $EXMM$ and a topological molecular lattice $\mathscr{L}_\eta^1$ on the lattices ${}^*E^\#X$, $E^\#X$?
(2) Are the membership functions defined on the lattices $EXMM$ by (5.12) and $E^\#X$ by (5.13) continuous from the topological space $(X,\mathscr{T}_\eta)$ to the topological molecular lattices $(EXMM,\mathscr{L}_\eta^2)$, $({}^*EXMM,\mathscr{L}_\eta^2)$, $({}^*E^\#X,\mathscr{L}_\eta^1)$, $(E^\#X,\mathscr{L}_\eta^1)$?

Problem 7.4. With a metric in the topological space on X, solving the pattern recognition problems will be possible for the database with various data types. Though we can have different choices from the topological theory for the metrics, what is suitable metric for this data of the pattern recognition problem?

Problem 7.5. Are the similarity degrees defined by Definition 7.7, Definition 7.8 and Definition 7.9 equivalent?

Problem 7.6. Let $(X, \mathscr{T}_\eta)$ be the topological space induced by η. Where η is the topological molecular lattice generated by some fuzzy concepts in EM. So far, we cannot obtain the differential degree and the similarity degree if η is the topological molecular lattice generated by more than 12 fuzzy concepts in EM. The more effective algorithm for the computation of the differential degree and the similarity degree in $(X, \mathscr{T}_\eta)$ are the most required.

Appendix A

Table 7.4 Evaluate results of 30 companies

Company	Factor					
	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
C1	H	H	H	B.H & VH	VH	B.M & H	B.M & H
C2	H	H	B.M & H	B.M & H	H	B.M & H	B.M & H
C3	H	H	B.M & H	H	H	B.M & H	B.M & H
C4	H	B.H & VH	H	B.H & VH	H	B.H & VH	B.M & H
C5	H	B.H & VH	H	B.H & VH	B.H & VH	B.H & VH	B.M & H
C6	H	H	B.M & H	M	B.M & H	B.M & H	B.M & H
C7	H	H	M	B.H & VH	B.M & H	B.M & H	M
C8	B.M & H	H	M	B.M & H	B.M & H	H	B.L& M
C9	B.M & H	H	H	H	B.H & VH	B.M & H	M
C10	H	VH	H	B.M & H	B.H & VH	H	B.M & H
C11	M	H	M	H	B.M & H	B.H & VH	B.H & VH
C12	VH	VH	H	B.H & VH	VH	H	M
C13	B.M & H	H	B.M & H	H	B.H & VH	B.M & H	B.M & H
C14	H	H	B.M & H	B.H & VH	B.H & VH	H	M
C15	H	H	H	H	B.H & VH	VH	M
C16	H	H	H	B.H & VH	B.H & VH	B.H & VH	B.H & VH
C17	B.M & H	H	M	H	B.H & VH	B.M & H	B.M & H
C18	M	H	B.M & H	B.H & VH	B.H & VH	M	M
C19	B.M & H	H	B.M & H	B.M & H	VH	B.M & H	B.H & VH
C20	B.M & H	M	M	H	B.H & VH	B.M & H	B.L& M
C21	B.H & VH	VH	B.H & VH	B.H & VH	B.H & VH	B.H & VH	VH
C22	H	B.H & VH	B.M & H	B.H & VH	B.H & VH	H	M
C23	B.H & VH	VH	H	B.H & VH	H	B.M & H	B.M & H
C24	H	B.M & H	M	M	B.H & VH	M	M
C25	VH	VH	H	B.H & VH	B.H & VH	B.H & VH	B.M & H
C26	H	M	H	B.H & VH	B.H & VH	B.H & VH	L
C27	B.M & H	B.M & H	H	H	B.H & VH	H	B.M & H
C28	B.H & VH	B.H & VH	B.M & H	H	B.H & VH	B.M & H	B.H & VH
C29	H	B.M & H	M	H	B.M & H	B.M & H	L
C30	H	B.H & VH	H	B.H & VH	H	H	M

Appendix B

The following figures show the plots of $R_\eta(x_k,x)$ and $R_E(x_k,x)$, $\forall x \in X.k=71,\ldots,130$.

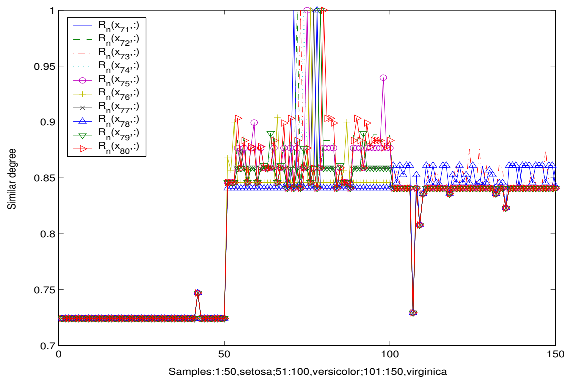


Fig. 7.3 The degrees of similarity relation of x_k to $x \in X$ based on topology, i.e., $R_\eta(x_k,x)$, $k=71,72,\ldots,80$

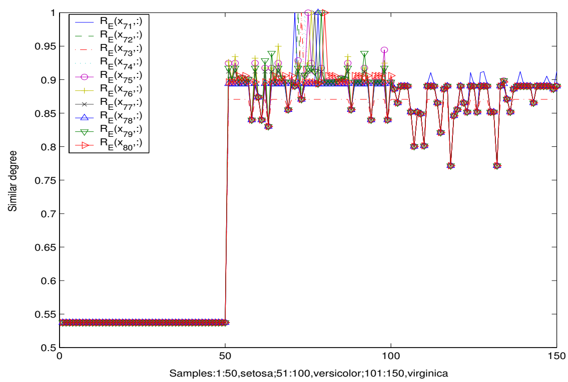


Fig. 7.4 The degrees of similarity relation of x_k to $x \in X$ based Euclidean distance, i. e., $R_E(x_k,x)$, $k=71,72,\ldots,80$

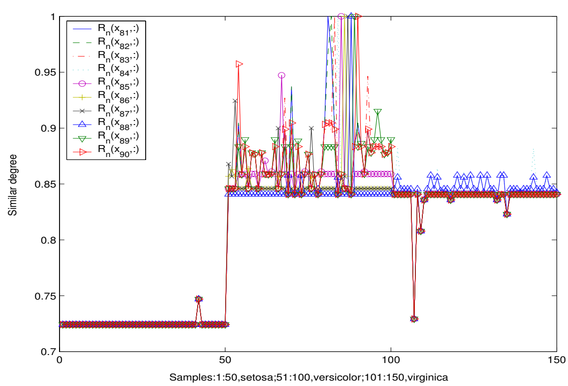


Fig. 7.5 The degrees of similarity relation of x_k to $x \in X$ based on topology, i.e., $R_\eta(x_k,x)$, $k=81,82,\ldots,90$

Fig. 7.6 The degrees of similarity relation of x_k to $x \in X$ based Euclidean distance, i. e., $R_E(x_k, x)$, $k = 81, 82, \ldots, 90$

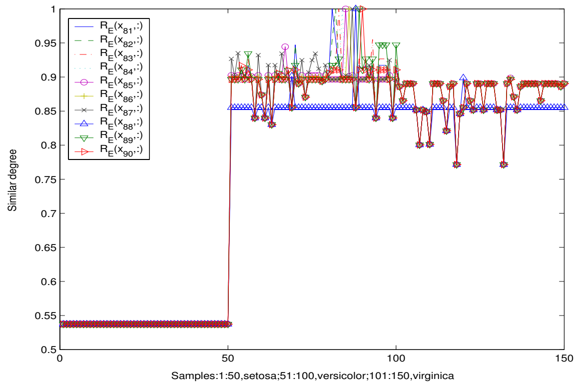


Fig. 7.7 The degrees of similarity relation of x_k to $x \in X$ based on topology, i.e., $R_\eta(x_k, x)$, $k = 91, 92, \ldots, 100$

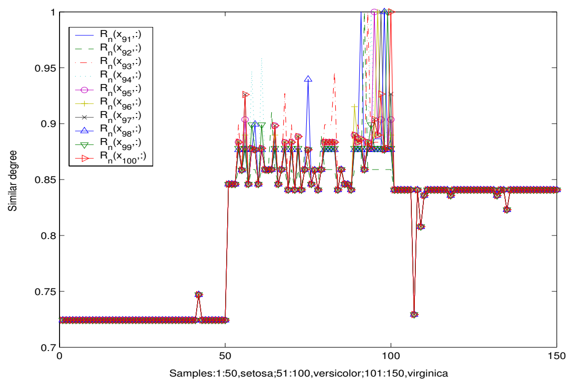
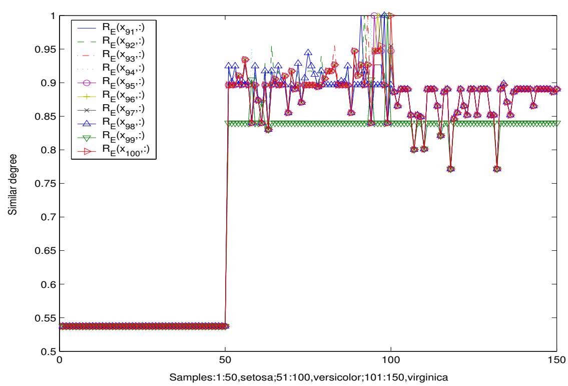


Fig. 7.8 The degrees of similarity relation of x_k to $x \in X$ based Euclidean distance, i. e., $R_E(x_k, x)$, $k = 91, 92, \ldots, 100$



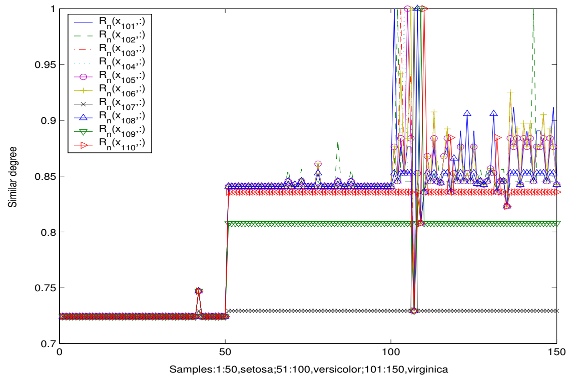


Fig. 7.9 The degrees of similarity relation of x_k to $x \in X$ based on topology, i.e., $R_\eta(x_k, x)$, $k = 101, 102, \ldots, 110$

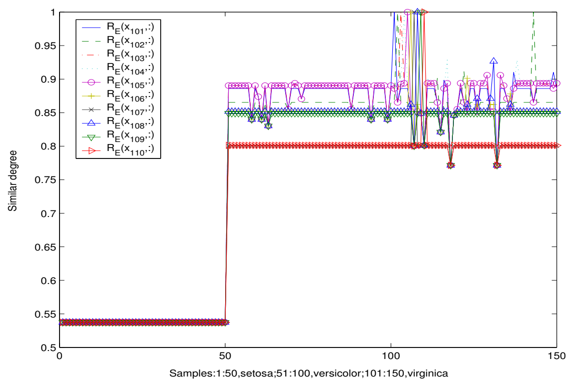


Fig. 7.10 The degrees of similarity relation of x_k to $x \in X$ based Euclidean distance, i. e., $R_E(x_k, x)$, $k = 101, 102, \ldots, 110$

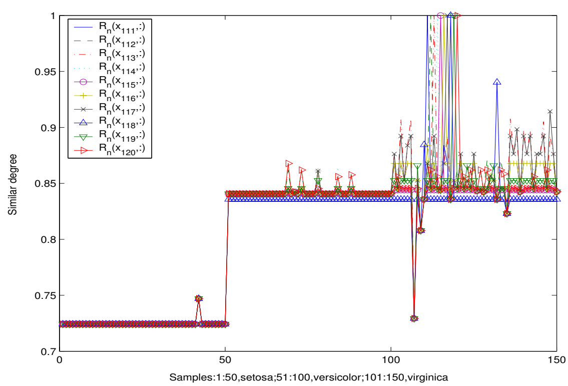


Fig. 7.11 The degrees of similarity relation of x_k to $x \in X$ based on topology, i.e., $R_\eta(x_k, x)$, $k = 111, 112, \ldots, 120$

Fig. 7.12 The degrees of similarity relation of x_k to $x \in X$ based Euclidean distance, i. e., $R_E(x_k, x)$, $k = 111, 112, \ldots, 120$

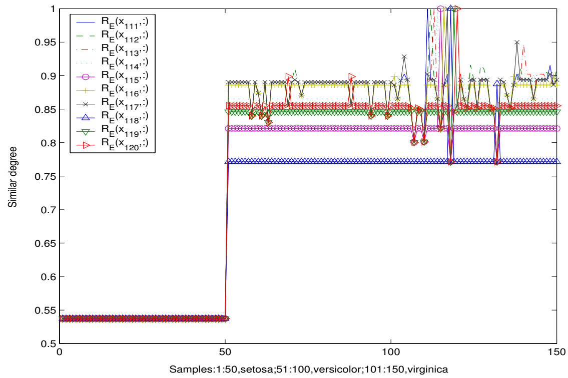


Fig. 7.13 The degrees of similarity relation of x_k to $x \in X$ based on topology, i.e., $R_\eta(x_k, x)$, $k = 121, 122, \ldots, 130$

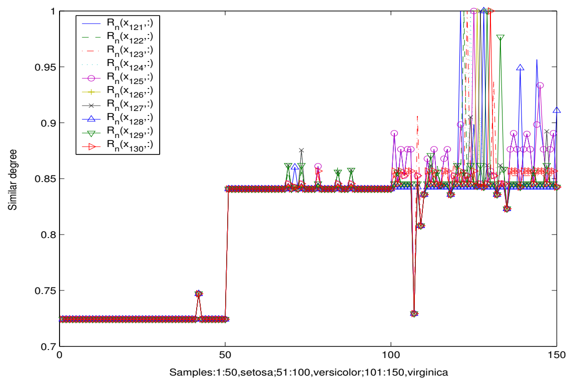
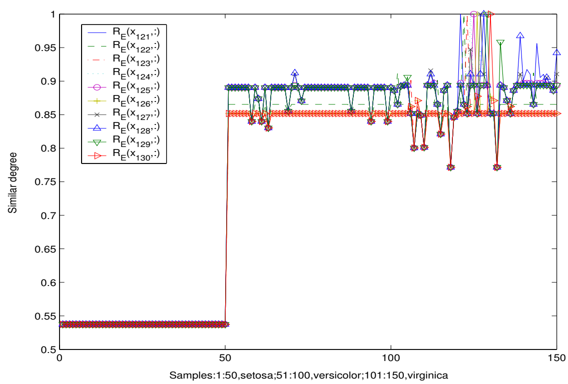


Fig. 7.14 The degrees of similarity relation of x_k to $x \in X$ based Euclidean distance, i. e., $R_E(x_k, x)$, $k = 121, 122, \ldots, 130$



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